TRIANGULAR NUMBERS

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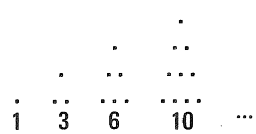
1. INTRODUCTION

To Fibonacci is attributed the arithmetic triangle of odd numbers, in which the n^{th} row has n entries, the center element is n^2 for even n, and the row sum is n^3. (See Stanley Bezuszka [11].)

FIBONACCI'S TRIANGLE	SUMS
1	$1 = 1^3$
3 5	$8 = 2^3$
7 9 11	$27 = 3^3$
13 15 17 19	$64 = 4^3$
21 23 25 27 29	$125 = 5^3$
....	....

We wish to derive some results here concerning the triangular numbers $1, 3, 6, 10, 15, \cdots, T_n, \cdots, \cdots$. If one observes how they are defined geometrically,



one easily sees that

$$T_n = 1+2+3+\cdots+n = \frac{n(n+1)}{2} \tag{1.1}$$

and

$$T_{n+1} = T_n + (n+1)\,. \tag{1.2}$$

By noticing that two adjacent arrays form a square, such as

$3+6 = 9$ 

we are led to

$$n^2 = T_n + T_{n-1}\,, \tag{1.3}$$

which can be verified using (1.1). This also provides an identity for triangular numbers in terms of subscripts which are also triangular numbers,

$$T_n^2 = T_{T_n} + T_{T_n-1}\,. \tag{1.4}$$

Since every odd number is the difference of two consecutive squares, it is informative to rewrite Fibonacci's triangle of odd numbers:

FIBONACCI'S TRIANGLE	SUMS
(1^2-0^2)	$T_1^2-T_0^2$
$(2^2-1^2)\quad(3^2-2^2)$	$T_2^2-T_1^2$
$(4^2-3^2)\quad(5^2-4^2)\quad(6^2-5^2)$	$T_3^2-T_2^2$
$(7^2-6^2)\quad(8^2-7^2)\quad(9^2-8^2)\quad(10^2-9^2)$	$T_4^2-T_3^2$
....	.....

Upon comparing with the first array, it would appear that the difference of the squares of two consecutive triangular numbers is a perfect cube. From (1.2),

$$T_{n+1}^2 = (T_n+n+1)^2 = T_n^2+2(n+1)T_n+(n+1)^2$$

But, from (1.1), $T_n = n(n+1)/2$, so that

$$\begin{aligned} T_{n+1}^2 - T_n^2 &= 2(n+1)[n(n+1)/2] + (n+1)^2 \\ &= n(n+1)^2+(n+1)^2 = (n+1)^3 \quad . \end{aligned}$$

Thus, we do indeed have

$$(1.5)\qquad T_{n+1}^2 - T_n^2 = (n+1)^3 ,$$

which also follows by simple algebra directly from (1.1).

Further,

$$\begin{aligned} T_n^2 &= (T_n^2-T_{n-1}^2)+(T_{n-1}^2-T_{n-2}^2)+\cdots+(T_2^2-T_1^2)+(T_1^2-T_0^2) \\ &= \quad n^3 \quad + \quad (n-1)^3 \quad + \cdots + \quad 2^3 \quad + \quad 1^3 \quad , \end{aligned}$$

or, again returning to (1.1),

$$(1.6)\qquad T_n^2 = (1+2+3+\cdots+n)^2 = \sum_{k=1}^{n} k^3 \quad .$$

For a wholly geometric discussion, see Martin Gardner [10].

Suppose that we now make a triangle of consecutive whole numbers.

WHOLE NUMBER TRIANGLE	SUMS
0	0
1 2	3
3 4 5	12
6 7 8 9	30
10 11 12 13 14	60
....	...

