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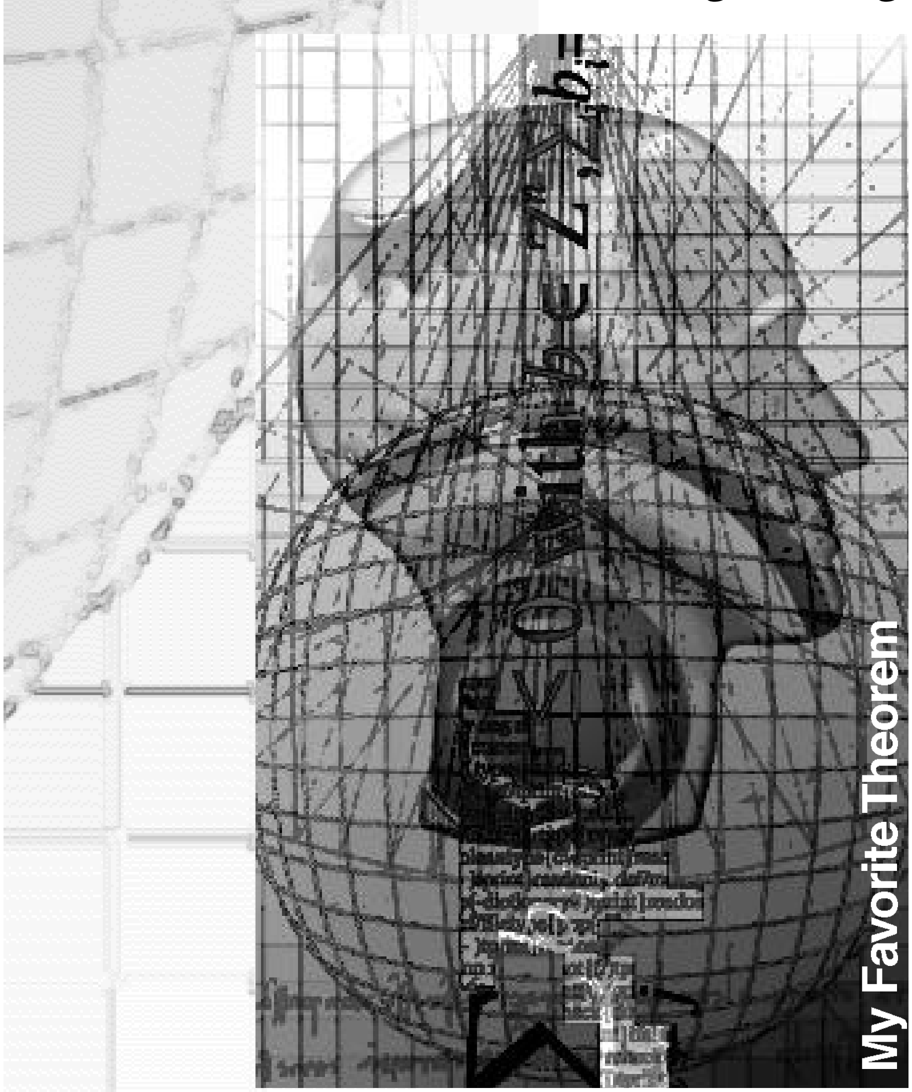


Characterizations of Perfect Graphs

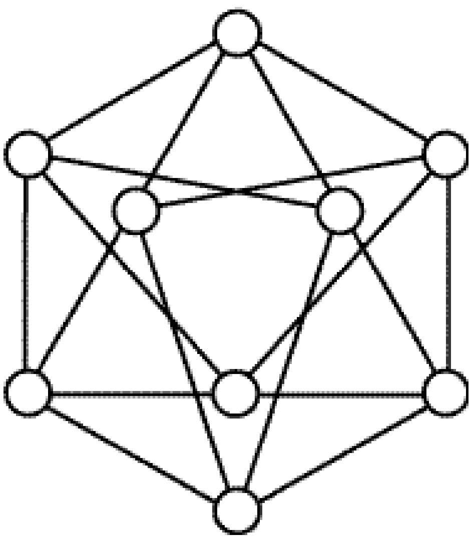
by Martin Grötschel

NUMBER 62

My Favorite Theorem:

Characterizations of Perfect Graphs

By Martin Grötschel



The favorite topics and results of a researcher change over time, of course. One area that I have always kept an eye on is that of perfect graphs. These graphs, introduced in the late '50s and early '60s by Claude Berge, link various mathematical disciplines in a truly unexpected way: graph theory, combinatorial optimization, semidefinite programming, polyhedral and convexity theory, and even information theory.

This is not a survey of perfect graphs. It's just an appetizer. To learn about the origins of perfect graphs, I recommend reading the historical papers [1] and [2]. The book [3] is a collection of important articles on perfect graphs. Algorithmic aspects of perfect graphs are treated in [13]. A comprehensive survey of graph classes, including perfect graphs, can be found in [5]. Hundreds of classes of perfect graphs are known; 96 important classes and the inclusion relations among them are described in [16].

So, what is a perfect graph? Complete graphs are perfect; bipartite, interval, comparability, triangulated, parity, and unimodular graphs are perfect as well. The following beautiful perfect graph is the line graph of the complete bipartite graph $K_{3,3}$.

Due to the evolution of the theory, definitions of perfection (and versions thereof) have changed over time. To keep this article short, I do not follow the historical development of the notation. I use definitions that streamline the presentation. Berge defined

G is a *perfect* graph,

if and only if

(1) $\omega(G') = \chi(G')$ for all node-induced subgraphs $G' \quad G$,

where $\omega(G)$ denotes the *clique number* of G (= largest cardinality of a *clique* of G, i.e., a set of mutually adjacent nodes) and $\chi(G)$ is the chromatic number of G (= smallest number of colors needed to color the nodes of G). Berge discovered that all classes of perfect graphs he found also have the property that

(2) $\alpha(G') = \overline{\chi}(G)$ for all node-induced subgraphs $G' \quad G$,

where $\alpha(G)$ is the *stability number* of G (= largest cardinality of a *stable set* of G, i.e., a set of mutually nonadjacent nodes) and $\overline{\chi}(G')$ denotes the *clique covering* number of G (= smallest number of cliques needed to cover all nodes of G exactly once).

Note that complementation (two nodes are adjacent in the *complement* $\overline{G}$ of a graph G iff they are nonadjacent in G) transforms a clique into a stable set and a coloring into a clique covering, and vice versa. Hence, the complement of a perfect graph satisfies (2). This observation and his discovery mentioned above led Berge to conjecture that G is a perfect graph if and only if

(3) $\overline{G}$ is a perfect graph.

Developing the antiblocking theory of polyhedra, Fulkerson launched a massive attack on this conjecture (see [10], [11], and [12]). The conjecture was solved in 1972 by Lovász [17], who gave two short and elegant proofs. Lovász [18], in addition, characterized perfect graphs as those graphs $G = (V, E)$ for which the following holds:

(4) $\omega(G') \cdot \alpha(G') \geq |V(G')|$ for all node-induced subgraphs $G' \quad G$.

A link to geometry can be established as follows. Given a graph $G = (V, E)$, we associate with G the vector space $\mathbf{R}^V$ where each component of a vector of $\mathbf{R}^V$ is indexed by a node of G. With every subset $S \quad V$, we can associate its incidence vector $\chi^S = (\chi^S_v)_{v \quad V} \quad \mathbf{R}^V$ defined by

$$\chi_v^S := 1 \text{ if } v \quad S,\ \chi_v^S := 0 \text{ if } v \quad S.$$

The convex hull of all the incidence vectors of stable sets in G is denoted by STAB(G), i.e.,

$$\text{STAB}(G) = \text{conv}\,\{\chi^S \quad \mathbf{R}^V \mid S \quad V \text{ stable}\}$$

and is called the *stable set polytope* of G. Clearly, a clique and a stable set of G can have at most one node in common. This observation yields that, for every clique $Q \quad V$, the so-called *clique inequality*

$$x(Q) := \sum_{v \quad Q} x_v \le 1$$

is satisfied by every incidence vector of a stable set. Thus, all clique inequalities are valid for STAB(G). The polytope

$$\text{QSTAB}(G) := \{x \quad \mathbf{R}^V \mid 0 \le x_v \quad v \quad V,\ x(Q) \le 1 \quad \text{cliques } Q \quad V\}$$

called *fractional stable set polytope* of G, is therefore a polyhedron containing STAB(G), and trivially,

$$\text{STAB}(G) = \text{conv}\,\{x \quad \{0,1\}^V \mid x \quad \text{QSTAB}(G)\}.$$

Knowing that computing $\alpha(G)$ (and its weighted version) is $\mathcal{NP}$-hard, one is tempted to look at the LP relaxation

$$\max c^T x,\ x \quad \text{QSTAB}(G),$$

where $c \quad \mathbf{R}_+^V$ is a vector of node weights. However, solving LPs of this type is also $\mathcal{NP}$-hard for general graphs G (see [14]).

For the class of perfect graphs G, though, these LPs can be solved in polynomial time – albeit via an involved detour (see below).

Let us now look at the following chain of inequalities and equations, typical for IP/LP approches to combinatorial problems. Let $G = (V,E)$ be some graph and $c \ge 0$ a vector of node weights:

$$\begin{aligned}
&\max\,\{\sum_{v \quad S} c_v \mid S \quad V \text{ stable set of } G\} = \\
&\max\,\{c^T x \mid x \quad \text{STAB}(G)\} = \\
&\max\,\{c^T x \mid x \ge 0,\ x(Q) \le 1 \ \text{ cliques} Q \quad V,\ x \quad \{0,1\}^V\} \le \\
&\max\,\{c^T x \mid x \ge 0,\ x(Q) \le 1 \ \text{ cliques} Q \quad V\} = \\
&\min\,\{\sum_{Q \text{clique}} y_Q \mid \sum_{Q \quad v} y_Q \ge c_v \quad v \quad V,\ y_Q \ge 0 \ \text{ cliques} Q \quad V\} \le \\
&\min\,\{\sum_{Q \text{clique}} y_Q \mid \sum_{Q \quad v} y_Q \ge c_v \quad v \quad V,\ y_Q \quad \mathbb{Z}_+ \ \text{ cliques} Q \quad V\}
\end{aligned}$$

The inequalities come from dropping or adding integrality constraints, the last equation is implied by LP duality. The last program can be interpreted as an IP formulation of the weighted clique covering problem. It follows from (2) that equality holds throughout the whole chain for all 0/1 vectors c iff G is a perfect graph. This, in turn, is equivalent to

(5) The value $\max\,\{c^T x \mid x \quad \text{QSTAB}(G)\}$ is integral for all $c \quad \{0,1\}^V$.

