ASYMPTOTICS FOR THE NUMBER OF ROW-FISHBURN MATRICES

KATHRIN BRINGMANN, YINGKUN LI, AND ROBERT C. RHOADES

ABSTRACT. In this paper, we provide an asymptotic for the number of row-Fishburn matrices of size n which settles a conjecture by Vit Jelínek. Additionally, using q-series constructions we provide new identities for the generating functions for the number of such matrices, one of which was conjectured by Peter Bala.

1. INTRODUCTION AND STATEMENT OF RESULTS

A *Fishburn matrix* is an upper-triangular matrix with non-negative integer entries such that every row and column contains at least one non-zero entry. The size of the matrix is the sum of the entries. For example, there are five Fishburn matrices of size three, namely

$$(3), \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}.$$

Fishburn matrices are in bijection with many combinatorial objects, including interval orders defined by Fishburn [13] and Stoimenow matchings [21]. These correspondences and others are discussed in Section 2. Let g_n be the number of Fishburn matrices of size n. Zagier established the following asymptotic for g_n.

Theorem 1.1 (Zagier [24]). *As $n \to \infty$*

$$g_n \sim \left(\frac{6}{\pi}\right)^n n! \sqrt{n} \left(C_0 + \frac{C_1}{n} + \frac{C_2}{n^2} + \cdots\right),$$

where $C_0 = \frac{12\sqrt{3}}{\pi^{5/2}} e^{\pi^2/12} = 2.70433249006\ldots$ and the C_j are effectively computable.

A Fishburn matrix is called *primitive* if its entries are all 0 or 1. There are only two primitive Fishburn matrices of size 3. Let p_n be the number of primitive Fishburn matrices of size n.

Theorem 1.2 (Jelínek, Drmota [15]). *As $n \to \infty$*

$$\lim_{n \to \infty} \frac{p_n}{g_n} = e^{-\frac{\pi^2}{6}} = 0.193025289139\ldots.$$

The following table contains the first few values of p_n and g_n, where we set $p_0 = g_0 = 1$.

n	0	1	2	3	4	5	6	7	8	9	10
p_n	1	1	1	2	5	16	61	271	1372	7795	49093
g_n	1	1	2	5	15	53	217	1014	5335	31240	201608

A *row-Fishburn matrix* is an upper-triangular matrix with non-negative integer entries such that every row contains at least one non-zero entry. As before the size of the row-Fishburn matrix is the sum of the entries and such a matrix is primitive if it contains only 0s and 1s. Let r_n be the number of primitive row-Fishburn matrices and f_n the total number of row-Fishburn matrices. The following table contains the first few values of r_n and f_n, where we set $r_0 = 0, f_0 = 1$.

n	0	1	2	3	4	5	6	7	8	9	10
r_n	0	1	2	7	33	197	1419	11966	115575	1257718	15223822
f_n	1	1	3	12	61	380	2815	24213	237348	2612681	31915787

Jelínek [15] established generating function identities for these sequences. Using a numerical technique of Zagier [24] to compute an asymptotic expansion for these sequences, he gave the following conjecture [15, Conjecture 5.3].

Conjecture (Jelínek [15]). *As $n \to \infty$*

$$f_n = n! \left(\frac{12}{\pi^2}\right)^n \left(\beta + O\left(\frac{1}{n}\right)\right)$$

with $\beta := \frac{6\sqrt{2}}{\pi^2} e^{\pi^2/24} = 1.29706861206\ldots$ and

$$\lim_{n \to \infty} \frac{r_n}{f_n} = e^{-\frac{\pi^2}{12}} = 0.439346434081\ldots.$$

We note that Jelínek stated his conjecture in terms of primitive self-dual interval orders of reduced size, resulting in the different value for β (see Cor. 4.2 in [15]).

In this paper we prove this conjecture.

Theorem 1.3. *The conjecture of Jelínek is true.*

Zagier proved Theorem 1.1 by establishing a relationship between the generating function for g_n and the "half derivative" of the Dedekind eta-function. Moreover, he showed that this function is a so-called quantum modular form [25]. Theorem 1.3 is proved by exploiting the connection between the generating function for f_n and a different quantum modular form, which is associated to a Maass wave form attached to a Hecke character. To prove the conjecture, we use the properties of the L-function and a certain twist of the L-function associated with this Hecke character. This is in contrast to Zagier's result which requires only the L-function associated to a Dirichlet character.

Besides the conjecture above, Jelínek made another conjecture concerning the asymptotics of the number of self-dual interval orders [15, Conjecture 5.4]. Since there is no clear relationship between its generating series and any modular object, it seems that the technique in this paper may not be the right tool to attack that conjecture.

In addition to the asymptotic above, we give some new expressions for the generating function for the number of row-Fishburn matrices and Fishburn matrices. It is easy to see that

$$\sum_{n=0}^{\infty} r_n x^n = \sum_{n=0}^{\infty} \prod_{j=0}^{n} ((1+x)^{j+1} - 1).$$

Zagier [24] showed that

$$\sum_{n=0}^{\infty} g_n x^n = \sum_{n=0}^{\infty} \prod_{j=1}^{n} (1 - (1-x)^j).$$

Theorem 1.4. *In the notation above,*

$$\sum_{n=0}^{\infty} f_n x^n = \sum_{n=0}^{\infty} (1-x)^{n+1} \prod_{i=1}^{n} \left(1 - (1-x)^{2i}\right)$$

$$= \sum_{n=0}^{\infty} \prod_{i=1}^{n} \left(1 - (1-x)^{2i-1}\right)$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \prod_{i=1}^{n} \frac{1}{1 + (1-x)^i}.$$

Additionally,

$$\sum_{n=0}^{\infty} p_n x^n = \sum_{n=0}^{\infty} (1+x)^{n+1} \prod_{i=1}^{n} \left(1 - (1+x)^i\right)^2$$

$$\sum_{n=0}^{\infty} g_n x^n = \sum_{n=0}^{\infty} \frac{1}{(1-x)^{n+1}} \prod_{i=1}^{n} \left(1 - \frac{1}{(1-x)^i}\right)^2.$$

Remark. The above identities should be understood as formal power series.