If we observe carefully, the row sum of the n^{th} row is nT_{n+1}, or $(n+2)T_n$, which we can easily derive by studying the form of each row of the triangle. Notice that the triangular numbers appear sequentially along the left edge. The n^{th} row, then, has elements

$$T_n \quad T_n+1 \quad T_n+2 \quad T_n+3 \quad \cdots \quad T_n+n$$

so that its sum is

$$(n+1)T_n+(1+2+3+\cdots+n) = (n+1)T_n+T_n = (n+2)T_n \ .$$

Also, the n^{th} row can be written as

$$T_n \quad T_{n+1}-n \quad \cdots \quad T_{n+1}-3 \quad T_{n+1}-2 \quad T_{n+1}-1$$

with row sum

$$T_n + nT_{n+1} - (1+2+3+\cdots+n) = T_n + nT_{n+1} - T_n = nT_{n+1}\ .$$

Then,

(1.7) $$nT_{n+1} = (n+2)T_n\ ,$$

which also follows from (1.1), since

$$nT_{n+1} = \frac{n(n+1)(n+2)}{2} = (n+2)T_n\ .$$

The row sums are also three times the binomial coefficients $1, 4, 10, 20, \cdots$, the entries in the third column of Pascal's left-justified triangle, since

$$nT_{n+1} = \frac{n(n+1)(n+2)}{2} = 3 \cdot \left[\frac{n(n+1)(n+2)}{3\cdot 2\cdot 1}\right] = 3 \cdot \binom{n+2}{3}\ .$$

The numbers $1, 4, 10, 20, \cdots$, are the triangular pyramidal numbers, the three-dimensional analog of the triangular numbers. Of course, the triangular numbers themselves are the binomial coefficients appearing in the second column of Pascal's triangle, so that, by mathematical induction or by applying known properties of binomial coefficients, we can sum the triangular numbers:

(1.8) $$T_n = \binom{n+1}{2}\ ; \qquad \sum_{k=0}^{n} T_k = \binom{n+2}{3}\ .$$

Finally, by summing over n rows of the whole number triangle and observing that the number on the right of the n^{th} row is $T_{n+1} - 1$,

(1.9) $$\sum_{j=1}^{n} jT_{j+1} = T_{T_{n+1}-1}\ ,$$

since, by (1.1), summing all elements of the triangle through the n^{th} row gives

$$0 + 1 + 2 + 3 + \cdots + (T_{n+1} - 1) = T_{T_{n+1}-1}\ .$$

Let us start again with

```
         1
       2   3
     4   5   6
   7   8   9   10
 . . . . . . . . . . . . . . .
```

This time we observe the triangular numbers are along the right edge. Each row sum, using our earlier process, is

$$nT_n - T_{n-1} = (n-1)T_{n-1} + n^2 = (n+1)T_n - n\ .$$

Clearly, the sum over n rows gives us

(1.10) $$T_{T_n} = T_{T_{n}-1} + T_n$$

or, referring again to the row sum of $(n-1)T_{n-1} + n^2$ and to Equation (1.3),

$$T_{T_n} = \sum_{j=1}^{n} [(j-1)T_{j-1} + j^2] = \sum_{j=1}^{n} [(j-1)T_{j-1} + T_j + T_{j-1}]$$

$$= \sum_{j=1}^{n-1} jT_j + \sum_{j=1}^{n} T_j + \sum_{j=1}^{n-1} T_j = \sum_{j=1}^{n-1} (j+2)T_j + T_n\ .$$

Therefore, from (1.10),

(1.11) $$T_{T_{n}-1} = \sum_{j=1}^{n-1} (j+2)T_j\ .$$

It is also easy to establish that

(1.12) $$T_{2n} = 3T_n + T_{n-1}\,,$$

and

(1.13) $$T_{2n} - 2T_n = n^2\,,$$

(1.14) $$T_{2n-1} - 2T_{n-1} = n^2\,.$$

2. GENERATING FUNCTIONS

Consider the array A

$$\begin{array}{lllll} 1 \\ 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

We desire to find the generating functions for the columns. The first column entries are clearly one more than the triangular numbers T_n, $(n = 0, 1, 2, \cdots)$. Thus, since the generating function for triangular numbers (as well as for the other columns of Pascal's triangle) is known,

$$G_0(x) = \sum_{n=0}^{\infty} (T_n + 1)x^n = \frac{x}{(1-x)^3} + \frac{1}{1-x} = \frac{1-x+x^2}{(1-x)^3}\,.$$

