PROBLEMS IN ELEMENTARY NUMBER THEORY

Hojoo Lee, Version 0.795 [2003/12/07]

God does arithmetic.   **C. F. Gauss**

Contents

1. Introduction

The heart of Mathematics is its problems.   Paul Halmos

1. Introduction *Number Theory* is a beautiful branch of Mathematics. The purpose of this book is to present a collection of interesting questions in *Number Theory*. Many of the problems are mathematical competition problems all over the world including IMO, APMO, APMC, Putnam, etc. I have given sources of the problems at the end of the book. The book is available at

http://my.netian.com/~ideahitme/eng.html

2. How You Can Help This is an unfinished manuscript. I would greatly appreciate hearing about any errors in the book, even minor ones. I also would like to hear about

　　　a) challenging problems in *Elementary* Number Theory,
　　　b) interesting problems concerned with the *History* of Number Theory,
　　　c) *beautiful* results that are *easily* stated,
　　　d) *remarks* on the problems in the book.

You can send all comments to the author at **hojoolee@korea.com** .

3. Acknowledgments The author would like to thank the following people for sending me suggestions and corrections, etc. : *Alexander A. Zenkin, Arne Smeets, Curt A. Monash, Don Coppersmith, Edward F. Schaefer, Edwin Clark, George Baloglou, Ha Duy Hung, Keith Matthews, Leonid G. Fel, Orlando Doehring*

2. Notations and Abbreviations

Notations

$\mathbf{Z}$ is the set of integers
$\mathbf{N}$ is the set of positive integers
$\mathbf{N}_0$ is the set of nonnegative integers
$m|n$ n is a multiple of m.
$\sum_{d|n} f(d) = \sum_{d \in D(n)} f(d)$ $(D(n) = \{d \in \mathbf{N} : d|n\})$
$[x]$ the greatest integer less than or equal to x
$\{x\}$ the fractional part of x $(\{x\} = x - [x])$
$\phi(n)$ the number of positive integers less than n that are relatively prime to n
$\pi(x)$ the number of primes p with $p \leq x$

Abbreviations

AIME American Invitational Mathematics Examination
APMO Asian Pacific Mathemats Olympiads
IMO International Mathematical Olympiads

3. Divisibility Theory I

Why are numbers beautiful? It's like asking why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is. Paul Erdös

A 1. *(Kiran S. Kedlaya) Show that if x, y, z are positive integers, then $(xy + 1)(yz + 1)(zx + 1)$ is a perfect square if and only if $xy + 1$, $yz + 1$, $zx + 1$ are all perfect squares.*

A 2. *The integers a and b have the property that for every nonnegative integer n the number of $2^n a + b$ is the square of an integer. Show that $a = 0$.*

A 3. *Let n be a positive integer such that $2 + 2\sqrt{28n^2 + 1}$ is an integer. Show that $2 + 2\sqrt{28n^2 + 1}$ is the square of an integer.*

A 4. *Let a and b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that*

$$\frac{a^2 + b^2}{ab + 1}$$

is the square of an integer. [1]

A 5. *Let x and y be positive integers such that xy divides $x^2 + y^2 + 1$. Show that* [2]

$$\frac{x^2 + y^2 + 1}{xy} = 3.$$

A 6. *Prove that among any ten consecutive positive integers at least one is relatively prime to the product of the others.*

A 7. *Show that 1994 divides $10^{900} - 2^{1000}$.*

A 8. *Let n be a positive integer with $n \geq 3$. Show that*

$$n^{n^{n^n}} - n^{n^n}$$

is divisible by 1989.

A 9. *Let n be an integer with $n \geq 2$. Show that n does not divide $2^n - 1$.*

A 10. *Let $k \geq 2$ and $n_1, n_2, \cdots, n_k \geq 1$ be natural numbers having the property $n_2 | 2^{n_1} - 1$, $n_3 | 2^{n_2} - 1$, $\cdots$, $n_k | 2^{n_{k-1}} - 1$, $n_1 | 2^{n_k} - 1$. Show that $n_1 = n_2 = \cdots = n_k = 1$.*

A 11. *Determine if there exists a positive integer n such that n has exactly 2000 prime divisors and $2^n + 1$ is divisible by n.*

---

[1] See **I6**

[2] See **B15** and **I34**

A 12. *Let m and n be natural numbers such that*

$$A = \frac{(m+3)^n + 1}{3m}.$$

is an integer. Prove that A is odd.

A 13. *Let m and n be natural numbers and let $mn + 1$ be divisible by 24. Show that $m + n$ is divisible by 24, too.*

A 14. *Let $f(x) = x^3 + 17$. Prove that for each natural number $n \geq 2$, there is a natural number x for which $f(x)$ is divisible by 3^n but not 3^{n+1}.*

A 15. *Determine all positive integers n for which there exists an integer m so that $2^n - 1$ divides $m^2 + 9$.*

A 16. *Let n be a positive integer. Show that the product of n consecutive integers is divisible by $n!$*

A 17. *Prove that the number*

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$$

is not divisible by 5 for any integer $n \geq 0$.

A 18. *(Wolstenholme's Theorem) Prove that if*

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}$$

is expressed as a fraction, where $p \geq 5$ is a prime, then p^2 divides the numerator.

A 19. *If p is a prime number greater than 3 and $k = [\frac{2p}{3}]$. Prove that the sum*

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

is divisible by p^2.

A 20. *Show that $\binom{2n}{n} | LCM[1, 2, \cdots, 2n]$ for all positive integers n.*

A 21. *Let m and n be arbitrary non-negative integers. Prove that*

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer. $(0! = 1)$.

A 22. *Show that the coefficients of a binomial expansion $(a + b)^n$ where n is a positive integer, are odd, if and only if n is of the form $2^k - 1$ for some positive integer k.*

A 23. *Prove that the expression*

$$\frac{gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of positive integers $n \geq m \geq 1$.

A 24. *For which positive integers k, is it true that there are infinitely many pairs of positive integers (m,n) such that*

$$\frac{(m+n-k)!}{m!n!}$$

is an integer ?

A 25. *Show that if $n \geq 6$ is composite, then n divides $(n-1)!$.*

A 26. *Show that there exist infinitely many positive integers n such that $n^2 + 1$ divides $n!$.*

A 27. *Let p and q be natural numbers such that*

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}.$$

Prove that p is divisible by 1979.

A 28. *Let $b > 1$, a and n be positive integers such that $b^n - 1$ divides a. Show that in base b, the number a has at least n non-zero digits.*

A 29. *Let $p_1, p_2, \cdots, p_n$ be distinct primes greater than 3. Show that $2^{p_1 p_2 \cdots p_n} + 1$ has at least 4^n divisors.*

A 30. *Let $p \geq 5$ be a prime number. Prove that there exists an integer a with $1 \leq a \leq p - 2$ such that neither $a^{p-1} - 1$ nor $(a+1)^{p-1} - 1$ is divisible by p^2.*

A 31. *An integer $n > 1$ and a prime p are such that n divides $p - 1$, and p divides $n^3 - 1$. Show that $4p + 3$ is the square of an integer.*

A 32. *Let n and q be integers with $n \geq 5$, $2 \leq q \leq n$. Prove that $q - 1$ divides $\left[\frac{(n-1)!}{q}\right]$.*

A 33. *If n is a natural number, prove that the number $(n+1)(n+2)\cdots(n+10)$ is not a perfect square.*

A 34. *Let p be a prime with $p > 5$, and let $S = \{p - n^2 | n \in \mathbf{N}, n^2 < p\}$. Prove that S contains two elements a, b such that $1 < a < b$ and a divides b.*

