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FREE ACTIONS OF FINITE GROUPS ON SPHERES

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Just about a century ago it was realised (by Killing, Klein and others) that the sphere and real projective space offered two different global models for geometry which were locally the same. In 1891, Killing formulated (in terminology that now appears vague) the problem of determining all such models. In 1926 Hopf revived the problem, gave a clear statement of its topological setting, and also raised the more general question of studying manifolds covered by spheres. This problem lies deep, and though much was done in the 3-dimensional case serious progress in general is comparatively recent, and can be said to start with Cartan and Eilenberg [**1956**] and Milnor [**1957**].

I will discuss the problem of classifying the homotopy types of such manifolds; in the first half I outline a reduction of this problem to two others, less complex though perhaps not less deep; then I will discuss the current state of knowledge on these. This includes a virtually complete determination of which groups can act freely on which spheres. This work is part of a collaboration with Charles Thomas and Ib Madsen. To save time, I will omit discussion of the smooth case, except to state one of our main conclusions.

Examples. *Orthogonal space-forms.* Since the isometries of the sphere S^{n-1} are all linear, the solution to the original problem is to be sought as follows: Seek linear representations ϕ of the given group π such that for $1 \neq g \in \pi$, $\phi(g)$ has no fixed points on S^{n-1}, or equivalently, 1 is not an eigenvalue of $\phi(g)$. In the present state of representation theory, this is not too hard and a full solution can be found in the book of Wolf [**1967**], with references to much earlier work.

Let us call representations ϕ satisfying the above condition $\mathscr{F}$-representations. Then the sum $\phi_1 \oplus \phi_2$ is an $\mathscr{F}$-representation if and only if ϕ_1 and ϕ_2 are (this construction corresponds to the equivariant join of the corresponding spheres), so it is enough to look at irreducible ϕ. The simplest examples are the (cyclic) sub-

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groups of SO_2: taking sums of these representations yields the lens spaces of de Rham [**1931**]: the 3-dimensional case goes back to Tietze [**1908**]. Another early example was that of binary polyhedral groups, which can be considered as subgroups of the group S^3 of unit quaternions, acting on S^3 by right translations. This includes the case of generalised quaternion groups Q_{4k} of order $4k$.

Finally if $\pi = \rho\sigma$ is a split extension of a cyclic group ρ by a cyclic group σ of coprime order, the faithful irreducible representations are all obtained by taking a faithful 1-dimensional (complex) representation of ρ, extending (faithfully) to the centraliser ζ of ρ in σ, and inducing up to π. If π admits an $\mathscr{F}$-representation, ζ meets each Sylow subgroup of σ: if this holds, all faithful irreducible representations are $\mathscr{F}$-representations (Burnside, [**1905**]). An equivalent condition is that every subgroup of π of order pq (p, q primes not necessarily distinct) is cyclic. It was also shown by Vincent [**1947**] that any soluble group π satisfying this latter condition possesses $\mathscr{F}$-representations.

Homotopy theory. We shall follow the basic method of surgery theory (assumed known) and accordingly treat the problem in three stages: homotopy classification, normal invariant and surgery obstruction.

Let us first observe that by the fixed point theorem of Brouwer [**1912**], an orientation-preserving homeomorphism of S^{n-1} has a fixed point if n is odd. Thus if π acts freely, the orientation-preserving subgroup (of index 1 or 2) is trivial. As $\boldsymbol{Z}/2$ can act freely on any sphere, and the orbit space is homotopy equivalent to real projective space, we shall not discuss this case further. So suppose n even. By the same argument, π now respects orientation, so the quotient manifold is orientable. We also ignore the trivial case of actions on S^1 ($n = 2$), so $n \geqq 4$ throughout. And for our positive results (using surgery) we have to omit the case $n = 4$, and so $n \geqq 6$ is even.

