TERMINATING DECIMALS IN THE CANTOR TERNARY SET

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1. Introduction

The classical Cantor set is usually constructed by beginning with the interval [0, 1], deleting the middle third, and then continuing to delete the middle third of each interval remaining after the previous step. Another characterization is that the Cantor set consists of all numbers between 0 and 1 that can be written in base three using only 0 and 2 as digits. In this paper, we show that there are only 14 terminating decimals in the Cantor set, namely,

$$\frac{1}{4}, \frac{3}{4}, \frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}, \frac{1}{40}, \frac{3}{40}, \frac{9}{40}, \frac{13}{40}, \frac{27}{40}, \frac{31}{40}, \frac{37}{40}, \frac{39}{40}.$$

Clearly, we may restrict our attention to fractions N/M where $M = 2^a 5^b$ $(a \geq 0,\ b \geq 0,\ ab \neq 0)$ and gcd $(N, M) = 1$. If N/M is a fraction in the Cantor set, then so is $1 - N/M$ and so is $3N/M$, provided $3N$ is reduced modulo M: the former is the 2's complement, and the latter is the fractional part after shifting the ternary point. In what follows, it will be convenient to write $M = \mu p$ and $\phi(M) = \gamma q$, where ϕ is Euler's function (the numbers μ and γ will be specified).

The claim above will be established by examining eight infinite classes of denominators and eight special cases. We will show that no fractions in the eight infinite classes are in the Cantor set; the eight special cases will yield the fourteen terminating decimal fractions listed above.

For j relatively prime to M, we will find it convenient to use the notation

$$[j] = \{j \cdot 3^k \pmod{M} : k = 0, 1, 2, \ldots\}.$$

If g is the smallest positive exponent for which $3^g \equiv 1 \pmod{M}$ and $(j, M) = 1$, then each set $[j]$ contains g elements, and there are $\phi(M)/g$ distinct sets $[j]$. Note that either all elements of $[j]$ are numerators of fractions in the Cantor set or none are, and that $[j]$ is eliminated if and only if $[-j]$ is.

Note that:

$$\begin{array}{llll} 3^{4k} \equiv 1 \pmod{80} & & 7^{4k} \equiv 1 \pmod{80} \\ 3^{4k+1} \equiv 3 \pmod{80} & & 7^{4k+1} \equiv 7 \pmod{80} \\ 3^{4k+2} \equiv 9 \pmod{80} & & 7^{4k+2} \equiv 49 \pmod{80} \\ 3^{4k+3} \equiv 27 \pmod{80} & & 7^{4k+3} \equiv 23 \pmod{80} \end{array}$$

Therefore,

Lemma 1: If $80|M$, then the sets $[j]$ are pairwise disjoint for $j = \pm 1, \pm 7, \pm 49$, and ± 343.

Reduction of the congruences yields

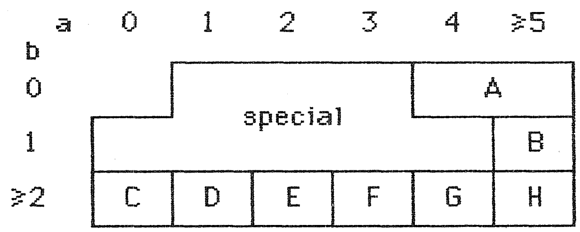
Lemma 2: If $40|M$, then the sets $[j]$ are pairwise disjoint for $j = \pm 1$ and ± 7.

Lemma 3: If $4|M$, then the sets $[j]$ are pairwise disjoint for $j = \pm 1$.

2. General Cases

In this section, we will examine eight infinite classes of denominators $M > 1$. For each class, we will describe the behavior of the numbers 3^k

(mod M). In each case, the congruence for 3^q may be proved by mathematical induction, and the others follow from it. No induction proofs will be presented because they are all easy (the hard part is spotting the patterns, not proving them). Then, we will show how each set $[j]$ with $(j, M) = 1$ contains an element N for which N/M is between 1/3 and 2/3, thus proving that the class contains no elements of the Cantor set. The scheme of proof is summarized by the following chart:



Class A: Suppose $M = 2^a$ with $a \geq 4$. Then $\phi(M) = 2^{a-1}$ and we write $M = 2p$ and $\phi(M) = 4q$. We observe that

$$3^q \equiv p + 1 \pmod{M} \qquad 3^{2q} \equiv 1 \pmod{M}$$

Then, by Lemma 3, the sets [1] and [-1] are disjoint, but [1] contains $p + 1$, which is obviously in the middle third. Details may be found in Reference 1, where it was proved that 1/4 and 3/4 are the only dyadic rationals in the Cantor set.

