Chapter 4

Bernoulli Polynomials

4.1 Bernoulli Numbers

The "generating function" for the *Bernoulli numbers* is

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n. \tag{4.1}$$

That is, we are to expand the left-hand side of this equation in powers of x, i.e., a Taylor series about $x = 0$. The coefficient of x^n in this expansion is $B_n/n!$. Note that we can write the left-hand side of this expression in an alternative form

$$\begin{aligned} \frac{x}{e^x - 1} &= \frac{x}{e^{x/2}\left(e^{x/2} - e^{-x/2}\right)} \\ &= \frac{x\,e^{-x/2}}{2\sinh\frac{x}{2}} \\ &= \frac{x}{2}\frac{\cosh\frac{x}{2} - \sinh\frac{x}{2}}{\sinh\frac{x}{2}} \\ &= \frac{x}{2}\coth\frac{x}{2} - \frac{x}{2}. \end{aligned} \tag{4.2}$$

Note that $\frac{x}{2}\coth\frac{x}{2}$ is an even function of x, while $\frac{x}{2}$ is odd. Therefore we conclude that all but one of the Bernoulli numbers of odd order are zero:

$$B_1 = -\frac{1}{2} \tag{4.3a}$$

$$B_{2k+1} = 0, \quad k = 1, 2, 3, \ldots. \tag{4.3b}$$

By writing $x = iy$ and noting that

$$\coth\frac{iy}{2} = -i\cot\frac{y}{2}, \tag{4.4}$$

we conclude that

$$\frac{iy}{2}\coth\frac{iy}{2} = \frac{y}{2}\cot\frac{y}{2} = \sum_{n=0}^{\infty}\frac{B_{2n}(iy)^{2n}}{(2n)!}, \tag{4.5}$$

or

$$\frac{y}{2}\cot\frac{y}{2} = \sum_{n=0}^{\infty}(-1)^n\frac{B_{2n}}{(2n)!}y^{2n}. \tag{4.6}$$

By straightforward expansion in powers of x we can read off the first few Bernoulli numbers:

$$\begin{aligned}
\frac{x}{2}\coth\frac{x}{2} &= \frac{x}{2}\frac{\cosh\frac{x}{2}}{\sinh\frac{x}{2}} \\
&\approx \frac{x}{2}\frac{1+\frac{1}{2!}\left(\frac{x}{2}\right)^2+\frac{1}{4!}\left(\frac{x}{2}\right)^4+\frac{1}{6!}\left(\frac{x}{2}\right)^6+\frac{1}{8!}\left(\frac{x}{2}\right)^8+\dots}{\frac{x}{2}+\frac{1}{3!}\left(\frac{x}{2}\right)^3+\frac{1}{5!}\left(\frac{x}{2}\right)^5+\frac{1}{7!}\left(\frac{x}{2}\right)^7+\frac{1}{9!}\left(\frac{x}{2}\right)^9+\dots} \\
&\approx \left[1+\frac{1}{2!}\left(\frac{x}{2}\right)^2+\frac{1}{4!}\left(\frac{x}{2}\right)^4+\frac{1}{6!}\left(\frac{x}{2}\right)^6+\frac{1}{8!}\left(\frac{x}{2}\right)^8\right] \\
&\qquad\times\left\{1-\left[\frac{1}{3!}\left(\frac{x}{2}\right)^2+\frac{1}{5!}\left(\frac{x}{2}\right)^4+\frac{1}{7!}\left(\frac{x}{2}\right)^6+\frac{1}{9!}\left(\frac{x}{2}\right)^8\right]\right. \\
&\qquad+\left[\frac{1}{3!}\left(\frac{x}{2}\right)^2+\frac{1}{5!}\left(\frac{x}{2}\right)^4+\frac{1}{7!}\left(\frac{x}{2}\right)^6\right]^2 \\
&\qquad\left.-\left[\frac{1}{3!}\left(\frac{x}{2}\right)^2+\frac{1}{5!}\left(\frac{x}{2}\right)^4\right]^3+\left[\frac{1}{3!}\left(\frac{x}{2}\right)^2\right]^4\right\}+\mathcal{O}(x^{10}) \\
&= \qquad\dots \\
&= 1+\frac{x^2}{2!}\left(\frac{1}{6}\right)+\frac{x^4}{4!}\left(-\frac{1}{30}\right)+\frac{x^6}{6!}\left(\frac{1}{42}\right)+\frac{x^8}{8!}\left(-\frac{1}{30}\right)+\dots.
\end{aligned} \tag{4.7}$$