According to Smith [**1944**], $\boldsymbol{Z}/p \times \boldsymbol{Z}/p$ cannot act freely on any sphere. Thus if π can so act, all subgroups of π of order p^2 (p any prime) are cyclic. Call π an $\mathscr{R}$-group if it satisfies this condition. The structure of $\mathscr{R}$-groups is known: the soluble case is due to Zassenhaus [**1935**] and the rest essentially to Suzuki [**1955**]. The Sylow p-subgroups of π are cyclic, or perhaps (if $p = 2$) generalised quaternionic. If all are cyclic (e.g., if π has odd order), π is metabelian. In general the quotient of π by the maximal normal subgroup $O_{2'}(\pi)$ of π of odd order, belongs to one of six types:

I. cyclic,
II. generalised quaternion 2-group,
III. binary tetrahedral,
IV. binary octahedral,
V. $SL_2(p)$ ($p \geqq 5$ prime),
VI. $TL_2(p)$.

We can define $TL_2(p)$ as follows. Choose a nonsquare $\omega \in \boldsymbol{F}_p^{\times}$. Then $TL_2(p) \cong \{y \in GL_2(p)\colon \det y = 1 \text{ or } \omega\}$ with product given by

$$\begin{aligned} y_1 \circ y_2 &= y_1 y_2 \text{ (product in } GL_2) && \text{if } \det y_1 = 1 \text{ or } \det y_2 = 1,\\ &= \omega^{-1} y_1 y_2 && \text{if } \det y_1 = \det y_2 = \omega. \end{aligned}$$

π is soluble for types I—IV, not for V, VI. Types III, IV may be regarded as the case $p = 3$ of V, VI but the detailed descriptions are somewhat different at the prime 3.

A deeper study by Cartan and Eilenberg [**1956**] with later work by Swan [**1960**] gives converse results at the homotopy level. Using Tate cohomology, the cohomology of π is *periodic* if $g \in \hat{H}^n(\pi; \boldsymbol{Z})$ is such that for all $a \in \boldsymbol{Z}$ and all π-modules A, cup product with g gives an isomorphism $\hat{H}^a(\pi; A) \to \hat{H}^{n-a}(\pi; A)$. A cohomology class g satisfies this condition if and only if $\hat{H}^n(\pi; \boldsymbol{Z})$ is cyclic, having the same order as π, and generated by g; so we will call such g *generators*. Then π has periodic cohomology if and only if π is an $\mathscr{R}$-group.

Now define a *π-polarised space* to consist of a CW complex X, dominated by a finite complex, together with an isomorphism $\pi_1(X) \to \pi$ and a homotopy equivalance of the universal cover $\tilde{X}$ on S^{n-1} ($n \geqq 3$).

PROPOSITION 1. *Taking the first k-invariant yields a bijection between (polarised) homotopy classes of π-polarised spaces and generators $g \in \hat{H}^n(\pi; \boldsymbol{Z})$.*

Recall that such a space X determines an obstruction $\theta(X) = \theta(g) \in \tilde{K}_0(\boldsymbol{Z}\pi)$ to its being homotopy equivalent to a finite complex. We will discuss this more fully below.

Note that if X is π-polarised, there is a natural isomorphism $H^i(\pi; A) \to H^i(X; A)$ for any coefficient module A and integer $i < n - 1$: from now on we will identify these groups. Note also that any polarised space is a Poincaré complex.

Normal invariants. A Poincaré complex X has a Spivak normal fibration, classified by a map $X \to BG$. A homotopy class of liftings to $X \to B$ Top is called a *normal invariant*. It determines by transversality a normal cobordism class of normal maps $\phi: M^{n-1} \to X$ ($n \neq 5$) of degree 1.

PROPOSITION 2. *Any polarised space X has a normal invariant.*

PROOF. We need some preliminary remarks. It follows from work of Sullivan that the obstruction corresponding to the top cell of X is zero (for any Poincaré complex), so we may ignore obstructions in $H^{n-1}(X; A)$. Next, since Top $\subset G$ is an infinite loop map, the obstruction to existence of a normal invariant is a homotopy class of maps $X \to B(G/\text{Top})$, the image of $X \to BG$. It is enough to show this becomes nullhomotopic after localisation at any prime.

At odd primes, $G/\text{Top} \simeq BO$, so we get a class in $KO^{-1}(X; \boldsymbol{Z}_{(\text{odd})})$. This is a summand of $K^{-1}(X; \boldsymbol{Z}_{(\text{odd})})$, which is well known to vanish.