Remark. The second equality of this theorem proves an open conjecture by Bala in OEIS (see [15]).

Remark. After the research for this paper was completed, the authors were informed of independent results by Andrews and Jelínek [3], where the authors found related identities as power series expressions. The first two identities of Theorem 1.4 are included as special cases of their results.

Section 2 contains information about Fishburn matrices and bijections with other combinatorial objects. It also describes the generating functions for the sequences given above and their quantum modular form properties. Section 3 contains the proof of Theorem 1.3 and Section 4 contains the proof of Theorem 1.4. Section 5 contains some open problems.

2. Background and History

2.1. **Combinatorial objects.** Fishburn matrices are in bijection with many other combinatorial objects, including interval orders and $(\mathbf{2} + \mathbf{2})$-free posets. A strict partial order $\prec$ is a binary relation that is irreflexive ($a \prec a$ is always false), transitive (if $a \prec b$ and $b \prec c$, then $a \prec c$), and asymmetric (if $a \prec b$, then $b \prec a$ is not satisfied). A set with a strict partial order is called a *partially ordered set*, or a *poset*. A poset P with a strict order relation $\prec$ is a $(\mathbf{2} + \mathbf{2})$-free poset if it does not have an induced subposet isomorphic to the disjoint union of two chains of length two. Equivalently, the poset is $(\mathbf{2} + \mathbf{2})$-free if for each $x \in P$ we may associate a real closed interval $[\ell_x, r_x]$ in such a way that $x \prec y$ if and only if $r_x < \ell_y$. This is known as an interval representation and explains the terminology "interval order". In the standard representation where ℓ_x and r_x are chosen to be the smallest possible positive integers, the interval orders of size 3 are

$$\{[1, 1], [1, 1], [1, 1]]\}, \{[1, 1], [1, 2], [2, 2]\}, \{[1, 1], [1, 1], [2, 2]\}, \{[1, 1], [2, 2], [2, 2]\},$$
$$\{[1, 1], [2, 2], [3, 3]\}.$$

The *dual* of a poset P is the poset $\overline{P}$ with the same elements as P and an order relation $\overline{\prec}$ defined by $x \overline{\prec} y$ if and only if $y \prec x$. A poset is *self-dual* if it is isomorphic to its dual. A *self-dual* Fishburn matrix is a Fishburn matrix equal to itself after reflection over the diagonal from the bottom left to the top right. Self-dual Fishburn matrices are in correspondence with self-dual $(\mathbf{2}+\mathbf{2})$-free posets.

Jelínek [15] studied self-dual interval orders enumerated by *reduced size*. The *reduced size* of a self-dual Fishburn matrix is sum of all of the diagonal elements, all of the south-east cells, and those entries in the last column. The south-east cells of a $k \times k$ matrix are those cells (i, j) such that $i + j > k + 1$. Moreover, he proved a bijection between self-dual Fishburn matrices of reduced size n and row-Fishburn matrices of size n.

Besides the relationship to interval orders, Fishburn matrices are in bijection with ascent sequences. This was established by Bousquet-Mélou, Claesson, Dukes, and Kitaev [5]. A sequence of non-negative integers $(x_1, x_2, \cdots, x_n)$ is an *ascent sequence* if for each i

$$0 \le x_i \le 1 + \mathrm{asc}(x_1, x_2, \cdots, x_{i-1}),$$

where $\mathrm{asc}(X)$ is the number of ascents (or increases) in the sequence X. The five ascent sequences of length three are

$$(0, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1), \quad (0, 1, 1), \quad (0, 1, 2).$$

A *matching* of the set $[2n] = \{1, 2, \cdots, 2n\}$ is a partition of $[2n]$ into subsets of size exactly two. Each of the subsets is called an *arc*. There are 15 matchings of [6]. A matching is a *Stoimenow matching* [21] if it has no pair of arcs $\{a, b\}$ and $\{c, d\}$ with $a < b$ and $c < d$ satisfying one of

(1) $a = c + 1$ and $b < d$,
(2) $a < c$ and $b = d + 1$.

In other words, a Stoimenow matching has no pair of arcs such that one is nested inside the other and the openers (smaller number in the arc) or closers (larger numbers in the arc) differ by one. There are five Stoimenow matchings of [6] (see Figure 1).
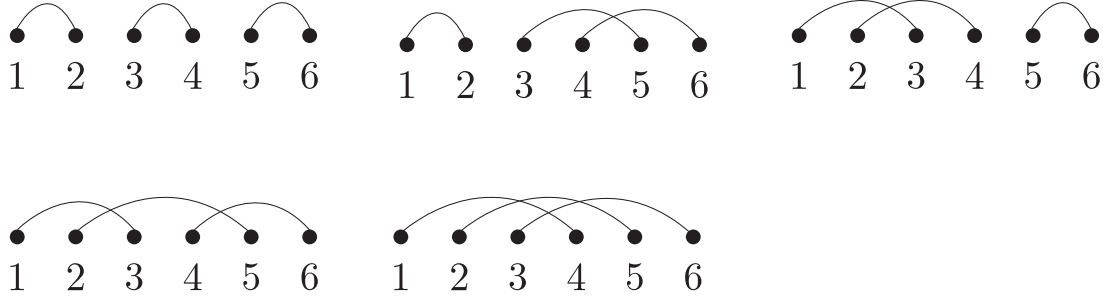
FIGURE 1. The five Stoimenow matchings of [6].

