Roots $x_k(y)$ of a formal power series $f(x,y) = \sum_{n=0}^{\infty} a_n(y)\, x^n$ with applications to graph enumeration and q-series

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Lectures at Queen Mary — 11, 18, 25 March and 8 April 2011

LECTURE #1

Some wonderful conjectures (but almost no theorems)
at the boundary between
analysis, combinatorics and probability:

The entire function $F(x,y) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\, y^{n(n-1)/2}$,

the polynomials $P_N(x,w) = \sum_{n=0}^{N} \binom{N}{n} x^n w^{n(N-n)}$,

and the generating polynomials of connected graphs

The entire function $F(x,y) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\, y^{n(n-1)/2}$

- Defined for complex x and y satisfying $|y| \le 1$
- Analytic in $\mathbb{C} \times \mathbb{D}$, continuous in $\mathbb{C} \times \overline{\mathbb{D}}$
- $F(\,\cdot\,, y)$ is entire for each $y \in \overline{\mathbb{D}}$
- Valiron (1938): "from a certain viewpoint the simplest entire function after the exponential function"

Applications:

- Statistical mechanics: Partition function of one-site lattice gas
- Combinatorics: Generating function for Tutte polynomials on K_n
  (also acyclic digraphs, inversions of trees, ...)
- Functional-differential equation: $F'(x) = F(yx)$ where $' = \partial/\partial x$
- Complex analysis: Whittaker and Goncharov constants

Application to Tutte polynomials of complete graphs

- Finite graph $G = (V, E)$
- Multivariate Tutte polynomial $Z_G(q, \mathbf{v}) = \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} v_e$

  where $k(A) = \#$ connected components in (V, A)
- Connected-spanning-subgraph polynomial $C_G(\mathbf{v}) = \lim_{q \to 0} q^{-1} Z_G(q, \mathbf{v})$
- Write $Z_G(q, v)$ and $C_G(v)$ if $v_e = v$ for all edges e

  [standard Tutte polynomial is $Z_G(q, v)$ in different variables]

Specialization to complete graphs K_n:

$$Z_n(q, v) = \sum_{m,k} a_{n,m,k} v^m q^k$$

$$C_n(v) = \sum_{m} c_{n,m} v^m$$

Exponential generating functions:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} Z_n(q, v) = F(x, 1+v)^q$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} C_n(v) = \log F(x, 1+v)$$

[see Tutte (1967) and Scott–A.D.S., arXiv:0803.1477]

- Usually considered as formal power series
- But series are *convergent* if $|1 + v| \le 1$

  [see also Flajolet–Salvy–Schaeffer (2004)]

Elementary analytic properties of $F(x,y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \, y^{n(n-1)/2}$

- $\boldsymbol{y = 0}$: $F(x,0) = 1 + x$

- $\boldsymbol{0 < |y| < 1}$: $F(\,\cdot\,, y)$ is a nonpolynomial entire function of order 0:
$$F(x,y) \;=\; \prod_{k=0}^{\infty} \left(1 \,-\, \frac{x}{x_k(y)}\right)$$
where $\sum |x_k(y)|^{-\alpha} < \infty$ for every $\alpha > 0$

- $\boldsymbol{y = 1}$: $F(x,1) = e^x$

- $\boldsymbol{|y| = 1}$ **with** $\boldsymbol{y \neq 1}$: $F(\,\cdot\,, y)$ is an entire function of order 1 and type 1:
$$F(x,y) \;=\; e^x \prod_{k=0}^{\infty} \left(1 \,-\, \frac{x}{x_k(y)}\right) e^{x/x_k(y)} \;.$$
where $\sum |x_k(y)|^{-\alpha} < \infty$ for every $\alpha > 1$

  [see also Ålander (1914) for y a root of unity; Valiron (1938) and Eremenko–Ostrovskii (2007) for y not a root of unity]

- $\boldsymbol{|y| > 1}$: The series $F(\,\cdot\,, y)$ has radius of convergence 0

Consequences for $C_n(v)$

- Make change of variables $y = 1 + v$:

$$\overline{C}_n(y) \;=\; C_n(y-1)$$

- Then for $|y| < 1$ we have

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} \overline{C}_n(y) \;=\; \log F(x,y) \;=\; \sum_k \log\Bigl(1 \,-\, \frac{x}{x_k(y)}\Bigr)$$

  and hence

$$\overline{C}_n(y) \;=\; -(n-1)! \sum_k x_k(y)^{-n} \qquad \text{for all } n \ge 1$$