We shall see that generally the column generators are

(2.1) $$G_k(x) = \frac{T_{k+1} - (k+1)^2x + (T_k+1)x^2}{(1-x)^3} = \frac{T_{k+1} - (T_{k+1}+T_k)x + (T_k+1)x^2}{(1-x)^3}$$

PROOF: Clearly, $G_o(x)$ is given by the formula above when $k = 0$. Assume that

$$G_k(x) = \frac{T_{k+1} - (k+1)^2x + (T_k+1)x^2}{(1-x)^3}\,.$$

Then, since each column is formed from the preceding by subtracting the first entry T_{k+1}, and adding one, the $(k+1)^{st}$ column generator is

$$G_{k+1}(x) = \left(\frac{T_{k+1} - (k+1)^2x + (T_k+1)x^2}{(1-x)^3} - T_{k+1} \right) \Big/ x + \frac{1}{1-x}$$

$$= \frac{T_{k+1} - (k+1)^2x + (T_k+1)x^2 - (1-3x+3x^2-x^3)T_{k+1}}{x(1-x)^3} + \frac{1}{1-x}$$

$$= \frac{(3T_{k+1} - (k+1)^2) + (T_k + 1 - 3T_{k+1})x + T_{k+1}x^2 + (1-2x+x^2)}{(1-x)^3}$$

Now, from $(k+1)^2 = T_k + T_{k+1}$ and $T_k = T_{k-1} + k$, this becomes

$$G_{k+1}(x) = [3T_{k+1} + 1 - (T_k + T_{k+1}) + (T_k - 1 - 3T_{k+1})x + (T_{k+1}+1)x^2]/(1-x)^3$$

$$= [(2T_{k+1} - T_k + 1) - (3T_{k+1} + 1 - T_k)x + (T_{k+1}+1)x^2]/(1-x)^3$$

$$= \frac{(T_{k+2}) - (T_{k+2} + T_{k+1})x + (T_{k+1}+1)x^2}{(1-x)^3} = \frac{T_{k+2} - (k+2)^2x + (T_{k+1}+1)x^2}{(1-x)^3}\,.$$

This may now be exploited as any triangular array.

We now proceed to another array B (Fibonacci's triangle).

$$\begin{array}{lllll} 1 \\ 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 & 19 \\ 21 & 23 & 25 & 27 & 29 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

We can tackle this immediately since we have already found the generators for array A, because each entry in array B is twice the corresponding entry in array A, less one. Thus the column generators are

(2.2)
$$G_k^*(x) = \frac{2[T_{k+1} - (k+1)^2x + (T_k+1)x^2]}{(1-x)^3} - \frac{1-2x+x^2}{(1-x)^3} = \frac{(2T_{k+1}-1) - 2[(k+1)^2-1]x + (2T_k+1)x^2}{(1-x)^3}.$$

Now since the row sums of Fibonacci's triangle are the cubes of successive integers, we can find a generating function for the cubes.

$$\sum_{k=0}^{\infty} x^k G_k^*(x) = \left(2\sum_{k=0}^{\infty} T_{k+1}x^k - \sum_{k=0}^{\infty} x^k - 2x\sum_{k=0}^{\infty}(k+1)^2x^k + 2x\sum_{k=0}^{\infty} x^k + 2x^2\sum_{k=0}^{\infty} T_kx^k + x^2\sum_{k=0}^{\infty} x^k\right)\Bigg/(1-x)^3 .$$

But

(2.3)
$$\sum_{k=0}^{\infty} T_{k+1}x^k = \frac{1}{(1-x)^3} \quad\text{and}\quad \sum_{k=0}^{\infty} T_kx^k = \frac{x}{(1-x)^3}$$

(2.4)
$$\sum_{k=0}^{\infty}(k+1)^2x^k = \frac{1+x}{(1-x)^3} = \sum_{k=0}^{\infty}(T_{k+1}+T_k)x^k$$