Stoimenow [21] defined such matchings in the language of *linearized chord diagrams* (LCDs), they are a fixed-point free involution τ on the set $[2n]$. The diagram is called *regular* if $[i, i+1] \subset [\tau(i+1), \tau(i)]$ whenever $\tau(i+1) < \tau(i)$. These are easily shown to be in bijection with the Stoimenow matchings. Claesson, Dukes, and Kitaev established a direct relationship between ascent sequences and Stoimenow matchings [9].

An important problem in knot theory is to determine the number $V(n)$ of linearly independent Vassiliev invariants of degree n. This is well known to be equivalent to counting the number of LCDs modulo a certain four term relation. The number of matchings on $[2n]$, and thus the number of LCDs on $[2n]$, is equal to

$$\frac{(2n)!}{2^n n!} \sim \frac{2^n n!}{\sqrt{\pi n}},$$

giving an upper bound for $V(n)$. The asymptotic follows from Stirling's approximation $n! \sim (n/e)^n \sqrt{2\pi n}$. The upper bound was improved by Chmutov and Duzhin [8] to $(n-1)!$.

2.2. **Generating functions.** Zagier provided a generating function for the numbers g_n of Stoimenow matchings on $[2n]$ and used it to prove Theorem 1.1, thus improving the above mentioned bound further. To be more precise, he showed that the generating function for the number of Stoimenow matchings of size n satisfies

$$\sum_{n=0}^{\infty} g_n x^n = F(1-x),$$

where $F(q)$ is Kontsevich's "strange function"

$$F(q) := \sum_{n=0}^{\infty} (q)_n,$$

where for $n \in \mathbb{N} \cup \{0\} \cup \{\infty\}$

$$(a; q)_n = (a)_n := \begin{cases} 1 & n = 0, \\ \prod_{j=0}^{n-1} (1 - aq^j) & n \in \mathbb{N} \cup \{\infty\}. \end{cases}$$

Zagier [24] proved that this function satisfies the "identity"

$$F(q) = -\frac{1}{2} \sum_{n=1}^{\infty} n \left(\frac{12}{n} \right) q^{\frac{n^2-1}{24}} =: \eta^*(q),$$

where $\left(\frac{12}{\bullet} \right)$ is the Kronecker symbol satisfying $\left(\frac{12}{n} \right) = 1$ if $n \equiv \pm 1 \pmod{12}$, $\left(\frac{12}{n} \right) = -1$ if $n \equiv \pm 5 \pmod{12}$, and $\left(\frac{12}{n} \right) = 0$ otherwise. The two sides of this identity do not make sense simultaneously. Indeed, the right side converges in the unit disk $|q| < 1$, but nowhere on the unit circle. The identity means that $F(q)$ at roots of unity agrees with the radial limit of the right hand side. That is,

$$F(\zeta) = \lim_{t \to 0} \eta^*(\zeta e^{-t}).$$

As Zagier pointed out in Section 6 of [24], the right hand side of the identity is essentially the "half-derivative" Dedekind's eta-function, ($q := e^{2\pi i z}$)

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = \sum_{n=1}^{\infty} \left(\frac{12}{n} \right) q^{\frac{n^2}{24}},$$

which is a weight $1/2$ holomorphic modular form. Moreover, Zagier showed that this function satisfies a "pseudo-modularity" property. With $\phi(x) := e^{\pi i x/12} F(e^{2\pi i x})$, the function $\phi : \mathbb{Q} \to \mathbb{C}$ satisfies

$$\phi(x) + (ix)^{-\frac{3}{2}} \phi \left(-\frac{1}{x} \right) = g(x),$$

where $g : \mathbb{R} \to \mathbb{C}$ is a C^{∞} function everywhere except at $x = 0$. This is one of the first examples of what Zagier has called a "quantum modular form" [25], which are functions on $\mathbb{Q}$ whose "obstruction to modularity" is nice.

Results of Khamis [16] or Dukes, Kitaev, Remmel, and Steingrímsson [12] give the following generating function for the number of primitive Fishburn matrices

$$\sum_{n=0}^{\infty} p_n x^n = \sum_{n=0}^{\infty} \prod_{i=1}^{n} \left(1 - \frac{1}{(1-x)^i} \right) = F \left(\frac{1}{1-x} \right).$$

The relationship to $F(q)$ and the asymptotics for g_n yield the asymptotic for p_n (Theorem 1.2).

