Roots $x_k(y)$ of a formal power series $$f(x,y) = \sum_{n=0}^{\infty} a_n(y)\, x^n$$ with applications to graph enumeration and q-series

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LECTURE #2

Applications of
the explicit implicit function formula
and the exponential formula

The basic set-up

Consider a formal power series

$$f(x,y) \;=\; \sum_{n=0}^{\infty} \alpha_n \, x^n \, y^{n(n-1)/2}$$

normalized to $\alpha_0 = \alpha_1 = 1$, or more generally

$$f(x,y) \;=\; \sum_{n=0}^{\infty} a_n(y) \, x^n$$

where

(a) $a_0(0) = a_1(0) = 1$;

(b) $a_n(0) = 0$ for $n \ge 2$; and

(c) $a_n(y) = O(y^{\nu_n})$ with $\lim\limits_{n\to\infty} \nu_n = \infty$.

Examples:

- The "partial theta function"

$$\Theta_0(x,y) \;=\; \sum_{n=0}^{\infty} x^n \, y^{n(n-1)/2}$$

- The "deformed exponential function" studied in Lecture #1:

$$F(x,y) \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!} \, y^{n(n-1)/2}$$

- More generally, consider

$$\widetilde{R}(x,y,q) \;=\; \sum_{n=0}^{\infty} \frac{x^n \, y^{n(n-1)/2}}{(1+q)(1+q+q^2) \, \cdots \, (1+q+\ldots+q^{n-1})}$$

which reduces to Θ_0 when $q=0$, and to F when $q=1$.

The leading root $x_0(y)$

- Start from a formal power series

$$f(x,y) \;=\; \sum_{n=0}^{\infty} a_n(y)\, x^n$$

  where

  (a) $a_0(0) = a_1(0) = 1$
  (b) $a_n(0) = 0$ for $n \ge 2$
  (c) $a_n(y) = O(y^{\nu_n})$ with $\lim\limits_{n\to\infty} \nu_n = \infty$

  and coefficients lie in a commutative ring-with-identity-element R.

- By (c), each power of y is multiplied by only *finitely many* powers of x.

- That is, f is a formal power series in y whose coefficients are *polynomials* in x, i.e. $f \in R[x][[y]]$.

- Hence, for *any* formal power series $X(y)$ with coefficients in R [not necessarily with zero constant term], the composition $f(X(y), y)$ makes sense as a formal power series in y.

- Not hard to see (by the implicit function theorem for formal power series or by a direct inductive argument) that there exists a unique formal power series $x_0(y) \in R[[y]]$ satisfying $f(x_0(y), y) = 0$.

- We call $x_0(y)$ the **leading root** of f.

- Since $x_0(y)$ has constant term -1, we will write $x_0(y) = -\xi_0(y)$ where $\xi_0(y) = 1 + O(y)$.

How to compute $\xi_0(y)$?

1. **Elementary method:** Insert $\xi_0(y) = 1 + \sum_{n=1}^{\infty} b_n y^n$ into $f(-\xi_0(y), y) = 0$ and solve term-by-term.

2. Method based on the explicit implicit function formula.

3. Method based on the exponential formula and expansion of $\log f(x, y)$.

- Methods #2 and #3 are computationally very efficient.
- Can they also be used to give *proofs*?

Tools I: The explicit implicit function formula

- See A.D.S., arXiv:0902.0069 or Stanley, vol. 2, Exercise 5.59
- (Almost trivial) generalization of Lagrange inversion formula
- Comes in analytic-function and formal-power-series versions
- Recall Lagrange inversion: If $f(x) = \sum_{n=1}^{\infty} a_n x^n$ with $a_1 \neq 0$ (as either analytic function or formal power series), then

$$f^{-1}(y) \;=\; \sum_{m=1}^{\infty} \frac{y^m}{m} \, [\zeta^{m-1}] \biggl(\frac{\zeta}{f(\zeta)} \biggr)^{m}$$

  where $[\zeta^n]g(\zeta)$ denotes the coefficient of ζ^n in the power series $g(\zeta)$. More generally, if $h(x) = \sum_{n=0}^{\infty} b_n x^n$, we have

$$h(f^{-1}(y)) \;=\; h(0) \,+\, \sum_{m=1}^{\infty} \frac{y^m}{m} \, [\zeta^{m-1}] \, h'(\zeta) \biggl(\frac{\zeta}{f(\zeta)} \biggr)^{m}$$