A 35. *Let n be a positive integer. Prove that the following two statements are equivalent.*

  ○ *n is not divisible by 4*
  ○ *There exist $a, b \in \mathbf{Z}$ such that $a^2 + b^2 + 1$ is divisible by n.*

A 36. *Determine the greatest common divisor of the elements of the set*
$$\{n^{13} - n | n \in \mathbf{Z}\}.$$

A 37. *Show that there are infinitely many composite n such that $3^{n-1} - 2^{n-1}$ is divisible by n*

A 38. *Suppose that $2^n + 1$ is an odd prime for some positive integer n. Show that n must be a power of 2.*

A 39. *Suppose that p is a prime number and is greater than 3. Prove that $7^p - 6^p - 1$ is divisible by 43.*

A 40. *Suppose that $4^n + 2^n + 1$ is prime for some positive integer n. Show that n must be a power of 3.*

A 41. *Let b, m, n be positive integers $b > 1$ and m and n are different. Suppose that $b^m - 1$ and $b^n - 1$ have the same prime divisors. Show that $b + 1$ must be a power of 2.*

A 42. *Show that a and b have the same parity if and only if there exist integers c and d such that $a^2 + b^2 + c^2 + 1 = d^2$.*

A 43. *Let n be a positive integer with $n > 1$. Prove that*
$$\frac{1}{2} + \cdots + \frac{1}{n}$$
is not an integer.

A 44. *Let n be a positive integer. Prove that*
$$\frac{1}{3} + \cdots + \frac{1}{2n+1}$$
is not an integer.

A 45. *Prove that there is no positive integer n such that, for $k = 1, 2, \cdots, 9$, the leftmost digit (in decimal notation) of $(n + k)!$ equals k.*

A 46. *Show that every integer $k > 1$ has a multiple less than k^4 whose decimal expansion has at most four distinct digits.*

A 47. *Let a, b, c and d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a + d = 2^k$ and $b + c = 2^m$ for some integers k and m, then $a = 1$.*

A 48. *Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.*

A 49. *Suppose that x, y, z are positive integers with $xy = z^2 + 1$. Prove that there exist integers a, b, c, d such that $x = a^2 + b^2$, $y = c^2 + d^2$, $z = ac + bd$.*

A 50. *A natural number n is said to have the property P, if whenever n divides $a^n - 1$ for some integer a, n^2 also necessarily divides $a^n - 1$.*

(a) Show that every prime number n has propery P.
*(b) Show that there are infinitely many composite numbers n
that possess property P.*

A 51. *Show that for every natural number n the product*

$$\left(4 - \frac{2}{1}\right)\left(4 - \frac{2}{2}\right)\left(4 - \frac{2}{3}\right)\cdots\left(4 - \frac{2}{n}\right)$$

is an integer.

A 52. *Let a, b, c be integers such that $a + b + c$ divides $a^2 + b^2 + c^2$. Prove
that there are infinitely many positive integers n such that $a + b + c$ divides
$a^n + b^n + c^n$.*

A 53. *Prove that for every positive integer n the following proposition holds
: The number 7 is a divisor of $3^n + n^3$ if and olny if 7 is a divisor of $3^n n^3 + 1$.*

A 54. *Let $k \geq 14$ be an integer, and let p_k be the largest prime number
which is strictly less than k. You may assume that $p_k \geq 3k/4$. Let n be a
composite integer. Prove :*

(a) if $n = 2p_k$, then n does not divide $(n - k)!$
(b) if $n > 2p_k$, then n divides $(n - k)!$.

A 55. *Suppose that n has (at least) two essentially distinct representations
as a sum of two squares. Specifically, let $n = s^2 + t^2 = u^2 + v^2$, where
$s \geq t \geq 0$, $u \geq v \geq 0$, and $s > u$. If $d = \gcd(su - tv, n)$, show that d is a
propor divisor of n.*

A 56. *Prove that there exist an infinite number of ordered pairs (a, b) of
integers such that for every positive integer t, the number $at + b$ is a triangular
number if and only if t is a triangular number. (The triangular numbers are
the $t_n = n(n + 1)/2$ with n in $\{0, 1, 2, \dots\}$.)*

4. Divisibility Theory II

Number theorists are like lotus-eaters - having tasted this food they can never give it up. Leopold Kronecker

B 1. *Determine all integers $n > 1$ such that*
$$\frac{2^n + 1}{n^2}$$
is an integer.

B 2. *Determine all pairs (n, p) of nonnegative integers such that*
- p *is a prime,*
- $n < 2p$, *and*
- $(p-1)^n + 1$ *is divisible by n^{p-1}.*

B 3. *Determine all pairs (n, p) of positive integers such that*
- p *is a prime, $n > 1$,*
- $(p-1)^n + 1$ *is divisible by n^{p-1}.* [3]

B 4. *Find an integer n, where $100 \le n \le 1997$, such that*
$$\frac{2^n + 2}{n}$$
is also an integer.

B 5. *Find all triples (a, b, c) such that $2^c - 1$ divides $2^a + 2^b + 1$.*

B 6. *Find all integers a, b, c with $1 < a < b < c$ such that*
$$(a-1)(b-1)(c-1) \quad \text{is a divisor of} \quad abc - 1.$$

B 7. *Find all positive integers, representable uniquely as*
$$\frac{x^2 + y}{xy + 1},$$
where x, y are positive integers.

B 8. *Determine all ordered pairs (m, n) of positive integers such that*
$$\frac{n^3 + 1}{mn - 1}$$
is an integer.

B 9. *Determine all pairs of integers (a, b) such that*
$$\frac{a^2}{2a^2 b - b^3 + 1}$$
is a positive integer.

---

[3] The anwser is $(n, p) = (2, 2), (3, 3)$. Note that this problem is a very *nice* generalization of the above two IMO problems B1 and B2 !

B 10. *Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that*

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

B 11. *Determine all triples of positive integers (a, m, n) such that $a^m + 1$ divides $(a + 1)^n$.*

B 12. *Find all $n \in \mathbf{N}$ such that $[\sqrt{n}]|n$.*

B 13. *Find all $n \in \mathbf{N}$ such that $2^{n-1}|n!$.*

B 14. *Find all positive integers (x, n) such that $x^n + 2^n + 1$ is a divisor of $x^{n+1} + 2^{n+1} + 1$.*

B 15. *Find all positive integers n such that $3^n - 1$ is divisible by 2^n.*

B 16. *Find all positive integers n such that $9^n - 1$ is divisible by 7^n.*

B 17. *Determine all pairs (a, b) of integers for which $a^2 + b^2 + 3$ is divisible by ab.[4]*

B 18. *Determine all pairs (x, y) of positive integers with $y|x^2+1$ and $x|y^3+1$.*

B 19. *Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2 b + a + b$.*

B 20. *Let a and b be positive integers. When $a^2 + b^2$ is divided by $a + b$, the quotient is q and the remainder is r. Find all pairs (a, b) such that $q^2 + r = 1977$.*

B 21. *Find the largest positive integer n such that n is divisible by all the positive integers less than $n^{1/3}$.*

B 22. *Find all $n \in \mathbf{N}$ such that $3^n - n$ is divisible by 17.*

B 23. *Suppose that a, b are natural numbers such that*

$$p = \frac{4}{b}\sqrt{\frac{2a - b}{2a + b}}$$

is a prime number. What is the maximum possible value of p?