Jelínek [15] proved the following generating function identities for the number of row-Fishburn matrices and the number of primitive row-Fishburn matrices.

$$J(1 + x) = \sum_{n=0}^{\infty} r_n x^n = \sum_{n=0}^{\infty} \prod_{i=0}^{n} \left((1+x)^{i+1} - 1 \right),$$

$$(2.1) \qquad J \left(\frac{1}{1-x} \right) = \sum_{n=0}^{\infty} f_n x^n = \sum_{n=0}^{\infty} \prod_{i=0}^{n} \left(\frac{1}{(1-x)^{i+1}} - 1 \right),$$

where the function $J(q)$ is analogous to $F(q)$ and defined by

$$(2.2) \qquad J(q) := \sum_{n=0}^{\infty} (-1)^n (q)_n.$$

At every root of unity, the function $J(q)$ exists and satisfies

$$2J(q) = 2 + 2\sum_{n=1}^{\infty}(q-1)(q^2-1)\cdots(q^n-1)$$

(2.3)
$$= 2 + \sum_{n=0}^{\infty}(q^{n+1}-1)(q^n-1)\cdots(q-1) + \sum_{n=1}^{\infty}(q^n-1)\cdots(q-1)$$

$$= 1 + q + \sum_{n=1}^{\infty}q^{n+1}(q-1)(q^2-1)\cdots(q^n-1) = \sigma(q),$$

where

$$\sigma(q) := \sum_{n=0}^{\infty}\frac{q^{\frac{n(n+1)}{2}}}{(-q)_n} = 1 + q + \sum_{n=1}^{\infty}q^{n+1}(-1)^n(q)_n.$$

Moreover, equation (2.3) also holds to infinite order at every root of unity, in particular at $q = 1$. That means we have the following Taylor series expansion

(2.4)
$$\frac{1}{2}\sigma(1+x) = \sum_{n=0}^{\infty}r_n x^n.$$

Surprisingly, it is known that the function $\sigma(q)$ is also a quantum modular form [25]. The series $\sigma(q)$ appears in Ramanujan's lost notebooks [1] and was studied by Andrews, Dyson, and Hickerson [2] and further studied by Cohen [10]. Andrews, Dyson, and Hickerson showed that the coefficients of $\sigma(q)$ are related to the arithmetic of $\mathbb{Q}(\sqrt{6})$, in particular leading to an explicit formula for those coefficients. For further examples of q-series associated to the arithmetic of real quadratic fields, we refer the reader to [4, 6, 11, 19].

2.3. **Quantum modularity and Maass forms.** To describe the quantum modular form property define a second q-series, also studied in [2, 10],

(2.5)
$$\sigma^*(q) := -2\sum_{n=0}^{\infty}q^{n+1}\left(q^2;q^2\right)_n.$$

Cohen showed that at every root of unity

(2.6)
$$-\sigma^*\left(q^{-1}\right) = \sigma(q).$$

Moreover, Zagier [25] describes how this identity holds to infinite order as an asymptotic expansion around any root of unity. Cohen also showed there the Fourier coefficients of $\sigma(q)$ and $\sigma^*(q)$ come from an even Hecke character χ_1 of $\mathbb{Q}(\sqrt{6})$ with order 2 and conductor $4(3 + \sqrt{6})$ (See Theorem 1.1 in [10]). It gives rise to a Maass cusp form $u(z)$ with Fourier expansion

(2.7)
$$u(z) = y^{\frac{1}{2}}\sum_{\substack{n\in\mathbb{Z}\\n\neq 0}}T(n)e^{\frac{2\pi inx}{24}}K_0\left(\frac{2\pi|n|y}{24}\right),$$

where $K_0(x)$ is the Bessel function, whose Mellin transform is $\Gamma(s/2)^2$ up to trivial factors.

In [25], Zagier defined the function

$$f(z) := \begin{cases} q^{\frac{1}{24}}\sigma(q) & \text{if } z \in \mathbb{H} \cup \mathbb{Q}, \\ -q^{\frac{1}{24}}\sigma^*(q^{-1}) & \text{if } z \in \mathbb{H}^- \cup \mathbb{Q}, \end{cases}$$

where $\mathbb{H}$ is the complex upper half-plane and $\mathbb{H}^-$ is the complex lower half-plane. Using the theory of period functions developed in [18], Zagier then showed that

$$(2z+1)f\left(\frac{z}{2z+1}\right) - e^{\frac{\pi i}{12}}f(z) = -\int_{\frac{1}{2}}^{\infty}[u(\tau), r_z(\tau)],$$

where $r_z(\tau) := (\mathrm{Im}\,(\tau)\,/(\tau - z)(\overline{\tau} - z))^{1/2}$ and $[\cdot, \cdot]$ denotes the Green's form

$$[u(\tau), v(\tau)] := \frac{\partial u(\tau)}{\partial \tau}v(\tau)d\tau + u(\tau)\frac{\partial v(\tau)}{\partial \overline{\tau}}d\overline{\tau}.$$

2.4. **L-functions and Dirichlet series.** In this paper we do not need the quantum modularity of $\sigma(q)$. Instead, we require the relationship between σ and the Maass wave form u given by the Hecke character χ_1. ¿From this, one could produce various L-series, whose analytic properties produce the desired asymptotics results. Recall that $\{T(n)\}_{n \in 24\mathbb{Z}+1}$ are the Fourier coefficients of $u(z)$ given in equation (2.7). By Theorem 1.1 in [10], we can write

$$q\sigma\left(q^{24}\right) = \sum_{n \geq 0}T(n)q^n, \quad \text{and} \quad q^{-1}\sigma^*\left(q^{24}\right) = \sum_{n < 0}T(n)q^{|n|}$$

and combine them into one q-series

$$\varphi(q) := q^{\frac{1}{24}}\sigma(q) + q^{-\frac{1}{24}}\sigma^*(q) = \sum_{n \in \mathbb{Z}}T(n)q^{\frac{|n|}{24}}.$$