The localisation of $B(G/\text{Top})$ at 2 is an Eilenberg-Mac Lane space, so the obstruction is given by a string of cohomology classes with coefficients $\boldsymbol{Z}_{(2)}$ or $\boldsymbol{Z}/2$. For the covering map $X(\pi_2) \to X$ corresponding to the Sylow 2-subgroup π_2, the corresponding cohomology groups map injectively, and the obstructions are natural, so it suffices to consider the problem for π_2.

But if $X(\pi_2)$ is homotopy equivalent to a manifold, it has a normal invariant. And we have orthogonal space-forms in every homotopy type if π_2 is cyclic, and (at least) in every relevant dimension if π_2 is generalised quaternionic. This is enough, since we can obtain all generators $g \in \hat{H}^n(\pi_2; \boldsymbol{Z})$ from one such by changing the attaching map of the top cell.

Up to this point, we have been surveying known results. The following, however, is new. We continue the above notations.

PROPOSITION 3. *Any normal invariant of $X(\pi_2)$ coming from a homotopy equivalence with an orthogonal space-form extends to a normal invariant for X.*

PROOF. By the above, X does have a normal invariant. Taking this as base point, we can identify normal invariants for X with homotopy classes of maps $X \to G/\mathrm{Top}$. We can use the induced lift also for the covering space $X(\pi_2)$. Then the given normal invariant here determines a map $\alpha: X(\pi_2) \to G/\mathrm{Top}$, which we wish to factor (up to homotopy) through X. Since (apart from the top cell) $X(\pi_2)$ is 2-local, it is sufficient to 2-localise throughout.

Now α determines classes $\alpha_{4k} \in H^{4k}(\pi_2; \boldsymbol{Z}_{(2)})$, $\alpha_{4k-2} \in H^{4k-2}(\pi_2; \boldsymbol{Z}/2)$ in dimensions $< n$. Since $H^{4k}(\pi; \boldsymbol{Z}_{(2)})$ maps onto $H^{4k}(\pi_2; \boldsymbol{Z}_{(2)})$ (this is obvious when one recalls that π has 2-period 2 or 4), the α_{4k} lead to no obstruction. As to the rest, $H^{4k-2}(\pi_2; \boldsymbol{Z}/2)$ is 0 (π_2 trivial), $\boldsymbol{Z}/2$ (π_2 nontrivial cyclic) or $\boldsymbol{Z}/2 \times \boldsymbol{Z}/2$ (π_2 generalised quaternionic) and—again since the 2-period divides 4—the image of $H^{4k-2}(\pi; \boldsymbol{Z}/2)$ is determined as follows:

Types I, II: surjective.

Types III, V: zero.

Types IV, VI: $\boldsymbol{Z}/2$.

There is thus no obstruction for types I, II.

For the remaining cases, we first reduce to considering the generalised binary tetrahedral groups T_v^* ($v \geq 1$): defined as the nontrivial (split) extension of a quaternion group of order 8 by a cyclic group of order $\boldsymbol{Z}/3^v$. This has type III, and for any group π of type III, π_2 is contained in a subgroup T_v^*. Thus if α extends to $X(T_v^*)$, the obstructions to extending to X must already vanish. If π has type IV, there is a subgroup π^0, of index 2, of type III. Set $\pi_2^0 = \pi^0 \cap \pi_2$. Then $H^{4k-2}(\pi_2; \boldsymbol{Z}/2) \to H^{4k-2}(\pi_2^0; \boldsymbol{Z}/2)$ has image $\boldsymbol{Z}/2$. If the restriction of α to $X(\pi_2^0)$ extends to $X(\pi^0)$, then the restriction of α_{4k-2} to π_2^0 must vanish. This single mod 2 obstruction must thus coincide with the obstruction to extending α_{4k-2} to π. So the type IV case reduces to the type III case. Finally for types V, VI we know by [II, Lemma 3.3] that α extends if and only if for each subgroup Q_8 of π_2 lying in a subgroup T_1^* of π, the restriction of α to Q_8 extends to T_1^*.