Results of Fulkerson [10] and Lovász [17] imply that (5) is in fact equivalent to

(6) The value $\max\,\{c^T x \mid x \quad \text{QSTAB}(G)\}$ is integral for all $c \quad \mathbf{Z}_+^V$.

and that, for perfect graphs, equality holds throughout the above chain for all $c \quad \mathbf{Z}_+^V$. This, as a side remark, proves that the constraint system defining QSTAB(G) in totally dual integral for perfect graphs G. Chvátal [6] observed that (6) holds iff

$$\text{STAB}(G) = \text{QSTAB}(G)$$

These three characterizations of perfect graphs provide the link to polyhedral theory (a graph is perfect iff certain polyhedra are identical) and integer programming (a graph is perfect iff certain LPs have integral solution values).

Another quite surprising road towards understanding properties of perfect graphs was paved by Lovász [19]. He introduced a new geometric representation of graphs linking perfectness to convexity and semidefinite programming.

An *orthonormal representation* of a graph $G = (V,E)$ is a sequence $(u_i \mid i \quad V)$ of $|V|$ vectors $u_i \quad \mathbf{R}^V$ such that $\|u_i\|=1$ for all $i \quad V$ and $u_i^T u_j = 0$ for all pairs i,j of nonadjacent nodes. For any orthonormal representation $(u_i \mid i \quad V)$ of G and any additional vector c of unit length, the so-called *orthonormal representation constraint*

$$\sum_{i \quad V} (c^T u_i)^2 x_i \le 1$$

is valid for STAB(G). Taking an orthonormal basis $B = \{e_1, \ldots, e_{|V|}\}$ of $\mathbf{R}^V$ and a clique Q of G, setting $c := u_i := e_1$ for all $i \quad Q$, and assigning different vectors of $B \backslash \{e_1\}$ to the remaining nodes $i \quad V \backslash Q$, one observes that every clique constraint is a special case of this class of infinitely many inequalities. The set

$$\begin{gathered}\text{TH}(G) := \{x \quad \mathbf{R}_+^V \mid x \text{ satisfies all} \\ \text{orthonormal representation constraints}\}\end{gathered}$$

is thus a convex set with

$$\text{STAB}(G) \quad \text{TH}(G) \quad \text{QSTAB}(G).$$

It turns out (see [14]) that a graph G is perfect if and only if any of the following conditions is satisfied:

(8) $\text{TH}(G) = \text{STAB}(G)$.

(9) $\text{TH}(G) = \text{QSTAB}(G)$.

(10) $\text{TH}(G)$ is a polytope.

The last result is particularly remarkable. It states that a graph is perfect if and only if a certain convex set is a polytope.

If $c \quad \mathbf{R}_+^V$ is a vector of node weights, the optimization problem (with infinitely many linear constraints)

$$\max c^T x,\ x \quad \text{TH}(G)$$

can be solved in polynomial time for any graph G. This implies, by (8), that the weighted stable set problem for perfect graphs can be solved in polynomial time, and by LP duality, that the weighted clique covering problem, and by complementation, that the weighted clique and coloring problem can be solved in polynomial time. These results rest on the fact that the value

$$\vartheta\,(G, c) := \max\ \{c^T x \mid x \quad \text{TH}(G)\}$$

can be characterized in many equivalent ways, e.g., as the optimum value of a semidefinite program, the largest eigenvalue of a certain set of symmetric matrices, or the maximum value of some function involving orthornormal representations.

Details of this theory can be found, e.g., in Chapter 9 of [14]. The algorithmic results involve the ellipsoid method. It would be nice to have "more combinatorial" algorithms that solve the four optimization problems for perfect graphs in polynomial time.

Let us now move into information theory. Given a graph $G = (V,E)$, we call a vector $p \quad \mathbf{R}_+^V$ a *probability distribution* on V if its components sum to 1. Let $G^{(n)} = (V^n, E^{(n)})$ denote the so-called *n-th conormal power* of G, i.e., V^n is the set of all n-vectors $x = (x_1,\ldots, x_n)$ with components $x_i \quad V$, and

$$E^{(n)} := \{xy \mid x, y \quad V^n \text{ and } \quad i \text{ with } x_i y_i \quad E\}$$

Each probability distribution p on V induces a probability distribution p^n on V^n as follows: $p^n(x) := p(x_1) \cdot p(x_2) \cdot \ldots \cdot p(x_n)$. For any node set $U \quad V^n$, let $G^{(n)}[U]$ denote the subgraph of $G^{(n)}$ induced by U and $\chi(G^{(n)}[U])$ its chromatic number. Then one can show that, for every $0 < \varepsilon < 1$, the limit

$$H(G,p) := \lim_{n\to\infty} \frac{1}{n} \min_{p^n(U)\geq 1-\varepsilon} \log \chi(G^{(n)}[U])$$

exists and is independent of ε (the logs are taken to base 2). $H(G,p)$ is called the graph entropy of the graph G with respect to the probability distribution p. If $G = (V,E)$ is the complete graph, we get the well-known *Shannon entropy*

$$H(p) = -\sum_{i \quad V} p_i \log p_i.$$

Let us call a graph $G = (V,E)$ strongly splitting if for every probability distribution p on V

$$H(p) = H(G,p) + H(\bar{G}, p)$$

holds. Csiszár et. al [9] have shown that a graph is perfect if and only if

G is strongly splitting.

I.e., G is perfect iff, for every probabiltity distribution, the entropies of G and of its complement $\bar{G}$ add up to the entropy of the complete graph (the Shannon entropy). I recommend [9] for the study of graph entropy and related topics.

Given all these beautiful characterizations of perfect graphs and polynomial time algorithms for many otherwise hard combinatorial optimization problems, it is really astonishing that nobody knows to date whether perfectness of a graph can be recognized in polynomial time. There are many ways to prove that, deciding whether a graph is not perfect, is in $\mathcal{NP}$. But that's all we know!

Many researchers hope that a proof of the most famous open problem in perfect graph theory, the *strong perfect graph conjecture*:

A graph G is perfect if and only if G neither contains an odd hole nor an odd antihole as an induced subgraph.

results in structural insights that lead to a polynomial time algorithm for recognizing perfect graphs. It is trivial that every odd hole (= chordless cycle of length at least five) and every odd antihole (= complement of an odd hole) are not perfect. Whenever Claude Berge encountered an imperfect graph G he discovered that G contains an odd hole or an odd antihole and, thus, came to the strong perfect graph conjecture. In his honor, it is customary to call graphs without odd holes and odd antiholes *Berge graphs.* Hence, the strong perfect graph conjecture essentially reads: every Berge graph is perfect.

This conjecture stimulated a lot of research resulting in fascinating insights into the structure of graphs that are in some sense nearly perfect or imperfect. E.g., Padberg [20], [21] (introducing perfect matrices and using proof techniques from linear algebra) showed that, for an imperfect graph $G = (V,E)$ with the property that the deletion of any node results in a perfect graph, satisfies the following:

- $|V| = \alpha(G) \cdot \omega(G) + 1$,
- G has exactly $|V|$ maximum cliques, and every node is contained in exactly $\omega(G)$ such cliques.
- G has exactly $|V|$ maximum stable sets, and every node is contained in exactly $\alpha(G)$ such stable sets.
- QSTAB(G) has exactly one fractional vertex, namely the point $x_v = 1/\omega(G) \quad v \quad V$, which is contained in exactly $|V|$ facets and adjacent to exactly $|V|$ vertices, the incidence vectors of the maximum stable sets.

Similar investigations (but not resulting in such strong structural results) have recently been made by Annegret Wagler [24] on graphs which are perfect and have the property that deletion (or addition) of any edge results in an imperfect graph. The graph of Figure 1 is from Wagler's Ph.D. thesis. It is the smallest perfect graph G such that whenever any edge is added to G or any edge is deleted from G the resulting graph is imperfect.

Particular efforts have been made to characterize perfect graphs "constructively" in the following sense. One first establishes that a certain class C_∞ of graphs is perfect and considers, in addition, a finite list C of special perfect graphs. Then one defines a set of "operations" (e.g., replacing a node by a stable set or a perfect graph) and "compositions" (e.g., take two graphs G and H and two nodes $u \quad V(G)$ and $v \quad V(H)$, define $V(G \circ H) := (V(G) \quad V(H)) \setminus \{u,v\}$ and $E(G \circ H) := E(G - u) \quad E(G - v) \quad \{x,y \mid xu \quad E(G), yv \quad E(H)\}$ and shows that every perfect graph can be constructed from the basic classes C_∞ and C by a sequence of operations and compositions. Despite ingenious constructions (that were very helpful in proving some of the results mentioned above) and lots of efforts, this route of research has not led to success yet. A paper describing many compositions that construct perfect graphs from perfect graphs is, e.g., [8].

Chvátal [7] initiated research into another "secondary structure" related to perfect graphs in order to come up with a (polynomial time recognizable) certificate of perfection. For a given graph $G = (V,E)$, its *P_4-structure* is the 4-uniform hypergraph on V whose hyperedges are all the 4-element node sets of V that induce a P_4 (path on four nodes) of G. Chvátal observed that any graph whose P_4-structure is that of an odd hole is an odd hole or its complement and, thus, conjectured that perfection of a graph depends solely on its P_4-structure. Reed [23] solved Chvátal's semi-strong perfect graph conjecture by showing that a graph G is perfect iff

(12) G has the P_4-structure of a perfect graph.

There are other such concepts, e.g., the *partner-structure*, that have resulted in further characterizations of perfect graphs through secondary structures. We recommend [15] for a thorough investigation of this topic. But the polynomial-time-recognition problem for perfect graphs is still open.