B 24. *Find all positive integer N which have the following properties*

    ◦ *N has exactly 16 positive divisors $1 = d_1 < d_2 < \cdots < d_{15} < d_{16} = N$,*
    ◦ *The divisor with d_2 is equal to $(d_2 + d_4)d_6$.*

B 25. *Find all positive integers n that have exactly 16 positive integral divisors $d_1, d_2 \cdots, d_{16}$ such that $1 = d_1 < d_2 < \cdots < d_{16} = n$, $d_6 = 18$, and $d_9 - d_8 = 17$.*

---

[4]See **B15** and **I34**

B 26. *Suppose that n is a positive integer and let*

$$d_1 < d_2 < d_3 < d_4$$

be the four smallest positive integer divisors of n. Find all integers n such that

$$n = d_1{}^2 + d_2{}^2 + d_3{}^2 + d_4{}^2.$$

B 27. *Let $1 = d_1 < d_2 < \cdots < d_k = n$ be all different divisors of positive integer n written in ascending order. Determine all n such that $d_7{}^2 + d_{10}{}^2 = \left(\frac{n}{d_{22}}\right)^2.$*

B 28. *Let $n \geq 2$ be a positive integer, with divisors*

$$1 = d_1 < d_2 < \cdots < d_k = n .$$

Prove that

$$d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k$$

is always less than n^2, and determine when it is a divisor of n^2.

B 29. *Find all positive integers n such that*

(a) *n has exactly 6 positive divisors $1 < d_1 < d_2 < d_3 < d_4 < n$,*

(b) *$1 + n = 5(d_1 + d_2 + d_3 + d_4)$.*

B 30. *Find all composite numbers n, having the property : each divisor d of n $(d \neq 1, n)$ satisfies inequalties $n - 20 \leq d \leq n - 12$.*

B 31. *Determine all three-digit numbers N having the property that N is divisible by 11, and $\frac{N}{11}$ is equal to the sum of the squares of the digits of N.*

B 32. *When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B. (A and B are written in decimal notation.)*

B 33. *A wobbly number is a positive integer whose digits in base 10 are alternatively non-zero and zero the units digit being non-zero. Determine all positive integers which do not divide any wobbly number.*

B 34. *Let n be a composite natural number and p be a proper divisor of n. Find the binary representatoin of the smallest natural number N such that*

$$\frac{(1 + 2^p + 2^{n-p})N - 1}{2}$$

is an integer.

B 35. *Find the smallest positive integer n such that*

(i) *n has exactly 144 distinct positive divisors, and*

(ii) *there are ten consecutive integers among the positive divisors of n.*

B 36. *Determine the least possible value of the natural number n such that $n!$ ends in exactly* 1987 *zeros.*

B 37. *Find four positive integers, each not exceeding* 70000 *and each having more than* 100 *divisors.*

B 38. *For each integer $n > 1$, let $p(n)$ denote the largest prime factor of n. Determine all triples (x, y, z) of distinct positive integers satisfying*

> *(i) x, y, z are in arithmetic progression, and*
> *(ii) $p(xyz) \leq 3$.*

B 39. *Find all positive integers a and b such that*

$$\frac{a^2 + b}{b^2 - a} \quad and \quad \frac{b^2 + a}{a^2 - b}$$

are both integers.

B 40. *For each positive integer n, write the sum $\sum_{m=1}^{n} 1/m$ in the form p_n/q_n, where p_n and q_n are relatively prime positive integers. Determine all n such that 5 does not divide q_n.*

5. Arithmetic in Z_n

Mathematics is the queen of the sciences and number theory is the queen of Mathematics. Johann Carl Friedrich Gauss

5.1. Primitive Roots.

C 1. *Let n be a positive integer. Show that there are infinitely many primes p such that the smallest positive primitive root of p is greater than n.*

C 2. *Let p be a prime with $p > 4 \left(\frac{p-1}{\phi(p-1)} \right)^2 2^{2k}$, where k denotes the number of distinct prime divisors of $p-1$, and let M be an intger. Prove that the set of integers $M+1$, $M+2$, $\cdots$, $M+2 \left[\frac{p-1}{\phi(p-1)} 2^k \sqrt{p} \right] - 1$ contains a primitive root to modulus p.*

C 3. *Show that for each odd prime p, there is an integer g such that $1 < g < p$ and g is a primitive root modulo p^n for every positive integer n.*

C 4. *Let g be a Fibonacci primitive root (mod p). i.e. g is a primitive root (mod p) satisfying $g^2 \equiv g + 1 \pmod{p}$. Prove that*
    (a) $g - 1$ is also a primitive root (mod p).
    (b) if $p = 4k+3$, then $(g-1)^{2k+3} \equiv g - 2 \pmod{p}$ and deduce that $g - 2$ is also a primitive root (mod p).

C 5. *If $g_1, \cdots, g_{\phi(p-1)}$ are the primitive roots mod p in the range $1 < g \le p - 1$, prove that*

$$\sum_{i=1}^{\phi(p-1)} g_i \equiv \mu(p-1) \pmod{p}.$$

C 6. *Suppose that m does not have a primitive root. Show that*

$$a^{\frac{\phi(m)}{2}} \equiv -1 \pmod{m}$$

for every a relatively prime m.

C 7. *Suppose that $p > 3$ is prime. Prove that the products of the primitive roots of p between 1 and $p - 1$ is congruent to 1 modulo p.*

5.2. Qudratic Residues.

C 8. *The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?*

C 9. *Let p be an odd prime number. Show that the smallest positive quadratic nonresidue of p is smaller than $\sqrt{p} + 1$.*

C 10. *Let M be an integer, and let p be a prime with $p > 25$. Show that the sequence M, $M+1$, $\cdots$, $M+3[\sqrt{p}]-1$ contains a qudratic non-residue to modulus p.*

C 11. *Let p be an odd prime and let Z_p denote (the field of) integers modulo p. How many elements are in the set*

$$\{x^2 : x \in Z_p\} \cap \{y^2 + 1 : y \in Z_p\}?$$

5.3. Congruences.

C 12. *If p is an odd prime, prove that*

$$\binom{k}{p} \equiv \left[\frac{k}{p}\right] \pmod{p}.$$

C 13. *Suppose p is an odd prime. Prove that*

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

C 14. *(Morley) Show that*

$$(-1)^{\frac{p-1}{2}} \binom{p-1}{\frac{p-1}{2}} \equiv 4^{p-1} \pmod{p^3}$$

for all prime numbers p with $p \geq 5$.

C 15. *Let n be a positive integer. Prove that n is prime if and only if*

$$\binom{n-1}{k} \equiv (-1)^k \pmod{n}$$

for all $k \in \{0, 1, \cdots, n-1\}$.

C 16. *Prove that for $n \geq 2$,*

$$\overbrace{2^{2^{\cdots^2}}}^{n \ terms} \equiv \overbrace{2^{2^{\cdots^2}}}^{n-1 \ terms} \pmod{n}.$$

C 17. *Show that, for any fixed integer $n \geq 1$, the sequence*

$$2,\ 2^2,\ 2^{2^2},\ 2^{2^{2^2}}, \ldots \pmod{n}$$

is eventually constant.

