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# Gompertz Constant, Gregory Coefficients and a Series of the Logarithm Function

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**Abstract:** The Gompertz constant appears in the evaluation of several improper integrals and infinite series. In the present paper we give a new series representation of this constant.

**Keywords:** Gompertz constant, Gregory coefficients, harmonic numbers, digamma function, incomplete gamma function

## 1 Introduction

Our goal is to prove the following identity:

$$G=\sum_{n=0}^{\infty}\frac{\ln(n+1)}{n!}-\sum_{n=0}^{\infty}C_{n+1}\{e\cdot n!\}-\frac{1}{2}.$$

Here $G=0.596347362323194\ldots$ is the *Gompertz constant*, $C_n$ is the sequence of the *Gregory coefficients*, and $\{x\}$ denotes the *fractional part* of $x$.

## 2 Preliminaries

The Gompertz constant [5] appears as a value of several improper integrals, like

$$G=\int_0^\infty \ln(1+x)e^{-x}dx=\int_0^\infty\frac{e^{-x}}{1+x}dx, \tag{1}$$

and as a product of the Napier constant $e$ and the *exponential integral* $\mathrm{Ei}(x)$ at $x=-1$. More precisely, $\mathrm{Ei}(x)$ is defined as

$$\mathrm{Ei}(x)=-\int_{-x}^\infty\frac{e^{-t}}{t}dt,$$

and then $G=-e\,\mathrm{Ei}(-1)$, that is, $G$ also equals to

$$e\int_1^\infty\frac{e^{-t}}{t}dt \tag{2}$$

(which is just a slight modification of the integral on the far right side of (1)). A series representation of $G$ is also known:

$$G=e\sum_{n=1}^\infty\frac{(-1)^{n+1}}{n!n}-e\gamma.$$

Here $\gamma=0.577215664901533\ldots$ is the *Euler constant* [4] (sometimes called as *Euler-Mascheroni constant*).

We also note that $G$ can be expressed by the *incomplete gamma function*

$$\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt:$$

$G=e\cdot\Gamma(0,1)$, which follows from (2). This latter representation helps to find a continued fraction representation for $G$. Namely, it is known [13] that

$$\Gamma(0,x)=\cfrac{e^{-x}}{x+1-\cfrac{1}{x+3-\cfrac{4}{x+5-\cfrac{9}{x+7-\ddots}}}},$$

from which the next expression comes:

$$G=\cfrac{1}{2-\cfrac{1}{4-\cfrac{4}{6-\cfrac{9}{8-\ddots}}}}.$$

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(The nominators are the squares of the positive integers.)

In the present paper we would like to increase the number of the existing representations of the Gompertz constant with the new identity in the Introduction. To reach this aim, we need to introduce the Gregory coefficients. These are the Taylor series coefficients of the function $\frac{x}{\ln(1-x)}$:

$$\frac{x}{\ln(1-x)} = \sum_{n=0}^{\infty} C_n x^n \quad (|x|<1). \tag{3}$$

(If $x=0$ we can use the convention $0^0=1$ to match the right hand side with the limit on the left.)

The coefficients $C_n$ were first studied by James Gregory, who calculated the first several terms in 1670. This sequence is important in the theory of numerical integration (via Gregory's formula, see [1,10]). A table of the first terms of the sequence $C_n$ is presented here:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $C_n$ | $-1$ | $\frac{1}{2}$ | $\frac{1}{12}$ | $\frac{1}{24}$ | $\frac{19}{720}$ | $\frac{3}{160}$ | $\frac{863}{60480}$ | $\frac{275}{24192}$ |

For these coefficients an integral formula is known:

$$C_n = \frac{1}{n!}\left|\int_0^1 t(t-1)\cdots(t-n+1)dt\right| \quad (n=1,2,\dots).$$

There are various names of the sequence $C_n$ and of its variants. For example, the Gregory coefficients are also called the *logarithmic numbers* – because of the generating function (3). Moreover, the numbers $C_n\cdot n!$ are called the *non-alternating Cauchy numbers* [2] and the numbers $(-1)^{n+1}C_n\cdot n!$ are the *Bernoulli numbers of the second kind* or *Cauchy numbers (of the first kind)* [11, p. 114]. For the Gregory coefficients some interesting identities are known. For example [2],

$$\gamma = \sum_{n=1}^{\infty}\frac{C_n}{n}.$$

See also [9] for some additional identities.