We have thus reduced to the case $\pi = T_v^*$, $\pi_2 = Q_8$. Then the (unique) free orthogonal action of π_2 on S^{n-1} extends to a free orthogonal action of π. By the above, the obstruction to extension is a sequence of classes $\alpha_{4k+2} \in H^{4k-2}(\pi_2; \boldsymbol{Z}/2)$. The outer automorphism c of period 3 of π_2 preserves the orthogonal normal invariant (which is unique) and becomes inner in π. Thus the classes α_{4k+2} are invariant under c. But c permutes the three nonzero elements of $H^{4k+2}(Q_8; \boldsymbol{Z}/2)$: the only invariant class is zero. This completes the proof.

Surgery. Instead of going back to first principles, we appeal to [II, Theorem 4.2], which was devised for the present purpose, and now restate it as

PROPOSITION 4. *Let ϕ: $M \to Y$ be a normal map of degree 1 from the closed manifold M of odd dimension $m \geq 5$ to the Poincaré complex Y with finite fundamental group π. Then we can perform surgery on ϕ to obtain a homotopy equivalence if and only if*

(i) *Y is homotopy equivalent to a finite complex,*

(ii) *for every 2-hyperelementary subgroup ρ of π, the corresponding covering space $Y(\rho)$ of Y is homotopy equivalent to a manifold, and*

(iii) *surgery is possible for the covering normal map* $\phi(\pi_2)\colon M(\pi_2) \to Y(\pi_2)$, π_2 *the Sylow* 2-*subgroup of* π.

I recall that the main ingredients in the proof of this result were the induction theorem due to Andreas Dress **[1975]** and the transfer formula of the author **[1976]**. It is now easy to deduce our first main theorem on homotopy type.

THEOREM 1. *The polarised space X (with* $n \geqq 6$) *is homotopy equivalent to a manifold if and only if*

(i) *the finiteness obstruction* $\theta(X) \in \tilde{K}_0(Z\pi)$ *is zero, and*

(ii) *for all* 2-*hyperelementary* $\rho \subset \pi$, $X(\rho)$ *is homotopy equivalent to a manifold.*

PROOF. The conditions are clearly necessary. If they are satisfied, first consider $X(\pi_2)$. This too has zero finiteness obstruction. It follows (by listing cases) that $X(\pi_2)$ has the homotopy type of an orthogonal space-form $Y(\pi_2)$. This homotopy equivalence induces a normal invariant for $X(\pi_2)$, which by Proposition 3 extends to one for X. This determines a class of normal maps $\phi\colon M \to X$ of degree 1. We now apply Proposition 4. Conditions (i) and (ii) are assumed explicitly here, and (iii) holds by the choice of normal invariant. Hence we can do surgery to obtain a homotopy equivalence. This proves the result.

Concerning this result, it seems appropriate to make two remarks. First, we can prove an analogous result in the smooth case, as follows:

If the polarised space X is homotopy equivalent to a manifold, then it is homotopy equivalent to a smooth manifold whose universal cover is diffeomorphic to S^{n-1} ($n \geqq 6$).

Thus we do obtain free smooth actions of π on S^{n-1}.

Secondly, note that it does not provide an existence theorem for free actions of π on spheres. Indeed, the well-known necessary condition due to Milnor has not yet even been mentioned. Accordingly, we now turn to the second part of our problem: namely, the consideration of which homotopy types X—or equivalently, generators g—satisfy conditions (i), (ii) above. Here, I now have to admit that the only examples of manifolds $X(\rho)$ with ρ 2-hyperelementary that we yet possess come from orthogonal space forms. Since these also yield information about the finiteness obstruction, we now reconsider them.

Hyperelementary space-forms: homotopy types.

Type I. π is an extension of a cyclic group ρ of order a prime to p by a cyclic group σ of order p^r (p prime). The extension is split, and is determined by a homomorphism $\alpha\colon \sigma \to \operatorname{Aut} \rho$. Let Im α have order p^s. Then π has an $\mathscr{F}$-representation $\Leftrightarrow$ every subgroup of order pq is cyclic $\Leftrightarrow$ α is *not* injective $\Leftrightarrow s < r$.