We then define the following L-series arising from σ and σ^*, namely

$$D(s) := \sum_{m \geq 1}\frac{T(m)}{m^s}$$

and for $\varepsilon \in \{+, -\}$,

$$L_{\varepsilon}(s) := \sum_{m \geq 1}\frac{T(m) + \varepsilon T(-m)}{m^s}.$$

Clearly $D(s) = \frac{1}{2}\left(L_+(s) + L_-(s)\right)$ and is only "half" of the usual L-function associated to a Hecke character. Moreover, set

$$\Lambda_+(s) := (1152)^{\frac{s}{2}}\pi^{-s}\Gamma\left(\frac{s}{2}\right)^2 L_+(s),$$

$$\Lambda_-(s) := (1152)^{\frac{s+1}{2}}\pi^{-(s+1)}\Gamma\left(\frac{s+1}{2}\right)^2 L_-(s).$$

Theorem 2.1 of [10] gives

Proposition 2.1. *We have*

$$\Lambda_{\varepsilon}(1 - s) = \Lambda_{\varepsilon}(s).$$

Proof. The claim for Λ_+ is directly given in Theorem 2.1 of [10]. A key step is to realize

$$\varphi(q) = \sum_{\mathfrak{a} \subset \mathbb{Z}[\sqrt{3}]} \chi_2(\mathfrak{a}) q^{\frac{|N(\mathfrak{a})|}{24}},$$

where χ_2 is a certain ideal character and N is the usual norm of ideals and to then write L_+ as a Hecke L-function. The claim for Λ_- follows from the proof of Theorem 3.1 of [10] observing that $\left(\frac{-3}{N(\cdot)}\right)$ exactly weights the coefficients $n \equiv 1 \pmod{24}$ with $+$. $\square$

It is an easy consequence to derive the following lemma.

Lemma 2.2. *For $s > 0$, we have*

$$D(-s) = \frac{(1152)^{s+\frac{1}{2}}}{2\,\pi^{2s+1}} \left(\left(\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(-\frac{s}{2}\right)} \right)^2 L_+(1+s) + \left(\frac{\Gamma\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{-s+1}{2}\right)} \right)^2 L_-(1+s) \right).$$

In the next section we will show how the asymptotics of these L-values control the asymptotics of the coefficients r_n. For this, we require the following asymptotic bahvior of D at negative integers.

Proposition 2.3. *As $m \to \infty$, we have*

$$D(-m) = \left(\frac{288}{\pi^2} \right)^m \frac{12\sqrt{2}}{\pi^2} m!^2 \left(1 + O\left(\frac{1}{2^m} \right) \right).$$

Proof. We use Lemma 2.2 and observe that, due to the poles of the Γ-function at nonpositive integers, the first (resp. the second) summand vanishes if m is even (resp. odd). We may then easily compute that

$$D(-m) = \left(\frac{288}{\pi^2} \right)^m \frac{12\sqrt{2}}{\pi^2} m!^2 L_\pm(1+m),$$

where we have the $+$ sign for m odd and the $-$ sign for m even. To finish the proof, observe that $T(m) = O(\log m)$ since it is the value of a Hecke character, hence

$$L_\pm(1+m) = 1 + O\left(\sum_{\ell \geq 2} \frac{\ell^\varepsilon}{\ell^{m+1}} \right) = 1 + O\left(2^{-m+\varepsilon} \right),$$

for $\varepsilon > 0$. $\square$

Finally we require a general lemma relating Dirichlet series at negative integers to the coefficients of their associated exponential generating functions (see page 47 of [23]).

Lemma 2.4. *Assume that $\mathcal{D}(s) := \sum_{n \geq 1} \frac{a_n}{n^s}$ converges for at least one $s \in \mathbb{C}$ and that $f(t) := \sum_{n \geq 1} a_n e^{-nt}$ converges for all $t > 0$. Further assume the existence of an asymptotic expansion*

$$f(t) \sim b_0 + b_1 t + \cdots \qquad (t \to 0).$$

Then $\mathcal{D}(s)$ has a holomorphic continuation to $\mathbb{C}$ and

$$\mathcal{D}(-n) = (-1)^n\, n!\, b_n.$$

3. Proof of Theorem 1.3

This section uses the asymptotics for the L-values $D(-m)$ derived in Proposition 2.3 to prove Theorem 1.3.

We first determine the asymptotic behavior of the Taylor coefficients of $\sigma(e^{-t})$. We therefore set

$$(3.1) \qquad \sigma\left(e^{-t}\right) = \sum_{m=0}^{\infty} (-1)^m b_m \frac{t^m}{m!} = 2 - 2t + 5t^2 - \frac{55}{3}t^3 + \frac{1073}{12}t^4 - \frac{32671}{60}t^5 + \cdots.$$

Lemma 3.1. *We have as* $n \to \infty$

$$b_n = n!^2 \frac{12\sqrt{2}}{\pi^2} \left(\frac{12}{\pi^2}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right).$$

Proof. Applying Lemma 2.4 to the function $q\sigma\left(q^{24}\right) = \sum_{n \geq 1} T(n)q^n$ yields that

$$e^{-t}\sigma\left(e^{-24t}\right) \sim \sum_{m \geq 0} \alpha_m t^m,$$

with $\alpha_m := \frac{D(-m)(-1)^m}{m!}$ for $m \in \mathbb{N}$. Proposition 2.3 then gives that

$$\alpha_m = (-1)^m \left(\frac{288}{\pi^2}\right)^m \frac{12\sqrt{2}}{\pi^2} m! \left(1 + O\left(\frac{1}{2^m}\right)\right).$$

To determine the asymptotic behavior of σ itself, we insert the Taylor expansion of the exponential function, yielding

$$\sigma\left(e^{-t}\right) \sim \sum_{n \geq 0} \left(\frac{t}{24}\right)^n \sum_{0 \leq m \leq n} \frac{\alpha_{n-m}}{m!}.$$