A relatively recent line of research in the area of structural perfect graph theory is the use of the probability theory. I would like to mention just one nice result of Prömel und Steger [22]. Let us denote the number of perfect graphs on n nodes by *Perfect* (n) and the number of Berge graphs on n nodes by *Berge* (n), then

$$\lim_{n\to\infty} \frac{\text{Perfect}(n)}{\text{Berge}(n)} = 1.$$

In other words, almost all Berge graphs are perfect, which means that if there are counterexamples to the strong perfect graph conjecture, they are "very rare."

The theory of random graphs provides deep insights into the probabilistic behavior of graph parameters (see [4], for instance). To take a simple example, consider a random graph $G = (V,E)$ on n nodes where each edge is chosen with probality ½. It is well known that the expected values of $\alpha(G)$ and $\omega(G)$ are of order log n while $\chi(G)$ and $\bar{\chi}(G)$ both have expected values of order $n/\log n$. This implies that such random graphs are almost surely not perfect. An interesting question is to see whether the "LP-relaxation of $\alpha(G)$," the so-called fractional stability number $\alpha^*(G) = \max\{\mathbf{1}^T x \mid x \quad \text{QSTAB}(G)\}$, is a good approximation of $\alpha(G)$. Observing that the point $x = (x_v)_{v \quad V}$ with $x_v := 1/\omega(G)$, $v \quad V$, satisfies all clique constraints and is thus in QSTAB(G) and knowing that $\omega(G)$ is of order log n one can deduce that the expected value of $\alpha^*(G)$ is of order $n/\log n$, i.e., it is much closer to $\bar{\chi}(G)$ than to $\alpha(G)$. Hence, somewhat surprisingly, $\alpha^*(G)$ is a pretty bad approximation of $\alpha(G)$ in general – not so for perfect graphs, though.

To summarize this quick tour through perfect graph theory (omitting quite a number of the other interesting developments and important results), here is my favorite theorem:

Theorem *Let G be a graph. The following twelve conditions are equivalent and characterize G as a perfect graph*:

(1) $\omega(G') = \chi(G')$ *for all node-induced subgraphs* $G' \quad G$.

(2) $\alpha(G') = \bar{\chi}(G')$ *for all node-induced subgraphs* $G' \quad G$.

(3) $\bar{G}$ *is a perfect graph*.

(4) $\omega(G') \cdot \alpha(G') \geq |V(G')|$ *for all node-induced subgraphs* $G' \quad G$.

(5) The value $\max\{c^T x \mid x \quad QSTAB(G)\}$ *is integral for all* $c \quad \{0,1\}^V$.

(6) The value $\max\{c^T x \mid x \quad QSTAB(G)\}$ *is integral for all* $c \quad \mathbf{Z}_+^V$.

(7) $STAB(G) = QSTAB(G)$.

(8) $TH(G) = STAB(G)$.

(9) $TH(G) = QSTAB(G)$.

(10) $TH(G)$ *is a polytope*.

(11) *G is strongly splitting*.

(12) *G has the* P_4*-structure of a perfect graph*.

References

[1] C. Berge, The history of the perfect graphs, *Southeast Asian Bull. Math. 20*, No.1 (1996) 5-10.

[2] C. Berge, Motivations and history of some of my conjectures, *Discrete Mathematics* 165-166 (1997) 61-70.

[3] C. Berge, V. Chvátal (eds.), Topics on perfect graphs, *Annals of Discrete Mathematics 21*, North-Holland, Amsterdam, 1984.

[4] B. Bollobás, *Random Graphs*, Academic Press, 1985.

[5] A. Brandstädt, V. B. Le, J. P. Spinrad, *Graph Classes: A Survey*, SIAM, 1999.

[6] V. Chvátal, On certain polytopes associated with graphs, *Journal of Combinatorial Theory B* 18 (1975) 138-154.

[7] V. Chvátal, A semi-strong perfect graph conjecture, *Annals of Discrete Mathematics 21* (1984) 279-280.

[8] G. Cornuéjols, W. H. Cunningham, Compositions for perfect graphs, *Discrete Mathematics 55* (1985) 245-254.

[9] I. Csiszár, J. Körner, L. Lovász, K. Marton, G. Simony, Entropy splitting for anti-blocking corners and perfect graphs, *Combinatorica 10* (1) (1990) 27-40.

[10] J. D.R. Fulkerson, The perfect graph conjecture and pluperfect graph theorem *Proc. 2nd Chapel Hill Conf. Combin. Math. Appl.*, Univ. North Carolina (1970), 171-175.

[11] D.R. Fulkerson, Blocking and anti-blocking pairs of polyhedra, *Mathematical Programming 1* (1971) 168-194.

[12] D. R. Fulkerson, On the perfect graph theorem, Math. Programming, *Proc. Advanced Seminar*, Univ. Wisconsin, Madison (1972) 69-76.

[13] M. C. Golumbic, *Algorithmic graph theory and perfect graphs*, Academic Press, New York, 1980.

[14] M. Grötschel, L. Lovász, A. Schrijver, *Geometric algorithms and combinatorial optimization*, Springer, Berlin-Heidelberg, 1988.

[15] S. Hougardy, *On the P_4-structure of perfect graphs*, Shaker-Verlag, Aachen 1996.

[16] S. Hougardy, Inclusions Between Classes of Perfect Graphs, Preprint, Institut für Informatik, Humboldt-Universität zu Berlin, 1998, see <http://www.informatik.hu-berlin.de/~hougardy/paper/classes.html>.

[17] L. Lovász, Normal hypergraphs and the perfect graph conjecture, *Discrete Mathematics 2* (1972) 253-267.

[18] L. Lovász, A characterization of perfect graphs, *Journal of Combinatorial Theory (B)* 13 (1972) 95-98.

[19] L. Lovász, On the Shannon capacity of a graph, *IEEE Transf. Inform. Theory 25* (1979) 1-7.

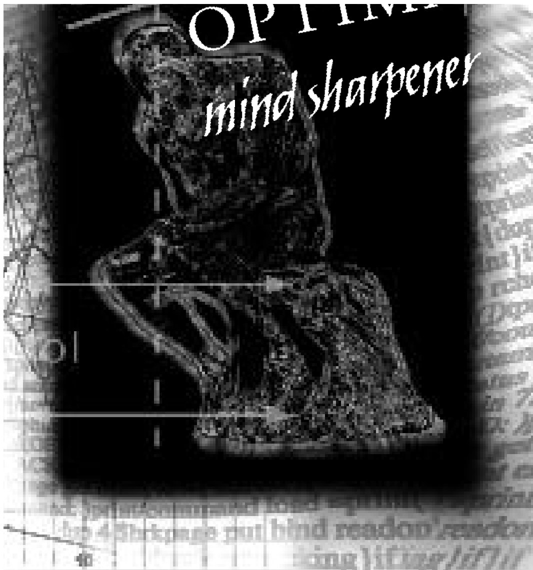
[20] M.W. Padberg, Perfect zero-one matrices, *Math. Programming 6* (1974) 180-196.

[21] M.W. Padberg, Almost integral polyhedra related to certain combinatorial optimization problems, *Lin. Alg. Appl. 15* (1976) 69-88.

[22] H.J. Prömel, A. Steger, Almost all Berge graphs are perfect, *Combin. Probab. Comput. 1* (1992) 53-79.

[23] B. Reed, A semi-strong perfect graph theorem, *Journal of Comb. Theory, Ser. B 43* (1987) 223-240.

[24] A. Wagler, *Critical edges in perfect graphs*, Ph.D. thesis, TU Berlin, 1999.



We invite OPTIMA readers to submit solutions to the problems to Robert Bosch (bobb@cs.oberlin.edu). The most attractive solutions will be presented in a forthcoming issue.

Peaceably Coexisting Armies of Queens

Robert A. Bosch

June 14, 1999

The 8-queens problem requires that eight queens be placed on a chessboard in such a way that no two can attack each other (i.e., lie in the same row, column, or diagonal). See Figure 1 for a solution. The 8-queens problem was proposed by a chessplayer named Max Bezzel in 1848. During the next two years, it captured the attention of Carl Friedrich Gauss (albeit briefly) and other prominent scholars. In September 1850, Franz Nauck presented 92 solutions. See [2] for a detailed history.

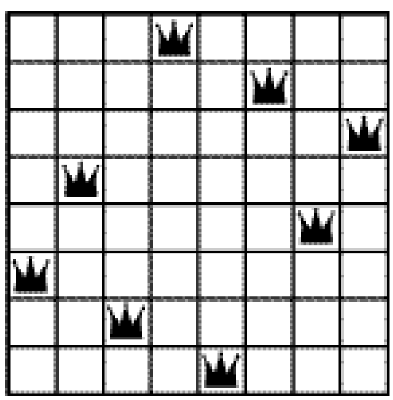


Figure 1

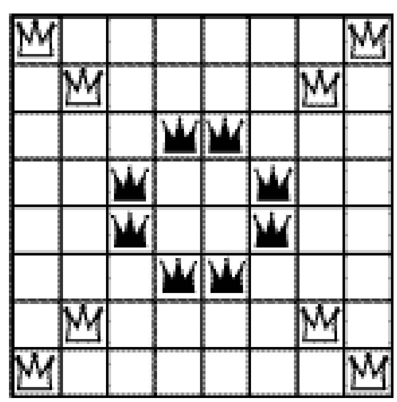


Figure 2

Today, the problem continues to fascinate. A web page maintained by Walter Kosters lists 66 recent books and articles that refer to the 8-queens problem or its generalization, the n-queens problem [5]. One article, by L.R. Foulds and D.G. Johnston, describes two strategies for solving various "chessboard non-attacking puzzles": maximal cliques and integer programming [3].

Here, we pose a variant.

Problems

1. Figure 2 demonstrates that two armies of eight queens can peaceably coexist on a chessboard (i.e., be placed on the board in such a way that no two queens from opposing armies can attack each other). Formulate an integer program to find the maximum size of two equal-sized, peaceably coexisting armies of queens.
2. How many optimal solutions does the integer program have?

Please send solutions and/or comments to <bobb@cs.oberlin.edu>. The most attractive solutions will be presented in a forthcoming issue of OPTIMA.

Pentomino Exclusion Revisited

In the December 1998 issue of OPTIMA, we presented an integer programming formulation of the $n \times n$ pentomino exclusion problem. This problem requires that monominoes (1×1 square) be placed on an $n \times n$ board in such a way that there is no room left for any pentominoes (objects formed by joining five 1×1 squares in an edge-to-edge fashion). The goal is to use as few monominoes as possible.