(2.5)
$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} .$$

Thus, applying (2.3), (2.4), and (2.5),

(2.6)
$$\sum_{k=0}^{\infty} x^k G_k^*(x) = \frac{2-(1-x)^2-2x(1+x)+2x(1-x)^2+2x^3+x^2(1-x)^2}{(1-x)^3(1-x)^3}$$

$$= \frac{(1+4x+x^2)(1-x)^2}{(1-x)^6} = \frac{1+4x+x^2}{(1-x)^4} = \sum_{k=0}^{\infty}(k+1)^3x^k .$$

Further extensions of arrays A and B will be found in a thesis by Robert Anaya [1].

Equation (2.6) also says that, for any three consecutive members of the third column of Pascal's triangle, the sum of the first and third, and four times the second, is a cube, or

$$\binom{n}{3} + 4\binom{n-1}{3} + \binom{n-2}{3} = n^3 .$$

Observe that

$$\binom{n}{2} + \binom{n-1}{2} = n^2 \quad\text{and}\quad \binom{n}{1} = n.$$

We can find

$$1\binom{n}{4} + 11\binom{n-1}{4} + 11\binom{n-2}{4} + 1\binom{n-3}{4} = n^4$$

by solving for the coefficients in the beginning values, using column 4 $(1, 5, 15, 35, \cdots)$, in the order given:

$$1 \cdot x_1 = 1^4$$
$$5 \cdot x_1 + 1 \cdot x_2 = 2^4$$
$$15 \cdot x_1 + 5 \cdot x_2 + 1 \cdot x_3 = 3^4$$
$$35 \cdot x_1 + 15 \cdot x_2 + 5 \cdot x_3 + 1 \cdot x_4 = 4^4$$

In the same manner,

$$\binom{n}{5}+26\binom{n-1}{5}+66\binom{n-2}{5}+26\binom{n-3}{5}+\binom{n-4}{5}=n^5 .$$

Applying this method to the k^{th} column, we obtain

$$n^k=\sum_{i=1}^{k}\left[\sum_{j=0}^{i}(i-j)^k(-1)^j\binom{k+1}{k+1-j}\right]\binom{n+1-i}{k} \quad . \tag{2.7}$$

Returning to generating functions, (2.3) is a generating function for the triangular numbers. The triangular numbers generalize to the polygonal numbers $P(n,k)$,

$$P(n,k) = [k(n-1)-2(n-2)]n/2 , \tag{2.8}$$

the n^{th} polygonal number of k sides. Note that $P(n,3)=T_n$, the n^{th} triangular number, and $P(n,4)=n^2$, the n^{th} square number. A generating function for $P(n,k)$ is

$$\frac{1+(k-3)x}{(1-x)^3}=\sum_{n=0}^{\infty}P(n,k)x^n \quad . \tag{2.9}$$

The sums of the corresponding polygonal numbers are the pyramidal numbers [9] which are generated by

$$\frac{1+(k-3)x}{(1-x)^4}=\sum_{n=0}^{\infty}P^*(n,k)x^n , \tag{2.10}$$

where $P^*(n,k)$ is the n^{th} pyramidal number of order k. Notice that $k=3$ gives the generating function for the triangular numbers and for the triangular pyramidal numbers, which are the sums of the triangular numbers.

3. SOME MORE ARITHMETIC PROGRESSIONS

It is well known that the k^{th} column sequence of Pascal's left-adjusted triangle is an arithmetic progression of order k with common difference of 1. In this section, we discuss subsequences of these whose subscripts are triangular numbers. To properly set the stage, we need first to discuss polynomials whose coefficients are the Eulerian numbers. (See Riordan [2].)