C 18. *Show that there exists a composite number n such that $a^n \equiv a \pmod{n}$ for all $a \in \mathbf{Z}$.*

C 19. *Let p be a prime number of the form $4k+1$. Suppose that $2p+1$ is prime. Show that there is no $k \in \mathbf{N}$ with $k < 2p$ and $2^k \equiv 1 \pmod{2p+1}$*

C 20. *During a break, n children at school sit in a circle around their teacher to play a game. The teacher walks clockwise close to the children and hands out candies to some of them according to the following rule. He selects one child and gives him a candy, then he skips the next child and gives a candy to the next one, then he skips 2 and gives a candy to the next one, then he skips 3, and so on. Determine the values of n for which eventually, perhaps after many rounds, all children will have at least one candy each.*

C 21. *Suppose that $m > 2$, and let P be the product of the positive integers less than m that are relatively prime to m. Show that $P \equiv -1 (mod\ m)$ if $m = 4$, p^n, or $2p^n$, where p is an odd prime, and $P \equiv 1(mod\ m)$ otherwise.*

C 22. *Let Γ consist of all polynomials in x with integer coefficienst. For f and g in Γ and m a positive integer, let $f \equiv g$ (mod m) mean that every coefficient of $f - g$ is an integral multiple of m. Let n and p be positive integers with p prime. Given that f, g, h, r and s are in Γ with $rf + sg \equiv 1$ (mod p) and $fg \equiv h$ (mod p), prove that there exist F and G in Γ with $F \equiv f$ (mod p), $G \equiv g$ (mod p), and $FG \equiv h$ (mod p^n).*

C 23. *Determine the number of integers $n \geq 2$ for which the congruence*

$$x^{25} \equiv x\ (mod\ n)$$

is true for all integers x.

C 24. *Let $n_1, \cdots, n_k$ and a be positive integers which satify the following condtions :*

       i) for any $i \neq j$, $(n_i, n_j) = 1$,
       ii) for any i, $a^{n_i} \equiv 1(mod\ n_i)$,
       iii) for any i, $n_i \nmid a - 1$.
Show that there exist at least $2^{k+1} - 2$ integers $x > 1$ with $a^x \equiv 1(mod\ x)$.

C 25. *Determine all positive integers $n \geq 2$ that satisfy the following condition ; For all integers a, b relatively prime to n,*

$$a \equiv b\ (mod\ n) \Longleftrightarrow ab \equiv 1\ (mod\ n).$$

C 26. *Determine all positive integers n such that $xy+1 \equiv 0(mod\ n)$ implies that $x + y \equiv 0(mod\ n)$.*

C 27. *Let p be a prime number. Determine the maximal degree of a polynomial $T(x)$ whose coefficients belong to $\{0, 1, \cdots, p - 1\}$ whose degree is less than p, and which satisfies*

$$T(n) = T(m)\ (mod\ p) \Longrightarrow n = m\ (mod\ p)$$

for all integers n, m.

6. Primes and Composite Numbers

Wherever there is number, there is beauty. Proclus Diadochus

D 1. *Prove that the number $512^3 + 675^3 + 720^3$ is composite.* [5]

D 2. *Show that there are infinitely many primes.*

D 3. *Find all natural numbers n for which every natural number whose decimal representation has $n-1$ digits 1 and one digit 7 is prime.*

D 4. *Prove that there do not exist polynomials P and Q such that*

$$\pi(x) = \frac{P(x)}{Q(x)}$$

for all $x \in \mathbf{N}$.

D 5. *Show that there exist two consecutive integer squares such that there are at least 1000 primes between them.*

D 6. *Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that $ac + bd = (b + d + a - c)(b + d - a + c)$. Prove that $ab + cd$ is not prime.*

D 7. *Prove that there is no noncontant polynomial $f(x)$ with integral coefficients such that $f(n)$ is prime for all $n \in \mathbf{N}$.*

D 8. *A prime p has decimal digits $p_n p_{n-1} \cdots p_0$ with $p_n > 1$. Show that the polynomial $p_n x^n + p_{n-1} x^{n-1} + \cdots + p_1 x + p_0$ cannot be represented as a product of two nonconstant polynomials with integer coefficients*

D 9. *Let $n \geq 2$ be an integer. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \sqrt{\frac{n}{3}}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n - 2$.*

D 10. *Prove that for any prime p in the interval $\left(n, \frac{4n}{3}\right]$, p divides*

$$\sum_{j=0}^{n} \binom{n}{j}^4$$

D 11. *Let a, b, and n be positive integers with $gcd(a, b) = 1$. Without using the Dirichlet's theorem[6], show that there are infinitely many $k \in \mathbf{N}$ such that $gcd(ak + b, n) = 1$.*

D 12. *Without using the Dirichlet's theorem, show that there are infintely many primes ending in the digit 9.*

D 13. *Let p be an odd prime. Without using the Dirichlet's theorem, show that there are infintely many primes of the form $2pk + 1$.*

---

[5]Note that there is a hint at chapter 15.

[6]For any $a, b \in \mathbf{N}$ with $gcd(a, b) = 1$, there are infinitely many primes of the form $ak + b$.

D 14. *Show that, for each $r \geq 1$, there are infintely many primes $p \equiv 1 \pmod{2^r}$.*

D 15. *Prove that if p is a prime, then $p^p - 1$ has a prime factor that is congruent to 1 modulo p.*

D 16. *Let p be a prime number. Prove that there exists a prime number q such that for every integer n, $n^p - p$ is not divisible by q.*

D 17. *Let $p_1 = 2, p_2 = 3, p_3 = 5, \cdots, p_n$ be the first n prime numbers, where $n \geq 3$. Prove that*

$$\frac{1}{p_1{}^2} + \frac{1}{p_2{}^2} + \cdots + \frac{1}{p_n{}^2} + \frac{1}{p_1 p_2 \cdots p_n} < \frac{1}{2}.$$

D 18. *Let p_n be the nth prime : $p_1 = 2, p_2 = 3, p_3 = 5, \cdots$. Show that the infinite series*

$$\sum_{n=1}^{\infty} \frac{1}{p_n}$$

diverges.

D 19. *Prove that $\log n \geq k \log 2$, where n is a natural number and k is the number of distinct primes that divide n.*

D 20. *Find the smallest prime which is not the difference (in some order) of a power of 2 and a power of 3.*

D 21. *Find the sum of all distinct positive divisors of the number 104060401.*

D 22. *Prove that 1280000401 is composite.*

D 23. *Prove that $\frac{5^{125}-1}{5^{25}-1}$ is a composite number.*

D 24. *Find the factor of $2^{33} - 2^{19} - 2^{17} - 1$ that lies between 1000 and 5000.*

D 25. *Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number.*

D 26. *Show that there exists a positive integer k such that $k \cdot 2^n + 1$ is composite for all $n \in \mathbf{N}_0$.*

D 27. *Show that $n^{\pi(2n)-\pi(n)} < 4^n$ for all positive integer n.*

D 28. *Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and next number on the circle in a given direction (that is, the numbers a, b, c, d are replaced by $a - b$, $b - c$, $c - d$, $d - a$). Is it possible after 1996 such steps to have numbers a, b, c, d such that the numbers $|bc - ad|$, $|ac - bd|$, $|ab - cd|$ are primes ?*

D 29. *Let s_n denote the sum of the first n primes. Prove that for each n there exists an integer whose square lies between s_n and s_{n+1}.*

D 30. *Given an odd integer $n > 3$, let k and t be the smallest positive integers such that both $kn + 1$ and tn are squares. Prove that n is prime if and only if both k and t are greater than $\frac{n}{4}$*

D 31. *Represent the number $989 \cdot 1001 \cdot 1007 + 320$ as the product of primes.*

7. Rational and Irrational Numbers

God made the integers, all else is the work of man. Leopold Kronecker

E 1. *Suppose that a rectangle with sides a and b is arbitrily cut into squares with sides $x_1, \cdots, x_n$. Show that $\frac{x_i}{a} \in \mathbf{Q}$ and $\frac{x_i}{b} \in \mathbf{Q}$ for all $i \in \{1, \cdots, n\}$.*