Finally, we remark that the Gregory coefficients are strongly related to Nørlund polynomials [3]. These polynomials are defined by the exponential generating function

$$\sum_{n=0}^{\infty} B_n^{(a)}\frac{x^n}{n!} = \left(\frac{x}{e^x-1}\right)^a.$$

Since the diagonal generating function of $B_n^{(n)}$ reads as

$$\sum_{n=0}^{\infty} B_n^{(n)}\frac{x^n}{n!} = \frac{x}{(x+1)\ln(x+1)},$$

it immediately comes that

$$C_n = (-1)^{n+1}\left(\frac{B_{n-1}^{(n-1)}}{(n-1)!}+\frac{B_n^{(n)}}{n!}\right) \quad (n\ge 1).$$

(The above generating function of $B_n^{(n)}$ comes from a result of H. W. Gould [6, Section 3.]

## 3 The proof

The formula we begin with is an integral representation of the *digamma function* [7]

$$\psi(x) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+x}\right),$$

where $x\in\mathbb{R}\setminus\{0,-1,-2,\dots\}$. Namely, for $n=1,2,\dots$

$$\psi(n)-\ln n = \int_0^\infty\left(\frac{1}{1-e^x}+\frac{1}{x}-1\right)e^{-nx}dx. \tag{4}$$

At positive integers the digamma function can be represented as a simple finite sum [7]:

$$\psi(n) = -\gamma+1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}.$$

The sum excluding the term $-\gamma$ is the $(n-1)$-th *harmonic number* denoted by $H_{n-1}$:

$$H_{n-1} = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}.$$

With this we can rewrite (4) as

$$\frac{H_n}{n!}-\frac{\gamma}{n!}-\frac{\ln(n+1)}{n!} =$$

$$\int_0^\infty\left(\frac{1}{(1-e^x)e^x}+\frac{1}{xe^x}-\frac{1}{e^x}\right)\frac{e^{-nx}}{n!}dx.$$

Summing over $n=0,1,2,\dots$ we have

$$\sum_{n=0}^{\infty}\frac{H_n}{n!}-e\gamma-\sum_{n=0}^{\infty}\frac{\ln(n+1)}{n!} =$$

$$\int_0^\infty\left(\frac{1}{(1-e^x)e^x}+\frac{1}{xe^x}-\frac{1}{e^x}\right)e^{e^{-x}}dx.$$

The first sum on the left is well known and due to R. W. Gosper [8]:

$$\sum_{n=0}^{\infty}\frac{H_n}{n!} = e\gamma - e\,\mathrm{Ei}(-1) = e\gamma+G.$$

For the second sum we could not find a simple closed form expression neither in the literature nor by ourselves. However, as the reviewer kindly noted to us, this sum was discussed on a private math forum. According to this forum – as Wm. Cordwell noted –, this constant, multiplied by an integer $n$ and divided by $\log(2)$, is the expected information loss from a random map on the set $\{1,\dots,n\}$.

We continue with the evaluation of the integral on the right.