  (also holds for $n \ge 2$ when $|y| = 1$)

- This is a *convergent* expansion for $\overline{C}_n(y)$
- In particular, gives large-n asymptotic behavior

$$\overline{C}_n(y) \;=\; -(n-1)!\; x_0(y)^{-n} \left[1 \,+\, O(e^{-\epsilon n})\right]$$

  whenever $F(\,\cdot\,, y)$ has a unique root $x_0(y)$ of minimum modulus

Question: What can we say about the roots $x_k(y)$?

Small-y expansion of roots $x_k(y)$

- For small $|y|$, we have $F(x,y) = 1 + x + O(y)$, so we expect a convergent expansion

$$x_0(y) \;=\; -1 \;-\; \sum_{n=1}^{\infty} a_n y^n$$

(easy proof using Rouché: valid for $|y| \lesssim 0.441755$)

- More generally, for each integer $k \geq 0$, write $x = \xi y^{-k}$ and study

$$F_k(\xi, y) \;=\; y^{k(k+1)/2} F(\xi y^{-k}, y) \;=\; \sum_{n=0}^{\infty} \frac{\xi^n}{n!}\, y^{(n-k)(n-k-1)/2}$$

Sum is dominated by terms $n = k$ and $n = k+1$; gives root

$$x_k(y) \;=\; -(k+1) y^{-k} \left[1 + \sum_{n=1}^{\infty} a_n^{(k)} y^n \right]$$

Rouché argument valid for $|y| \lesssim 0.207875$ uniformly in k:
all roots are simple and given by convergent expansion $x_k(y)$

- Can also use theta function in Rouché (Eremenko)

Might these series converge for all $|y| < 1$?

Two ways that $x_k(y)$ could fail to be analytic for $|y| < 1$:

1. Collision of roots ($\to$ branch point)
2. Root escaping to infinity

Theorem (Eremenko): No root can escape to infinity for y in the open unit disc $\mathbb{D}$.

In fact, for any compact subset $K \subset \mathbb{D}$ and any $\epsilon > 0$, there exists an integer k_0 such that for all $y \in K \setminus \{0\}$ we have:

(a) The function $F(\,\cdot\,, y)$ has exactly k_0 zeros (counting multiplicity) in the disc $|x| < k_0|y|^{-(k_0-\frac{1}{2})}$, and

(b) In the region $|x| \geq k_0|y|^{-(k_0-\frac{1}{2})}$, the function $F(\,\cdot\,, y)$ has a simple zero within a factor $1+\epsilon$ of $-(k+1)y^{-k}$ for each $k \geq k_0$, and no other zeros.

- Proof is based on comparison with a theta function (whose roots are known by virtue of Jacobi's product formula)
- *Conjecture* that roots cannot escape to infinity even in the *closed* unit disc except at $y = 1$

Big Conjecture #1. All roots of $F(\,\cdot\,, y)$ are simple for $|y| < 1$. [and also for $|y| = 1$, I suspect]

Consequence of Big Conjecture #1. Each root $x_k(y)$ is analytic in $|y| < 1$.

But I conjecture more ...

Big Conjecture #2. The roots of $F(\,\cdot\,, y)$ are non-crossing *in modulus* for $|y| < 1$:

$$|x_0(y)| \;<\; |x_1(y)| \;<\; |x_2(y)| \;<\; \ldots$$

[and also for $|y| = 1$, I suspect]

Consequence of Big Conjecture #2. The roots are actually separated in modulus by a factor at least $|y|$, i.e.

$$|x_k(y)| \;<\; |y|\,|x_{k+1}(y)| \qquad \text{for all } k \geq 0$$

PROOF. Apply the Schwarz lemma to $x_k(y)/x_{k+1}(y)$.

Consequence for the zeros of $\overline{C}_n(y)$

Recall

$$\overline{C}_n(y) \;=\; -(n-1)! \sum_k x_k(y)^{-n}$$

and use a variant of the Beraha–Kahane–Weiss theorem [A.D.S., arXiv:cond-mat/0012369, Theorem 3.2] $\implies$ the limit points of zeros of $\overline{C}_n$ are the values y for which the zero of minimum modulus of $F(\,\cdot\,, y)$ is *nonunique*.