E 2. *(K. Mahler, 1953) Prove that for any $p, q \in \mathbf{N}$ with $q > 1$ the following inequality holds.* [7]
$$\left| \pi - \frac{p}{q} \right| \geq q^{-42}$$

E 3. *(K. Alladi, M. Robinson, 1979) Suppose that $p, q \in \mathbf{N}$ satisfy the inequality $e(\sqrt{p+q} - \sqrt{q})^2 < 1$.[8] Show that the number $\ln\left(1 + \frac{p}{q}\right)$ is irrational.*

E 4. *Find the smallest positive integer n such that*
$$0 < n^{\frac{1}{4}} - [n^{\frac{1}{4}}] < 0.00001.$$

E 5. *Prove that for any positive integers a and b*
$$\left| a\sqrt{2} - b \right| > \frac{1}{2(a+b)}.$$

E 6. *Prove that there exist positive integers m and n such that*
$$\left| \frac{m^2}{n^3} - \sqrt{2001} \right| < \frac{1}{10^8}.$$

E 7. *Let a, b, c be integers, not all zero and each of absolute value less than one million. Prove that*
$$\left| a + b\sqrt{2} + c\sqrt{3} \right| > \frac{1}{10^{21}}.$$

E 8. *(Hurwitz) Prove that for any irrational number ξ, there are infinitely many rational numbers $\frac{m}{n}$ $((m, n) \in \mathbf{Z} \times \mathbf{N})$ such that*
$$\left| \xi - \frac{n}{m} \right| < \frac{1}{\sqrt{5}m^2}.$$

E 9. *Show that π is irrational.*

E 10. *Show that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ is irrational.*

E 11. *Show that $\cos \frac{\pi}{7}$ is irrational.*

---

[7]This is a deep theorem in *transcendental number theory*. Note that it follows from this result that π is irrational ! In fact, it's known that for sufficiently large q, the exponent 42 can be replaced by 30. Here is a similar result due to A. Baker : For any rationals $\frac{p}{q}$, one has $|\ln 2 - \frac{p}{q}| \geq 10^{-100000}q^{-12.5}$. [AI, pp. 106]

[8]Here, $e = \sum_{n \geq 0} \frac{1}{n!}$.

E 12. *Show that $\frac{1}{\pi} \arccos\left(\frac{1}{\sqrt{2003}}\right)$ is irrational.*

E 13. *Show that $\cos 1°$ is irrational.*

E 14. *Prove that there cannot exist a positive rational number x such that*

$$x^{[x]} = \frac{9}{2}$$

holds.

E 15. *Let x, y, z non-zero real numbers such that xy, yz, zx are rational.*
    (a) Show that the number $x^2 + y^2 + z^2$ is rational.
    (b) If the number $x^3 + y^3 + z^3$ is also rational, show that x,
y, z are rational.

E 16. *Show that the cube roots of three distinct primes cannot be terms in an arithmetic progression.*

E 17. *Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with rational area.*

E 18. *Let a be a rational number with $0 < a < 1$ and suppose that $\cos(3\pi\alpha) + 2\cos(2\pi\alpha) = 0$. (Angle meauserments are in radians.) Prove that $a = \frac{2}{3}$.*

E 19. *Suppose $\tan\alpha = \frac{p}{q}$, where p and q are integers and $q \neq 0$. Prove the number $\tan\beta$ for which $\tan 2\beta = \tan 3\alpha$ is rational only when $p^2 + q^2$ is the square of an integer.*

E 20. *If x is a positive rational number show that x can be uniquely expressed in the form*

$$x = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots,$$

where a_1, a_2, $\cdots$ are integers, $0 \leq a_n \leq n - 1$, for $n > 1$, and the series terminates. Show also that x can be expressed as the sum of reciprocals of different integers, each of which is greater than 10^6.

E 21. *Find all polynomials W with real coefficients possessing the following property : if $x + y$ is a rational number, then $W(x) + W(y)$ is rational as well.*

E 22. *Show that any positive rational number can be represented as the sum of three positive rational cubes.*

E 23. *Prove that every positive rational number can be represented in the form*

$$\frac{a^3 + b^3}{c^3 + d^3}$$

for some positive integers a, b, c, d.

E 24. *The set S is a finite subset of $[0, 1]$ with the following property : for all $s \in S$, there exist a, $b \in S \bigcup \{0, 1\}$ with $a, b \neq x$ such that $x = \frac{a+b}{2}$. Prove that all the numbers in S are rational.*

E 25. *Let $S = \{x_0, x_1, \cdots, x_n\} \subset [0, 1]$ be a finite set of real numbers with $x_0 = 0$ and $x_1 = 1$, such that every distance between pairs of elements occurs at least twice, except for the distance 1. Prove that all of the x_i are rational.*

E 26. *You are given three lists A, B, and C. List A contains the numbers of the form 10^k in base 10, with k any integer greater than or equal to 1. Lists B and C contain the same numbers translated into base 2 and 5 respectively:*

A	B	C
10	1010	20
100	1100100	400
1000	1111101000	13000
$\vdots$	$\vdots$	$\vdots$

Prove that for every integer $n > 1$, there is exactly one number in exactly one of the lists B or C that has exactly n digits.

E 27. *(Beatty) Prove that if α and β are positive irrational numbers satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then the sequences*

$$[\alpha], [2\alpha], [3\alpha], \cdots$$

and

$$[\beta], [2\beta], [3\beta], \cdots$$

together include every positive integer exactly once.

E 28. *For a positive real number α, define*

$$S(\alpha) = \{[n\alpha] | n = 1, 2, 3, \cdots\}.$$

Prove that $\mathbf{N}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$.

8. Diophantine Equations I

In the margin of his copy of Diophantus' Arithmetica, Pierre de Fermat wrote : *To divide a cube into two other cubes, a fourth power or in general any power whatever into two powers of the same denomination above the second is impossible, and I have assuredly found an admirable proof of this, but the margin is too narrow to contain it.*

F 1. *One of Euler's conjecture[9] was disproved in the 1980s by three American Mathematicians[10] when they showed that there is a positive integer n such that*
$$n^5 = 133^5 + 110^5 + 84^5 + 27^5.$$
Find the value of n.

F 2. *The number 21982145917308330487013369 is the thirteenth power of a positive integer. Which positive integer?*

F 3. *Does there exist a solution to the equation*
$$x^2 + y^2 + z^2 + u^2 + v^2 = xyzuv - 65$$
in integers x, y, z, u, v greater than 1998?

F 4. *Find all pairs (x, y) of positive rational numbers such that $x^2 + 3y^2 = 1$.*

F 5. *Find all pairs (x, y) of rational numbers such that $y^2 = x^3 - 3x + 2$.*

F 6. *Show that there are infinitely many pairs (x, y) of rational numbers such that $x^3 + y^3 = 9$.*

F 7. *Determine all pairs (x, y) of positive integers satisfying the equation*
$$(x + y)^2 - 2(xy)^2 = 1.$$

F 8. *Show that the equation*
$$x^3 + y^3 + z^3 + t^3 = 1999$$
has infinitely many integral solutions. [11]

F 9. *Determine all integers a for which the equation*
$$x^2 + axy + y^2 = 1$$
has infinitely many distinct integer solutions x, y.