The most simple case is the term

$$-\int_0^\infty\frac{e^{e^{-x}}}{e^x}dx.$$

which equals to $1-e$ (since the primitive function of $e^{e^{-x}}/e^x$ is $-e^{e^{-x}}$). Hence, at this point we have

$$G=\sum_{n=0}^{\infty}\frac{\ln(n+1)}{n!}+1-e+\int_0^{\infty}\left(\frac{1}{(1-e^x)e^x}+\frac{1}{xe^x}\right)e^{e^{-x}}dx. \tag{5}$$

Now, we deal with the remaining improper integral. The substitutions $x\mapsto -x$ and then $x\mapsto \ln x$ leads to the equality

$$\int_0^{\infty}\left(\frac{1}{(1-e^x)e^x}+\frac{1}{xe^x}\right)e^{e^{-x}}dx=-\int_0^1 e^x\left(\frac{1}{\ln x}+\frac{x}{1-x}\right)dx.$$

The two terms cannot be separated, becaue they are divergent. The reason of the divergence of the integrals

$$\int_0^1 e^x\frac{1}{\ln x}\quad\text{and}\quad\int_0^1 e^x\frac{x}{1-x}dx$$

can be seen if we consider the Laurent series of $\frac{1}{\ln x}$ around $x=1$. Namely, by using (3):

$$\frac{1}{\ln x}=\frac{1}{x-1}+\sum_{n=0}^{\infty}C_{n+1}(-1)^n(x-1)^n,$$

which holds in the interval $x\in]0,2[$. On this open interval the series on the right is absolutely convergent, so integration term by term is possible. The only one problem arises when we try to integrate $1/(x-1)$, but this singularity is cancelled by the other term $x/(1-x)$, because of the trivial equality $1/(x-1)+x/(1-x)=-1$. Therefore the next simplification is possible:

$$-\int_0^1 e^x\left(\frac{1}{\ln x}+\frac{x}{1-x}\right)dx=$$

$$e-1-\sum_{n=0}^{\infty}C_{n+1}(-1)^n\int_0^1 e^x(x-1)^n.$$

At the same time, employing (5), we obtain an intermediate formula for $G$:

$$G=\sum_{n=0}^{\infty}\frac{\ln(n+1)}{n!}-\sum_{n=0}^{\infty}C_{n+1}(-1)^n\int_0^1 e^x(x-1)^n. \tag{6}$$

Now we use the binomial theorem to determine the integral. Easy induction shows that

$$\int_0^1 e^x(x-1)^n=\int_0^1\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}e^x x^k=$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\left((-1)^{k+1}k!+(-1)^k\cdot e\cdot D_k\right),$$

where $D_k$ is the $k$th *derangement number* which is equal to

$$D_k=\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}l!.$$

See [12, p. 199] for other properties of these significant combinatorial numbers. What is important for us is that

$$\sum_{k=0}^{n}\binom{n}{k}D_k=n!\quad(n\geq 0)$$

(see (2.9) and (2.10) of [12, p. 198] for the general formulas on binomial transformation.) Hence

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\left((-1)^{k+1}k!+(-1)^k\cdot e\cdot D_k\right)=$$

$$(-1)^{n+1}\sum_{k=0}^{n}\binom{n}{k}k!+(-1)^n\cdot e\cdot n!,$$

Moreover, we utilize the simple fact that

$$\sum_{k=0}^{n}\binom{n}{k}k!=\lfloor e\cdot n!\rfloor\quad(n\geq 1).$$

By using these, we have for all $n\geq 1$ that

$$\int_0^1 e^x(x-1)^n=(-1)^n(e\cdot n!-\lfloor e\cdot n!\rfloor)=(-1)^n\{e\cdot n!\}.$$

If $n=0$, however, we have that $\int_0^1 e^x dx=e-1$, and $C_1=\frac{1}{2}$. Since $C_1(e-1)$ is half more than $C_1\{e\cdot 0!\}$, we have to add $\frac{1}{2}$ to match this case to the general term $C_{n+1}\{e\cdot n!\}$ of the sum.

Our main formula has been proved after a straightforward substitution of the value of the integral.

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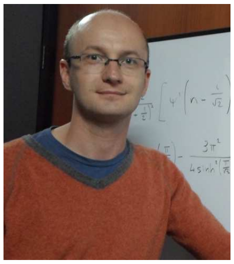

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