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A formula for $\pi(x)$ applied to a result of Koninck-Ivić

We are going to give an approximate formula for $\pi(x)$ which is better than the well known $\pi(x) \sim \frac{x}{\log x}$, or than the more precise formula from [2]: $\pi(x) \sim \frac{x}{\log x - 1}$, meaning that $\pi(x) = \frac{x}{\log x - \alpha(x)}$, where $\lim_{x\to\infty} \alpha(x) = 1$. We will prove

Theorem 1.

$$\pi(x) = \frac{x}{\log x - 1 - \frac{k_1}{\log x} - \frac{k_2}{\log^2 x} - \ldots - \frac{k_n(1+\alpha_n(x))}{\log^n x}},$$

where $k_1, k_2, \ldots, k_n$ are given by the recurrence relation

$$k_n + 1!k_{n-1} + 2!k_{n-2} + \ldots + (n-1)!k_1 = n \cdot n!, \quad n = 1, 2, 3, \ldots$$

and $\lim_{x\to\infty} \alpha_n(x) = 0$.

Proof. The following asymptotic formula

$$\pi(x) = \mathrm{Li}(x) + O(x \exp(-a \log x)^{\alpha}),$$

where a and α are positive constants and $\alpha < \frac{3}{5}$ is well known [3]. Integrating by parts and taking into account that

$$x \exp(-a \log x)^{\alpha} = o(\frac{x}{\log^{n+2} x}),$$

where $n \geq 1$, it follows that

$$\pi(x) = x\left(\frac{1}{\log x} + \frac{1!}{\log^2 x} + \ldots + \frac{n!}{\log^{n+1} x}\right) + O\left(\frac{x}{\log^{n+2} x}\right) \tag{1}$$

We define the constants $k_1, k_2, \ldots, k_n$ by the recurrence

$$k_m + 1!k_{m-1} + 2!k_{m-2} + \ldots + (m-1)!k_1 = (m+1)! - m!,$$

for $m = 1, 2, \ldots, n$. For $y > 0$ we consider

$$f(y) = (\sum_{i=0}^{n} \frac{i!}{y^{i+1}})(y - 1 - \sum_{i=1}^{n} \frac{k_i}{y^i}),$$

and we have

$$f(y) = 1 + \frac{2! - 1! - k_1}{y^2} + \frac{3! - 2! - 1!k_1 - k_2}{y^3} + \ldots + \frac{n! - (n-1)! - k_1(n-2)! - \ldots - k_{n-1}}{y^n} + O(\frac{1}{y^{n+1}})$$

for $y \to \infty$. It follows that $f(y) = 1 + O(\frac{1}{y^{n+1}})$, i.e.

$$\sum_{i=0}^{n} \frac{i!}{y^{i+1}} = \frac{1 + O(\frac{1}{y^{n+1}})}{y - 1 - \sum_{i=1}^{n} \frac{k_i}{y^i}} = \frac{1}{y - 1 - \sum_{i=1}^{n} \frac{k_i}{y^i}} + O(\frac{1}{y^{n+2}}).$$

We denote $y = \log x$, and using the relations of type (1) it follows that

$$\pi(x) = \frac{x}{\log x - 1 - \sum_{i=1}^{n} \frac{k_i}{(\log x)^i}} + O(\frac{x}{\log^{n+2}(x)}) \tag{2}$$

Consider

$$\pi(x) = \frac{x}{\log x - 1 - \frac{k_1}{\log x} - \frac{k_2}{\log^2 x} - \ldots - \frac{k_n(1+\alpha_n(x))}{\log^n x}}.$$

Combining this formula with (2) yields $k_n \alpha_n(x) = O(\frac{1}{\log x})$, from which it follows that $\lim_{x\to\infty} \alpha_n(x) = 0$. □

Remark 2. It can be shown immediately that $k_1 = 1$, $k_2 = 3$, $k_3 = 13$, $k_4 = 71$.