Let

$$\frac{A_k(x)}{(1-x)^{k+1}}=\sum_{n=0}^{\infty}n^kx^n \quad . \tag{3.1}$$

Differentiate and multiply by x, to obtain

$$\frac{x(1-x)A_k'(x)+x(k+1)A_k(x)}{(1-x)^{k+2}}=\sum_{n=0}^{\infty}n^{k+1}x^n \quad .$$

But, by definition,

$$\frac{A_{k+1}(x)}{(1-x)^{k+2}}=\sum_{n=0}^{\infty}n^{k+1}x^n$$

so that

$$A_{k+1}(x) = x(1-x)A_k'(x)+x(k+1)A_k(x) . \tag{3.2}$$

Since, from Section 2,

$$\sum_{n=0}^{\infty}n^1x^n=\frac{x}{(1-x)^2}, \qquad A_1(x)=x$$

$$\sum_{n=0}^{\infty}n^2x^n=\frac{x+x^2}{(1-x)^3}, \qquad A_2(x)=x+x^2$$

$$\sum_{n=0}^{\infty}n^3x^n=\frac{x+4x^2+x^3}{(1-x)^4}, \qquad A_3(x)=x+4x^2+x^3 .$$

From the recurrence it is easy to see that by a simple inductive argument,

$$A_k(1) = k! \quad .$$

Also, we can easily write $A_4(x) = x^4 + 11x^3 + 11x^2 + x$, which allows us to demonstrate Eq. (1.6) in a second way. Thus, using $T_n = n(n+1)/2$, and the generating functions just listed,

$$\sum_{n=0}^{\infty} T_n^2 x^n = \sum_{n=0}^{\infty} \frac{(n^4+2n^3+n^2)}{4} x^n$$

$$= \frac{1}{4} \cdot \left[\frac{x^4+11x^3+11x^2+x}{(1-x)^5} + \frac{2(1-x)(x^3+4x^2+x)}{(1-x)^5} + \frac{(1-x)^2(x^2+x)}{(1-x)^5} \right] = \frac{x^3+4x^2+x}{(1-x)^5}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} k^3 x^n$$

so that

$$T_n^2 = (1+2+3+\cdots+n)^2 = \sum_{k=0}^{n} k^3 \quad .$$

Now we can write

$$A_k(x) = \sum_{n=1}^{k} \left[\sum_{j=0}^{n} (n-j)^k (-1)^j \binom{k+1}{k+1-j} \right] x^n \quad , \tag{3.3}$$

from (2.4) by applying the generating function to Pascal's triangle. Notice that $A_1(x)$, $A_2(x)$, $A_3(x)$, and $A_4(x)$ all have the form given in (3.3).

Next, from a thesis by Judy Kramer [3], we have the following theorem.

Theorem 57. If generating function

$$A(x) = \frac{N(x)}{(1-x)^{r+1}} \quad ,$$

where $N(x)$ is a polynomial of maximum degree r, then $A(x)$ generates an arithmetic progression of order r, and the constant of the progression is $N(1)$.

We desire now to look at

$$\frac{1}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \sum_{n=0}^{\infty} \frac{(n+k)(n+k-1)\cdots(n+1)}{k!} x^n = \sum_{n=0}^{\infty} Q(n,k)x^n .$$

Now consider

$$G(x) = \sum_{n=0}^{\infty} Q(T_n, k)x^n \quad ,$$

where T_n is the n^{th} triangular number. Clearly, this is a polynomial in n of degree $2k$. Let us assume it is expanded

$$Q(T_n,k) = \sum_{j=0}^{2k} b_j n^j \qquad \text{and} \qquad \frac{A_j(x)}{(1-x)^{j+1}} = \sum_{n=0}^{\infty} n^j x^n$$

so that

$$G(x) = \sum_{j=0}^{2k} \frac{b_j A_j(x)}{(1-x)^{j+1}} = \frac{N_k(x)}{(1-x)^{2k+1}} \quad .$$

All of the $A_j(x)$ are multiplied by powers of $(1-x)$ in $N_k(x)$ except $A_{2k}(x)$; thus,

$$N_k(1) = A_{2k}(1) = (2k)!/2^k k! \ ,$$

which is, of course, an integer. Thus $Q(T_n,k)$ is an arithmetic progression of order $2k$ and common difference $d = (2k)!/2^k k!$. The general result is that, for

$$G^*(x) = \sum_{n=0}^{\infty} Q\left(Q(n,m),k\right)x^n$$

$Q(Q(n,m),k)$ is an arithmetic progression of order mk and common difference $d = (mk)!/m^k k!$ which thus must be an integer.