We also challenged readers to find a better formulation, and we were pleased to receive a couple of excellent submissions. One, by Frank Plastria, contains a description of several families of valid inequalities for our original formulation as well as proofs that some of these inequalities define facets [6]. The second, by Kurt Anstreicher, does not discuss the formulation at all. Instead, it gives a simple, direct proof that at least 24 monominoes are needed to exclude all pentominoes from an 8 × 8 board [1].

In the next section, we discuss Plastria's valid inequalites. And then, in the three sections that follow, we describe a completely different formulation of the problem, present evidence that it is vastly superior to the original formulation, and show how its constraints can be used to construct a simple proof of a result concerning toroidal pentomino exclusion problems.

Valid Inequalities for the Original Formulation

In [6], Plastria begins by noting that the original formulation can be written very concisely:

$$\begin{array}{lll} \text{minimize} & \sum_{s \in S} x_s & \\ \text{subject to} & \sum_{s \in P} x_s \geq 1 & \text{for all } P \in \mathcal{P} \\ & x_s \in \{0,1\} & \text{for all } s \in S. \end{array}$$

Here, S stands for the set of all squares of the board, and $\mathcal{P}$ stands for the set of all pentomino placements. Plastria considers a pentomino placement to be a 5-element subset of S that consists of the five squares that are covered when some pentomino is placed – either in its "standard" orientation or in some rotated or reflected orientation – somewhere on the board.

Plastria goes on to construct several families of "hexomino" constraints. He proves that his hexomino constraints are valid inequalities and that some are facet-generating. Figure 3 contains graphical displays of three families of hexomino constraints. All three are facet-generating. The one on the left states that at least two of the six squares in any 2 x 3 rectangular-shaped region should receive monominoes. The center one states that if the central square of a region that forms a "ribbon-shaped" hexomino does not receive a monomino, then at least two of the other five squares of the region must receive monominoes.

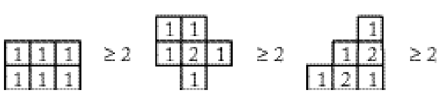


Figure 3

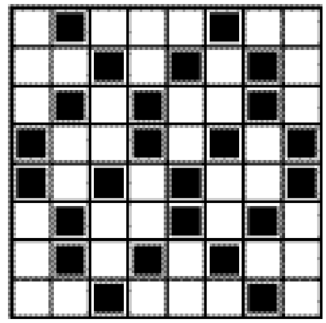


Figure 4

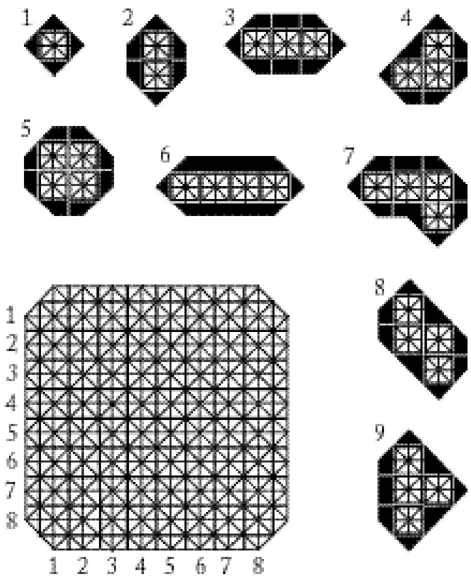


Figure 5

A Completely Different Formulation

Figure 4 displays an optimal solution to the 8 × 8 pentomino exclusion problem. Note that in the solution, empty squares (squares without monominoes) appear in "clumps." In fact, each clump of empty squares is a polyomino ("skew" tetrominoes, T tetrominoes, and L tetrominoes).

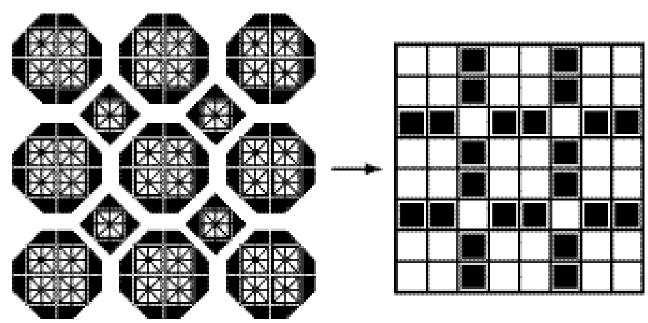


Figure 6

This simple fact can serve as the foundation of a set packing formulation of polyomino exclusion problems. (The original formulation is a set covering formulation.) Here, we form solutions not by placing individual monominoes on individual squares of the board, as in the original, set covering approach, but by placing "tiles" on a modified version of the board. Each tile consists of an entire clump of empty squares as well as portions of neighboring squares (and portions of monominoes).

The lower lefthand corner of Figure 5 displays the modified board. We created it by expanding the original board (by adding a border of partial squares) and then "triangulating" the expansion (by dividing each square and partial square into equal-sized triangular pieces).

The rest of Figure 5 displays each tile in its "standard orientation." Note that there is a tile for each possible clump of empty squares. Each of tiles 1 and 5 has only one orientation (henceforth referred to as "number 1"). Each of tiles 2, 3, and 6 has two: its standard orientation (number 1), and the orientation obtained by rotating the standard orientation 90 degrees clockwise (number 2). Similarly, each of tiles 4 and 9 has four orientations: its standard orientation (number 1), and the orientations obtained by rotating the standard orientation 90, 180, and 270 degrees clockwise (numbers 2, 3, and 4). Tile 7 has eight orientations, and tile 8 has four. In each of these last two cases, half of the orientations are reflections (orientations that result from "picking up" another orientation and "flipping it over").

Figure 6 demonstrates how some of these tiles can be assembled to form the solution presented in [4].

We now present the details of the set packing formulation. We begin by defining a set

$\mathcal{A} = \{(t,o,i,j)$: tile t lies entirely on the board if placed there in orientation o with is reference point (the small circle) in the center of square $i,j\}$

and, for each triangular piece τ of the board, a set

$C_\tau = \{(t,o,i,j) \in \mathcal{A}$: tile t covers triangular piece τ if placed on the board in orientation o with is reference point in the center of square $i,j\}$

Note that the set $\mathcal{A}$ tells us how and where we can place the tiles on the board. We refer to the elements of $\mathcal{A}$ as "admissible placements." (Note that (4,4,1,1) is an admissible placement, but (4,1,1,1), (4,2,1,1), and (4,3,1,1) are not.) And note that the set C_τ contains all admissible placements that cover triangular piece τ. We call C_τ the covering set for τ.

The set packing formulation has a binary variable for each admissible placement. In particular, for each $(t,o,i,j) \in \mathcal{A}$,

$$y_{t,o,i,j} = \begin{cases} 1 \text{ if admissible placement } (t,o,i,j) \text{ is used,} \\ 0 \text{ otherwise.} \end{cases}$$

And the set packing formulation contains one constraint for each triangular piece τ of the board:

$$\sum_{(t,o,i,j) \in C_\tau} y_{t,o,i,j} \leq 1.$$

These constraints make sure that each triangular piece of the board is covered by at most one tile.

We minimize the number of monominoes placed on the board by maximizing the total number of empty squares on the board. If, for each tile t, v_t denotes the number of empty squares contained within (note that $v_1 = 1$, $v_2 = 2$, $v_3 = v_4 = 3$, and $v_5 = v_6 = \cdots = v_9 = 4$) then our objective is to maximize

$$\sum_{(t,o,i,j) \in \mathcal{A}} v_t y_{t,o,i,j}.$$

Comparing the Formulations

We tested the two formulations on a number of $n \times n$ pentomino exclusion problems, using a 200 MHz Pentium PC and CPLEX (version 4.0.9, with all parameters at default settings).

Table 1 makes it clear that the set packing formulation is far superior to the original, set covering formulation. The set packing formulation was able to solve many more of the problems tested, and it required considerably fewer branch-and-bound nodes on all problems tested, considerably fewer iterations of the simplex method on all but one of the problems tested (the smallest, the 6×6 problem), and much less CPU time on all but one of the problems tested (again, the smallest).

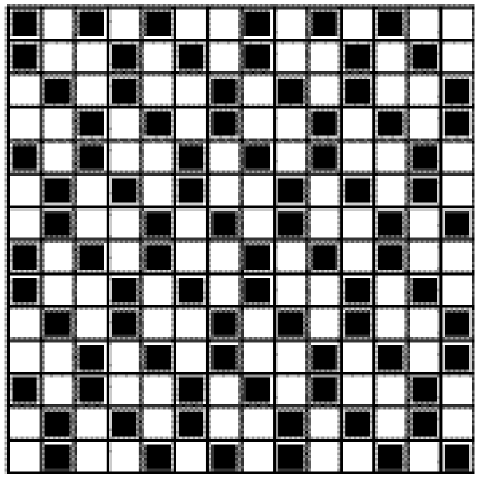


Figure 7

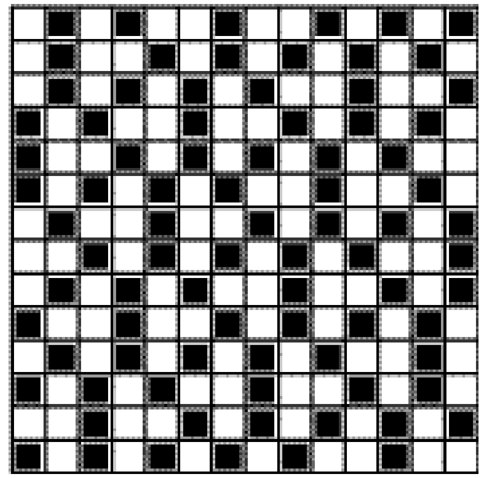


Figure 8

Toroidal Problems

Figures 7 and 8 display two optimal solutions to the 14×14 toroidal pentomino exclusion problem. (An $n \times n$ toroidal board is constructed by gluing together both pairs of opposite borders of a standard $n \times n$ board.) Note that each solution has a density of 3/7. (We consider the density of a solution to be the fraction of squares of the board that have monominoes in them.) Also note that the Figure 7 solution is made entirely of skew tetromino tiles (tile 8 in Figure 5), while the Figure 8 solution is made entirely of T tetromino tiles (tile 9 in Figure 5).