We give now a formula for k_m (although not suitable for a direct computation).

Theorem 3. *The coefficient k_m is given by the relation:*

$$k_m = det \begin{pmatrix} m \cdot m! & 1! & 2! & \cdots & (m-1)! \\ (m-1)\cdot(m-1)! & 0! & 1! & \cdots & (m-2)! \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 2 \cdot 2! & 0 & 0 & \cdots & 1! \\ 1 \cdot 1! & 0 & 0 & \cdots & 0! \end{pmatrix}$$

Proof. The recurrence relations giving the coefficients k_m are:

$$\begin{aligned} k_m + k_{m-1}1! + \ldots + k_1(m-1)! &= m \cdot m! \\ k_{m-1} + \ldots + k_1(m-2)! &= (m-1)\cdot(m-1)! \\ &\cdots \\ k_2 + k_1 1! &= 2 \cdot 2! \\ k_1 &= 1 \cdot 1! \end{aligned}$$

The determinant of this linear system is 1 and the result follows by Cramer's rule. □

As an application of the above results we are going to improve the following approximation, due to J.-M. de Koninck and A. Ivić, [1]:

$$\sum_{n=2}^{[x]} \frac{1}{\pi(n)} = \frac{1}{2}\log^2 x + O(\log x).$$

Using Theorem 1 we are going to prove

Theorem 4.

$$\sum_{n=2}^{[x]} \frac{1}{\pi(n)} = \frac{1}{2}\log^2 x - \log x - \log\log x + O(1).$$

Proof. It is enough to take

$$\pi(x) = \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{k(x)}{\log^2 x}},$$

where $\lim_{x\to\infty} k(x) = 3$, and it follows that

$$\frac{1}{\pi(n)} = \frac{\log n}{n} - \frac{1}{n} - \frac{1}{n\log n} - \frac{k(n)}{n\log^2 n},$$

for $n \geq 2$. Therefore we get that

$$\sum_{n=2}^{[x]} \frac{1}{\pi(n)} = \sum_{n=2}^{[x]} \frac{\log n}{n} - \sum_{n=2}^{[x]} \frac{1}{n} - \sum_{n=2}^{[x]} \frac{1}{n\log n} - \sum_{n=2}^{[x]} \frac{k(n)}{n\log^2 n}.$$

For $x \geq e$, $f(x) = \frac{\log x}{x}$ is decreasing and thus

$$\frac{\log(k+1)}{k+1} \leq \int_k^{k+1} \frac{\log x}{x} dx \leq \frac{\log k}{k},$$

for $k \geq 3$. It follows immediately that

$$\sum_{n=3}^{[x]} \frac{\log n}{n} = \int_{n=3}^{[x]} \frac{\log t}{t} dt + O(\frac{\log x}{x}),$$

and so

$$\sum_{n=2}^{[x]} \frac{\log n}{n} = \frac{1}{2}\log^2 x + O(1).$$

Similar arguments lead us to the relations

$$\sum_{n=2}^{[x]} \frac{1}{n} = \log x + O(1),$$

and

$$\sum_{n=2}^{[x]} \frac{1}{n\log n} = \log\log x + O(1).$$

As there exists $M > 0$ with $|\ k(x)\ | \leq M$, and $\sum_{n=2}^{\infty} \frac{1}{n\log^2 n}$ is convergent, it follows that $\sum_{n=2}^{[x]} \frac{k(n)}{n\log^2 n} = O(1)$, and the proof is complete. □

References

1 J.-M. de Koninck, A. Ivić, *Topics in arithmetical functions*, North-Holland, Amsterdam, New York, Oxford, 1980.

2 J. B. Rosser, L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math., **6** (1962), 64-94.

3 I. M. Vinogradov, *A new estimate for* $\zeta(1+it)$, Izv. Akad. Nauk. SSSR, Ser. Mat. **22** (1958).