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Modular rings and the integer 3

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Abstract. The characteristics within modular rings of the integer three are discussed. This integer has unique row structures in modular rings which appear to underlie restraints on various aspects of triples, particularly the factors and powers structure of the components. The function $N = x^m+2^n$, with m even and n odd but x not divisible by 3, always has 3 as a factor, and a majority of elements of the sequence of triangular numbers $\{N_T\}$ are such that $3|\ N_T$. The modular ring Z_3 and the distribution of primes within its structure are also discussed.

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AMS Subject Classification: 11A41, 11A07

1 Introduction

The integer 3 can be described as historically interesting since it commonly features in mathematical games such as noughts and crosses and three-in-a-row [1]. Pythagorean triples have been studied since antiquity, and 3 is the first odd prime. There is also Bogolmy's asymptotic result [2] that almost every integer has a 3 in it. This was established by showing that K_n, the number of positive integers below 10^n whose decimal numeral contains a digit 3, satisfies the first order recurrence relation

$$K_{n+1} = 9K_n + 10^n,\ n > 0, \tag{1.1}$$

with initial condition $K_1 = 1$, which has a solution

$$K_n = 10^n - 9^n, \tag{1.2}$$

and $K_n/10^n \to 1$ as $n \to \infty$. Obviously, what is true for the digit 3 is also true for other non-zero digits. So, there "are more" numbers which contain a given digit than those that do not [4]. While this is a caveat and a bit of fun it does connect with Section 4 later. In any case there is no ambiguity about the integer 3 in what follows.

More germane to this paper, 3 is significant in integer structure [5], and unlike other integers, the squares of those which are divisible by 3 are elements of the sequence of triangular numbers in the modular ring Z_4, whereas the squares of other integers are elements of the sequence of pentagonal numbers in the same ring [5]. This functional disparity appears to be the critical constraint to the formation of triples when the power is greater than 2 [7].

Furthermore, one of the minor components of primitive Pythagorean triples always has a factor of 3, whereas the major component is not divisible by 3. This is because the integers in Class $\bar{3}_4$ ($3 \in \bar{3}_4$) can never be a sum of squares [5,6,8,9].

2 Modular Ring Z_6

This modular ring (Table 2) has the advantage that integers divisible by 3 are located in one odd $(\bar{6}_6)$ and one even $(\bar{3}_6)$ class. Such classifications facilitate the analysis of functions such as

$$N = x^m + 2^n \tag{2.1}$$

(m even, n odd).

Row	$f(r)$ / Class	$4r_0$ / $\bar{0}_4$	$4r_1+1$ / $\bar{1}_4$	$4r_2+2$ / $\bar{2}_4$	$4r_3+3$ / $\bar{3}_4$
0		0	1	2	3
1		4	5	6	7
2		8	9	10	11
3		12	13	14	15
4		16	17	18	19
5		20	21	22	23
6		24	25	26	27
7		28	29	30	31

Table 1: Rows of Z_4

Row	$f(r)$ / Class	$6r_1-2$ / $\bar{1}_6$	$6r_2-1$ / $\bar{2}_6$	$6r_3$ / $\bar{3}_6$	$6r_4+1$ / $\bar{4}_6$	$6r_5+2$ / $\bar{5}_6$	$6r_6+3$ / $\bar{6}_6$
0		-2	-1	0	1	2	3
1		4	5	6	7	8	9
2		10	11	12	13	14	15
3		16	17	18	19	20	21
4		22	23	24	25	26	27
5		28	29	30	31	32	33
6		34	35	36	37	38	39
7		40	41	42	43	44	45

Table 2: Rows of Z_6

(a) If x is odd but not divisible by 3, then $x^m \in \overline{4}_6$ since $\overline{2}_6$ has no even powers, so that

$$\begin{aligned} N &= 6R_4 + 1 + 2^n \\ &= 6(R_4 + k) + 3 \end{aligned} \tag{2.2}$$

with

$$k = \tfrac{1}{3}\left(2^{n-1} - 1\right)$$

n (odd) and k non-negative integers, k = 0,1,5,21,85… . Hence 3|N because $N \in \overline{6}_6$. k is a repunit of base 4 [11] when for any non-negative integer m:

$$k = \tfrac{1}{3}\left(4^{m-1} - 1\right)$$

(b) If x is even but not divisible by 3, then $x^m \in \overline{1}_6$ since 5_6 has no even powers, so that

$$\begin{aligned} N &= 6R_1 - 2 + 2^n \\ &= 6(R_1 + k) \end{aligned} \tag{2.3}$$

with k as before. Hence 3|N because $N \in \overline{3}_6$.