We conclude this section with a theorem about toroidal pentomino exclusion problems.

Theorem: For *all n*, *all* solutions to the $n \times n$ toroidal pentomino exclusion problem have density greater than or equal to 3/7.

Proof: First, add up all constraints of the set packing formulation of the $n \times n$ toroidal problem. Since the $n \times n$ toroidal board was divided into $8n^2$ triangular pieces, the resulting inequality is

$$\sum_{\tau} \sum_{(t,o,i,j) \in C_\tau} y_{t,o,i,j} \leq 8n^2. \tag{1}$$

(The first summation is over all triangular pieces τ.) Now let δ_t denote the number of triangular pieces in tile t. Note that $\delta_1 = 16$, $\delta_2 = 32$, $\delta_3 = 48$, $\delta_4 = 44$, $\delta_5 = 56$, $\delta_6 = 64$, $\delta_7 = 60$, and $\delta_8 = \delta_9 = 56$. Furthermore, note that

$$\sum_{\tau} \sum_{(t,o,i,j) \in C_\tau} y_{t,o,i,j} = \sum_{(t,o,i,j) \in \mathcal{A}} \delta_t y_{t,o,i,j}. \tag{2}$$

(Both sides count the number of triangular pieces of the board that are covered with tiles. The left side forms the sum by examining each triangular piece and adding a one to the total if the piece is covered by a tile, and a zero to the total if it isn't. The right side forms the sum by considering each tile used in the solution – recall that each admissible placement corresponds to the placement of a certain tile in a certain orientation in a certain position on the board – and adding the number of triangular pieces it covers to the total.) Substituting (4) into (3) and then dividing both sides of the resulting inequality by 14 yields.

	Set Covering			Set Packing		
Board	nodes	iterations	seconds	nodes	iterations	seconds
6 x 6	68	1,310	1.80	5	1,217	1.30
7 x 7	4,232	50,285	122.66	0	1,008	1.20
8 x 8	5,106	139,745	451.34	15	4,608	12.05
9 x 9	ran out of memory			19	7,921	27.58
10 x 10	ran out of memory			19	10,237	42.25
11 x 11	ran out of memory			32	21,935	152.52
12 x 12	ran out of memory			55	26,895	274.81
13 x 13	ran out of memory			402	194,724	2,976.30

Table 1

$$\sum_{(t,o,i,j)\in\mathcal{A}} \frac{\delta_t}{14} y_{t,o,i,j} \le \frac{8n^2}{14} = \frac{4n^2}{7}.$$

And since $\lfloor \delta/14 \rfloor \ge v_t$ for each tile t, it follows that

$$\sum_{(t,o,i,j)\in A} v_t y_{t,o,i,j} \le \frac{4n^2}{7}$$

Hence the density is greater than or equal to 3/7. □

The Pentomino Spanning Problem

In the December 1998 issue of OPTIMA, we also challenged readers to devise an integer programming formulation for finding the smallest subset of pentominoes that spans an $n \times n$ board. (A subset of pentominoes spans a board if its members can be placed on the board in such a way that they exclude the remaining pentominoes.) In this section, we present a formulation due to Frank Plastria [6]. For the sake of brevity, instead of Plastria's notation we use notation very similar to that used in the previous three sections.

We begin by numbering the pentominoes, numbering the orientations of each pentomino, and selecting one square of each pentomino to serve as that pentomino's reference square. We then define a set of admissible placements

$\mathcal{A}^S = \{(p,o,i,j)$: pentomino p lies entirely on the board if placed there in orientation o with its reference square on square $i,j\}$

and, for each square i,j of the board, a covering set

$C^S_{i,j} = \{(p,o,i',j') \in \mathcal{A}^S$: pentomino p covers square i,j, if place on the board in orientation o with its reference square on square $i',j'\}$.

In addition, for each admissible placement $(p,o,i,j) \in \mathcal{A}^S$, we define a set

$S_{p,o,i,j} = \{(p',o',i',j') \in \mathcal{A}^S$: (p',o',i',j') covers at least one of the squares covered by $(p,o,i,j)\}$.

Using this notation, we can write Plastria's formulation as follows:

$$\begin{array}{llr}
\min & \displaystyle\sum_{(p,o,i,j)\in\mathcal{A}^s} y_{p,o,i,j} & \\
\text{s.t.} & \displaystyle\sum_{(p,o,i',j')\in C^s_{i,j}} y_{p,o,i',j'} \le 1 & 1 \le i,j \le n \\
& \displaystyle\sum_{\substack{o,i,j:\\(p,o,i,j)\in\mathcal{A}^s}} y_{p,o,i,j} \le 1 & 1 \le p \le 12 \\
& \displaystyle\sum_{\substack{o',i',j':\\(p,o',i',j')\in\mathcal{A}^s}} y_{p,o',i',j'} + \sum_{(p',o',i',j')\in S_{p,o,i,j}} y_{p',o',i',j'} \le 1 & (p,o,i,j)\in\mathcal{A}^s \\
& y_{p,o,i,j} \in \{0,1\} & (p,o,i,j)\in\mathcal{A}^s
\end{array}$$

The variables are basically the same as in the set packing formulation of the pentomino exclusion problem: for each $(p,o,i,j) \in \mathcal{A}^S$,

$$y_{p,o,i,j} = \begin{cases} 1 \text{ if admissible placement } (p,o,i,j) \text{ is used,} \\ 0 \text{ otherwise.} \end{cases}$$

The first set of constraints ensures that each square is covered by at most one pentomino. The second set guarantees that each pentomino is placed on the board at most once. The third set makes sure that the pentominoes that are placed on the board actually span the board. In particular, the constraint corresponding to admissible placement (p,o,i,j) makes sure that if pentomino p is not placed on the board, then at least one of the squares that ***would have*** been covered by (p,o,i,j) is covered. To see this, note that

1. the leftmost summation takes on a value of 0 if and only if pentomino p is ***not*** placed on the board, and
2. the rightmost summation takes on a positive value if and only if at least one of the squares that ***would have been*** covered by (p,o,i,j) is covered.

References

[1] K.M. Anstreicher, A pentomino exclusion problem, preprint, University of Iowa, February 1999.

[2] P.J. Campbell, Gauss and the eight queens problem: a study in miniature of the propogation of historical error, *Historia Mathematica*, 4 (1977), pp. 397-404.

[3] L.R. Fould and D.G. Johnston, An application of graph theory and integer programming: chessboard non-attacking puzzles, *Mathematics Magazine*, 57 (1984), pp. 95-104.

[4] S.W. Golomb, Polyominoes: Puzzles, Patterns, Problems, and Packings (Princeton University Press, Princeton, NJ, 1994).

[5] W. Kosters, n-Queens web page, <http://www.wi.leidenuniv.nl/home/kosters/nqueens.html>.

[6] F. Plastria, On IP formulations for the pentomino exclusion and spanning problems, preprint, Vrije Universiteit Brussel, May 1999.

Conference CALENDAR

▶ **Second International Workshop on Approximation Algorithms for Combinatorial Optimization Problems, and Third International Workshop on Randomization and Approximation Techniques in Computer Science**
August 8-11, 1999,Berkeley, CA,USA
http://cuiwww.unige.ch/~rolim/approx;http://cuiwww.unige.ch/~rolim/random

▶ **Symposium on Operations Research 1999, SOR '99**
September 1-3, 1999,Magdeburg,Germany
http://www.uni-magdeburg.de/SOR99/

▶ **Sixth International Conference on Parametric Optimization and Related Topics**
October 4-8, 1999,Dubrovnik,Croatia
http://www.math.hr/dubrovnik/index.htm

▶ **INFORMS National Meeting**
November 7-10, 1999,Philadelphia, PA,USA
http://www.informs.org/Conf/Philadelphia99/

▶ **7th INFORMS Computing Society Conference on Computing and Optimization: Tools for the New Millenium**
January 5-7, 2000,Cancun,Mexico
http://www-bus.colorado.edu/Faculty/Laguna/cancun2000.html

▶ **DIMACS 7th Implementation Challenge:Semidefinite and Related Optimization Problems Workshop**
January 24-26, 2000,Rutgers University;Piscataway, NJ
http://dimacs.rutgers.edu/Workshops/7thchallenge

▶ **Seventh International Workshop on Project Management and Scheduling (PMS 2000)**
April 17-19, 2000,University of Osnabrueck,Germany
http://www.mathematik.uni-osnabrueck.de/research/OR/pms2000/

▶ **Applied Mathematical Programming and Modelling Conference (APMOD 2000)**
17-19 April 2000, Brunel University, London
http://www.apmod.org.uk

▶ **ISMP 2000 17th International Symposium on Mathematical Programming**
August 7-11, 2000,Georgia Institute of Technology, Atlanta,GA,USA
http://www.isye.gatech.edu/ismp2000

First Announcement
General Information
Call For Papers

The 17th International Symposium on Mathematical Programming

August 7-11, 2000

Georgia Institute of Technology, Atlanta, Georgia, USA.

http://www.isye.gatech.edu/ismp2000

Papers on all theoretical, computational and practical aspects of mathematical programming are welcome. The presentation of very recent results is encouraged.

Deadlines March 31, 2000: deadline for submission of titles and abstracts, and deadline for early registration.

Site The symposium will take place at the Georgia Institute of Technology located in Atlanta, Georgia, USA. Hotels of various categories will be available as well as low priced accommodations on the Georgia Tech campus.

Organizing Committee
General Co-Chairs: *G.L. Nemhauser, M.W.P. Savelsbergh*
Program Committee Co-Chairs: *E.L. Johnson, A. Shapiro, J. VandeVate, R. Monteiro, R. Thomas, C. Tovey, V. Vazarani*;
Local Committee Chair: *D. Llewellyn;*
Committee Members: *F. Al-Khayyal, E. Barnes, P. Keskinocak, A. Kleywegt, J. Sokol*

Structure of the Meeting A large number of people will be invited to organize sessions. Others may nominate themselves as session organizers. To do so, they must write us and, upon our program committee's aproval, they will be included in the list.
During the plenary opening session, the following prizes will be awarded: Dantzig Prize (for original research having a major impact on mathematical programming); Fulkerson Prizes (for outstanding papers in discrete mathematics); Orchard-Hays Prize (for excellence in computational mathematical programming); and A.W. Tucker Prize (for an outstanding paper by a student).