So if $F(\,\cdot\,, y)$ has a *unique* zero of minimum modulus for all $y \in \mathbb{D}$ (a weakened form of Big Conjecture #2), then the zeros of $\overline{C}_n$ do not accumulate anywhere in the open unit disc.

I actually conjecture more (based on computations up to $n \approx 80$):

Big Conjecture #3. For each n, $\overline{C}_n(y)$ has no zeros with $|y| < 1$. [and, I suspect, no zeros with $|y| = 1$ except the point $y = 1$]

What is the evidence for these conjectures?

Evidence #1: Behavior at real y.

Theorem (Laguerre): For $0 \leq y < 1$, all the roots of $F(\,\cdot\,, y)$ are simple and negative real.

Corollary: Each root $x_k(y)$ is analytic in a complex neighborhood of the interval $[0, 1)$.

[Real-variables methods give further information about the roots $x_k(y)$ for $0 \leq y < 1$: see Langley (2000).]

Now combine this with

Evidence #2: From numerical computation of the series $x_k(y)$ ...

Three methods for computing the series $x_k(y)$

1. Insert $x_k(y) = -(k+1)y^{-k}\left[1+\sum_{n=1}^{\infty} a_n^{(k)} y^n\right]$ and solve term-by-term

2. Use "explicit implicit function theorem" (generalization of Lagrange inversion formula) given in arXiv:0902.0069:

   solve $z = G(z, w)$ with $G(0,0) = 0$ and $\left|\frac{\partial G}{\partial z}(0,0)\right| < 1$ by

$$\varphi(w) \;=\; \sum_{m=1}^{\infty} \frac{1}{m}\,[\zeta^{m-1}] G(\zeta, w)^m$$

   and more generally

$$H(\varphi(w), w) \;=\; H(0, w) \,+\, \sum_{m=1}^{\infty} \frac{1}{m}\,[\zeta^{m-1}] \frac{\partial H(\zeta, w)}{\partial \zeta}\, G(\zeta, w)^m$$

Methods 1 and 2 work *symbolically* in k.

3. Use

$$\overline{C}_n(y) \;=\; -(n-1)! \sum_k x_k(y)^{-n}$$

   together with recursion

$$\overline{C}_n(y) \;=\; y^{n(n-1)/2} \;-\; \sum_{j=1}^{n-1} \binom{n-1}{j-1} \overline{C}_j(y)\, y^{(n-j)(n-j-1)/2}$$

   [cf. Leroux (1988) and Scott–A.D.S., arXiv:0803.1477]
   — can go to very high n, at least for small k

And let MATHEMATICA run for a weekend …

$$\begin{aligned}
-x_0(y) \;=\; & 1 + \tfrac{1}{2}y + \tfrac{1}{2}y^2 + \tfrac{11}{24}y^3 + \tfrac{11}{24}y^4 + \tfrac{7}{16}y^5 + \tfrac{7}{16}y^6 \\
& + \tfrac{493}{1152}y^7 + \tfrac{163}{384}y^8 + \tfrac{323}{768}y^9 + \tfrac{1603}{3840}y^{10} + \tfrac{57283}{138240}y^{11} \\
& + \tfrac{170921}{414720}y^{12} + \tfrac{340171}{829440}y^{13} + \tfrac{22565}{55296}y^{14} \\
& + \ldots + \text{terms through order } y^{899}
\end{aligned}$$

and all the coefficients (so far) are nonnegative!

Big Conjecture #4. For each k, the series $-x_k(y)$ has all nonnegative coefficients.

Combine this with the known analyticity for $0 \le y < 1$, and Vivanti–Pringsheim gives:

Consequence of Big Conjecture #4. Each root $x_k(y)$ is analytic in the open unit disc.

NEED TO DO: Extended computations for $k = 1, 2, \ldots$ and for symbolic k.

But more is true . . .

Look at the *reciprocal* of $x_0(y)$:

$$
\begin{aligned}
-\frac{1}{x_0(y)} \;=\; & 1 - \tfrac{1}{2}y - \tfrac{1}{4}y^2 - \tfrac{1}{12}y^3 - \tfrac{1}{16}y^4 - \tfrac{1}{48}y^5 - \tfrac{7}{288}y^6 \\
& - \tfrac{1}{96}y^7 - \tfrac{7}{768}y^8 - \tfrac{49}{6912}y^9 - \tfrac{113}{23040}y^{10} - \tfrac{17}{4608}y^{11} \\
& - \tfrac{293}{92160}y^{12} - \tfrac{737}{276480}y^{13} - \tfrac{3107}{1658880}y^{14} \\
& - \ldots - \text{terms through order } y^{899}
\end{aligned}
$$

and all the coefficients (so far) beyond the constant term are *nonpositive*!