Note that the 'k sequence' $\{k_n\}$ satisfies the second order recurrence relation

$$k_{n+1} = 5k_n - 4k_{n-1},\ n > 1,$$

with initial conditions $k_1 = 0$, $k_2 = 1$. The k_n are elements of the sequence {n(3n-2)} sometimes called generalized octagonal numbers [10], but $\{k_n\}$ is more famous as the binary sequence {0,1,101,10101,1010101,…}.

3 Powers of 3 in Z_6

The rows R_6 of 3^n (n odd or even) are given by the recurrence relations:

$$R_{6(j+1)} - R_{6(j)} = \begin{cases} 12 \times 9^{(j-1)} & n \text{ is even} \quad (3.1) \\ 4 \times 9^{(j-1)} & n \text{ is odd}. \quad (3.2) \end{cases}$$

4 Triangular Numbers

The triangular numbers, N_T, are given by

$$N_T = \tfrac{1}{2}n(n+1) \tag{4.1}$$

T	1	2	3	4	5	6	7	8	9	10	11
N_T	1	3	6	10	15	21	28	36	45	55	66
3 as a factor		X	X		X	X		X	X		X

Table 3: Triangular numbers divisible by 3

Divisibility occurs for

$$n = 3t + \begin{cases} 2 \\ 3 \end{cases}. \qquad \begin{matrix} (4.2) \\ (4.3) \end{matrix}$$

The remaining triangular numbers (uncrossed in Table 3) have $3 | (N_{T_{n+1}} - N_{T_n})$ with $n = 3t + 1$.

5 Modular Ring Z_3

If we limit ourselves to 3 columns (Table 4), then each class has even and odd integers.

Row	$f(r)$	$3r_0$	$3r_1 + 1$	$3r_2 + 2$
	Class	$\bar{0}_3$	$\bar{1}_3$	$\bar{2}_3$
0		0	1	2
1		3	4	5
2		6	7	8
3		9	10	11
4		12	13	14
5		15	16	17
6		18	19	20
7		21	22	23

Table 4: Rows of Z_3

When 3 divides N, the integers all belong to Class $\bar{0}_3$.

When 3 does not divide N, odd powers and primes belong to Classes $\bar{1}_3, \bar{2}_3$.

When 3 does not divide N^m (m even), N^m belongs to Class $\bar{1}_3$; for example:

$$(3r_2 + 2)^2 = 3(3r_3^2 + 4r_3 + 1) + 1 . \qquad (5.1)$$

In Class $\bar{1}_3$ odd numbers fall in even rows of Table 4, while in Class $\bar{2}_3$ odd numbers fall in odd rows of Table 4.

Rows with the right-end-digit (RED) equal to $8\,(r_1^* = 8)$ in $\bar{1}_3$ cannot contain primes, and rows with $r_2^* = 1$ in $\bar{2}_3$ cannot contain primes. This is because one of the integers in these rows has a factor of 5.

In a sense because $\bar{2}_3$ has no even powers there will be more vacancies for primes and it can be expected that more primes fall in Class $\bar{2}_3$, just as in Z_4 more primes fall in Class $\bar{3}_4$ than $\bar{1}_4$ as $\bar{3}_4$ has no even powers [5]. For example, for the range up to 500 there are 44 primes in $\bar{1}_3$ and 48 in $\bar{2}_3$.

Other modular rings could be similarly investigated, just as the 'Chess' ring, Z_8, has been discussed previously [4,5].

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