Morning Historical Perspectives Plenaries Nonlinear Programming: *Roger Fletcher*; Integer Programming: *Martin Grötschel*; Combinatorial Optimization: *William Pulleyblank*; Linear Programming: *Michael Todd*; Developments in the Soviet Union: *Boris Polyak*.

Afternoon Semi-Plenaries *Aharon Ben-Tal, William Cunningham, Don Goldfarb, Alan King, Jim Orlin, James Renegar, Alexander Schrijver, David Williamson, Yinyue Ye.*

Social Program
Sunday, August 6: Welcome reception, 18:00 – 19:30
Monday, August 7: Opening ceremony, 9:00 – 12:00
Wednesday, August 9: Evening reception, 18:00 – 23:00
A special program for accompanying persons is being organized.

Important Addresses
Mailing address: ISMP 2000, c/o A. Race, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0205, USA;
Fax: (+1) 404 894 0390;
E-Mail: <ismp2000@isye.gatech.edu>;
Web site: <http://www.isye.gatech.edu/ismp2000>.

List of Topics
Sessions on the following topics are planned.
Suggestions for further areas to be included are welcome.
(A) Approximation Algorithms,
(B) Combinatorial Optimization,
(C) Complementarity and Variational Inequalities,
(D) Computational Biology,
(E) Computational Complexity,
(F) Computational Geometry,
(G) Convex Programming and Nonsmooth Optimization,
(H) Dynamic Programming and Optimal Control,
(I) Finance and Economics,
(J) Game Theory,
(K) Global Optimization,
(L) Graphs and Networks,
(M) Integer and Mixed Integer Programming,
(N) Interior Point Algorithms and Semi-Definite Programming,
(O) Linear Programming,
(P) Logistics and Transportation,
(Q) Multicriteria Optimization,
(R) Nonlinear Programming,
(S) Parallel Computing,
(T) Production Planning and Manufacturing,
(U) Scheduling,
(V) Semi-Infinite and Infinite Dimensional Programming,
(W) Software,
(X) Statistics,
(Y) Stochastic Programming, and
(Z) Telecommunications and Network Design.

Call for Proposals to Host
I S M P 2 0 0 3

The time has come for all interested parties to make proposals for hosting the 2003 International Symposium on Mathematical Programming. Following tradition, a university site outside the US will host the 2003 Symposium.
All proposals are welcome and will be examined by the Symposium Advisory Committee, composed of Karen Aardal, John Dennis, Martin Grötschel and Thomas Liebling (Chair). It will make its recommendation based on criteria such as professional reputation of the local organizers, facilities, accommodations, accessibility and funding. Based on the recommendations of the Advisory Committee, the final decision will be made and announced by the MPS Council during the 2000 Symposium in Atlanta.
Detailed proposal letters should be addressed to: Prof. Thomas M. Liebling, DMA-EPFL, CH-1015 Lausanne, Switzerland (E-mail: Thomas.Liebling@epfl.ch).

Workshop on Discrete Optimization (DO'99)

DIMACS/RUTCOR, Rutgers University

July 25-30, 1999

http://rutcor.rutgers.edu/~do99

The updated announcement below includes, besides the information contained in the first announcement, new information concerning: 1. Registration (including deadlines); 2. Support for participants (including the availability of an NSF grant); 3. Accommodations (including housing request deadlines); and 4. Publications.

Goals Discrete optimization underwent a tumultuous development in the last half century and had a particularly spectacular growth in the last few decades. The main goal of this workshop is to survey the state of the art in discrete optimization. This goal will be achieved by the presentation of expository lectures presenting the major subareas of the field, including its theoretical foundations, its methodology and applications. The surveys will be presented by some of the most prominent researchers in the field. DO'99 will also provide a forum for the presentation of new developments in discrete optimization. To accomplish this goal, DO'99 will feature a series of sessions for the presentation of contributed talks, presenting the latest research of the participants.

DO'99 is being held 22 years after DO'77, which had very similar goals, and whose collection of surveys (Discrete Optimization I and II, Annals of Discrete Mathematics, vols. 4 and 5, P.L. Hammer, E.L. Johnson and B.H. Korte, eds., North Holland, Amsterdam, New York, Oxford, 1979) is still frequently used. It is hoped that DO'99 will present the latest state of the art in discrete optimization and will provide a similarly useful source of information and inspiration to the community of discrete optimizers as DO'77 did 22 years ago.

Venue The DO'99 workshop will take place on the Busch Campus of Rutgers, The State University of New Jersey, July 25-30, 1999.

Invited Survey Talks *Egon Balas* (Carnegie Mellon University), Lift-and-project: progress and some open questions; *Peter Brucker* (University of Osnabrueck), Complex scheduling problems; *Rainer Burkard* (Technical University of Graz), Assignment problems; *Vasek Chvatal* (Rutgers University), On the solution of traveling salesman problems; *Gerard Cornuejols* (Carnegie Mellon University), Packing and covering; *Yves Crama* (University of Liege), Optimization models in production planning; *Fred Glover* (University of Colorado), Tabu search & evolutionary methods: unexpected developments; *Alan Hoffman* (IBM Research Center), Greedy algorithms in linear programming problems; *Karla Hoffman* (George Mason University), Applications of column-generation and constraint-generation methods; *Toshihide Ibaraki* (Kyoto University), Graph connectivity & its augmentation; *Bernhard Korte* (University of Bonn), Gigahertz-processors need discrete optimization; *Jakob Krarup* (University of Copenhagen), Locational decisions with friendly and obnoxious facilities; *Tom Magnanti* (M.I.T.), Network design; *Silvano Martello* (University of Bologna), Bin packing problems in two and three dimensions; *George L. Nemhauser* (Georgia Institute of Technology), Discrete Optimization in Air Transportation; *Jim Orlin* and *Ravi Ahuja* (M.I.T.), Neighborhood search made difficult; *Bill Pulleyblank* (IBM Research Center), Hilbert bases, Caratheodory's theorem and integer programming; *Andras Recski* (Technical University of Budapest), Combinatorics of grid-like frameworks; *Paolo Toth* (University of Bologna), Vehicle routing; *David Williamson* (IBM Research Center), Approximation algorithms; and *Laurence Wolsey* (Catholic University of Louvain), Survey on mixed integer programming.

Workshop on the Theory and Practice of Integer Programming *in Honor of Ralph E. Gomory on the Occasion of his 70th Birthday*

We are pleased to announce a workshop in celebration of Ralph Gomory's 70th birthday. The focus of the workshop will be on integer linear programming. The workshop is sponsored by DIMACS, as part of the 1998-99 Special Year on Large-Scale Discrete Optimization, and by IBM. The workshop will be held August 2-4, 1999, at the IBM Watson Research Center in Yorktown Heights, New York. The workshop will include lectures by leading international experts covering all aspects of integer programming. We hope that the lecture program will be of particular interest to young researchers in the field, including Ph.D. students and post-doctoral fellows.

A conference banquet will be held with *Alan Hoffman* (IBM) as the Master of Ceremonies. The banquet speakers will include *Paul Gilmore* (University of British Columbia), *Ellis Johnson* (Georgia Tech), and *Herb Scarf* (Yale).

Invited Lecturers include: *Karen I. Aardal*, Utrecht University; *Egon Balas*, Carnegie Mellon University; *Francisco Barahona*, IBM Watson Research Center; *Imre Barany*, Hungarian Academy of Sciences; *Daniel Bienstock*, Columbia University; *Robert Bixby*, Rice University; *Charles E. Blair*, University of Illinois; *Vasek Chvatal*, Rutgers University; *Sebastian Ceria*, Columbia University; *Gerard Cornuéjols*, Carnegie Mellon University; *William H. Cunningham*, University of Waterloo; *John J. Forrest*, IBM Watson Research Center; *Michel X. Goemans*, Université Catholique de Louvain; *Ralph Gomory*, Sloan Foundation; *Peter Hammer*, Rutgers University; *T.C. Hu*, University of California at San Diego; *Ellis Johnson*, Georgia Tech; *Mike Juenger*, Universitat zu Koeln; *Berhard Korte*, University of Bonn; *Thomas L. Magnanti*, Massachusetts Institute of Technology; *George L. Nemhauser*, Georgia Institute of Technology; *Gerd Reinelt*, Universität Heidelberg; *Martin W.P. Savelsbergh*, Georgia Institute of Technology; *Herbert E. Scarf*, Yale University; *Andras Sebö*, University of Grenoble; *Bruce Shepherd*, Lucent Bell Laboratories; *Bernd Sturmfels*, University of California at Berkeley; *Mike Trick*, Carnegie Mellon University; *Leslie Earl Trotter, Jr.*, Cornell University; *Robert Weismantel*, University of Magdeburg; *David P. Williamson*, IBM Watson Research Laboratory; *Laurence Alexander Wolsey*, Université Catholique de Louvain; and *Günter Ziegler*, Technische Universität Berlin.

Conference Organizers: *William Cook*, Rice University; and *William Pulleyblank*, IBM Watson Research Center.

For more details, please see <http://dimacs.rutgers.edu/Workshops/Gomory/>.

DIMACS 7th Implementation Challenge: Semidefinite and Related Optimization Problems Workshop

DIMACS Center, CoRE Building, Rutgers University; Piscataway, NJ
January 24-26, 2000

Organizers *Farid Alizadeh*, RUTCOR, Rutgers University; *David Johnson*, AT&T Labs – Research; *Gabor Pataki*, Columbia University

Presented under the auspices of the Special Year on Large Scale Discrete Optimization.

The purpose of DIMACS computational challenges has been to encourage the experimental evaluation of algorithms, in particular those with efficient performance from a theoretical point of view. The past Challenges brought together researchers to test time-proven, mature, and novel experimental approaches on a variety of problems in a given subject. As the subject of the last Challenge of this century, one could hardly think of a better choice than Semidefinite Programming (SDP), one of the most interesting and challenging areas in optimization theory to emerge in the last decade. In the past few years, much has been learned on both the kinds of problem classes that SDP can tackle, and the best SDP algorithms for the various classes. In addition, a great deal has been learned about the limits of the current approaches to solving SDP's.