---

[9]In 1769, Euler, by generalising Fermat's Last Theorem, conjectured that "it is impossible to exhibit three fourth powers whose sum is a fourth power", "four fifth powers whose sum is a fifth power, and similarly for higher powers" [Rs]

[10]L. J. Lander, T. R. Parkin, J. L. Selfridge

[11]More generally, the following result is known : let n be an integer, then the equation $x^3 + y^3 + z^3 + w^3 = n$ has infinitely many integral solutions (x, y, z, w) if there can be found one solution $(x, y, z, w) = (a, b, c, d)$ with $(a + b)(c + d)$ negative and with either $a \neq b$ and $c \neq d$. [Eb2, pp.90]

F 10. *Prove that there are unique positive integers a and n such that*

$$a^{n+1} - (a+1)^n = 2001.$$

F 11. *Find all $(x, y, n) \in \mathbf{N}^3$ such that $gcd(x, n+1) = 1$ and $x^n + 1 = y^{n+1}$.*

F 12. *Find all $(x, y, z) \in \mathbf{N}^3$ such that $x^4 - y^4 = z^2$.*

F 13. *Find all pairs (x, y) of positive integers that satisfy the equation* [12]

$$y^2 = x^3 + 16.$$

F 14. *Show that the equation $x^2 + y^5 = z^3$ has infinitely many solutions in integers x, y, z for which $xyz \neq 0$.*

F 15. *Prove that there are no integers x, y satisfying $x^2 = y^5 - 4$.*

F 16. *Find all pairs (a, b) of different positive integers that satisfy the equation $W(a) = W(b)$, where $W(x) = x^4 - 3x^3 + 5x^2 - 9x$.*

F 17. *Find all positive integers n for which the equation*

$$a + b + c + d = n\sqrt{abcd}$$

has a solution in positive integers.

F 18. *Determine all positive integer solutions (x, y, z, t) of the equation*

$$(x+y)(y+z)(z+x) = xyzt$$

for which $gcd(x, y) = gcd(y, z) = gcd(z, x) = 1$.

F 19. *Find all $(x, y, z, n) \in \mathbf{N}^4$ such that $x^3 + y^3 + z^3 = nx^2y^2z^2$.*

F 20. *Determine all positive integers n for which the equation*

$$x^n + (2+x)^n + (2-x)^n = 0$$

has an integer as a solution.

F 21. *Prove that the equation*

$$6(6a^2 + 3b^2 + c^2) = 5n^2$$

has no solutions in integers except $a = b = c = n = 0$.

F 22. *Find all integers (a, b, c, x, y, z) such that*

$$a + b + c = xyz, \; x + y + z = abc, \; a \geq b \geq c \geq 1, \; x \geq y \geq z \geq 1.$$

F 23. *Find all $(x, y, z) \in \mathbf{N}^3$ such that $x^3 + y^3 + z^3 = x + y + z = 3$.*

---

[12]It's known that there are infinitely many integers k so that the equation $y^2 = x^3 + k$ has no integral solutions. For example, if k has the form $k = (4n - 1)^3 - 4m^2$, where m and n are integers such that no prime $p \equiv -1 \pmod 4$ divides m, then the equation $y^2 = x^3 + k$ has no integral solutions. For a proof, see [Tma, pp. 191].

F 24. *Prove that if n is a positive integer such that the equation*
$$x^3 - 3xy^2 + y^3 = n.$$
has a solution in integers (x, y), then it has at least three such solutions.
   Show that the equation has no solutions in integers when $n = 2891$.

F 25. *What is the smallest positive integer t such that there exist integers $x_1, x_2, \cdots, x_t$ with*
$$x_1{}^3 + x_2{}^3 + \cdots + x_t{}^3 = 2002^{2002} \ ?$$

F 26. *Solve in integers the following equation*
$$n^{2002} = m(m + n)(m + 2n) \cdots (m + 2001n).$$

F 27. *Prove that there exist infinitely many positive integers n such that $p = nr$, where p and r are respectively the semiperimeter and the inradius of a triangle with integer side lengths.*

F 28. *Let a, b, c be positive integers such that a and b are relatively prime and c is relatively prime either to a and b. Prove that there exist infinitely many triples (x, y, z) of distinct positive integers x, y, z such that*
$$x^a + y^b = z^c.$$

F 29. *Find all pairs of integers (x, y) satisfying the equality*
$$y(x^2 + 36) + x(y^2 - 36) + y^2(y - 12) = 0$$

F 30. *Let a, b, c be given integers $a > 0$, $ac - b^2 = P = P_1 P_2 \cdots P_n$, where $P_1, \cdots, P_n$ are (distinct) prime numbers. Let $M(n)$ denote the number of pairs of integers (x, y) for which $ax^2 + bxy + cy^2 = n$. Prove that $M(n)$ is finite and $M(n) = M(p^k \cdot n)$ for every inegers $k \geq 0$.*

F 31. *Determine integer solutions of the system*
$$2uv - xy = 16,$$
$$xv - yu = 12.$$

F 32. *Let n be a natural number. Solve in whole numbers the equation $x^n + y^n = (x - y)^{n+1}$.*

F 33. *Does there exist an integer such that its cube is equal to $3n^2 + 3n + 7$, where n is integer?*

F 34. *Are there integers m and n such that $5m^2 - 6mn + 7n^2 = 1985$?*

F 35. *Find all cubic polynomials $x^3 + ax^2 + bx + c$ admitting the rational numbers a, b and c as roots.*

F 36. *Prove that the equation $a^2 + b^2 = c^2 + 3$ has infinitely many integer solutions (a, b, c).*

9. Diophantine Equations II

The positive integers stand there, a continual and inevitable challenge to the curiousity of every healthy mind. Godfrey Harold Hardy

G 1. *Given that*

$$34! = 95232799cd96041408476186096435ab000000,$$

determine the digits a, b, c, d.

G 2. *Prove that the equation $(x_1-x_2)(x_2-x_3)(x_3-x_4)(x_4-x_5)(x_5-x_6)(x_6-x_7)(x_7-x_1) = (x_1-x_3)(x_2-x_4)(x_3-x_5)(x_4-x_6)(x_5-x_7)(x_6-x_1)(x_7-x_2)$ has a solution in natural numbers where all x_i are different.*

G 3. *Solve in positive integers the equation $10^a + 2^b - 3^c = 1997$.*

G 4. *Solve the equation $28^x = 19^y + 87^z$, where x, y, z are integers.*

G 5. *(Erdös) Show that the equation $\binom{n}{k} = m^l$ has no integral solution with $l \geq 2$ and $4 \leq k \leq n - 4$.*

G 6. *Find all positive integers x, y such that $7^x - 3^y = 4$.*

G 7. *Show that $|12^m - 5^n| \geq 7$ for all $m, n \in \mathbf{N}$.*

G 8. *Show that there is no positive integer k for which the equation*

$$(n - 1)! + 1 = n^k$$

is true when n is greater than 5.

G 9. *Determine all integers a and b such that*

$$(19a + b)^{18} + (a + b)^{18} + (19b + a)^{18}$$

is a positive square.

G 10. *Let b be a positive integer. Determine all 200-tuple integers of non-negative integers $(a_1, a_2, \cdots, a_{2002})$ satisfying*

$$\sum_{j=1}^{n} a_j{}^{a_j} = 2002b^b.$$

G 11. *Is there a positive integers m such that the equation*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{m}{a + b + c}$$

has infintely many solutions in positive integers a, b, c ?

G 12. *Consider the system*

$$x + y = z + u$$
$$2xy = zu$$

Find the greatest value of the real constant m such that $m \leq \frac{x}{y}$ for any positive integer solution (x, y, z, u) of the system, with $x \geq y$.