4. PALINDROMIC TRIANGULAR NUMBERS

There are 27 triangular numbers T_n, $n < 151340$, which are palindromes in base 10, as given by Trigg [8]. However, borrowing from Leonard [4] and Merrill [5], every number in array C is a triangular number:

(C)
1
11
111
1111
11111
.....

Clearly, base 10 is ruled out, but array C indeed provides triangular numbers in base 9. Below we discuss some interesting consequences including a proof.

Let $T_{U_n} = (11111 \cdots 1)_9 = C_n$ (n one's) so that

$$C_n = 9^n + 9^{n-1} + 9^{n-2} + \cdots + 9 + 1 = (9^{n+1} - 1)/(9 - 1) .$$

Now

$$T_{U_n} = \frac{U_n(U_n + 1)}{2} ,$$

where U_n, written in base 3 notation, has n one's,

$$U_n = (1111 \cdots 1)_3 = (3^{n+1} - 1)/(3 - 1).$$

Then

$$T_{U_n} = \frac{1}{2}\cdot\left(\frac{3^{n+1}-1}{3-1}\right)\left(\frac{3^{n+1}-1}{3-1}+1\right) = \frac{(3^{n+1}-1)(3^{n+1}+1)}{8} = \frac{9^{n+1}-1}{9-1} = C_n .$$

Also, it is simple to show that if T_n is any triangular number, then so is

(4.1) $$9T_n + 1 = T_{3n+1}$$

since

$$9T_n + 1 = \frac{9n(n+1)}{2} + 1 = \frac{9n^2 + 9n + 2}{2} = \frac{(3n+1)(3n+2)}{2} = T_{3n+1} .$$

This means that, if T_n is any triangular number written in base 9 notation, annexing any number of 1's on the right provides another triangular number, and the new subscript can be found by annexing the same number of 1's to the subscript of T_n, where n is written in base 3 notation. The numbers in array C, then, are a special case of Eq. (4.1).

Three other interesting sets of palindromic triangular numbers occur in bases 3, 5, and 7. In each case below, the triangular number as well as its subscript are expressed in the base given.

Base 3	Base 5	Base 7
$T_1 = 1$	$T_2 = 3$	$T_3 = 6$
$T_{11} = 101$	$T_{22} = 303$	$T_{33} = 606$
$T_{111} = 10101$	$T_{222} = 30303$	$T_{333} = 60606$
$T_{1111} = 1010101$	$T_{2222} = 3030303$	$T_{3333} = 6060606$
....	....	....

Now, base 3 uses only even powers of 3, so the base 9 proof applies. For base 5, if T_n is any triangular number, then

(4.2) $$25T_n + 3 = T_{5n+2}$$

since

$$25T_n + 3 = \frac{25n(n+1)}{2} + 3 = \frac{25n^2 + 25n + 6}{2} = \frac{(5n+2)(5n+3)}{2} = T_{5n+2}$$

so that annexing 03 to any triangular number written in base 5 notation provides another triangular number whose subscript can be found by annexing 2 to the right of the original subscript in base 5 notation. Base 7 is demonstrated similarly from the identity

$$(4.3) \qquad 49T_n + 6 = T_{7n+3} .$$

Using similar reasoning, if any triangular number is written in base 8, annexing 1 to the right will provide a square number, since

$$(4.4) \qquad 8T_n + 1 = (2n+1)^2 .$$

For example, $T_6 = (25)_8$ and $(251)_8 = 169 = 13^2$.

Any odd base $(2k+1)$ has an "annexing property" for triangular numbers, for (4.3) generalizes to

$$(4.5) \qquad T_{(2k+1)n+k} = (2k+1)^2 T_n + T_k ,$$

but other identities of the pleasing form given may require special digit symbols, and T_k must be expressed in base $(2k+1)$. Some examples follow, where both numbers and subscripts are expressed in the base given.