So by comparison with Eq. (4.1) we find

$$B_0 = 1,\quad B_2 = \frac{1}{6},\quad B_4 = -\frac{1}{30},\quad B_6 = \frac{1}{42},\quad B_8 = -\frac{1}{30}. \tag{4.8}$$

What is the radius of convergence of the series

$$\frac{z}{e^z-1} = -\frac{z}{2}+\sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}z^{2n}? \tag{4.9}$$

Recall that a power series converges everywhere within its circle of convergence, and diverges outside that circle. Since a uniformly convergent series must converge to a continuous function, the power series must converge to a well-behaved function within the circle of convergence. That is, the limit function must have a *singularity* somewhere on the circle of convergence, but must be singularity-free within the circle of convergence. The precise theorem, proved in Chapter

n	B_{2n}	Asymptotic value	Relative error
0	1	-2	300%
1	$\frac{1}{6}$	$\frac{1}{\pi^2}$	39%
2	$-\frac{1}{30}$	$-\frac{3}{\pi^4}$	7.6%
3	$\frac{1}{42}$	$\frac{45}{2\pi^6}$	1.7%
4	$-\frac{1}{30}$	$-\frac{315}{\pi^8}$	0.41%
5	$\frac{5}{66}$	$\frac{14175}{2\pi^{10}}$	0.099%
6	$-\frac{691}{2730}$	$-\frac{467775}{2\pi^{12}}$	0.025%
7	$\frac{7}{6}$	$\frac{42567525}{4\pi^{14}}$	0.0061%
8	$-\frac{3617}{510}$	$-\frac{638512875}{\pi^{16}}$	0.0015%
9	$\frac{43867}{798}$	$\frac{97692469875}{2\pi^{18}}$	0.00038%
10	$-\frac{174611}{330}$	$-\frac{9280784638125}{2\pi^{20}}$	0.0000095%

Table 4.1: The Bernoulli numbers B_{2n} for n from 0 to 10, compared with the asymptotic values (4.12). The last column shows the relative error of the asymptotic estimate. Note that the later rather rapidly approaches the true value.

5, is that the radius of convergence of a power series is the distance from the origin to the nearest singularity of the function the series represents.

In this case, it is clear that the generating function is singular wherever $e^z = 1$, except for $z = 0$. Thus the closest singularities to the real axis occur at $\pm 2\pi i$, so that the radius of convergence is 2π. On the other hand

$$(2\pi)^2 = \rho^2 = \lim_{n\to\infty} \frac{|B_{2n}|}{(2n)!} \frac{[2(n+1)]!}{|B_{2(n+1)}|} = \lim_{n\to\infty} (2n+2)(2n+1) \left| \frac{B_{2n}}{B_{2n+2}} \right|, \quad (4.10)$$

from which we can infer the fact that the Bernoulli numbers grow rapidly with n,

$$|B_{2n}| \sim \frac{(2n)!}{(2\pi)^{2n}}, \quad n \to \infty. \quad (4.11)$$

We cannot deduce the sign or overall constant from this analysis: The true asymptotic behavior of B_{2n} is

$$B_{2n} \sim 2(-1)^{n+1} \frac{(2n)!}{(2\pi)^{2n}}. \quad (4.12)$$

The table shows the relative accuracy of the asymptotic approximation (4.12).

4.2 Bernoulli Polynomials

The *Bernoulli polynomials* are defined by the generating function

$$F(x,s) = \frac{x\,e^{xs}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(s)\frac{x^n}{n!}, \tag{4.13}$$

that is, according to Eq. (2.94),

$$B_n(s) = \left(\frac{\partial}{\partial x}\right)^n F(x,s)\bigg|_{x=0}. \tag{4.14}$$

From the properties of $F(x,s)$ we can deduce all the properties of these polynomials:

1. Note that

$$F(x,0) = \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \tag{4.15}$$

   Therefore, we conclude that the Bernoulli polynomials at zero are equal to the Bernoulli numbers,

$$B_n(0) = B_n. \tag{4.16}$$

2. Next we notice that

$$F(x,1) = \frac{x\,e^x}{e^x - 1} = \frac{x}{1 - e^{-x}} = \frac{-x}{e^{-x} - 1} = F(-x,0), \tag{4.17}$$

   so that by comparing corresponding terms in the generating function expansion, we find

$$B_n(1) = (-1)^n B_n(0) = (-1)^n B_n. \tag{4.18}$$

3. If we differentiate the generating function with respect to its second argument, we obtain the relation

$$\frac{\partial}{\partial s}F(x,s) = \frac{x^2 e^{xs}}{e^x - 1} = \sum_{n=0}^{\infty} B_n'(s)\frac{x^n}{n!}. \tag{4.19}$$

   But obviously

$$\frac{x^2 e^{xs}}{e^x - 1} = xF(x,s) = \sum_{n=0}^{\infty} B_n \frac{x^{n+1}}{n!}, \tag{4.20}$$

   so equating coefficients of $x^n/n!$ we conclude that

$$B_n'(s) = nB_{n-1}(s). \tag{4.21}$$

(Note that $B_0'(s) = 0$ is consistent with this if $B_{-1}(s)$ is finite.)

Again, by direct power series expansion of the generating function we can read off the first few Bernoulli polynomials:

$$\begin{aligned} F(x,s) &\approx x\frac{1 + xs + \frac{1}{2}(xs)^2 + \frac{1}{6}(xs)^3}{x + \frac{1}{2}x^2 + \frac{1}{3!}x^3} \\ &\approx 1 + x\left(s - \frac{1}{2}\right) + x^2\left(\frac{s^2}{2} - \frac{s}{2} - \frac{1}{6} + \frac{1}{4}\right) + \dots, \end{aligned} \tag{4.22}$$

from which we read off

$$B_0(s) = 1, \tag{4.23a}$$

$$B_1(s) = s - \frac{1}{2}, \tag{4.23b}$$

$$B_2(s) = s^2 - s + \frac{1}{6}. \tag{4.23c}$$

By keeping two more terms in the expansion we find

$$B_3(s) = s^3 - \frac{3}{2}s^2 + \frac{1}{2}s, \tag{4.23d}$$

$$B_4(s) = s^4 - 2s^3 + s^2 - \frac{1}{30}. \tag{4.23e}$$

Note that the properties (4.16) and (4.18) are satisfied. Note further we can use the property (4.21) to derive higher Bernoulli polynomials from lower ones. Thus from Eq. (4.23c) we know that

$$B_3'(s) = 3s^2 - 3s + \frac{1}{2}. \tag{4.24}$$

The expression for $B_3(s)$, (4.23d) is recovered, when it is recalled that $B_3 = 0$.

4.3 Euler-Maclaurin Summation Formula

Using the above recursion relation (4.21) we can deduce a very important formula which allows a precise relation between a discrete sum and a continuous integral. First note that since $B_0 = B_0(s) = 1$ we can write

$$\int_0^1 f(x)B_0(x)\,dx = \int_0^1 f(x)\,dx, \tag{4.25}$$

valid for any function f. But now we can integrate by parts using

$$B_1'(x) = B_0(x) : \tag{4.26}$$

$$\int_0^1 f(x)\,dx = \int_0^1 f(x)B_1'(x)\,dx$$

$$
\begin{aligned}
&= f(x)B_1(x)\Big|_{x=0}^{1} - \int_0^1 f'(x)B_1(x)\,dx \\
&= \frac{1}{2}\left[f(1)+f(0)\right] - \int_0^1 f'(x)B_1(x)\,dx. \qquad (4.27)
\end{aligned}
$$

Here, we have used the facts that

$$B_1(0) = B_1 = -\frac{1}{2}, \qquad (4.28a)$$

$$B_1(1) = -B_1 = \frac{1}{2}. \qquad (4.28b)$$

Now we can continue integrating by parts by noting that

$$B_1(x) = \frac{1}{2}B_2'(x), \qquad (4.29)$$

so that

$$
\begin{aligned}
\int_0^1 f(x)\,dx &= \frac{1}{2}\left[f(1)+f(0)\right] - \frac{1}{2}\left[f'(1)B_2(1) - f'(0)B_2(0)\right] \\
&\quad + \frac{1}{2}\int_0^1 f''(x)B_2(x)\,dx \\
&= \frac{1}{2}\left[f(1)+f(0)\right] - \frac{1}{2}B_2\left[f'(1)-f'(0)\right] \\
&\quad + \frac{1}{2}\int_0^1 f''(x)B_2(x)\,dx. \qquad (4.30)
\end{aligned}
$$