- Rewrite this in terms of $g(x) = x/f(x)$: then $f(x) = y$ becomes $x = g(x)y$, and its solution $x = \varphi(y) = f^{-1}(y)$ is given by the power series

$$\varphi(y) \;=\; \sum_{m=1}^{\infty} \frac{y^m}{m} \, [\zeta^{m-1}] g(\zeta)^m$$

  and

$$h(\varphi(y)) \;=\; h(0) \,+\, \sum_{m=1}^{\infty} \frac{y^m}{m} \, [\zeta^{m-1}] h'(\zeta) g(\zeta)^m$$

- There is also an alternate form

$$h(\varphi(y)) \;=\; h(0) \,+\, \sum_{m=1}^{\infty} y^m \, [\zeta^m] h(\zeta) \bigl[g(\zeta)^m \,-\, \zeta g'(\zeta) g(\zeta)^{m-1} \bigr]$$

The explicit implicit function formula, continued

- Generalize $x = g(x)y$ to $x = G(x, y)$, where
  - $G(0,0) = 0$ and $|(\partial G/\partial x)(0,0)| < 1$ (analytic-function version)
  - $G(0,0) = 0$ and $(\partial G/\partial x)(0,0) = 0$ (formal-power-series version)
- Then there is a unique $\varphi(y)$ with zero constant term satisfying $\varphi(y) = G(\varphi(y), y)$, and it is given by

$$\begin{aligned}\varphi(y) &= \sum_{m=1}^{\infty} \frac{1}{m} [\zeta^{m-1}] G(\zeta, y)^m \\ &= \sum_{m=1}^{\infty} [\zeta^{m-1}] \Big[G(\zeta, y)^m - \zeta \frac{\partial G(\zeta, y)}{\partial \zeta} G(\zeta, y)^{m-1} \Big]\end{aligned}$$

  More generally, for any $H(x, y)$ we have

$$\begin{aligned}H(\varphi(y), y) &= H(0, y) + \sum_{m=1}^{\infty} \frac{1}{m} [\zeta^{m-1}] \frac{\partial H(\zeta, y)}{\partial \zeta} G(\zeta, y)^m \\ &= H(0, y) + \sum_{m=1}^{\infty} [\zeta^m] H(\zeta, y) \Big[G(\zeta, y)^m - \zeta \frac{\partial G(\zeta, y)}{\partial \zeta} G(\zeta, y)^{m-1} \Big]\end{aligned}$$

- First versions are slightly more convenient but require R to contain the rationals as a subring.
- Proof imitates standard proof of the Lagrange inversion formula: the variables y simply "go for the ride".
- Alternate interpretation: Solving fixed-point problem for the family of maps $x \mapsto G(x, y)$ parametrized by y. Variables y again "go for the ride".

A possible extension [open problem]

- Conditions on G and φ in the explicit implicit function formula seem natural:
  - If $G(x, y)$ is a formal power series, it ordinarily makes sense to substitute $x = \varphi(y)$ only when $\varphi(y)$ is a formal power series *with zero constant term.*
  - Then a solution to the fixed-point equation $\varphi(y) = G(\varphi(y), y)$ with $\varphi(y)$ having zero constant term can exist only if $G(0, 0) = 0$.
- But there is one important case where these conditions can be weakened: namely, if $G(x, y)$ belongs to $R[x][[y]]$, i.e. if the coefficient of each power of y is a *polynomial* in x.
  - In this case it makes sense to substitute for x an *arbitrary* formal power series $\varphi(y)$, *not necessarily with zero constant term.*
  - The result $G(\varphi(y), y)$ is a well-defined formal power series in y.
  - What can be said about existence and uniqueness of solutions to $\varphi(y) = G(\varphi(y), y)$?
  - And is there an explicit "Lagrange-like" formula for $\varphi(y)$?
  - I suspect that the answer is yes, but I haven't worked out the details.
  - And it looks like this may be useful in our application.