The cohomology period is $2p^s$. If $s < r$, the irreducible $\mathscr{F}$-representations all have (complex) degree p^s. If one of them has k-invariant g_0, the others have k-invariants $b^{p^s}g_0$, and any b prime to $|\pi|$ can occur. Since adding representations corresponds to multiplying k-invariants, it remains only to characterise g_0. This we can do via its restrictions to Sylow subgroups (or to ρ and to σ). These are cyclic, so have cohomology rings which are polynomial on a 2-dimensional generator. It is not hard to see

PROPOSITION 5 (i). *The k-invariants which arise from* $\mathscr{F}$-*representations are those cohomology classes whose restriction to each Sylow subgroup is a nonzero* p^s*th power.*

Type II. π is an extension of a cyclic group ρ of odd order by a generalised quaternion group σ of order $2^k \geqq 8$. Again the extension splits, and we have α: $\sigma \to \text{Aut } \rho$. Now Aut ρ is abelian, so the commutator subgroup of σ (in particular the element of order 2) is in Ker α: every subgroup of order $2p$ is automatically cyclic. Moreover, σ^{ab} is a four-group, so we can decompose ρ into eigenspaces under Im α. Explicitly, if $\sigma = \langle x, y / x^{2^{k-1}} = 1, y^2 = x^{2^{k-2}}, y^{-1}xy = x^{-1} \rangle$, we can write $\rho = A \times B \times C \times D$ as Cartesian product of (cyclic) groups of coprime orders a, b, c and d such that

transformation by x centralises A and B, but inverts C and D;

transformation by y centralises A and C, but inverts B and D.

In the notation of Milnor [**1957**], this appears as $\boldsymbol{Z}/a \times Q(2^k b; c, d)$. If $k = 3$, σ has outer automorphisms permuting x, y and xy; so b, c and d play a symmetrical role. We will suppose $b \geqq c \geqq d$. If $k > 3$, we can only permute y and xy, and normalise by $c \geqq d$.

All these groups have cohomological period 4.

Though all have $\mathscr{F}$-representations, the degrees no longer match up. If $c = d = 1$, all irreducible $\mathscr{F}$-representations have real degree 4. Otherwise, all have real degree 8. There is a corresponding dichotomy in determining their homotopy types. As before, we restrict to Sylow subgroups. The part of $H^*(\sigma; \boldsymbol{Z})$ in dimensions divisible by 4 is a polynomial ring. The Chern class (alias k-invariant as above) of an irreducible 2-dimensional representation (necessarily an $\mathscr{F}$-representation) yields a generator $\gamma_0 \in H^4$, unique modulo squares.

PROPOSITION 5 (ii). *If $c = d = 1$, the k-invariants of $\mathscr{F}$-representations are the classes which restrict to nonzero squares at odd Sylow subgroups, and to nonzero squares times powers of γ_0 on σ.*

If $c > 1$, we have nonzero fourth powers at odd Sylow subgroups, and nonzero fourth powers times powers of γ_0^2 on σ. (*Note, however, that changing the orientation of S^{n-1} will change the sign of g.*)

It is now time to discuss

Condition (ii) in Theorem 1. We observed above that the only homotopy types of $X(\pi)$, π 2-hyperelementary, currently known to contain manifolds, are those which contain orthogonal space-forms. While I do not expect this to be the complete result, this gives already a fairly large and representative set of examples.

A 2-hyperelementary $\mathscr{R}$-group π has an $\mathscr{F}$-representation if every subgroup of order $2p$ is cyclic (and this can only fail for π of Type I). But a theorem of Milnor [**1957**] shows that this condition is already necessary for the existence of free topological actions. We may thus impose it.

Next, in 1973 Ronnie Lee reproved Milnor's result by exhibiting an explicit surgery obstruction. He went on to show that if $Q(m; c, d)$ acts freely on S^{n-1}, $16|m$ and $c > 1$, then $8|n$. Thus 2-hyperelementary groups of type II can be placed in three categories:

K, $c = 1$. Free orthogonal action on any S^{4r-1}.

L, $k \geqq 4$, $c > 1$. Free orthogonal action on any S^{8r-1}. No free action on any S^{8r+3}.