A general pattern is emerging. Let us assume the following formula holds for some integer k (we have just proved it for $k = 1$):

$$
\begin{aligned}
\int_0^1 f(x)\,dx &= \frac{1}{2}\left[f(1)+f(0)\right] \\
&\quad - \sum_{m=1}^{k}\frac{B_{2m}}{(2m)!}\left[f^{(2m-1)}(1) - f^{(2m-1)}(0)\right] \\
&\quad + \frac{1}{(2k)!}\int_0^1 f^{(2k)}(x)B_{2k}(x)\,dx. \qquad (4.31)
\end{aligned}
$$

We shall then prove that the same formula holds for $k \to k+1$, thereby establishing this formula, the *Euler-Maclaurin summation formula*, for all k. We proceed as follows. Note that

$$B_{2k}(x) = \frac{B_{2k+1}'(x)}{2k+1} = \frac{B_{2k+2}''(x)}{(2k+1)(2k+2)}, \qquad (4.32)$$

so that by integrating by parts, we rewrite the last term in Eq. (4.31) as

$$\frac{1}{(2k)!}\int_0^1 f^{(2k)}(x)\frac{1}{(2k+1)(2k+2)}B_{2k+2}''(x)\,dx$$

$$
\begin{aligned}
&= \frac{1}{(2k+2)!}\Bigg[f^{(2k)}(1)B'_{2k+2}(1) - f^{(2k)}(0)B'_{2k+2}(0) \\
&\qquad - \int_0^1 f^{(2k+1)}(x)B'_{2k+2}(x)\,dx\Bigg] \\
&= \frac{1}{(2k+2)!}\Bigg[-f^{(2k+1)}(1)B_{2k+2}(1) + f^{(2k+1)}(0)B_{2k+2}(0) \\
&\qquad + \int_0^1 f^{(2k+2)}(x)B_{2k+2}(x)\,dx\Bigg], \qquad (4.33)
\end{aligned}
$$

where we have noted that for $k > 0$

$$
\begin{aligned}
B'_{2k+2}(0) &= (2k+2)B_{2k+1}(0) = 0, \qquad (4.34a) \\
B'_{2k+2}(1) &= (2k+2)B_{2k+1}(1) = -(2k+2)B_{2k+1}(0) = 0. \qquad (4.34b)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_0^1 f(x)\,dx &= \frac{1}{2}[f(1)+f(0)] \\
&\quad - \sum_{m=1}^{k+1}\frac{B_{2m}}{(2m)!}\left[f^{(2m-1)}(1) - f^{(2m-1)}(0)\right] \\
&\quad + \frac{1}{(2k+2)!}\int_0^1 f^{(2k+2)}(x)B_{2k+2}(x)\,dx. \qquad (4.35)
\end{aligned}
$$

This is exactly Eq. (4.31) with k replaced by $k+1$; so since the formula is true for $k = 1$ it is true for all integers $k \geq 1$. Notice that the last term in this formula, the remainder, can also be written in the form

$$
-\frac{1}{(2k+3)!}\int_0^1 f^{(2k+3)}(x)B_{2k+3}(x)\,dx. \qquad (4.36)
$$

Now consider the integral (N a positive integer)

$$
\int_0^N f(s)\,ds = \sum_{k=0}^{N-1}\int_k^{k+1} f(s)\,ds = \sum_{k=0}^{N-1}\int_0^1 f(k+t)\,dt, \qquad (4.37)
$$

where we have introduced a local variable t. For the latter integral, we can use the Euler-Maclaurin sum formula, which here reads

$$
\begin{aligned}
\int_0^1 f(k+t)\,dt &= \frac{1}{2}[f(k+1)+f(k)] \\
&\quad - \sum_{m=1}^{n}\frac{B_{2m}}{(2m)!}\left[f^{(2m-1)}(k+1) - f^{(2m-1)}(k)\right] \\
&\quad + \frac{1}{(2n)!}\int_0^1 f^{(2n)}(k+t)B_{2n}(t)\,dt. \qquad (4.38)
\end{aligned}
$$