Using that

$$\frac{\alpha_{n-m}}{\alpha_n} \sim (-1)^m \left(\frac{\pi^2}{288}\right)^m \frac{(n-m)!}{n!}$$

gives

$$\sum_{0 \leq m \leq n} \frac{\alpha_{n-m}}{m!} \sim \alpha_n \sum_{0 \leq m \leq n} (-1)^m \left(\frac{\pi^2}{288}\right)^m \frac{(n-m)!}{m!n!}.$$

Now

$$\left| \sum_{1 \leq m \leq n} (-1)^m \left(\frac{\pi^2}{288}\right)^m \frac{(n-m)!}{m!n!} \right| \leq \frac{1}{n} + \sum_{1 \leq m \leq n} \frac{1}{n^2} \ll \frac{1}{n}.$$

This directly gives the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Proof of Theorem 1.3. To prove the theorem, we first note that

$$\frac{t^m}{m!} = \sum_{n=m}^{\infty} S_{n,m} \frac{(1 - e^{-t})^n}{n!}$$

where $S_{n,m}$ are the Stirling numbers of the first kind (see p.954 of [24]). Along with equation (2.4), this gives that

$$\sum_{n=0}^{\infty} r_n x^n = \frac{1}{2}\sigma(1+x) = \frac{1}{2}\sum_{m=0}^{\infty}(-1)^m b_m \sum_{n=m}^{\infty} S_{n,m}\frac{(-x)^n}{n!}$$

(3.2)
$$= \frac{1}{2}\sum_{n=0}^{\infty} b_n \frac{x^n}{n!} \sum_{k=0}^{n}(-1)^k S_{n,n-k}\frac{b_{n-k}}{b_n}.$$

As a consequence of the recursion formula $S_{n+1,m+1} = S_{n,m}+nS_{n,m+1}$, the Stirling numbers satisfy

$$S_{n,n-k} = \frac{P_k(n)}{2^k k!}$$

for a monic polynomial $P_k(x)$ of degree $2k$. Furthermore for each j, the coefficient of x^{2k-j} in $P_k(x)$ is the value at k of a polynomial $(-1)^j c_j(y)$, which has degree $2j$ and is independent of k or n. Thus, we can write

$$S_{n,n-k} = \frac{n^{2k}}{2^k k!}\left(1 - \frac{c_1(k)}{n} + \frac{c_2(k)}{n^2} + \cdots\right),$$

for all $n \in \mathbb{N}$ and $0 \le k \le n$.

Moreover, Lemma 2.4 gives that

$$\frac{b_{n-k}}{b_n} \sim \left(\frac{(n-k)!}{n!}\right)^2 \left(\frac{\pi^2}{12}\right)^k \left(1 + O\left(\frac{k}{n}\right)\right).$$

Some simple manipulation gives us that for all $k, n \in \mathbb{N}$ with $k \le n$,

$$\left(\frac{(n-k)!}{n!}\right)^2 = n^{-2k}\left(1 + \frac{d_1(k)}{n} + \frac{d_2(k)}{n^2} + \cdots\right)$$

where $d_j(x)$ are polynomials in x independent of k and n. Inserting these into (3.2) and using the asymptotic for b_n from Lemma 3.1 results in

$$r_n \sim \frac{b_n}{2n!}\sum_{k=0}^{n}\frac{(-1)^k}{k!}\left(\frac{\pi^2}{24}\right)^k\left(1 - \frac{c_1(k)}{n} + \cdots\right)\left(1 + \frac{d_1(k)}{n} + \cdots\right)\left(1 + O\left(\frac{k}{n}\right)\right)$$

$$\sim \frac{b_n}{2n!}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\pi^2}{24}\right)^k + O\left(\frac{1}{n}\right)\right)$$

$$\sim \frac{6\sqrt{2}}{\pi^2}n!\left(\frac{12}{\pi^2}\right)^n e^{-\frac{\pi^2}{24}}\left(1 + O\left(\frac{1}{n}\right)\right).$$

To obtain the asymptotic for f_n, recall that

$$\sum_{n=0}^{\infty} f_n x^n = J\left(\frac{1}{1-x}\right) = -\frac{1}{2}\sigma^*(1-x).$$

The second equality follows from equation (2.6). By the same reasoning, equation (3.1) implies that

$$-\sigma^*(e^{-t}) = \sum_{m=0}^{\infty} b_m \frac{t^m}{m!}.$$

This gives

$$\sum_{n=0}^{\infty} f_n x^n = \frac{1}{2} \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} \sum_{k=0}^{n} S_{n,n-k} \frac{b_{n-k}}{b_n}.$$

The same argument for r_n then leads to the desired asymptotic

$$f_n \sim \frac{6\sqrt{2}}{\pi^2} n! \left(\frac{12}{\pi^2}\right)^n e^{\frac{\pi^2}{24}} \left(1 + O\left(\frac{1}{n}\right)\right).$$

$\square$

4. GENERATING FUNCTION IDENTITIES

This section contains the proofs of new identities for the generating functions for the sequence $\{r_n\}$ (the number of primitive row-Fishburn matrices), $\{p_n\}$ (the number of primitive Fishburn matrices), and $\{g_n\}$ (the number of Fishburn matrices).

Proof of Theorem 1.4. Jelínek proved the generating function for the sequence $\{f_n\}$ is $-\frac{1}{2}\sigma\left(\frac{1}{1-x}\right)$ (see Section 2.2). Moreover, (2.6) gives $\sigma(q^{-1}) = -\sigma^*(q)$ at every root of unity, where $\sigma^*(q) = -2\sum_{n=0}^{\infty} q^{n+1}(q^2; q^2)_n$ (see (2.5)). Additionally, the identity $-\sigma^*(q^{-1}) = \sigma(q)$ holds to infinite order at $q = 1$ (see [25]). It follows that as power series in x, $-\frac{1}{2}\sigma\left(\frac{1}{1-x}\right) = \frac{1}{2}\sigma^*(1-x)$.