M, $k = 3$, $c > 1$. Free orthogonal action on any S^{8r-1} but on no S^{8r+3}. Unknown if free actions exist on any S^{8r+3}.

The most obvious outstanding problem in this area is deciding this last case.

General discussion. The 'generators' are the units of the Tate cohomology ring $\hat{H}^*(\pi; \mathbf{Z})$, and thus form a multiplicative group $\mathscr{G}$, say. Those of degree 0 form a subgroup $\mathscr{G}_0$ isomorphic to the group of units in $(\mathbf{Z}/N)^\times$, where $N = |\pi|$. We have an exact sequence $1 \to \mathscr{G}_0 \to \mathscr{G} \to^{\deg} \mathbf{Z}$, and the image of "deg" is $d\mathbf{Z}$, where d is the period.

The generators of positive degree form a subsemigroup $\mathscr{G}^-$, and by Proposition 1 determine polarised spaces and hence finiteness obstructions. This gives a map $\theta\colon \mathscr{G}^- \to \tilde{K}_0(\mathbf{Z}\pi)$ which, using the algebraic interpretation, is easily seen to be a homomorphism and hence extends to $\mathscr{G}$. The restriction of θ to $\mathscr{G}_0$ was described by Swan **[1960]**: If $r \in (\mathbf{Z}/N)^\times$, then $\theta(r)$ is the class of the projective ideal $\langle r, \sum\{g \in \pi\} \rangle$ in $\mathbf{Z}\pi$. Abstractly, $\mathscr{G} \cong \mathscr{G}_0 \oplus \mathbf{Z}$: can we choose the splitting so that $\theta(\mathbf{Z}) = 0$?

To study this question, first observe that by the induction theorem of Swan **[1960]** the restriction

$$K_0(\mathbf{Z}\pi) \to \bigoplus \{K_0(\mathbf{Z}\rho) : \rho \subset \pi \text{ hyperelementary}\}$$

is injective. This map takes the finiteness obstruction for the (polarised) space X to those for its covering spaces $X(\rho)$. So again it suffices to consider the case where π is hyperelementary.

Here again, orthogonal space-forms yield examples of manifolds, and hence of vanishing finiteness obstructions, for π of Type II, or of Type I with $s < r$. This gap is filled by our next main result.

THEOREM 2. *Let π be p-hyperelementary of Type* I *with $s = r$, g a cohomology generator whose restriction to each Sylow subgroup is a p^sth power. Then $\theta(g) = 0$.*

I will outline the proof of this result below. The following is an easy consequence of Theorems 1 and 2.

THEOREM 3. *Let π be an $\mathscr{R}$-group such that all subgroups of order $2p$ are cyclic. Let $2d_1$ be the cohomology period of π, and let $d_2 = 2d_1$ if $d_1 \equiv 4 \pmod 8$ and π has a hyperelementary subgroup of type* IIL *or* IIM; *$d_2 = d_1$ otherwise.*

Then if X is a polarised space corresponding to a generator g whose restriction to each cyclic Sylow subgroup is a d_2th power, and to each quaternionic Sylow subgroup a d_2th power times a power of γ_0, X is homotopy equivalent to a manifold.

By Theorem 1, with the induction theorem for $K_0(\mathbf{Z}\pi)$, it is enough to consider hyperelementary subgroups. Condition (i) is satisfied by Proposition 5—note that the hypothesis on g is preserved on restriction to subgroups ρ, and that the cohomology period of ρ divides that of π. Condition (ii) is satisfied by Proposition 5 and Theorem 2.

Observe that we can choose $g \in H^{2d_2}(\pi : \mathbf{Z})$. Thus π can act freely—and, in fact, smoothly—on S^{2d_2-1}, and on any S^{2rd_2-1} (this holds even if $d_2 = 1$ or 2, by checking Milnor's 1957 list of cases). These are the only spheres on which π can act freely, except perhaps in case $d_2 = 2d_1$, and π has subgroups of type IIM but none of type IIL. This happens only rarely—for example, if π has type I, III or V, then $d_2 = d_1$.

Though the result on dimensions is almost the best possible, that on homotopy types is definitely not.