A closely related problem to semidefinite programming is that of convex quadratically constrained quadratic programming (QCQP). This problem resides in between linear and semidefinite programming. It also arises in a variety of applications from statistics to engineering; and a number of combinatorial optimization problems, in particular in the Steiner tree problems and plant location problems, have found QCQP as a subproblem. Similar to, and indeed by an extension from, semidefinite programming, a great deal is known about optimization with convex quadratic constraints as well as limitation of current methods. Finally, this knowledge has been extended to problems containing variables and constraints with some or all of linear, convex quadratic or semidefinite constraints.

This Challenge attempts to distill and expand upon this accumulated knowledge.

We have collected a variety of interesting and challenging SDP instances in the following classes. We have made an effort to create a collection containing instances that are as "real" as possible, are presently on, or beyond the limits of solvability, and whose solution would expand our knowledge on the applicability of SDP.

More precisely, we included: MAXCUT problems from theoretical physics currently solvable by polyhedral, but not by semidefinite methods; the Lovász-Schrijver semidefinite relaxations of 0-1 MIP's, which are unsolvable by either cutting plane methods, or branch and bound; truss topology, and Steiner tree problems lacking a strictly complementary solution.

We invite papers dealing with all computational aspects of semidefinite programming and related problems. In particular, the following classes of problems are of special interest: (1) Cut, and partition problems; (2) Theta function, and graph entropy problems; (3) SDP relaxation of very difficult, (currently unsolvable) 0-1 mixed integer programming problems; (4) Problems in convex quadratically constrained quadratic programs from engineering; (5) Problems from statistics and finance; (6) SDP instances from engineering, for example truss topology design, and control theory problems; (7) Difficult, randomly generated problems designed to challenge algorithms on performance and numerical stability; (8) In addition to the classes above, we invite investigations which focus on using SDP and related codes to test out behavior of heuristics and other new applications. The SDP code can be developed by the investigators or they may choose off-the-shelf codes.

All communications regarding the challenge should be directed to <challenge@dimacs.rutgers.edu>; in particular, preliminary proposal submission, extended abstracts, and possible submission of software and problem instances should be sent to the above address.

Important Dates *September 15 - Preliminary proposals due for comment and feedback; November 15 - Extended abstracts due date for consideration for the workshop; January 24-26 - The workshop will take place; Final drafts due date for appearance in the workshop proceedings will be determined.*

Further Information There will be a $40/day, $5/day workshop registration fee for postdocs and graduate students. For information on registration, travel and accommodations, please visit the workshop web site <http://dimacs.rutgers.edu/Workshops/7thchallenge>; the conference e-mail address is <challenge@dimacs.rutgers.edu>.

APMOD 2000

The Applied Mathematical Programming and Modelling Conference (APMOD 2000)

Brunel University, London,
17-19 April 2000

The objective of APMOD 2000 is to bring together active researchers, research students and practitioners from various countries and provide a forum for discussing and presenting established as well as new techniques in Operational Research/Management Science. APMOD 2000 provides a bridge between emerging research and their applicability in industry. The theme of APMOD 2000 is chosen to be Corporate Application of Mathematical Optimisation. The conference embraces a broad range of classical OR topics including linear programming, integer programming, combinatorial optimisation, stochastic programming and nonlinear programming. Emphasis is placed on novel techniques of approaching these problems and their application in industries such as finance, the utilities, transport and many other diverse areas of interest.

The three-day event will have plenary sessions by leading researchers and also four parallel streams. The written contributions will be refereed and published in the Annals of Operations Research.

For further information and pre-registration, please refer to <http://www.apmod.org.uk> or contact <apmod@brunel.ac.uk>; alternatively, write to: Mrs Gail Woodley, APMOD 2000, Department of Mathematics and Statistics Brunel University, Uxbridge, Middlesex, UB8 3PH, United Kingdom.

Symposium on Operations Research 1999, SOR '99

During September 1-3, 1999, an International Symposium, SOR '99, organized by the German Operations Research Society (GOR) will take place in Magdeburg, Germany. All areas of Operations Research will be covered at this conference. For more information, contact: *G. Schwödiauer* (general chair), University of Magdeburg, Faculty of Economics and Management, P.O. Box 41 20, D-39016 Magdeburg, Germany; phone +49 391 6718739; fax +49 391 6711136; e-mail <schwoediauer@wiwi.uni-magdeburg.de>. Additional information about the conference can be found online at <http://www.uni-magdeburg.de/SOR99/>.

First Announcement and Call for Papers

Seventh International Workshop on Project Management and Scheduling (PMS 2000)

April 17-19, 2000
University of Osnabrueck, Germany

Following the six successful workshops in Lisbon (Portugal), Como (Italy), Compiegne (France), Leuven (Belgium), Poznan (Poland), and in Istanbul (Turkey), the Seventh International Workshop on Project Management and Scheduling is to be held in Osnabrück, a small, charming city located halfway between Cologne and Hamburg.

The main objectives of PMS 2000 are to bring together researchers in the area of project management and scheduling in order to provide a medium for discussions of research results and research ideas and to create an opportunity for researchers and practitioners to get involved in joint research.

Another objective is to attract new recruits to the field of project management and scheduling to make them feel a part of a larger network. For this aim there will be special sessions on railway scheduling, timetabling, batch scheduling in the chemical industry, and robot scheduling.

Program Committee *Peter Brucker*, Chair (University of Osnabrueck), *Lucio Bianco* (IASI, Rome), *Jacek Blazewicz* (Poznan University of Technology), *Fayez Boctor* (Laval University), *Jacques Carlier* (Université de Technologie Compiegne), *Eric Demeulemeester* (Katholieke Universiteit Leuven), *Andreas Drexl* (Christian-Albrechts-Universität zu Kiel), *Salak E. Elmaghraby* (North Carolina State University), *Selcuk Erenguc* (University of Florida), *Willy Herroelen* (Katholieke Universiteit Leuven), *Wieslaw Kubiak* (Memorial University of Newfoundland), *Chung-Yee Lee* (Texas A&M University), *Klaus Neumann* (University of Karlsruhe), *Linet Ôzdamar* (Istanbul Kultur University), *James Patterson* (Indiana University), *Erwin Pesch* (University of Bonn), *Marie-Claude Portmann* (Ecole des Mines de Nancy, INPL), *Avraham Shtub* (Technion Israel Institute of Technology), *Roman Slowinski* (Poznan University of Technology), *Luis Valadares Tavares* (Instituto Superior Technico, Lisbon), *Gunduz Ulusoy* (Bogazici University, Istanbul), *Vicente Valls* (University of Valencia), *Jan Weglarz* (Poznan University of Technology), and *Robert J. Willis* (Monash University).

Preregistration If you are interested in participating, please visit our web site <http://www.mathematik.uni-osnabrueck.de/research/OR/pms2000/> and complete the pre-registration form, or contact us by e-mail <pms2000@mathematik.uni-osnabrueck.de>.

Pre-registration does not involve any obligations, but helps us to plan the schedule and keep you informed. In your e-mail please include your surname, first name(s), affiliation and e-mail address, and whether or not you intend to give a talk. Presentations will be selected on the basis of a three-page extended abstract to be submitted no later than September 15, 1999.

Important Dates **Abstract submission: September 15, 1999; Notification of acceptance: November 1, 1999; Workshop registration deadline: December 15, 1999.**

Registration Costs include the conference fee, a welcoming party, coffee breaks, and three lunches. The following prices are provisional: Early registration fee, DM 300; Late registration fee, DM 350; Excursion and dinner, to be announced.

The deadline for early registration is **December 15, 1999**. Please consult the conference web site to register.

Information Sources For up-to-date information, including information on hotels and the city of Osnabrück, please visit our web site <http://www.mathematik.uni-osnabrueck.de/research/OR/pms2000/>

The George B. Dantzig Prize 2000

CALL FOR NOMINATIONS

Nominations are solicited for the George B. Dantzig Prize, administered jointly by the Mathematical Programming Society (MPS) and the Society for Industrial and Applied Mathematics (SIAM). This prize is awarded to one or more individuals for original research which by its originality, breadth and depth, is having a major impact on the field of mathematical programming. The contribution(s) for which the award is made must be publicly available and may belong to any aspect of mathematical programming in its broadest sense. Strong preference will be given to candidates that have not reached their 50th birthday in the year of the award.

The prize will be presented at the Mathematical Programming Society's triennial symposium, to be held 7-11 August 2000, in Atlanta, Georgia, USA. Past prize recipients are listed on the MPS web site <http://www.caam.rice.edu/~mathprog/>. The members of the prize committee are William H. Cunningham, Claude Lemaréchal, Stephen M. Robinson (Chair), and Laurence A. Wolsey.

Nominations should consist of a letter describing the nominee's qualifications for the prize, and a current curriculum vitae of the nominee including a list of publications. They should be sent to: Stephen M. Robinson, Department of Industrial Engineering, University of Wisconsin-Madison, 1513 University Avenue, Madison, WI 53706-1572, USA, E-mail: <smrobins@facstaff.wisc.edu>.

Nominations must be received by **15 October 1999**. Any nominations received after that date will not be considered. Submission of nomination materials in electronic form (e-mail with attachments as needed) is strongly encouraged.

Linear Semi-Infinite Optimization

Miguel A. Goberna and Marco A. Lopez

John Wiley & Sons, 1998

ISBN 0-471-97040-9

Linear semi-infinite programming deals with the problem of minimizing (maximizing) a linear objective function of a finite number of variables with respect to an (possibly and generally) infinite number of linear constraints. There is a great variety of applications of semi-infinite optimization, including problems in approximation theory (using polyhedral norms), operation research, optimal control, boundary value problems and others. These applications and appealing theoretical properties of semi-infinite problems gave rise to intensive (and up to now undiminished) research activities in this field since their inception in the 1960s.

The book under review is, according to the authors, intended "... as a monograph as well as a textbook …" It fills a gap in the present literature about optimization.

The authors organize their material into four parts:

Part I, **Modeling**, deals with numerous examples of occurring linear semi-infinite optimization problems, divided into two chapters (1) **Modeling with the primal problem** and (2) **Modeling with the dual problem**. Most of the models described in Part I arise from other fields of applied mathematics.

In Part II, **Linear Semi-Infinite Systems**, the feasible sets of (primal) linear semi-infinite optimization problems are investigated. It provides the necessary fundamentals for the forthcoming theory and contains the chapters (3) **Alternative Systems**, (4) **Consistency**, (5) **Geometry** and (6) **Stability**.