Base 9	Base 17	Base 25 $(t)_{25} = (12)_{10}$
$T_4 = 11$	$T_8 = 22$	$T_t = 33$
$T_{44} = 1111$	$T_{88} = 2222$	$T_{tt} = 3333$
$T_{444} = 111111$	$T_{888} = 222222$	$T_{ttt} = 333333$
... ...	... ...	... ...

Base 33 $(s)_{33} = (16)_{10}$	Base 41 $(q)_{41} = (20)_{10}$	Base 49 $(r)_{49} = (24)_{10}$
$T_s = 44$	$T_q = 55$	$T_r = 66$
$T_{ss} = 4444$	$T_{qq} = 5555$	$T_{rr} = 6666$
$T_{sss} = 444444$	$T_{qqq} = 555555$	$T_{rrr} = 666666$
... ...	... ...	... ...

Base 57 $(m)_{57} = (28)_{10}$	Base 65 $(n)_{65} = (32)_{10}$	Base 73 $(p)_{73} = (36)_{10}$
$T_m = 77$	$T_n = 88$	$T_p = 99$
$T_{mm} = 7777$	$T_{nn} = 8888$	$T_{pp} = 9999$
$T_{mmm} = 777777$	$T_{nnn} = 888888$	$T_{ppp} = 999999$
... ...	... ...	... ...

Base 19 $(t)_{19} = (12)_{10}$
$T_9 = tt$
$T_{99} = tttt$
$T_{999} = tttttt$
... ...

5. GENERALIZED BINOMIAL COEFFICIENTS FOR TRIANGULAR NUMBERS

Walter Hansell [6] formed generalized binomial coefficients from the triangular numbers,

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{T_m T_{m-1} \cdots T_{m-n+1}}{T_n T_{n-1} \cdots T_1}, \qquad 0 < n \leqslant m .$$

That these are integers doesn't fall within the scope of Hoggatt [7]. However, it is not difficult to show. Since $T_m = m(m+1)/2$,

$$\begin{bmatrix} m \\ n \end{bmatrix} = \binom{m}{n}\binom{m+1}{n+1}\frac{1}{m-n+1},$$

where $\binom{m}{n}$ are the ordinary binomial coefficients, so that $\begin{bmatrix} m \\ n \end{bmatrix}$ are indeed integers if one defined

$$\begin{bmatrix} m \\ 0 \end{bmatrix} = \begin{bmatrix} m \\ m \end{bmatrix} = 1,$$

as will be seen in the next paragraph or two.

The generalized binomial coefficients for the triangular numbers are

$$\begin{array}{llllll} 1 \\ 1 & 1 \\ 1 & 3 & 1 \\ 1 & 6 & 6 & 1 \\ 1 & 10 & 20 & 10 & 1 \\ 1 & 15 & 50 & 50 & 15 & 1 \\ 1 & 21 & 105 & \cdots & \cdots & \cdots \\ \cdots \end{array}$$

If the Catalan numbers $C_n = 1, 1, 2, 5, 14, 42, 132, \cdots,$ are given by

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n,$$

then we note that the row sums are the Catalan numbers, C_{n+1}.

We compare elements in corresponding positions in Pascal's triangle of ordinary binomial coefficients and in the triangular binomial coefficient array:

$$\begin{array}{lllll} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & \textcircled{3} & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ \cdots \end{array} \qquad \begin{array}{lllll} 1 \\ 1 & 1 \\ 1 & 3 & 1 \\ 1 & \textcircled{6} & 6 & 1 \\ 1 & 10 & 20 & 10 & 1 \\ \cdots \end{array}$$

Let us examine

$$\begin{bmatrix} m \\ n \end{bmatrix} = \begin{vmatrix} \binom{m}{n} & \binom{m}{n+1} \\ \binom{m+1}{n} & \binom{m+1}{n+1} \end{vmatrix} = \binom{m}{n}\binom{m+1}{n+1}\cdot\frac{1}{m-n+1}$$

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