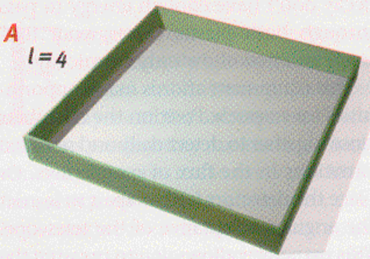
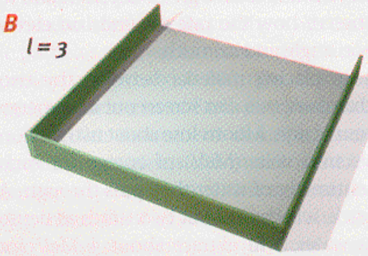
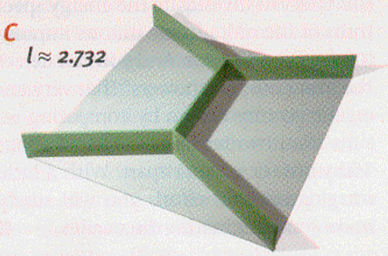
GEOMETRY_OPAQUE FENCES

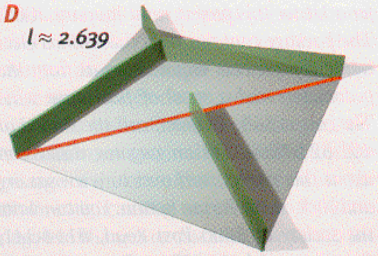
Pursuing Polygonal Privacy

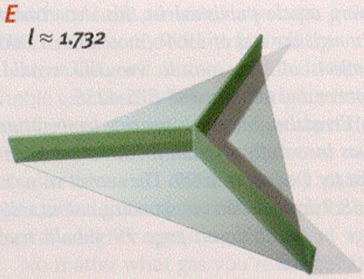
Ian Stewart proves that good fences make good neighbors











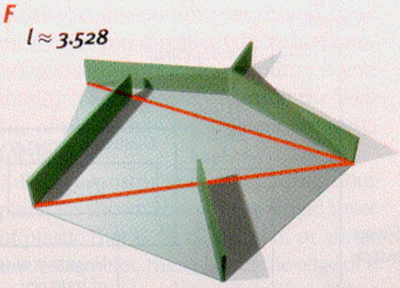
Combinatorial geometry is one of the most appealing areas of mathematics, full of simple problems whose solutions are unknown. The aim of these problems is to find arrangements of lines, curves or other geometric shapes that achieve some objective in the most efficient manner. This month I want to concentrate on a puzzle known as the Opaque Square Problem, along with several fascinating variations. Bernd Kawohl of the University of Cologne in Germany brought the puzzle to my attention, and my discussion is based on an article he sent me.

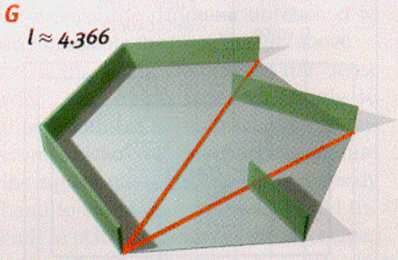
Suppose you own a square plot of land whose sides, for the sake of simplicity, are each one mile long. To ensure your privacy, you want to build an opaque fence—a barrier that will block any straight line of sight passing through the square plot. Moreover, to save money, you want the fence to be as short as possible. How should you build it? The fence can be as complicated as you like, with lots of different pieces that can be curved or straight.

Perhaps the most obvious solution is to build a fence around the perimeter of the square plot, with a total length of four miles [*see illustration A at left*]. A few moments' thought reveals an improvement: leave out one side to create a square-cornered U shape [*see illustration B*]. Now the length reduces to three miles. This is, in fact, the shortest fence possible if we impose the additional condition that the fence must be a single polygonal or curved line. Why? Because every opaque fence must contain all four corners of the square, and the three-sided U is the shortest single curve that contains all the corners.

We can build a shorter fence, however, that consists of more than one curve. Illustration C shows a fence with a length of $1 + \sqrt{3}$ (about 2.732) miles. The angles between the lines are all 120 degrees. Arrangements of this kind are called Steiner trees; the 120-degree angles minimize the length of the tree. This is the shortest fence in which the curves are connected. If we allow the fence to have several disconnected pieces, the total length can be reduced to about 2.639 miles [*see illustration D*]. The three lines in the upper half of the diagram also meet at angles of 120 degrees. This last example is widely believed to be the shortest opaque fence for a square plot, but nobody has proved this yet.

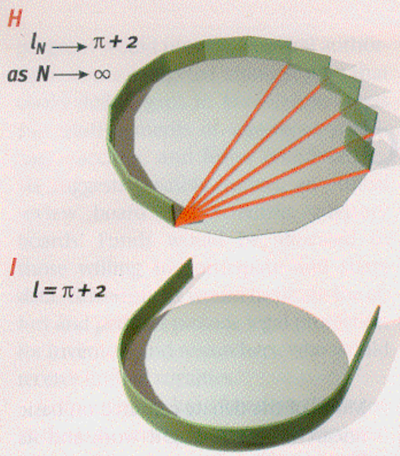
Indeed, mathematicians are not sure whether a shortest opaque fence exists. It may be possible to keep shortening the length by making the fence more and more complicated. For any given number of connected components, it has been proved that a shortest opaque fence does exist. What is not known is whether the minimal length keeps shrinking as the number of components increases without limit or whether a fence with an infi-





OPAQUE FENCES are barriers that block any straight line of sight passing through a given figure. For a square, a perimeter fence (*A*) and a three-sided U shape (*B*) are opaque, but a Steiner tree (*C*) and a two-component fence (*D*) are shorter. The shortest opaque fence for an equilateral triangle is also a Steiner tree (*E*). The best-known opaque fences for the regular pentagon (*F*) and hexagon (*G*) each have three components. All fence lengths (*l*) are approximate except those for *A* and *B*.