For the second claim of the theorem, begin by noting that Theorem 3 of Andrews, Jiménez-Urroz, and Ono [4] contains the identity

$$\sum_{n=0}^{\infty} \left((q; q^2)_n - (q; q^2)_{\infty}\right) = -\frac{1}{2}\sigma^*(q) - (q; q^2)_{\infty} \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}}.$$

Set $q = 1 - x$ and consider the resulting formal power series in x. Appealing to (2.5) and the first part of the theorem yields the desired result. Note that $((1-x); (1-x)^2)_{\infty}$ is zero as a formal power series in x.

The third identity is similar. Theorem 3 of [4] contains

$$\sum_{n=0}^{\infty} \left(\frac{1}{(-q)_n} - \frac{1}{(-q)_{\infty}}\right) = -\sigma^*(q) - \frac{1}{(-q)_{\infty}} \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n}.$$

Use $(-q; q)_{\infty} = \frac{1}{(q; q^2)_{\infty}}$. Then, set $q = 1 - x$. Analogously to the second identity, the third identity follows by considering the formal power series.

Recall that Zagier proved the generating function for $\{p_n\}$ can be expressed as $F(1 + x)$ where $F(q) = \sum_{n=0}^{\infty} \prod_{i=1}^{n} (1 - q^i)$. Moreover, $F(1/(1-x))$ is the generating function for g_n. The final two claims in the theorem follow from a q-series identity for $F(q)$. To be more precise, Bryson, Ono, Pitman and the third author [7] showed that

$$U(q) := \sum_{n=0}^{\infty} q^{n+1} (q)_n^2$$

satisfies

$$U\left(q^{-1}\right) = F(q)$$

at every root of unity to infinite order. For the first of the final two claims take $q = 1 + x$ and the formal power series expansion to obtain the result. The final result follows analogously. □

5. Open Problems and Relations to q-hypergeometric series

This section presents some open problems. There are many refinements of the generating functions considered in this paper. For instance, let $p_{n,k}$ be the number of $(\mathbf{2}{+}\mathbf{2})$-free posets of size n with k minimal elements. The numbers $p_{n,k}$ also count

(1) ascent sequences of length n with k elements;
(2) permutations of length n avoiding a certain pattern whose left-most decreasing run is of size k;
(3) regularized linear chord diagrams on $2n$ points with initial run of openers of size k;
(4) Fishburn matrices of size n such that the sum of the first row is k;

Kitaev and Remmel [17] proved that

$$\sum_{n,k} p_{n,k} x^n z^k = 1 + \sum_{n=0}^{\infty} \frac{zx}{(1 - zx)^{n+1}} \prod_{i=1}^{n} \left(1 - (1 - x)^i\right)$$

or alternatively

$$\sum_{n,k} p_{n,k} x^n z^k = \sum_{n=0}^{\infty} \prod_{i=1}^{n} (1 - (1 - x)^{i-1}(1 - xz)).$$

The second of these formulas was conjectured in [17] and proved by Yan [22]

Open Problem. *What is the asymptotic distribution of the number of minimal elements in $(\mathbf{2}{+}\mathbf{2})$-free posets of size n?*

Open Problem. *What quantum modular form properties, if any, exist for the two variable generating function of the sequence $\{p_{n,k}\}$?*

For other refinements and generating functions for statistics on $(\mathbf{2}{+}\mathbf{2})$-free posets see [5, 12, 15, 17]. Analogous questions apply to the statistics in those works as well.

Stoimenow [21] introduced the notion of connected regular LCDs and denoted the number of such of size n by λ_n^c. He conjectured that

$$\lim_{n \to \infty} \frac{\lambda_n^c}{g_n} = \frac{1}{e}.$$

The generating function is

$$\Lambda(x) = \sum_{n=1}^{\infty} \lambda_n^c x^n = x + x^2 + 2x^3 + 5x^4 + 16x^5 + 63x^6 + 293x^7 + 1561x^8 + \cdots$$

Zagier [24] showed

Theorem 5.1. *The generating function $\Lambda(x)$ equals $\Phi_x^{-1}(1)$, where*

$$\Phi_x(z) := \sum_{n=0}^{\infty} \frac{(q)_n}{(-z)_{n+1}}$$

with $q = 1 - x$.

Curiously, $\Phi_x(1)$ is a quantum modular form related to the half-derivative of a weight $1/2$ modular form, similar to the situation arising for $F(q)$. In particular, Theorem 2 in [4] implies that

$$2\sum_{n=1}^{\infty}(-1)^n n q^{n^2} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{(q;q)_n}{(-q;q)_n} - \frac{(q;q)_\infty}{(-q;q)_\infty}\right) - \frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{j=1}^{\infty}\frac{q^j}{1-q^{2j}}.$$

So that

$$\Phi_x(1) = 2\sum_{n=1}^{\infty}(-1)^n n q^{n^2},$$

where the meaning is that the value at any valid root of unity of the left hand side is equal to the radial limit of the right hand side as q tends toward that root of unity. This is completely analogous to the strange identity of Zagier for $F(q)$ and the half-derivative of the Dedekind eta-function. In this case the right hand side is the half-derivative of the weight $1/2$ theta function $\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}$.