Application to leading root of $f(x, y)$

- Start from a formal power series $f(x, y) = \sum_{n=0}^{\infty} a_n(y)\, x^n$ satisfying properties (a)–(c) above.
- Write out $f(-\xi_0(y), y) = 0$ and add $\xi_0(y)$ to both sides:

$$\xi_0(y) \;=\; a_0(y) \;-\; [a_1(y) - 1]\xi_0(y) \;+\; \sum_{n=2}^{\infty} a_n(y)\, (-\xi_0(y))^n$$

- Insert $\xi_0(y) = 1+\varphi(y)$ where $\varphi(y)$ has zero constant term. Then $\varphi(y) = G(\varphi(y), y)$ where

$$G(z, y) \;=\; \sum_{n=0}^{\infty} (-1)^n\, \widehat{a}_n(y)\, (1 + z)^n$$

  and

$$\widehat{a}_n(y) \;=\; \begin{cases} a_n(y) - 1 & \text{for } n = 0, 1 \\ a_n(y) & \text{for } n \geq 2 \end{cases}$$

  And $\varphi(y)$ is the *unique* formal power series with zero constant term satisfying this fixed-point equation.

- Since this G satisfies $G(0, 0) = 0$ and $(\partial G/\partial z)(0, 0) = 0$ [indeed it satisfies the stronger condition $G(z, 0) = 0$], we can apply the explicit implicit function formula to obtain an explicit formula for $\xi_0(y)$:

$$\xi_0(y) \;=\; 1 \;+\; \sum_{m=1}^{\infty} \frac{1}{m}\, [\zeta^{m-1}] \left(\sum_{n=0}^{\infty} (-1)^n\, \widehat{a}_n(y)\, (1 + \zeta)^n \right)^{m}$$

  More generally, for any formal power series $H(z, y)$, we have

$$\begin{aligned} &H(\xi_0(y) - 1, y) \\ &\;=\; H(0, y) \;+\; \sum_{m=1}^{\infty} \frac{1}{m}\, [\zeta^{m-1}] \frac{\partial H(\zeta, y)}{\partial \zeta} \left(\sum_{n=0}^{\infty} (-1)^n\, \widehat{a}_n(y)\, (1 + \zeta)^n \right)^{m} \end{aligned}$$

Application to leading root of $f(x, y)$, continued

- In particular, by taking $H(z, y) = (1+z)^\beta$ we can obtain an explicit formula for an arbitrary power of $\xi_0(y)$:

$$\xi_0(y)^\beta \;=\; 1 + \sum_{m=1}^{\infty} \frac{\beta}{m} \sum_{n_1,\ldots,n_m \ge 0} \binom{\beta - 1 + \sum n_i}{m-1} \prod_{i=1}^{m} (-1)^{n_i} \widehat{a}_{n_i}(y)$$

- Important special case: $a_0(y) = a_1(y) = 1$ and $a_n(y) = \alpha_n\, y^{\lambda_n}$ $(n \ge 2)$ where $\lambda_n \ge 1$ and $\lim\limits_{n\to\infty} \lambda_n = \infty$. Then

$$[y^N] \frac{\xi_0(y)^\beta - 1}{\beta} \;=\; \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{n_1,\,\ldots,\,n_m \ge 2 \\ \sum_{i=1}^{m} \lambda_{n_i} = N}} (-1)^{\sum n_i} \binom{\beta - 1 + \sum n_i}{m-1} \prod_{i=1}^{m} \alpha_{n_i}$$

- Can this formula be used for proofs of nonnegativity???
- *Empirically* I know that the RHS is ≥ 0 when $\lambda_n = n(n-1)/2$:
  - For $\beta \ge -2$ with $\alpha_n = 1$ (partial theta function)
  - For $\beta \ge -1$ with $\alpha_n = 1/n!$ (deformed exponential function)
  - For $\beta \ge -1$ with $\alpha_n = (1-q)^n/(q;q)_n$ and $q > -1$
- And I can *prove* this (by a *different* method!) for the partial theta function with $\beta \ge -1$
- **How can we see these facts from this formula???**
  [open combinatorial problem]