EXAMPLE 1. π p-hyperelementary of order pq ($p \neq q$ odd primes). Then $X(g)$ has zero finiteness obstruction—and hence is homotopy equivalent to a manifold—if and only if the restriction of g to the Sylow q-subgroup is a pth power. This is an easy deduction from the calculation by Galovitch, Reiner and Ullom **[1972]** of $K_0(\mathbf{Z}\pi)$.

EXAMPLE 2. π the binary tetrahedral group $SL_2(3)$. In this case, hyperelementary subgroups are cyclic or equal to π_2 (quaternion of order 8, and normal). Thus $\theta(g) = 0 \Leftrightarrow g|\pi_2$ is a square times a power of γ_0. For such g, the corresponding X is homotopy equivalent to a manifold. The case $g = 7g_0$ does not come from an orthogonal action, or from Theorem 3.

The projective class group. PROOF OF THEOREM 2. The localisation square

$$\begin{array}{ccc} A = \mathbf{Z}\pi & \rightarrow & B = \mathbf{Z}\left[\frac{1}{p}\right]\pi \\ \downarrow & & \downarrow \\ C = \mathbf{Z}_{(p)}\pi & \rightarrow & D = \mathbf{Q}\pi \end{array}$$

satisfies the conditions (Bass, **[1968]**) for existence of a natural exact sequence

$$K_1B \oplus K_1C \rightarrow K_1D \xrightarrow{\partial} K_0A \xrightarrow{i_*} K_0B \oplus K_0C.$$

Suppose we have a finite chain complex P_* of finitely generated projective A-modules such that $\chi(P) = \sum(-1)^i[P_i] \in K_0(A)$ is in Ker $i_* = \operatorname{Im} \partial$. Then $P_* \otimes B$, $P_* \otimes C$ are chain homotopy equivalent to free, based complexes F_B, F_C over B, C respectively. We now have a preferred homotopy class of chain homotopy equivalences η: $F_B \otimes D \rightarrow F_C \otimes D$ of based free D-complexes, defining a torsion element $\tau(\eta) \in K_1(D)$. Then $\chi(P) = \partial\tau(\eta)$. The proof is straightforward, reducing to the definition of the homomorphism ∂.

For our problem, the algebra and topology are related as follows (cf. Swan **[1960]**). An exact sequence

$$0 \rightarrow \mathbf{Z} \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow \mathbf{Z} \rightarrow 0$$

with P_i a finitely generated projective $\mathbf{Z}\pi$-module determines a class g in $\operatorname{Ext}^n_{\mathbf{Z}\pi}(\mathbf{Z}; \mathbf{Z}) \cong H^n(\pi; \mathbf{Z})$ which is a generator. Conversely, g determines the chain-homotopy type of P_*, and $\theta(g) = \chi(P_*)$. Corresponding statements hold on localising $\mathbf{Z}$. Though it is difficult to write down P_* explicitly, such complexes can be produced after suitable localisations. We proceed indirectly as follows.

Write $\pi = \rho\sigma$ as usual, with $\alpha: \sigma \rightarrow \operatorname{Aut} \rho$. Choose an epimorphism $\sigma' \rightarrow \sigma$, with σ' cyclic of order p^{s+1} (so the kernel ζ has order p), and let the composition α': $\sigma' \rightarrow \sigma \rightarrow \operatorname{Aut} \rho$ define the split extension $\pi' = \rho\sigma'$. We have an epimorphism $\pi' \rightarrow \pi$ with kernel ζ. Since α' has nontrivial kernel, π' has $\mathscr{F}$-representations; choose one whose k-invariant g' is the image of g at primes other than p (such exist by the conditions on g). This yields a *free* $\mathbf{Z}\pi'$-chain complex F'_* defining g'.