Class B: Suppose $M = 2^a5$ with $a \geq 5$. We write $M = 2p$ and $\phi(M) = 2^{a+1} = 16q$. Then we may prove that:

$$3^q \equiv p + 1 \pmod{M} \qquad 3^{2q} \equiv 1 \pmod{M}$$

By Lemma 1, it suffices to examine the sets $[j]$ for $j = \pm 1, \pm 7, \pm 49$, and ± 343. Because $p + 1$ is in the middle third, sets [1] and [-1] do not qualify. Note that $7(p + 1) \equiv p + 7$, which is in the middle third, so [±7] is eliminated. Similarly, $49(p + 1) \equiv p + 49$ and $p + 49$ is in the middle third except for $p = 80$ ($M = 160$); but $243(49) \equiv 67 \pmod{160}$ and $67/160 = 0.41\ldots$, eliminating [±49]. Note that $343(p + 1) \equiv p + 343$ and $p + 343$ is in the middle third unless $p \leq 1029$, so [±343] is eliminated except possibly for $M = 160, 320, 640$, and 1280. But each of these possibilities includes an element of the middle third:

$M = 160$: $3 \cdot 343 \equiv 69 = 0.43\ldots M$
$M = 320$: $9 \cdot 343 \equiv 207 = 0.64\ldots M$
$M = 640$: $343 = 0.53\ldots M$
$M = 1280$: $9 \cdot 343 \equiv 527 = 0.41\ldots M$

Therefore, Class B is eliminated.

Class C: Suppose $M = 5^b$ with $b \geq 2$. We write $M = 5p$ and $\phi(M) = 4 \cdot 5^{b-1} = 10q$. Then:

$$3^q \equiv 2p - 1 \pmod{M} \qquad 3^{5q} \equiv -1 \pmod{M}$$
$$3^{2q} \equiv p + 1 \pmod{M} \qquad 3^{10q} \equiv 1 \pmod{M}$$

But then the numbers 3^j for $0 < j \leq \phi(M)$ are distinct, so none of the numbers can be in the Cantor set, since $2p - 1$ obviously is not.

Class D: Suppose $M = 2 \cdot 5^b$ with $b \geq 2$. We write $M = 5p$ and $\phi(M) = 4 \cdot 5^{b-1} = 10q$. Then:

$$3^q \equiv p - 1 \pmod{M} \qquad 3^{5q} \equiv -1 \pmod{M}$$
$$3^{2q} \equiv 3p + 1 \pmod{M} \qquad 3^{10q} \equiv 1 \pmod{M}$$

As in Class C, we cannot have all the numbers ($3p + 1$ in particular), so we have none of them.

Class E: Suppose $M = 2^2 5^b$ with $b \geq 2$. We write $M = 10p$ and $\phi(M) = 8 \cdot 5^{b-1} = 20q$. Then:

$$3^q \equiv p - 1 \pmod{M} \qquad 3^{5q} \equiv 5p - 1 \pmod{M}$$
$$3^{2q} \equiv 8p + 1 \pmod{M} \qquad 3^{10q} \equiv 1 \pmod{M}$$

We have only the sets [±1] to check, but they are eliminated because $5p - 1$ is in the middle third.

Class F: Suppose $M = 2^3 5^b$ with $b \geq 2$. We write $M = 20p$ and $\phi(M) = 16 \cdot 5^{b-1} = 40q$. Then:

$$3^q \equiv p - 1 \pmod{M} \qquad 3^{5q} \equiv 5p - 1 \pmod{M}$$
$$3^{2q} \equiv 8p + 1 \pmod{M} \qquad 3^{10q} \equiv 1 \pmod{M}$$

By Lemma 2, there are the four sets [±1] and [±7] to check. We quickly eliminate [±1] because $8p + 1$ is in the middle third. If $p > 21$, we eliminate [±7] because $7(p - 1)$ is in the middle third. If $p \leq 21$, then $p = 10$ and $M = 200$, but $3^5 7 \equiv 101 \pmod{200}$; thus, Class F yields no members of the Cantor set.