Tools II: Variants of the exponential formula

- Let R be a commutative ring containing the rationals.
- Let $A(x) = \sum_{n=0}^{\infty} a_n\, x^n$ be a formal power series (with coefficients in R) satisfying $a_0 = 1$.
- Now consider $C(x) = \log A(x) = \sum_{n=1}^{\infty} c_n\, x^n$.
- It is well known (and easy to prove) that
$$a_n \;=\; \sum_{k=1}^{n} \frac{k}{n}\, c_k\, a_{n-k} \qquad \text{for } n \ge 1$$
This allows $\{a_n\}$ to be calculated given $\{c_n\}$, or vice versa.
- Sometimes useful to introduce $\widetilde{c}_n = nc_n$, which are the coefficients in
$$\frac{x\, A'(x)}{A(x)} \;=\; \sum_{n=1}^{\infty} \widetilde{c}_n\, x^n$$
- See Scott–Sokal, arXiv:0803.1477 for generalizations to $A(x)^{\lambda}$ and applications to the multivariate Tutte polynomial

- Now specialize to $R = R_0[[y]]$ and $A(x,y) = \sum_{n=0}^{\infty} a_n(y)\, x^n$ where $a_0(y) = 1$
- Assume further that $a_1(0) = 1$ and $a_n(0) = 0$ for $n \ge 2$ [conditions (a) and (b) for our $f(x,y)$]
- Then
$$\frac{x\, A'(x,y)}{A(x,y)} \;=\; \sum_{n=1}^{\infty} \widetilde{c}_n(y)\, x^n$$
where $'$ denotes $\partial/\partial x$ and $\widetilde{c}_n(y)$ has constant term $(-1)^{n-1}$.

Application to leading root of $f(x, y)$

- Start from a formal power series $f(x,y) = 1 + x + \sum_{n=2}^{\infty} a_n(y)\, x^n$ satisfying
$$a_n(y) \;=\; O(y^{\alpha(n-1)}) \qquad \text{for } n \geq 2$$
for some real $\alpha > 0$. [This is a bit stronger than (a)–(c).]

- Define $\{\widetilde{c}_n(y)\}_{n=1}^{\infty}$ by
$$\frac{x\, f'(x,y)}{f(x,y)} \;=\; \sum_{n=1}^{\infty} \widetilde{c}_n(y)\, x^n$$
where $'$ denotes $\partial/\partial x$.

- **Theorem:** We have
$$\widetilde{c}_n(y) \;=\; (-1)^{n-1}\, \xi_0(y)^{-n} \;+\; O(y^{\alpha n})$$
or equivalently
$$\xi_0(y) \;=\; [(-1)^{n-1} \widetilde{c}_n(y)]^{-1/n} \;+\; O(y^{\alpha n})$$

- This theorem provides an extraordinarily efficient method for computing the series $\xi_0(y)$:
  - Compute the $\widetilde{c}_n(y)$ inductively using the recursion
  $$\widetilde{c}_n \;=\; n a_n \;-\; \sum_{k=1}^{n-1} \widetilde{c}_k\, a_{n-k}$$
  - Take the power $-1/n$ to extract $\xi_0(y)$ through order $y^{\lceil \alpha n \rceil - 1}$

- This abstracts the recursive method shown in Lecture #1 for the special case $F(x,y) \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}\, y^{n(n-1)/2}$.

Proof of Theorem (via complex analysis)

- Use complex-analysis argument to prove Theorem when $R = \mathbb{C}$ and f is a polynomial.
- Infer general validity by some abstract nonsense.

Lemma. Fix a real number $\alpha > 0$, and let $P(x, y) = 1 + x + \sum_{n=2}^{N} a_n(y)x^n$ where the $\{a_n(y)\}_{n=2}^{N}$ are polynomials with complex coefficients satisfying $a_n(y) = O(y^{\alpha(n-1)})$. Then there exist numbers $\rho > 0$ and $\gamma > 0$ such that $P(\,\cdot\,, y)$ has precisely one root in the disc $|x| < \gamma|y|^{-\alpha}$ whenever $|y| \le \rho$.