Now when we sum the first term here on the right-hand side over k we obtain

$$\sum_{k=0}^{N-1} \frac{1}{2}[f(k+1)+f(k)] = \sum_{k=0}^{N} f(k) - \frac{1}{2}[f(0)+f(N)], \tag{4.39}$$

while the second term when summed on k involves

$$\sum_{k=0}^{N-1} \left[f^{(2m-1)}(k+1) - f^{(2m-1)}(k)\right] = f^{(2m-1)}(N) - f^{(2m-1)}(0). \tag{4.40}$$

Thus we find

$$\begin{aligned}\int_0^N f(s)\,ds &= \sum_{k=0}^{N} f(k) - \frac{1}{2}[f(0)+f(N)] \\ &\quad - \sum_{m=1}^{n} \frac{1}{(2m)!} B_{2m} \left[f^{(2m-1)}(N) - f^{(2m-1)}(0)\right] \\ &\quad + \frac{1}{(2n)!}\int_0^1 \sum_{k=0}^{N-1} f^{(2n)}(t+k) B_{2n}(t)\,dt.\end{aligned} \tag{4.41}$$

Equivalently, we can write this as a relation between a finite sum and an integral, with a remainder R_n:

$$\begin{aligned}\sum_{k=0}^{N} f(k) &= \int_0^N f(s)\,ds + \frac{1}{2}[f(0)+f(N)] \\ &\quad + \sum_{m=1}^{n} \frac{1}{(2m)!} B_{2m} \left[f^{(2m-1)}(N) - f^{(2m-1)}(0)\right] + R_n,\end{aligned} \tag{4.42}$$

where the remainder

$$R_n = -\frac{1}{(2n)!}\int_0^1 \sum_{k=0}^{N-1} f^{(2n)}(t+k) B_{2n}(t)\,dt. \tag{4.43}$$

is often assumed to vanish as $n \to \infty$. Note that the remainder can also be written as

$$R_n = -\frac{1}{(2n)!}\int_0^N f^{(2n)}(t) B_{2n}(t - \lfloor t \rfloor)\;dt, \tag{4.44}$$

where $\lfloor t \rfloor$ signifies the greatest integer less than or equal to t.

4.3.1 Examples

1. Use the Euler-Maclaurin formula to evaluate the sum $\sum_{n=0}^{N} \cos(2\pi n/N)$.

$$\sum_{n=0}^{N} \cos\frac{2\pi n}{N} = \int_0^N dn \cos\frac{2\pi n}{N} + \frac{1}{2}(1+1) + 0 = 1, \tag{4.45}$$

because

$$f^{(2m-1)}(0) = f^{(2m-1)}(N) = 0 \tag{4.46}$$

and

$$\int_0^N dn\,\cos\frac{2n\pi}{N} = \frac{N}{2\pi}\int_0^{2\pi} dx\,\cos x = 0. \tag{4.47}$$

Of course, the sum may be carried out directly,

$$\begin{aligned}\sum_{n=0}^{N}\cos\frac{2\pi n}{N} &= \frac{1}{2}\sum_{0}^{N}\left(e^{i2\pi n/N} + e^{-i2\pi n/N}\right)\\ &= \frac{1}{2}\left[\frac{1-e^{2\pi i(N+1)/N}}{1-e^{2\pi i/N}} + \frac{1-e^{-2\pi i(N+1)/N}}{1-e^{-2\pi i/N}}\right]\\ &= \frac{1}{2}(1+1) = 1.\end{aligned} \tag{4.48}$$

2. The following sum occurs, for example, in computing the vacuum energy in a cosmological model:

$$\sum_{l=0}^{\infty}(2l+1)e^{-l(l+1)t}. \tag{4.49}$$

How does this behave as $t \to 0$? We will answer this question by using the Euler-Maclaurin formula assuming that the remainder R_n tends to zero as $n \to \infty$. Thus we will write the limiting form of that sum formula as

$$\begin{aligned}\sum_{l=0}^{\infty} f(l) &= \int_0^\infty dl\, f(l) + \frac{1}{2}[f(\infty) + f(0)]\\ &\quad + \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(\infty) - f^{(2k-1)}(0)\right].\end{aligned} \tag{4.50}$$