Big Conjecture #5. For each k, the series $-(k+1)y^{-k}/x_k(y)$ has all *nonpositive* coefficients after the constant term 1.

[This implies the preceding conjecture, but is stronger.]

- Relative simplicity of the coefficients of $-1/x_0(y)$ compared to those of $-x_0(y) \longrightarrow$ simpler combinatorial interpretation?
- Note that $x_k(y) \to -\infty$ as $y \uparrow 1$ (this is fairly easy to prove). So $1/x_k(y) \to 0$. Therefore:

Consequence of Big Conjecture #5. For each k, the coefficients (after the constant term) in the series $-(k+1)y^{-k}/x_k(y)$ are the *probabilities* for a positive-integer-valued random variable.

What might such a random variable be???
Could this approach be used to *prove* Big Conjecture #5?

AGAIN NEED TO DO: Extended computations for $k = 1, 2, \ldots$ and for symbolic k.

But I conjecture that even more is true ...

Define $D_n(y) = \dfrac{\overline{C}_n(y)}{(-1)^{n-1}(n-1)!}$ and recall that $-x_0(y) = \lim_{n\to\infty} D_n(y)^{-1/n}$

Big Conjecture #6. For each n,

(a) the series $D_n(y)^{-1/n}$ has all nonnegative coefficients,

and even more strongly,

(b) the series $D_n(y)^{1/n}$ has all nonpositive coefficients after the constant term 1.

Since $D_n(y) > 0$ for $0 \le y < 1$, Vivanti–Pringsheim shows that Big Conjecture #6a implies Big Conjecture #3:

For each n, $\overline{C}_n(y)$ has no zeros with $|y| < 1$.

Moreover, Big Conjecture #6b $\implies$ for each n, the coefficients (after the constant term) in the series $D_n(y)^{1/n}$ are the *probabilities* for a positive-integer-valued random variable.

Such a random variable would generalize the one for $-1/x_0(y)$ in roughly the same way that the binomial generalizes the Poisson.

Roots $x_k(y)$ computed *symbolically* in k

$$x_k(y) \;=\; -(k+1)y^{-k}\left[1+\sum_{n=1}^{\infty}\frac{P_n(k)}{Q_n(k)}\,y^n\right]$$

where I have computed up to $n = 21$:

$$\begin{aligned}
P_1(k) &= 1 \\
P_2(k) &= 2+6k+3k^2 \\
P_3(k) &= 11+29k+63k^2+65k^3+28k^4+4k^5 \\
P_4(k) &= 22+146k+273k^2+359k^3+355k^4+211k^5+63k^6+7k^7 \\
&\;\vdots \\
Q_n(k) &= (k+1)^n(k+2)^n\prod_{j=3}^{\infty}(k+j)^{\lfloor n/\binom{j}{2}\rfloor}
\end{aligned}$$

- $P_n(k)$ has nonnegative coefficients for $n \le 9$ but not for $n = 10, 15, 16, 18, 19, 20, 21$
- $P_n(k) \ge 0$ for all *real* $k \ge 0$ for $n \le 14$ but not for $n = 15, 18, 19, 21$
- But … $P_n(k) \ge 0$ for all *integer* $k \ge 0$ at least for $n \le 21$

which gives evidence that Big Conjecture #4 holds for all k:

For each k, the series $-x_k(y)$ has all nonnegative coefficients.