Idea of proof: Apply Rouché's theorem to $f(x) = x$ and $g(x) = 1 + \sum_{n=2}^{N} a_n(y)x^n$ on the circle $|x| = \gamma|y|^{-\alpha}$.

Proof of Theorem when $R = \mathbb{C}$ and f is a polynomial: Write

$$P(x, y) \;=\; \prod_{i=1}^{k(y)} \Bigl(1 - \frac{x}{X_i(y)}\Bigr)$$

with $k(y) \le N$. Therefore

$$\frac{x\,P'(x, y)}{P(x, y)} \;=\; \sum_{i=1}^{k(y)} \frac{-x/X_i(y)}{1 \,-\, x/X_i(y)}$$

and hence

$$\widetilde{c}_n(y) \;=\; -\sum_{i=1}^{k(y)} X_i(y)^{-n} \,.$$

Now, for small enough $|y|$, one of the roots is given by the *convergent* series $-\xi_0(y)$ and is smaller than $\gamma|y|^{-\alpha}$ in magnitude, while the other roots have magnitude $\geq \gamma |y|^{-\alpha}$ by the Lemma. We therefore have

$$\left| \widetilde{c}_n(y) \;-\; (-1)^{n-1} \xi_0(y)^{-n} \right| \;\leq\; (N-1) \gamma^{-n} |y|^{\alpha n}$$

for small enough $|y|$, as claimed. $\square$

Proof of Theorem in general case: Write

$$a_n(y) \;=\; \sum_{m=\lceil \alpha(n-1) \rceil}^{\infty} a_{nm}\, y^m$$

Work in the ring $R = \mathbb{Z}[\boldsymbol{a}]$ where $\boldsymbol{a} = \{a_{nm}\}_{n \geq 2,\, m \geq \lceil \alpha(n-1) \rceil}$ are treated as indeterminates. Then the claim of the Theorem amounts to a series of identities between polynomials in $\boldsymbol{a}$ with integer coefficients. We have verified these identities when evaluated on collections $\boldsymbol{a}$ of complex numbers of which only finitely many are nonzero; and this is enough to prove them as identities in $\mathbb{Z}[\boldsymbol{a}]$. $\square$

There is also a direct formal-power-series proof (due to Ira Gessel) at least in the case $\alpha = 1$. I don't know whether it extends to arbitrary real $\alpha > 0$.

Computational use of Theorem

- Can compute $\xi_0(y)$ through order y^{N-1} by computing $\widetilde{c}_N(y)$
- Do this by computing $\widetilde{c}_n(y)$ for $1 \leq n \leq N$ using recursion
- Observe that all $\widetilde{c}_n(y)$ can be truncated to order y^{N-1}
  [no need to keep the full polynomial of degree $n(n-1)/2$]

- For F, have done $N = 900$
  [$N = 400$ takes a minute, $N = 900$ takes less than 6 hours;
  but $N = 900$ needs 24 GB memory!]

- For Θ_0, have done $N = 7000$
  [$N = 500$ takes a minute, $N = 1500$ takes less than an hour;
  $N = 7000$ took 11 days and 21 GB memory]

- For $\widetilde{R}$, have done $N = 350$
  [$N = 50$ takes a minute, $N = 100$ takes less than an hour;
  $N = 350$ took a month and 10 GB memory]

Some positivity properties of formal power series

- Consider formal power series with real coefficients

$$f(y) \;=\; 1 + \sum_{m=1}^{\infty} a_m \, y^m$$

- For $\alpha \in \mathbb{R}$, define the class $\mathcal{S}_\alpha$ to consist of those f for which

$$\frac{f(y)^\alpha - 1}{\alpha} \;=\; \sum_{m=1}^{\infty} b_m(\alpha) \, y^m$$

  has all nonnegative coefficients (with a suitable limit when $\alpha = 0$).