ALL ILLUSTRATIONS BY BRYAN CHRISTIE



EVEN-SIDED POLYGON with many sides has an opaque fence with many components (*H*). Their combined length approximates the length of the shortest single-curve fence for a circle (*I*).

nite number of components can outperform all fences with a finite number of components. These possibilities seem unlikely, but neither has been ruled out.

Kawohl has provided a lovely proof that illustration D on the opposite page is the shortest fence having exactly two components. He shows that one component must contain three corners of the square and that the other must contain the remaining corner. The first component must therefore be the shortest Steiner tree linking three corners, which is the shape shown in the upper part of the figure. The convex hull of this shape—the smallest convex region that contains it—is the triangle formed by cutting the square in two along a diagonal. The second component must be the shortest curve that joins the fourth corner to this triangle: the diagonal line from that corner to the center of the square.

What about shapes other than the square? If the plot of land is an equilateral triangle, the shortest opaque fence is a Steiner tree formed by joining each corner to the center along a straight line [*see illustration E*]. If the plot is a regular pentagon, the best-known opaque fence comes in three pieces [*see illustration F*]. One piece of the fence is a Steiner tree linking three adjacent corners of the pentagon. The second piece is a straight line joining the fourth corner to the convex hull of the Steiner tree. The third piece is a straight line joining the fifth corner to the convex hull of the four other corners. Nobody has proved that this fence has a minimal length, but no shorter opaque fence has been found.

The best-known fence for the regular hexagon is similar [*see illustration G*]. Because the corner angles of the hexagon are 120 degrees, the Steiner tree consists of three consecutive sides of the figure itself, linking four adjacent corners. The second component of the fence is the shortest line joining a fifth corner to the convex hull of the Steiner tree, and the third component is the shortest line joining the sixth corner to the convex hull of the five other corners. Again, no one has proved that this fence has a minimal length.

You can use the same type of construction to draw a conjectured minimal fence for any regular polygon with an even number of sides [*see illustration H at left*]. Simply divide the polygon in two by a diameter joining two opposite corners. The first component of the fence is formed from all the edges that lie in that half, forming the polygonal analogue of a semicircle. The second component is the shortest line linking the next corner to the convex hull of the first component. The third component is the shortest line linking the next corner to the convex hull of the first two components, and so on.

A regular polygon with a large number of sides is very close to a circle. What is the shortest fence that makes a circle opaque? For simplicity, suppose that the circle has a radius of one mile. The simplest fence that comes to mind is the circumference of the circle, which has a length of 2π (about 6.283) miles. We can do better, however, if the fence is permitted to lie outside the circular plot. Run the fence along half the circumference, creating a semicircle, and extend it by adding two one-mile lines that are tangent to the circle at the ends of the semicircle [*see illustration I*]. The resulting U shape is an opaque fence for the circle, with a length of $\pi + 2$ (about 5.142) miles.

It can be proved that this figure is the shortest opaque fence if we insist that it be a single curve—all in one piece and with no branching points. Another way to describe the problem is to think of trenches instead of fences. Imagine that a straight underground pipe is known to pass within a mile of some specific point. What is the shortest trench we can dig that is guaranteed to find the pipe? We know that the pipe must cross a circle with a one-mile radius centered at that point and must therefore hit any opaque fence for that circle. So we should dig a trench in the form of an opaque fence.

In this version of the puzzle, it is natural to allow the trench to go outside the circle, but fences are typically built on the owner's land rather than on the neighbors'. Kawohl shows that the shortest opaque fence lying entirely inside the circle also cannot be longer than $\pi + 2$ miles. He does this by considering the conjectured fence for an even-sided polygon with a very large number of sides, thus closely approximating the circle. A trigonometric calculation proves that the length of the fence shown in illustration H approaches $\pi + 2$ as the number of sides increases without limit.

But are the conjectured fences truly the shortest, or is there a way to shorten them further? What about other shapes, such as irregular polygons (convex or not), ellipses and semicircles? And what about the same problem in three dimensions (the opaque cube and sphere)? Recreational mathematicians have much to investigate. ■

READER_FEEDBACK

Several readers objected to a calculation I did in the column on logical fractals ["A Fractal Guide to Tic-Tac-Toe," August 2000]. I stated that the number of possible games of tic-tac-toe is 362,880. I should have made it clear that this number is correct only under the assumption that the game continues until all the squares in the grid are filled, rather than stopping when someone wins. The total number of sequences leading to a completed grid is $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ (denoted as 9!), which equals 362,880.

But what is the number of actual games? John Stewart of Rockledge, Fla., pointed out that the number can be expressed as:

$$9! - 24M - 6N - 2P - Q + (M + N + P + Q)$$

where M, N, P and Q are the number of games completed after the fifth, sixth, seventh and eighth moves, respectively. The precise values of M, N, P and Q remain to be calculated. Any takers? John Stewart (no relation to myself, by the way) suggests that M might be 1,440.

—*I.S.*