Reciprocals of roots $x_k(y)$ computed *symbolically* in k

$$\frac{-(k+1)y^{-k}}{x_k(y)} \;=\; \left[1 - \sum_{n=1}^{\infty} \frac{\widehat{P}_n(k)}{Q_n(k)}\, y^n\right]$$

where I have computed up to $n = 21$:

$$\begin{aligned}
\widehat{P}_1(k) &= 1 \\
\widehat{P}_2(k) &= 1 + 6k + 3k^2 \\
\widehat{P}_3(k) &= 2 - 10k + 33k^2 + 59k^3 + 28k^4 + 4k^5 \\
\widehat{P}_4(k) &= 3 + 71k + 24k^2 + 82k^3 + 236k^4 + 194k^5 + 63k^6 + 7k^7 \\
&\vdots
\end{aligned}$$

and $Q_n(k)$ are the same as before

- $\widehat{P}_n(k)$ does not have nonnegative coefficients (except for $n = 1, 2, 4$)
- $\widehat{P}_n(k) \geq 0$ for all *real* $k \geq 0$ for $n = 1, 2, 3, 4, 5, 7, 8$ but not in general
- But ... $\widehat{P}_n(k) \geq 0$ for all *integer* $k \geq 0$ at least for $n \leq 21$

which gives evidence that Big Conjecture #5 holds for all k:

> For each k, the series $-(k+1)y^{-k}/x_k(y)$ has all *nonpositive* coefficients after the constant term 1.

Ratios of roots $x_k(y)/x_{k+1}(y)$

The series

$$\frac{x_0(y)}{x_1(y)} \;=\; \tfrac{1}{2}y \,+\, \tfrac{1}{6}y^2 \,+\, \tfrac{5}{72}y^3 \,+\, \tfrac{11}{216}y^4 \,+\, \tfrac{29}{1296}y^5 \,+\, \ldots$$

has nonnegative coefficients at least up to order y^{136}.
(But its reciprocal does not have any fixed signs.)

Big Conjecture #7. The series $x_0(y)/x_1(y)$ has all nonnegative coefficients.

Consequence of Big Conjecture #7. Since $\lim\limits_{y\uparrow 1} x_0(y)/x_1(y) = 1$, Big Conjecture #7 implies that $|x_0(y)| < |x_1(y)|$ for all $y \in \mathbb{D}$ (a special case of Big Conjecture #2 on the separation in modulus of roots).

- But unfortunately … the series

  $$\frac{x_1(y)}{x_2(y)} \;=\; \tfrac{2}{3}y \,+\, \tfrac{1}{18}y^2 \,+\, \tfrac{17}{216}y^3 \,+\, \tfrac{23}{810}y^4 \,+\, \tfrac{343}{17280}y^5 \,+\, \ldots$$

  has a negative coefficient at order y^{13}. This doesn't contradict the conjecture that $|x_1(y)/x_2(y)| < 1$ in the unit disc, but it does rule out the simplest method of proof.

- Symbolic computation of $x_k(y)/x_{k+1}(y)$ shows that, up to order y^{22}, the *only* cases of a negative coefficient for *integer* $k \ge 0$ are the coefficient of y^{13} for $k = 1, 2, 3$; y^{17} for $k = 2$; and y^{19}, y^{21} for $k = 2, 3, 4$.

- The series $y^{-k} x_0(y)/x_k(y)$ has nonnegative coefficients for all integer $k \ge 0$ through at least order y^{21}.

Asymptotics of roots as $y \to 1$

Write $y = e^{-\gamma}$ with $\mathrm{Re}\,\gamma > 0$.
Want to study $\gamma \to 0$ (non-tangentially in the right half-plane).

I *believe* I will be able to prove that

$$-x_k(e^{-\gamma}) \;\approx\; \frac{1}{e}\gamma^{-1} \,+\, c_k\gamma^{-1/3} \,+\, \ldots$$

for suitable constants $c_0 < c_1 < c_2 < \ldots$. But I have not yet worked out all the details.

Overview of method:

1. Develop an asymptotic expansion for $F(x, e^{-\gamma})$ when $\gamma \to 0$ and x is taken to be of order γ^{-1}, because this is the regime where the zeros will be found.
2. Use this expansion for $F(x, e^{-\gamma})$ to deduce an expansion for $x_k(e^{-\gamma})$.