Class G: Suppose $M = 2^4 5^b$ with $b \geq 2$. We write $M = 80$ and $\phi(M) = 32 \cdot 5^{b-1} = 80q$. A "leapfrog" induction shows that

$$3^q \equiv 2p - 1 \pmod{M} \qquad 3^{5q} \equiv 50p - 1 \pmod{M}$$
$$3^{2q} \equiv 16p + 1 \pmod{M} \qquad 3^{10q} \equiv 1 \pmod{M}$$

if b is even, while

$$3q \equiv 42p - 1 \pmod{M} \qquad 3^{5q} \equiv 10p - 1 \pmod{M}$$
$$3^{2q} \equiv 16p + 1 \pmod{M} \qquad 3^{10q} \equiv 1 \pmod{M}$$

if b is odd. In any event, we have to examine the sets [±1], [±7], [±49], and [±343].

Suppose b is even. Because $50p - 1$ is in the middle third, we eliminate [±1]. Note that $7(50p - 1) \equiv 30p - 7$, which is in the middle third, and that $49(50p - 1) \equiv 50p - 49$, which is also in the middle third, eliminating [±7] and [±49]. Now, $343(2p - 1) \equiv 46p - 343$, which is in the middle third except when $p = 5$ and $M = 400$. Coupling this with the fact that $3(343) \equiv 229 \pmod{400}$, we eliminate [±343] and, therefore, all of Class G.

Class H: Suppose $M = 2^a 5^b$ with $a \geq 5$ and $b \geq 2$. We write $M = 10p$ and $\phi(M) = 2^{a+1} 5^{b-1} = 80q$. Then double induction shows that:

$$3^q \equiv p + 1 \pmod{M} \qquad 3^{5q} \equiv 5p + 1 \pmod{M}$$
$$3^{2q} \equiv 2p + 1 \pmod{M} \qquad 3^{10q} \equiv 1 \pmod{M}$$

Once again, we must examine [±1], [±7], [±49], and [±343]. But $5p + 1$ is in the middle third, eliminating [±1]. Also, $7(2p + 1) \equiv 4p + 7$ and $49(5p + 1) \equiv 5p + 49$, so we may eliminate [±7] and [±49]. Because $343(5p + 1) \equiv 5p + 343$, we may eliminate [±343] except possibly for $p = 80$ ($M = 800$) and $p = 160$ ($M = 1600$). But $343/800 = 0.42\ldots$ and $3(343)/1600 = 0.64\ldots$, so the exceptional cases present no problem.

3. Special Cases

Classes A through H yield no terminating decimals in the Cantor set, so the only possible denominators are 2, 4, 5, 8, 10, 20, 40, and 80. If $M = 2$, the only fraction possible is 1/2, which is clearly in the middle third. For the other choices of M, we will simply list (in the order obtained) the elements of the sets $[j]$; an asterisk denotes a member of the middle third:

$M = 4 \qquad [1] = \{1, 3\}$

$M = 5 \qquad [1] = \{1, 3*, 4, 2*\}$

$$M = 8 \qquad \begin{aligned} [1] &= \{2, 3*\} \\ [-1] &= \{7, 5*\} \end{aligned}$$

$M = 10 \qquad [1] = \{1, 3, 9, 7\}$

$$M = 20 \qquad \begin{aligned} [1] &= \{1, 3, 9*, 7*\} \\ [-1] &= \{19, 17, 11*, 13*\} \end{aligned}$$

$$M = 40 \qquad \begin{aligned} [1] &= \{1, 3, 9, 27\} \\ [-1] &= \{39, 37, 31, 13\} \\ [7] &= \{7, 21*, 23*, 29\} \\ [-7] &= \{33, 19*, 17*, 11\} \end{aligned}$$

$$M = 80 \qquad \begin{aligned} [1] &= \{1, 3, 9, 27*\} \\ [-1] &= \{79, 77, 71, 53*\} \\ [7] &= \{7, 21, 63, 29*\} \\ [-7] &= \{73, 59, 17, 51*\} \\ [49] &= \{49*, 67, 41*, 43*\} \\ [-49] &= \{31*, 13, 39*, 37*\} \\ [343] &= \{23, 69, 47*, 61\} \\ [-343] &= \{57, 11, 33*, 19\} \end{aligned}$$

Thus, the terminating decimals in the Cantor set are precisely those claimed earlier.

Reference

1. C. R. Wall. Solution to Problem H-339. *Fibonacci Quarterly* *21.3* (1983):239.

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