- In other words:
  - For $\alpha > 0$ (resp. $\alpha = 0$), the class $\mathcal{S}_\alpha$ consists of those f for which f^α (resp. $\log f$) has all nonnegative coefficients.
  - For $\alpha < 0$, the class $\mathcal{S}_\alpha$ consists of those f for which f^α has all *nonpositive* coefficients after the constant term 1.

- Containment relations among the classes $\mathcal{S}_\alpha$ are given by the following fairly easy result:

  Proposition (Scott–A.D.S., unpublished):
  Let $\alpha, \beta \in \mathbb{R}$. Then $\mathcal{S}_\alpha \subseteq \mathcal{S}_\beta$ if and only if either

  (a) $\alpha \le 0$ and $\beta \ge \alpha$, or
  (b) $\alpha > 0$ and $\beta \in \{\alpha, 2\alpha, 3\alpha, \ldots\}$.

  Moreover, the containment is strict whenever $\alpha \neq \beta$.

Application to deformed exponential function F

As shown last week, it seems that $\xi_0(y) \in \mathcal{S}_1$:

$$\begin{aligned}\xi_0(y) \;=\; & 1 + \tfrac{1}{2}y + \tfrac{1}{2}y^2 + \tfrac{11}{24}y^3 + \tfrac{11}{24}y^4 + \tfrac{7}{16}y^5 + \tfrac{7}{16}y^6 \\ & + \tfrac{493}{1152}y^7 + \tfrac{163}{384}y^8 + \tfrac{323}{768}y^9 + \tfrac{1603}{3840}y^{10} + \tfrac{57283}{138240}y^{11} \\ & + \tfrac{170921}{414720}y^{12} + \tfrac{340171}{829440}y^{13} + \tfrac{22565}{55296}y^{14} \\ & + \ldots + \text{terms through order } y^{899}\end{aligned}$$

and indeed that $\xi_0(y) \in \mathcal{S}_{-1}$:

$$\begin{aligned}\xi_0(y)^{-1} \;=\; & 1 - \tfrac{1}{2}y - \tfrac{1}{4}y^2 - \tfrac{1}{12}y^3 - \tfrac{1}{16}y^4 - \tfrac{1}{48}y^5 - \tfrac{7}{288}y^6 \\ & - \tfrac{1}{96}y^7 - \tfrac{7}{768}y^8 - \tfrac{49}{6912}y^9 - \tfrac{113}{23040}y^{10} - \tfrac{17}{4608}y^{11} \\ & - \tfrac{293}{92160}y^{12} - \tfrac{737}{276480}y^{13} - \tfrac{3107}{1658880}y^{14} \\ & - \ldots - \text{terms through order } y^{899}\end{aligned}$$

But I have no proof of either of these conjectures!!!

- Note that $\xi_0(y)$ is analytic on $0 \le y < 1$ and diverges as $y \uparrow 1$ like $1/[e(1-y)]$.
- It follows that $\xi_0(y) \notin \mathcal{S}_\alpha$ for $\alpha < -1$.

Application to partial theta function Θ_0

It seems that $\xi_0(y) \in \mathcal{S}_1$:

$$\begin{aligned}\xi_0(y) \;&=\; 1 + y + 2y^2 + 4y^3 + 9y^4 + 21y^5 + 52y^6 + 133y^7 + 351y^8 \\ &\qquad +948y^9 + 2610y^{10} \;+\; \ldots \;+\; \text{terms through order } y^{6999}\end{aligned}$$

and indeed that $\xi_0(y) \in \mathcal{S}_{-1}$:

$$\begin{aligned}\xi_0(y)^{-1} \;&=\; 1 - y - y^2 - y^3 - 2y^4 - 4y^5 - 10y^6 - 25y^7 - 66y^8 \\ &\qquad -178y^9 - 490y^{10} \;-\; \ldots \;-\; \text{terms through order } y^{6999}\end{aligned}$$

and indeed that $\xi_0(y) \in \mathcal{S}_{-2}$:

$$\begin{aligned}\xi_0(y)^{-2} \;&=\; 1 - 2y - y^2 \qquad - y^4 - 2y^5 - 7y^6 - 18y^7 - 50y^8 \\ &\qquad -138y^9 - 386y^{10} \;-\; \ldots \;-\; \text{terms through order } y^{6999}\end{aligned}$$

Here I *do* have a proof of the first two properties (but not the third).
Coming next week!