Sketch of step #1: Insert Gaussian integral representation for $e^{-\frac{\gamma}{2}n^2}$ to obtain

$$F(x, e^{-\gamma}) \;=\; (2\pi\gamma)^{-1/2} \int\limits_{-\infty}^{\infty} \exp[g(t)]\, dt$$

with

$$g(t) \;=\; -\frac{t^2}{2\gamma} \,+\, xe^{\gamma/2}e^{it}$$

Saddle-point equation $g'(t) = 0$ is $-ite^{-it} = \gamma e^{\gamma/2}x$, so it makes sense to make the change of variables

$$x = \gamma^{-1}e^{-\gamma/2}we^{w} ,$$

which puts the saddle point at $t_0 = iw$. (Note that this brings in the Lambert W function, i.e. the inverse function to $w \mapsto we^w$.) We then have

$$F(\gamma^{-1}e^{-\gamma/2}we^{w}, e^{-\gamma}) = (2\pi\gamma)^{-1/2} \int_{-\infty}^{\infty} dt \, \exp\left[-\frac{t^2}{2\gamma} + \frac{we^w}{\gamma}e^{it}\right]$$

Now shift the contour to go through the saddle point (parallel to the real axis) and make the change of variables $t = s + iw$: we have

$$F(\gamma^{-1}e^{-\gamma/2}we^{w}, e^{-\gamma}) = (2\pi\gamma)^{-1/2} \exp\left[\frac{w^2}{2\gamma} + \frac{w}{\gamma}\right] \int_{-\infty}^{\infty} ds \, \exp[h(s)]$$

where

$$h(s) = -\frac{(1+w)}{2\gamma}s^2 + \frac{w}{\gamma}\left(e^{is} - 1 - is + \frac{s^2}{2}\right)$$

and the integration goes along the real s axis.

These formulae should allow computation of asymptotics

(a) $\gamma \to 0$ (in a suitable way) for (suitable values of) fixed w; and

(b) $w \to \infty$ (in a suitable direction) for (suitable values of) fixed γ.

Focus for now on (a).

Recall that

$$h(s) \;=\; -\frac{(1+w)}{2\gamma}s^2 \;+\; \frac{w}{\gamma}\Bigl(e^{is} - 1 - is + \frac{s^2}{2}\Bigr)$$

Consider for simplicity γ and x real. There seem to be three regimes:

- **"High temperature":** $w > -1$ (i.e. $we^w > -1/e$).

  Easiest case: $s = 0$ saddle point is Gaussian, and can compute the asymptotics to all orders in terms of 3-associated Stirling subset numbers $\left\{ {n \atop m} \right\}_{\ge 3}$. [Still need to justify this formal calculation by showing that only the $s = 0$ saddle point contributes.]

- **"Low temperature":** $w = -\eta \cot \eta + \eta i$ with $-\pi < \eta < \pi$ (i.e. $we^w < -1/e$).

  Saddle points at $s = 0$ and $s = 2\eta$ contribute; I *think* this is all.

- **"Critical regime":** $w = -(1 + \xi \gamma^{1/3})$ with ξ fixed, which corresponds to

  $$x \;=\; -\frac{1}{e\gamma}\left[1 \;-\; \frac{\xi^2}{2}\gamma^{2/3} \;+\; O(\gamma)\right]$$

  – *At* the "critical point" $\xi = 0$: Dominant behavior at $s = 0$ saddle point is no longer Gaussian (it vanishes) but rather the cubic term $is^3/(6\gamma)$. Can compute the asymptotics to all orders in terms of 4-associated Stirling subset numbers $\left\{ {n \atop m} \right\}_{\ge 4}$ (at least formally).

  – In the critical regime (ξ arbitrary): Expect to have Airy asymptotics as in Flajolet–Salvy–Schaeffer (2004). This is where the roots will lie.

I would appreciate help with the details!!!

The polynomials $P_N(x,w) = \sum_{n=0}^{N} \binom{N}{n} x^n w^{n(N-n)}$

- Partition function of Ising model on complete graph K_N, with $x = e^{2h}$ and $w = e^{-2J}$
- Related to binomial $(1+x)^N$ in same way as our $F(x,y)$ is related to exponential e^x
  [but we have written $w^{n(N-n)}$ instead of $y^{n(n-1)/2}$]
- $\lim_{N\to\infty} P_N\!\left(\frac{xw^{1-N}}{N},\, w\right) \;=\; F(x, w^{-2})$ when $|w| > 1$
- So results about zeros of P_N generalize those about F (just as results about the binomial generalize those about the exponential function)
- Lee–Yang theorem: In ferromagnetic case ($0 \le w \le 1$), all zeros are on the unit circle $|x| = 1$
- Laguerre: In antiferromagnetic case ($w \ge 1$), all zeros are real and negative
- What about "complex antiferromagnetic" case $|w| > 1$??