Here

$$f(l) = (2l+1)e^{-l(l+1)t}, \tag{4.51}$$

so that

$$f(\infty) = f^{(2k-1)}(\infty) = 0, \tag{4.52}$$

while a very simple calculation shows

$$\begin{aligned}f(0) &= 1, &\text{(4.53a)}\\ f'(0) &= 2-t, &\text{(4.53b)}\\ f'''(0) &= -12t + 12t^2 - t^3, &\text{(4.53c)}\\ f^{(5)}(0) &= 120t^2 - 180t^3 + 30t^4 - t^5, &\text{(4.53d)}\\ f^{(7)}(0) &= -1680t^3 + 3360t^4 - 840t^5 + 56t^6 - t^7, &\text{(4.53e)}\\ f^{(2k-1)}(0) &= \mathcal{O}(t^4), \quad k \ge 5. &\text{(4.53f)}\end{aligned}$$

Thus Eq. (4.50) yields

$$\begin{aligned}
\sum_{l=0}^{\infty}(2l+1)e^{-l(l+1)t} &= \int_0^\infty dl\,(2l+1)e^{-l(l+1)t} + \frac{1}{2} \\
&\quad - \frac{B_2}{2}f'(0) - \frac{B_4}{4!}f'''(0) - \dots \\
&= \frac{1}{t}\int_0^\infty du\, e^{-u} + \frac{1}{2} + \frac{1}{2}\left(\frac{1}{6}\right)(t-2) \\
&\quad + \frac{1}{4!}\left(-\frac{1}{30}\right)[12t + \mathcal{O}(t^2)] + \mathcal{O}(t^2) \\
&= \frac{1}{t} + \frac{1}{3} + \frac{t}{15} + \frac{4}{315}t^2 + \frac{1}{315}t^3 + \dots . \qquad (4.54)
\end{aligned}$$

Here the integral was evaluated by making the substitution $u = l(l+1)t$, $du = (2l+1)t\,dl$, and in the last line we have displayed the next two terms in this *asymptotic expansion* for small t.

3. The Riemann zeta function (2.50) is defined by

$$\zeta(\alpha) = \sum_{n=1}^{\infty}\frac{1}{n^\alpha}, \quad \operatorname{Re}\alpha > 1. \qquad (4.55)$$

Suppose we approximate this by the first M terms in the sum occurring in the Euler-Maclaurin formula (4.42):

$$\zeta(\alpha, M) = \frac{1}{\alpha-1} + \frac{1}{2} - \sum_{m=1}^{M}\frac{B_{2m}}{(2m)!}f^{(2m-1)}(1), \qquad (4.56)$$

where $f(n) = n^{-\alpha}$, and the first two terms here come from the integral and the $\frac{1}{2}f(1)$ terms in the EM formula. It is easy to see that

$$f^{(2m-1)}(1) = -\frac{\Gamma(\alpha+2m-1)}{\Gamma(\alpha)}. \qquad (4.57)$$

Given the asymptotic behavior of the Bernoulli numbers in (4.12), it is apparent that the limit $M \to \infty$ of $\zeta(\alpha, M)$ does not exist. This limit is an example of an asymptotic series. However, in Table 4.2 we compare the sum of the first N terms of the series in (4.55) with the first N terms in the series defined by (4.56), that is $\zeta(\alpha, N)$, for $\alpha = 3$, where $\zeta(3) = 1.2020569$. The original series converges monotonically to the correct limiting value, but not spectacularly fast. For $N = 9$ terms, the relative error is about -0.5%. The asymptotic series is divergent; however, the $N = 1$ term is in error by only 4%, and the average of the $N = 1$ and $N = 2$ is larger than the true value by only $+0.5\%$. This illustrates a characteristic feature of asymptotic series: A few terms in the series approximates the function rather well, but as more and more terms are included the series deviates from the true value by an ever increasing amount.

N	$\sum_{n=1}^{N} n^{-3}$	$\zeta(3,N)$	$\frac{1}{2}[\zeta(3,N)+\zeta(3,N+1)]$
1	1	1.25	1.208
2	1.125	1.1667	1.208
3	1.1620	1.25	1.175
4	1.1777	1.1	1.308
5	1.1857	1.5167	0.694
6	1.1903	−0.1286	
7	1.1932	8.6214	
8	1.1952	−51.6619	
9	1.1965	470.564	

Table 4.2: Two approximations compared for $\zeta(3) = 1.20206\ldots$: N terms in the defining series (4.55) and N terms (without the remainder) in the Euler-Maclaurin sum (4.56). The former converges monotonically to the limit from below, while the later diverges, yet approximates the true value to better than 1% for low values of N.