▶

Graph Theory

Reinhard Diestel

Springer Verlag

ISBN 0-387-98210-8

Have you heard of Szemerédi's regularity lemma? If you have been in touch with graph theory at least a little, the answer is almost surely yes. How about its proof? In case you were so far too afraid to go through the details and understand the basic ideas of this far-reaching result, the book *Graph Theory* by G. Diestel will prove an excellent guide. One of the main strengths of this book is making deep and difficult results of graph theory accessible.

Actually, the goal of this book is (at least) twofold: First, to give a solid introduction to basic notions and provide the standard material of graph theory, such as the matching theorems of König, Hall, Tutte, Petersen, the theorems of Dilworth and Menger on paths and chains, Kuratowski's characterization of planar graphs, coloring theorems of Brooks and Vizing, etc. But already the introductory chapters include results which have not yet appeared in textbooks: Mader's theorem (1.4.2) on the existence of *k*-connected subgraphs of a sufficiently dense graph, Tutte's characterization of 3-connected graphs. It is also refreshing to see the proof of Mader's theorem (3.6.1) on the existence of large topological complete graph as a minor in a dense graph, or of the pretty theorem of Jung and of Larman and Mani (3.6.2) stating the existence of *k* disjoint paths between any set of *k* pairs of nodes in a sufficiently highly connected graph.

The main novelty of this book, however, is that a great number of difficult and deep theorems, along with their full proofs, are exhibited. These include some older results like Szemerédi's above-mentioned regularity lemma, or Fleischner's theorem (10.3.1) on the Hamiltonicity of the square of a graph, with a full proof of over seven pages. A fundamental result

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The essential Part III, **Theory of Linear Semi-Infinite Programming**, presents in chapter (7) **Optimality** a natural extension of the classical Karush-Kuhn-Tucker theory to linear semi-infinite optimization problems and provides further information on some optimality conditions for the dual problem and for convex semi-infinite programming problems.

Chapter (8), **Duality**, develops a systematic approach to the main topics of duality theory in linear semi-infinite programming. Additionally, the (occasionally) occurring duality gaps are analyzed, as well as the connection between duality gaps and the applicability of discretization methods.

Chapter (9), **Extremality and Boundedness**, deals with extreme points and extreme directions of the feasible sets and optimal sets of a (fixed) pair of linear semi-infinite optimization problems.

Perturbations in the data set and their impact on the solutions are investigated in chapter (10), **Stability and Well-Posedness**, with an analysis of the optimal value function and the optimal-set mapping.

Part IV, **Methods of Linear Semi-Infinite Programming**, gives in the two chapters (11) **Local Reduction and Discretization Methods** and (12) **Simplex-Like and Exchange Methods** an overview of numerical methods for solving linear semi-infinite optimization problems. "They are described in a conceptual form... but omitting a detailed discussion of the numerical difficulties encountered in the auxiliary problems."

Each chapter contains a lot of historical and bibliographical notes and hints, and ends with a collection of exercises (without solutions). These exercises include routine tasks and applications as well as theoretical complements.

For convenience, some basic concepts and properties of convex sets and convex functions are collected in an **Appendix** (without proofs). So the text is nearly self-contained.

The book gives a good view of the topic. It is addressed to (graduate) students in mathematics and to scientists who are interested in the ideas behind the theory of linear optimization. The reader is assumed to be familiar with linear algebra and elementary calculus; he should have a certain knowledge in linear programming and also in elementary topology.

The text is carefully written, the exposition is clear and goes quite deeply into details. The book is more to provide a profound discussion of the subject than to get a first insight into the topic. It may be also used as a basis and a guideline for lectures on this subject; the authors give some proposals of how to arrange the material for several courses.

As a minor complaint it should be mentioned that the list of **Symbols and Abbreviations** (p. 321) unfortunately does not contain the place (page) where certain notation occurs or is defined, so the reader sometimes has trouble finding it.

All in all, the book leaves a remarkable impression of the concepts, tools and techniques in linear semi-infinite optimization. Students as well as professionals will profitably read and use it.

–FRIEDRICH JUHNKE, MAGDEBURG

(Theorem 9.3.1) (due to three groups of authors around the beginning of the '70s) from induced Ramsey theory is also completely proved (four pages). The classical results on algebraic flows, including Tutte's investigations and Seymour's 6-flow theorem, are fully covered as well. Not only Lovász's perfect graph theorem (5.5.3), stating the perfectness of the complement of a perfect graph, is discussed, but his second, significantly deeper characterization of perfect graphs, as well (Theorem 5.5.5). (Naturally, not every fundamental result of graph theory could be included with its proof; but perhaps – if the reviewer may make a suggestion – the author may find a way in the next edition to exhibit a proof of Mader's beautiful theorem (3.4.1) on the maximum number of independent H-paths.)

Beyond these classics, recent or even brand-new results are also discussed. For example, Galvin's charming list-colouring theorem (5.4.4) has appeared in 1995, or Thomassen's pearl on 5-choosability of planar graphs in 1994. An even more recent, very difficult result (Theoerem 8.1.1), due to Komlós and Szemerédi and to Bollobás and Thomason, state that every graph with average degree at least cr^2 contains K^r as a topological minor. The paper of Komlós and Szemerédi appeared in 1996 while the work of Bollobás and Szemerédi has appeared only this year. Isn't it nice that a proof (of more than five pages) is already accessible in Diestel's book?

A grand undertaking of the past 15 years has been the proof of the so-called minor theorem (what a contradiction between name and significance!) It reads: *in every infinite set of finite graphs there are two such that one is the minor of the other*. The supporting theory of tree-decompositions, well-quasi-orderings, minors has mainly been developed by N. Robertson and P. Seymour. The complete proof of the minor theorem is over 500 pages, so not surprisingly this is excluded here, but Chapter 12 on Minors, Trees and WQO offers an excellent overview of the framework of the proof along with its far-reaching consequences.

The book includes 12 chapters: (1) The Basics, (2) Matching, (3) Connectivity, (4) Planar Graphs, (5) Colouring, (6) Flows, (7) Substructures in Dense Graphs, (8) Substructures in Sparse Graphs, (9) Ramsey Theory for Graphs, (10) Hamiltonian Cycles, (11) Random Graphs, and (12) Minors, Trees, and WQU.

As far as the basic approach of the book is concerned, I just cannot state it any better than the back-page review of the book does: "Viewed as a branch of pure mathematics, the theory of finite graphs is developed as a coherent subject in its own right, with its own unifying questions and methods. The book thus seeks to complement, not replace, the existing more algorithmic treatments of the subject."

The book is written in a thorough and clear style. The author puts special emphasis on explaining the underlying ideas, and the technical details are made as painless as possible. Some typographic novelties are introduced: for example, there are little reminders on the margins to notations, definitions, references. These may be helpful in following the text more easily; the general outlook of a page, however, becomes sometimes a bit messy.

All in all, the book of R. Diestel is a smooth introduction to standard material and is a particularly rich source of deep results of graph theory. I can highly recommend it to graduate students as well as professional mathematicians.

ANDRÁS FRANK <frank@cs.elte.hu>

New Optimization Book Series

J.E. Dennis, Jr.

Our MPS Council and the SIAM Council and Board of Trustees have approved a new joint book series, the **MPS/SIAM Series on Optimization**. Both professional societies will be responsible jointly for the scientific content of the series for which SIAM will act as publisher. The series was announced at the May SIAM Optimization Meeting, and already the first manuscripts submitted to the series are in review. Our first volume should appear early in 2000. SIAM has agreed that MPS members will receive the SIAM member discount on all SIAM books, not just the books in this series.

We are in the process of naming the editorial board, which will be organized in the standard MPS format. The first appointment other than my own as Editor-in-Chief, was the appointment of Steve Wright as Co-Editor for Continuous Optimization. I considered Steve's agreement to undertake this new task a key issue for the success of the series. He is responsible, hard-working, and he has written a fine monograph on interior point methods. Thus, he understands the huge difference between writing a paper and writing a successful book. We are in the process of naming a Co-Editor for Discrete Optimization, and then we will make collective decisions concerning the Associate Editors. I call the new series MPC, and I think of it as the logical next step after MPA and MPB. I hope you will agree, and help to make the series a success by sending us your manuscripts and acting as a reviewer on request.

Our aim will be to publish three to five high quality books each year. The series will cover the entire spectrum of optimization. We welcome research monographs, textbooks at all levels, books on applications, and tutorials. Because our goal is to complement MPA and MPB, we do not intend to publish proceedings or collections of papers.

The requirements for the series are simple: a manuscript must advance the understanding and practice of optimization, and it must be clearly written appropriately to its level. The content must be of high scientific quality. Our review process will be somewhere between that of a journal and a commercial book publisher. The journal review process has a certain "gatekeeper" aspect to it, appropriate because of the significance that academic committees attach to journal publication records. Our referee process will also be judgmental, but we intend to be more consultative with authors than is common for journal editors. We will encourage and work with the authors if we feel that the manuscript belongs in the series. Because of our high standards, we hope that publication in our series will mean more to an author's vita than books generally mean.

The advantages of SIAM as a scientific partner are self-evident. However, I feel strongly that the case for SIAM as publisher is as compelling. SIAM publishes a wide variety of books, and the quality of books they publish has been always at least as high as the quality of commercial publishers, and now it seems generally higher. SIAM markets aggressively at meetings, online, and by mail. At the same time, books published with SIAM stay in print long after a commercial publisher would have allowed them to go out of print. The prices are consistently lower than for commercial publishers, though the royalties are competitive. SIAM will maintain a web site for misprints and comments on each book in the series. I believe that this service will be especially valuable for textbooks because instructors can share useful software and information about exercises.

Of course, it would have been possible to collaborate with a commercial publisher in this venture; however, frankly, I doubt the future of commercial publishers for scientific books. At the Atlanta meeting next year, compare the per page cost at the various booths and you will wonder, as I do, if commercial scientific publishers have a future except for elementary texts. Let me be clear that we hope to sell a lot of books, but we plan to publish fine books at low prices and to keep them in print. The point here is that a commercial publisher is accountable to investors, while SIAM is accountable to its members, people like us.

gallimaufry

The deadline for the next issue of OPTIMA is September15, 1999.

For the electronic version of OPTIMA, please see:

http://www.ise.ufl.edu/~optima/

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