G 13. *Determine all positive rational number $r \neq 1$ such that $r^{\frac{1}{r-1}}$ is rational.*

G 14. *Show that the equation $\{x^3\} + \{y^3\} = \{z^3\}$ has infinitely many rational non-integer solutions.*

G 15. *Let n be a positive integer. Prove that the equation*

$$x + y + \frac{1}{x} + \frac{1}{y} = 3n$$

does not have solutions in positive rational numbers.

G 16. *Find all pairs (x, y) of positive rational numbers such that $x^y = y^x$*

G 17. *Find all pairs (a, b) of positive integers that satisfy the equation*

$$a^{b^2} = b^a.$$

G 18. *Find all pairs (a, b) of positive integers that satisfy the equation*

$$a^{a^a} = b^b.$$

G 19. *Let x, a, b be positive integers such that $x^{a+b} = a^b b$. Prove that $a = x$ and $b = x^x$.*

G 20. *Find all pairs (m, n) of integers that satisfy the equation*

$$(m - n)^2 = \frac{4mn}{m + n - 1}$$

G 21. *Find all pairwise relatively prime positive integers l, m, n such that*

$$(l + m + n)\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)$$

is an integer.

G 22. *Let x, y, z be integers with $z > 1$. Show that*

$$(x + 1)^2 + (x + 2)^2 + \cdots + (x + 99)^2 \neq y^z.$$

G 23. *Find all values of the positive integers m and n for which*

$$1! + 2! + 3! + \cdots + n! = m^2$$

G 24. *Prove that if a, b, c, d are integers such that $d = (a + 2^{\frac{1}{3}}b + 2^{\frac{2}{3}}c)^2$ then d is a perfect square (i. e. is the square of an integer).*

G 25. *Find a pair of relatively prime four digit natural numbers A and B such that for all natural numbers m and n, $|A^m - B^n| \geq 400$.*

G 26. *Find all solutions in positive integers a, b, c to the equation*
$$a!b! = a! + b! + c!.$$

G 27. *Find positive integers a and b such that*
$$(\sqrt[3]{a} + \sqrt[3]{b} - 1)^2 = 49 + 20\sqrt[3]{6}.$$

10. Functions in Number Theory

Gauss once said "Mathematics is the queen of the sciences and number theory is the queen of mathematics." If this be true we may add that the Disauistiones is the Magna Charta of number theory.   M. Cantor

H 1. *Let α be the positive root of the equation $x^2 = 1991x + 1$. For natural numbers m, n define*

$$m * n = mn + [\alpha m][\alpha n],$$

where $[x]$ is the greatest integer not exceeding x. Prove that for all natural numbers p, q, r,

$$(p * q) * r = p * (q * r).$$

H 2. *Find the total number of different integer values the function*

$$f(x) = [x] + [2x] + \left[\frac{5x}{3}\right] + [3x] + [4x]$$

takes for real numbers x with $0 \le x \le 100$.

H 3. *Show that $[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+2}]$ for all positive integer n.*

H 4. *Let $((x)) = x - [x] - \frac{1}{2}$ if x is not an integer, and let $((x)) = 0$ otherwise. If n and k are integers, with $n > 0$, prove that*

$$\left(\left(\frac{k}{n}\right)\right) = -\frac{1}{2n} \sum_{m=1}^{n-1} \cot\frac{\pi m}{n} \sin\frac{2\pi km}{n}.$$

H 5. *Let $\sigma(n)$ denote the sum of the positive divisors of the positive integer n. and $\phi(n)$ the Euler phi-function. Show that $\phi(n) + \sigma(n) \ge 2n$ for all positive integers n.*

H 6. *Let n be an integer with $n \ge 2$. Show that $\phi(2^n - 1)$ is divisible by n.*

H 7. *Show that if the equation $\phi(x) = n$ has one solution it always has a second solution, n being given and x being the unknown.*

H 8. *Let $d(n)$ denote the number of positive divisors of the natural number n. Prove that $d(n^2 + 1)^2$ does not become monotonic from any given point onwards.*

H 9. *For any $n \in \mathbf{N}$, let $d(n)$ denote the number of positive divisors of n. Determine all positive integers n such that $n = d(n)^2$.*

H 10. *For any $n \in \mathbf{N}$, let $d(n)$ denote the number of positive divisors of n. Determine all positive integers k such that*

$$\frac{d(n^2)}{d(n)} = k$$

for some $n \in \mathbf{N}$.

H 11. *Show that for all positive integers m and n,*

$$gcd(m, n) = m + n - mn + 2\sum_{k=0}^{m-1} \left[\frac{kn}{m}\right].$$

H 12. *Show that for all primes p,*

$$\sum_{k=1}^{p-1} \left[\frac{k^3}{p}\right] = \frac{(p+1)(p-1)(p-2)}{4}$$

H 13. *Let p be a prime number of the form $4k + 1$. Show that*

$$\sum_{k=1}^{p-1} \left(\left[\frac{2k^2}{p}\right] - 2\left[\frac{k^2}{p}\right]\right) = \frac{p-1}{2}$$

H 14. *Let p be a prime number of the form $4k + 1$. Show that*

$$\sum_{i=1}^{k} \left[\sqrt{ip}\right] = \frac{p^2 - 1}{12}$$

H 15. *Let a, b, n be positive integers with $gcd(a, b) = 1$. Prove that*

$$\sum_{k} \left\{\frac{ak + b}{n}\right\} = \frac{n-1}{2},$$

where k runs through a complete system of residues modulo m.

H 16. *The function $\mu : \mathbf{N} \longrightarrow \mathbf{C}$ is defined by*

$$\mu(n) = \sum_{k \in R_n} \left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right),$$

where $R_n = \{k \in \mathbf{N} | 1 \leq k \leq n, gcd(k, n) = 1\}$. Show that for all positive integer n, $\mu(n)$ is an integer.

H 17. *(Gauss) Show that for all $n \in \mathbf{N}$,*

$$n = \sum_{d|n} \phi(d).$$

H 18. *Let m, n be positive integers. Prove that, for some positive integer a, each of $\phi(a), \phi(a + 1), \cdots, \phi(a + n)$ is a multiple of m.*

H 19. *For a positive integer n, let $d(n)$ be the number of all positive divisord of n. Find all positive integers n such that $d(n)^3 = 4n$.*

H 20. *Let n be a positive integers. Let $\sigma(n)$ be the sum of the natural divisors d of n (including 1 and n). We say that an integer $m \geq 1$ is superabondant if*

$$\frac{\sigma(m)}{m} > \frac{\sigma(k)}{k},$$

for all $k \in \{1, 2, \cdots, m - 1\}$. Prove that there exists an infinite number of superabondant numbers.

H 21. *Prove that there is a function f from the set of all natural numbers into itself such that for any natural number n, $f(f(n)) = n^2$.*

H 22. *Find all surjective function $f : \mathbf{N} \longrightarrow \mathbf{N}$ satisfying the condition $m|n \Longleftrightarrow f(m)|f(n)$ for all $m, n \in \mathbf{N}$.*

H 23. *Let $S(n)$ be the sum of all different natural divisors of an odd natural number $n > 1$ (including 1 and n). Prove that $S(n)^3 < n^4$.*

H 24. *If n is composite, prove that $\phi(n) \le n - \sqrt{n}$.*

11. SEQUENCES OF INTEGERS

A peculiarity of the higher arithmetic is the great difficulty which has often been experienced in proving simple general theorems which had been suggested quite naturally by numerical evidence. Harold Davenport

I 1. *Let a, b, c, d be integers. Show that the product*

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

is divisible by 12.

