Even and odd square-free numbers

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It is well known that the proportion of square-free numbers among all numbers is asymptotically $6/\pi^2$ (e.g. [1, p. 269], [2, section 2.5]). In a recent number of the *Gazette*, J.A. Scott [3] conjectures that the proportion of *odd* square-free numbers is asymptotically $4/\pi^2$, so that ratio of odd to even square-free numbers is asymptotically 2:1. We show that this is true, by a suitable adaptation of the standard proof of the $6/\pi^2$ result.

This proof is most efficiently presented in the language of Dirichlet series and convolutions. For any arithmetic function $a(n)$, there is a corresponding Dirichlet series $\sum_{n=1}^{\infty} a(n)/n^s$, defining a function $F_a(s)$ where it converges. If we multiply two Dirichlet series and collect the terms in the obvious way (which is valid provided that both series converge absolutely), we find

$$F_a(s)F_b(s) = F_{a*b}(s), \tag{1}$$

where the *convolution* (alias *Dirichlet product*) $a * b$ is defined by

$$(a*b)(n) = \sum_{jk=n} a(j)b(k) = \sum_{j|n} a(j)b(n/j).$$

Define e_1 by:

$$e_1(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, e_1 is the sequence having 1 in place 1 and 0 elsewhere. Then $a * e_1 = a$ for any arithmetic function a, so e_1 is the identity for convolution. The corresponding Dirichlet series function is the constant function 1. We also define u by: $u(n) = 1$ for all n.

Recall that the *Möbius function* μ takes the value 1 at 1 and $(-1)^k$ at a square-free integer with k prime factors. At all other integers its value is 0. Hence $|\mu(n)|$ is 1 when n is square-free and 0 otherwise. The basic property of the Möbius function (found in most books on number theory) is:

Lemma 1. We have $u * \mu = e_1$. Hence for all $n > 1$, $\sum_{j|n} \mu(j) = 0$.

Since $\sum_{n=1}^{\infty} 1/n^s = \zeta(s)$ for $s > 1$, it follows from (1) that $\sum_{n=1}^{\infty} \mu(n)/n^s = 1/\zeta(s)$. (Alternatively, this identity, together with the definition of $\mu(n)$ itself, can be derived directly from the Euler product; see [2].) In particular,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}. \tag{2}$$

It will help to introduce the following notation:

$$v(n) = \begin{cases} 1 & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

Clearly,

$$\sum_{n=1}^{\infty} \frac{v(n)}{n^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

Let $A(x)$ be the number of odd square-free numbers not greater than x. Clearly,

$$A(x) = \sum_{n \le x} |\mu(n)| v(n). \tag{3}$$

We write ab for the pointwise product of a and b, so $(ab)(n) = a(n)b(n)$.

Lemma 2: We have $(\mu v) * v = e_1$, hence $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)v(n)}{n^2} = \frac{8}{\pi^2}$.

Proof: Firstly, $[(\mu v) * v](e_1) = (\mu v)(1)v(1) = 1$. Take $n > 1$. Then

$$[(\mu v) * v](n) = \sum_{j|n} \mu(j)v(j)v(n/j).$$

If n is odd, then all its divisors are odd, so $v(j)v(n/j) = 1$ for divisors j, hence

$$[(\mu v) * v](n) = \sum_{j|n} \mu(j) = 0.$$

If n is even and $j|n$, then either j or n/j is even, so $v(j)v(n/j) = 0$ and again $[(\mu v)*v](n) = 0$. So $(\mu v) * v = e_1$. The series statement follows, by (1).

The proof of the $6/\pi^2$ result uses the fact that $|\mu(n)| = \sum_{m^2|n} \mu(m)$. The corresponding result for us is:

Lemma 3: For all n, we have

$$|\mu(n)|v(n) = \sum_{m^2|n} \mu(m)\, v(m)\, v(n/m).$$

Proof: Denote this sum by S. If n is even, then $v(m)\, v(n/m) = 0$ for all such m, so $S = 0$. If n is odd, then $v(m)\, v(n/m) = 1$ and $S = \sum_{m^2|n} \mu(m)$. If n is square-free, then the only such m is 1, so $S = 1$. If n is not square-free, express it as h^2k, where $h > 1$ and k is square-free. If $m^2|n$, then $m|h$. Hence $S = \sum_{m|h} \mu(m)$, which is 0, by Lemma 1. So $S = 1$ when n is odd and square-free, 0 otherwise, which equates it to $|\mu(n)|v(n)$.

Theorem: $A(x) = \dfrac{4}{\pi^2}x + q(x)$, where $|q(x)| \le 3x^{1/2}$.

Proof: By (3) and Lemma 3,

$$A(x) = \sum_{n \le x} \sum_{m^2 | n} \mu(m)\, v(m)\, v(n/m).$$

For a fixed $m \le x^{1/2}$, a term $\mu(m)v(m)$ will occur for each odd multiple $n = rm^2$ of m^2 with $rm^2 \le x$, so

$$A(x) = \sum_{m \le x^{1/2}} n_m \mu(m) v(m),$$

where n_m is the number of these odd multiples. This means that $(2n_m - 1)m^2 \le x < (2n_m + 1)m^2$, hence $n_m - \frac{1}{2} \le x/2m^2 \le n_m + \frac{1}{2}$, so

$$n_m = \frac{x}{2m^2} + r_m,$$

where $|r_m| \le \frac{1}{2}$. So

$$A(x) = \frac{x}{2} \sum_{m \le x^{1/2}} \frac{\mu(m)v(m)}{m^2} + q_1(x),$$

where $q_1(x) \le \frac{1}{2} x^{1/2}$. Now by Lemma 2,

$$\sum_{m \le x^{1/2}} \frac{\mu(m)v(m)}{m^2} = \frac{8}{\pi^2} - q_2(x),$$

where $q_2(x) = \sum_{m > x^{1/2}} \mu(m)v(m)/m^2$, hence

$$|q_2(x)| \le \sum_{m > x^{1/2}} \frac{1}{m^2}.$$

Comparison with the integral of $1/t^2$ on $[x^{1/2}, \infty)$ shows that this is no greater than $1/(x^{1/2} - 1)$, so less than $2/x^{1/2}$ if $x > 4$.

Further note. The heuristic reasoning given in [3] cannot be developed into a proof, for the following reason. Given a set A of positive integers, let $A[x] = \{n : n \in A,\ n \le x\}$, and let $A(x)$ be the number of members of $A[x]$ (watch the brackets!). If $A(x)/x$ tends to a limit as $x \to \infty$, this limit is called the *natural density* of A, denoted by $d(A)$. The *logarithmic density* $\delta(A)$ is defined similarly, with weighting $1/n$. In other words,

$$\delta(A) = \lim_{x \to \infty} \frac{1}{\log x} \sum_{n \in A[x]} \frac{1}{n}$$

if the limit exists. The existence of $d(A)$ implies the existence of $\delta(A)$, with the same value (see [4, chapter III.1]). However, the existence of $\delta(A)$ does not imply the existence of $d(A)$ (though of course it does show that $d(A)$ cannot take any *other* value). A counter-example is given by $A = \bigcup_{k=1}^{\infty} E_k$, where $E_k = \{n : 2^{k-1} < n \le 2^k\}$.

References

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