Open Problem. *What role, if any, do quantum modular forms play in the asymptotics for λ_n^c?*

Split interval orders add a distinguished point f_x to each interval $[\ell_x, r_x]$. Define $(X, \prec)$ as a split interval order if there are real $a_x \leq f_x \leq b_x$ for all $x \in X$ such that, for all $x, y \in X$

$$x \prec y \Leftrightarrow [f_x < \ell_y \text{ and } r_x < f_y].$$

See Fishburn and Trotter's paper [14] for general discussion about these objects. Reeds and Fishburn [20] computed the number of split interval orders of size n, denoted $\ell\ell_n$, for $n < 10$.

n	1	2	3	4	5	6	7	8	9
$\ell\ell_n$	1	2	5	16	63	315	1979	15576	151606

Open Problem. *Find a generating function for the sequence $\{\ell\ell_n\}_n$.*

REFERENCES

[1] G. E. Andrews, *Ramanujan's "Lost" Notebook V: Euler's Partition Identity*, Adv. Math. **61** (1986), 156–164.

[2] G. E. Andrews, F. Dyson, and D. Hickerson, *Partitions and indefinite quadratic forms*, Invent. Math. **91** (1988), 391-407.

[3] G. E. Andrews and V. Jelínek, *Some q-series identities related to interval orders*, in preparation.

[4] G. E. Andrews, J. Jiménez-Urroz, and K. Ono, *q-series identities and values of certain L-functions*, Duke Math. J. **108** (2001), 395–419.

[5] M. Bousquet-Mélou, A. Claesson, M. Dukes, and S. Kitaev, *(2+2)-free posets, ascent sequences and pattern avoiding permutations.* J. Comb. Theory Ser. A, 117 (**7**) (2010), 884–909.

[6] K. Bringmann and B. Kane, *Multiplicative q-hypergeometric series arising from real quadratic fields.* Trans. Amer. Math. Soc. **363** (2011), 2191-2209.

[7] J. Bryson, K. Ono, S. Pitman, and R. C. Rhoades, *Unimodal sequences and quantum and mock modular forms*, Proc. Natl. Acad. Sci. USA, to appear.

[8] S.V. Chmutov and S.V. Duzhin, *An upper bound for the number of Vassiliev knot invariants*, J. Knot Theory Ramifications **3** (1994), 141–151.

[9] A. Claesson, M. Dukes, and S. Kitaev, *A direct encoding of Stoimenow matchings as ascent sequences.* http://arxiv.org/abs/0910.1619.

[10] H. Cohen, *q-identities for Maass waveforms*, Invent. Math. **91** (1988), 409–422.

[11] D. Corson, D. Favero, K. Liesinger, and S. Zubairy, *Characters and q-series in* $\mathbb{Q}(\sqrt{2})$. J. Number Theory **107** (2004), 392-405

[12] M. Dukes, S. Kitaev, J. Remmel, and E. Steingrímsson, *Enumerating (**2+2**)-free posets by indistinguishable elements.* http://arxiv.org/abs/1006.2696.

[13] P. C. Fishburn, *Intransitive indifference with unequal indifference intervals.* Journal of Mathematical Psychology, **7** (1970), 144–149.

[14] P. C. Fishburn and W. T. Trotter, *Split semiorders.* Discrete Math. **195** (1999), 111–126.

[15] V. Jelínek, *Counting Self-Dual Interval Orders.* J. Comb. Theory. Series A, **119** (3), 599-614 (2012)

[16] S. M. Khamis, *Exact Counting of Unlabeled Rigid Interval Posets Regarding or Disregarding Height.* Order **29** (2012), 443-461.

[17] S. Kitaev and J. Remmel, *Enumerating (**2+2**)-free posets by the number of minimal elements and other statistics.* In 22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010), 2010.

[18] J. Lewis and D. Zagier, *Period functions for Maass wave forms. I.* Ann. Math. **153** (2001), 191–258.

[19] J. Lovejoy, *Overpartitions and real quadratic fiels.* J. Number Theory **106**, (2004), 178-186.

[20] J. A. Reeds and P. C. Fishburn, *Counting split interval orders.* Order **18** (2001), 129–135.

[21] A. Stoimenow, *Enumeration of chord diagrams and an upper bound for Vassiliev invariants.* J. Knot Theory Ramifications, **7** (1998), 93–114.

[22] S. H. F. Yan, *On a conjecture about enumeration (**2+2**)-free posets.* European Journal of Combinatorics **32** (2011), 282–287.

[23] D. Zagier, *Zetafunktionen und quadratische Körper: eine Einführung in die höhere Zahlentheorie.* Hochschultext, Springer-Verlag, Berlin-Heidelberg-New York (1981),

[24] D. Zagier, *Vassiliev invariants and a strange identity related to the Dedekind eta-function.* Topology **40** (2001), no. 5, 945–960.

[25] D. Zagier, *Quantum modular forms.* In Quanta of Maths: Conference in honor of Alain Connes, Clay Math. Proc. **11**, Amer. Math. Soc., Providence, (2010) 659–675.

MATHEMATICAL INSTITUTE, UNIVERSITY OF COLOGNE, 50931 COLOGNE, GERMANY
E-mail address: `kbringma@math.uni-koeln.de`

MATHEMATICAL INSTITUTE, UNIVERSITY OF COLOGNE, 50931 COLOGNE, GERMANY
E-mail address: `yli9@math.uni-koeln.de`

CENTER FOR COMMUNICATIONS RESEARCH, 805 BUNN DR., PRINCETON, NJ 08540,U.S.A.
E-mail address: `rob.rhoades@gmail.com`