Big Conjecture #8. For $|w| > 1$, all zeros of $P_N(\,\cdot\,, w)$ are separated in modulus (by at least a factor $|w|^2$).

Taking $N \to \infty$, this implies Big Conjecture #2 about the separation in modulus of the zeros of $F(\,\cdot\,, y)$.

Differential-equation approach to $P_N(x,w) = \sum_{n=0}^{N} \binom{N}{n} x^n w^{n(N-n)}$

On the space of polynomials $Q_N(x) = \sum_{n=0}^{N} a_n x^n$ of degree N with $a_0 \neq 0$, define the semigroup

$$(\mathcal{A}_t Q_N)(x) \;\equiv\; \sum_{n=0}^{N} a_n x^n e^{tn(N-n)}$$

Roots of $\mathcal{A}_t Q_N$ evolve according to an *autonomous* differential equation, which is best expressed in terms of *logarithms* of roots $\zeta_i = \log x_i$:

$$\frac{d\zeta_i}{dt} \;=\; \sum_{j \neq i} F(\zeta_i - \zeta_j)$$

where

$$F(z) \;=\; \coth(z/2)$$

These are first-order (“Aristotelian”) equations of motion for a system of n “particles” (in $\mathbb{R}$ or $\mathbb{C}$) with a translation-invariant “force” F.

Moreover, the specific force $F = \coth$ is a Calogero–Moser–Sutherland system, much studied in the theory of integrable systems.

For polynomials Q_N with *real* roots and *real* $t > 0$, this approach gives interesting results on separation of zeros. (In particular, it gives a new proof of Laguerre’s theorem.)

Is this approach useful for *complex* t with $\operatorname{Re} t > 0$???
Can it be used to prove Big Conjecture #8?

A more general approach to the leading root $x_0(y)$ [details to be given in the subsequent lectures!]

- Consider a formal power series

$$f(x,y) \;=\; \sum_{n=0}^{\infty} \alpha_n \, x^n \, y^{n(n-1)/2}$$

normalized to $\alpha_0 = \alpha_1 = 1$, or more generally

$$f(x,y) \;=\; \sum_{n=0}^{\infty} a_n(y) \, x^n$$

where

(a) $a_0(0) = a_1(0) = 1$;
(b) $a_n(0) = 0$ for $n \geq 2$; and
(c) $a_n(y) = O(y^{\nu_n})$ with $\lim_{n\to\infty} \nu_n = \infty$.

It makes sense to study the "leading root" $x_0(y)$ in this generality.

- Example: The "partial theta function"

$$\Theta_0(x,y) \;=\; \sum_{n=0}^{\infty} x^n \, y^{n(n-1)/2}$$

beloved of q-series practitioners (going back at least to Ramanujan).

- More generally, consider

$$\widetilde{R}(x,y,q) \;=\; \sum_{n=0}^{\infty} \frac{x^n \, y^{n(n-1)/2}}{(1+q)(1+q+q^2) \, \cdots \, (1+q+\ldots+q^{n-1})}$$

which reduces to Θ_0 when $q=0$, and to F when $q=1$.

A more general approach, continued ...

- A power series for the leading root $x_0(y)$ can be computed from the power-series expansion of $\log f(x,y)$, generalizing Method 3 above for $F(x,y)$. This is extremely efficient!
- Example: For Θ_0 we have

$$-x_0(y) \;=\; 1+y+2y^2+4y^3+9y^4+21y^5+52y^6+133y^7+351y^8+\ldots$$

with strictly positive coefficients at least through order y^{6999}.

- More generally, for $\widetilde{R}(x,y,q)$ it can be proven that

$$-x_0(y,q) \;=\; 1 \,+\, \sum_{n=1}^{\infty} \frac{P_n(q)}{Q_n(q)}\, y^n$$

where

$$Q_n(q) \;=\; \prod_{k=2}^{\infty} (1+q+\ldots+q^{k-1})^{\lfloor n/\binom{k}{2} \rfloor}$$

and $P_n(q)$ is a self-inversive polynomial with integer coefficients.

I have verified for $n \le 349$ that $P_n(q)$ has *two* interesting positivity properties:

(a) $P_n(q)$ has all nonnegative coefficients. Indeed, all the coefficients are strictly positive except $[q^1]\, P_5(q) = 0$.

(b) $P_n(q) > 0$ for $q > -1$.

Can any of this be proven???

YES!!! ... but please stay tuned for our next installment ...