Now $F'_* \otimes_{\mathbf{Z}\pi'} \mathbf{Z}\pi$ does not give a multiple extension of $\mathbf{Z}$ by $\mathbf{Z}$, but this fails only at the prime p. We can thus take $F_B = F'_* \otimes B$. As to C, this contains the idempotent $\varepsilon_\rho = \sum\{g \in \rho\}/|\rho|$, which is central. Hence C splits as $C\varepsilon_\rho \oplus C(1 - \varepsilon_\rho)$, and $C\varepsilon_\rho \cong \boldsymbol{Z}_p\sigma$. Since σ is cyclic, with generator T, say, we now have the resolution

$$
\begin{array}{lcccccccccccc}
 & \cdots & C\varepsilon_\rho & \xrightarrow{\Sigma T^i} & C\varepsilon_\rho & \xrightarrow{1-T} & C\varepsilon_\rho & \xrightarrow{\Sigma T_i} & C\varepsilon_\rho & \xrightarrow{1-T} & C\varepsilon_\rho & \to & \boldsymbol{Z}1/p \to 0 \\
F_C: & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
 & \cdots & C(1-\varepsilon_\rho) & \xrightarrow{0} & C(1-\varepsilon_\rho) & = & C(1-\varepsilon_\rho) & \xrightarrow{0} & C(1-\varepsilon_\rho) & = & C(1-\varepsilon_\rho) & \longrightarrow & 0
\end{array}
$$

To compute the torsion, define F'_C for π' similarly to F_C for π, using a generator T' for σ' which maps onto T. Observe that the corresponding cohomology classes evidently restrict to p^sth powers at σ, σ'. We can choose T' so that this restriction agrees with that of g'. Then there is a chain homotopy equivalence $F'_* \otimes C' \to F'_C$ whose torsion in $K_1(C')$ clearly maps to 0 in $K_0(A')$. We can thus factor our equivalence θ as

$$F_B \otimes D = F'_* \otimes D \xrightarrow{\theta_1} F'_C \otimes D \xrightarrow{\theta_2} F_C \otimes D.$$

Now $\partial\tau(\theta_1) = 0$ by an obvious commutative diagram. As to θ_2, observe that $F'_C \otimes C$ differs from F_C only in that each map $\sum T^i$ is replaced by $p\sum T^i$ (since the order of T' is p times that of T). Thus $\tau(\theta_2) \in K_1(D)$ is equal to 1 at summands of D corresponding to nontrivial representations of π, and to p^{p^s} at the trivial component.

Our obstruction is thus $\partial(p^{p^s} \oplus 1)$. Since π has Artin exponent p^s (see Bass [**1968**]), this vanishes provided $\partial(p \oplus 1)$ is trivial on cyclic subgroups τ of π. But if $\boldsymbol{Z}\tau \subset \boldsymbol{Q}\tau = \boldsymbol{Q} \oplus (\boldsymbol{Q}\tau)'$ has projections $\boldsymbol{Z}$, $(\boldsymbol{Z}\tau)'$, $\partial(p \oplus 1)$ is in $\mathrm{Ker}(K_0(\boldsymbol{Z}\tau) \to K_0((\boldsymbol{Z}\tau)'))$, which is known to vanish for τ cyclic.

To conclude the proof, it remains to note that any triple (F_B, F_C, θ) *does* determine a chain-homotopy type of P_A as above—this is a standard argument—so we obtain a complex P_* with $0 = \chi(P_*)$ and hence (after easy modifications) a free complex. The corresponding g'' is (by the above) equal to g at Sylow q-subgroups with $q \neq p$, and a p^sth power on σ: now by changing g' we see that we can make this any p^sth power, and so achieve $g'' = g$. This concludes our outline proof.

The proof of Theorem 3 shows that—apart from troubles with 2-hyperelementary groups of types IIL, IIM—we have succeeded in splitting $\mathscr{G} = \mathscr{G}_0 \oplus \boldsymbol{Z}$ so that $\theta(\boldsymbol{Z}) = 0$. Further progress therefore depends on a study of the ideals $\langle r, \sum\{g \in \pi\} \rangle$ in $\boldsymbol{Z}\pi$ for π p-hyperelementary. This depends only on r modulo $|\pi|$, and vanishes (as we have seen above) for p^sth powers, where p^s is the period. In the opposite direction, our best result so far is

PROPOSITION 6. *If $\theta(g) = 0$, then for each odd prime p, the restriction of g to the Sylow p-subgroup is a d_pth power, where $2d_p$ is the p-period of π.*

But even this (with the analogous result for $p = 2$) is not always sufficient. I hope to give a fuller discussion of this problem at a future date.

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