- Note that
$$\frac{\xi_0(y)^\alpha - 1}{\alpha} \;=\; y \;+\; \frac{\alpha+3}{2} y^2 \;+\; \frac{(\alpha+2)(\alpha+7)}{6} y^3 \;+\; O(y^4)$$
- So $\xi_0(y) \notin \mathcal{S}_\alpha$ for $\alpha < -2$.

Application to $\widetilde{R}(x,y,q) \;=\; \sum_{n=0}^{\infty} \frac{x^n\, y^{n(n-1)/2}}{(1+q)\,\cdots\,(1+q+\ldots+q^{n-1})}$

- Can use explicit implicit function formula to prove that

$$\xi_0(y;q) \;=\; 1 \,+\, \sum_{n=1}^{\infty} \frac{P_n(q)}{Q_n(q)}\, y^n$$

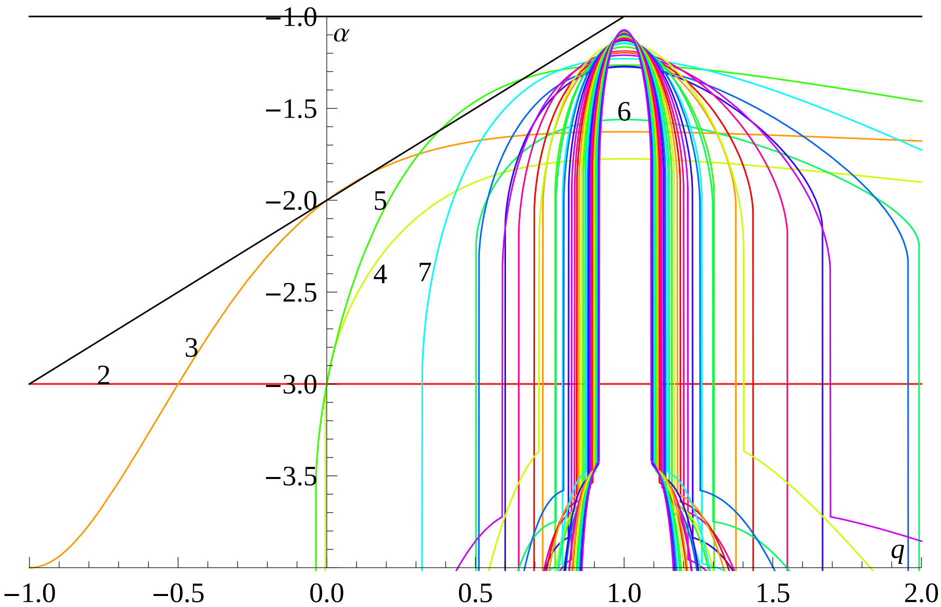
where

$$Q_n(q) \;=\; \prod_{k=2}^{\infty} (1+q+\ldots+q^{k-1})^{\lfloor n/\binom{k}{2} \rfloor}$$

and $P_n(q)$ is a self-inversive polynomial in q with integer coefficients.

- *Empirically* $P_n(q)$ has *two* interesting positivity properties:
  - (a) $P_n(q)$ has all nonnegative coefficients. Indeed, all the coefficients are strictly positive except $[q^1]\, P_5(q) = 0$.
  - (b) $P_n(q) > 0$ for $q > -1$.

- *Empirically* $\xi_0(y;q) \in \mathcal{S}_{-1}$ for all $q > -1$:



Can any of this be proven???

- It seems that $\widetilde{R}(x, y, q)$ is the right unification of Θ_0 and F.
- But thus far my proofs are only for $q = 0$ (i.e. Θ_0).
  Coming next week!
- Can anything be generalized to $q \neq 0$???

- **Open problem:** For $q = 0$, prove $\xi_0(y) \in \mathcal{S}_1$ or $\mathcal{S}_{-1}$ (or even $\mathcal{S}_{-2}$) *directly from the explicit implicit function formula.*
- If this works, it might be generalizable to $q \neq 0$.