I 2. *If $a_1 < a_2 < \cdots < a_n$ are integers, show that*

$$\prod_{1 \le i < j \le n} \frac{a_i - a_j}{i - j}$$

is an integer.[13]

I 3. *Let a, b be odd positive integers. Define the sequence (f_n) by putting $f_1 = a$, $f_2 = b$, and by letting f_n for $n \ge 3$ be the greatest odd divisor of $f_{n-1} + f_{n-2}$. Show that f_n is constant for n sufficiently large and determine the eventual value as a function of a and b.*

I 4. *Numbers $d(n, m)$ with m, n integers, $0 \le m \le n$, are defined by $d(n, 0) = d(n, n) = 1$ for all $n \ge 0$, and $md(n, m) = md(n - 1, m) + (2n - m)d(n - 1, m - 1)$ for $0 < m < n$. Prove that all the $d(n, m)$ are integers.*

I 5. *Show that the sequence $\{a_n\}_{n \ge 1}$ defined by $a_n = [n\sqrt{2}]$ contains an infinite number of integer powers of 2.*

I 6. *Let a_n be the last nonzero digit in the decimal representation of the number $n!$. Does the sequence $a_1, a_2, a_3, \cdots$ become periodic after a finite number of terms ?*

I 7. *Let m be a positive integer. Define the sequence $\{a_n\}_{n \ge 0}$ by*

$$a_0 = 0, \ a_1 = m, \ a_{n+1} = m^2 a_n - a_{n-1}.$$

Prove that an ordered pair (a, b) of non-negative integers, with $a \le b$, gives a solution to the equation

$$\frac{a^2 + b^2}{ab + 1} = m^2$$

if and only if (a, b) is of the form (a_n, a_{n+1}) for some $n \ge 0$. [14]

---

[13]The result follows immediately from the theory of Lie groups; the number turns out to be the dimension of an irreducible representation of $SU(n)$. [Rc]

[14]See **A4**

I 8. *Let x_n, y_n be two sequences defined recursively as follows*

$$x_0 = 1, \; x_1 = 4, \; x_{n+2} = 3x_{n+1} - x_n$$

$$y_0 = 1, \; y_1 = 2, \; y_{n+2} = 3y_{n+1} - y_n$$

for all $n = 0, 1, 2, \cdots$.

     a) Prove that $x_n{}^2 - 5y_n{}^2 + 4 = 0$ for all non-negative integers.
     b) Suppose that a, b are two positive integers such that $a^2 - b^2 + 4 = 0$. Prove that there exists a non-negative integer k such that $a = x_k$ and $b = y_k$.

I 9. *Let $P(x)$ be a nonzero polynomial with integral coefficients. Let $a_0 = 0$ and for $i \geq 0$ define $a_{i+1} = P(a_i)$. Show that $\gcd(a_m, a_n) = a_{\gcd(m,n)}$ for all $m, n \in \mathbf{N}$*

I 10. *An integer sequence $\{a_n\}_{n \geq 1}$ is defined by*

$$a_0 = 0, \; a_1 = 1, \; a_{n+2} = 2a_{n+1} + a_n$$

Show that 2^k divides a_n if and only if 2^k divides n.

I 11. *An integer sequence $\{a_n\}_{n \geq 1}$ is defined by*

$$a_1 = 1, \; a_{n+1} = a_n + [\sqrt{a_n}]$$

Show that a_n is a square if and only if $n = 2^k + k - 2$ for some $k \in \mathbf{N}$.

I 12. *Let $f(n) = n + [\sqrt{n}]$. Prove that, for every positive integer m, the sequence*

$$m, f(m), f(f(m)), f(f(f(m))), \cdots$$

contains at least one square of an integer.

I 13. *An integer sequence $\{a_n\}_{n \geq 1}$ is given such that*

$$2^n = \sum_{d \mid n} a_d$$

for all $n \in \mathbf{N}$. Show that a_n is divisible by n.

I 14. *Let k, m, n be natural numbers such that $m + k + 1$ is a prime greater than $n + 1$. Let $c_s = s(s + 1)$. Prove that the product $(c_{m+1} - c_k)(c_{m+2} - c_k) \cdots (c_{m+n} - c_k)$ is divisible by the product $c_1 c_2 \cdots c_n$.*

I 15. *Show that for all prime numbers p*

$$Q(p) = \prod_{k=1}^{p-1} k^{2k-p-1}$$

is an integer.

I 16. *The sequence $\{a_n\}_{n \geq 1}$ is defined by*

$$a_1 = 1, \; a_2 = 2, \; a_3 = 24, \; a_{n+2} = \frac{6a_{n-1}{}^2 a_{n-3} - 8a_{n-1}a_{n-2}{}^2}{a_{n-2}a_{n-3}} \; (n \geq 4)$$

Show that for all n, a_n is an integer.

I 17. *Show that there is a unique sequence of integers $\{a_n\}_{n\geq 1}$ with*

$$a_1 = 1,\ a_2 = 2,\ a_4 = 12,\ a_{n+1}a_{n-1} = a_n{}^2 + 1\ \ (n \geq 2).$$

I 18. *The sequence $\{a_n\}_{n\geq 1}$ is defined by*

$$a_1 = 1,\ a_{n+1} = 2a_n + \sqrt{3a_n{}^2 + 1}\ \ (n \geq 1)$$

Show that a_n is an integer for every n.

I 19. *Prove that the sequence $\{y_n\}_{n\geq 1}$ defined by*

$$y_0 = 1,\ y_{n+1} = \frac{1}{2}\left(3y_n + \sqrt{5a_n{}^2 - 4}\right)\ \ (n \geq 0)$$

consists only of integers.

I 20. *(C. von Staudt) The Bernoulli sequence[15] $\{B_n\}_{n\geq 0}$ is defined by*

$$B_0 = 1,\ B_n = -\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k\ \ (n \geq 1)$$

Show that for all $n \in \mathbf{N}$,

$$(-1)^n B_n - \sum \frac{1}{p},$$

is an integer where the summation being extended over the primes p such that $p\,|\,2k - 1$.

I 21. *Let n be a positive integer. Show that*

$$\sum_{i=1}^{n}\tan^2\frac{i\pi}{2n+1}$$

is an odd integer.

I 22. *An integer sequence $\{a_n\}_{n\geq 1}$ is defined by*

$$a_1 = 2,\ a_{n+1} = \left[\frac{3}{2}a_n\right]$$

Show that there are infinitely many even and infinitely many odd integers.

I 23. *Prove or disprove that there exists a positive real number u such that $[u^n] - n$ is an even integer for all positive integer n.*

I 24. *Let $\{a_n\}$ be a strictly increasing positive integers sequece such that $gcd(a_i, a_j) = 1$ and $a_{i+2} - a_{i+1} > a_{i+1} - a_i$. Show that the infinite series*

$$\sum_{i=1}^{\infty}\frac{1}{a_i}$$

converges.

---

[15]$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, \cdots$

I 25. *Let $\{n_k\}_{k \geq 1}$ be a sequence of natural numbers such that for $i < j$, the decimal representation of n_i does not occur as the leftmost digits of the decimal representation of n_j. Prove that*

$$\sum_{k=1}^{\infty} \frac{1}{n_k} \leq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{9}.$$

I 26. *An integer sequence satisfies $a_{n+1} = a_n^3 + 1999$. Show that it contains at most one square.*

I 27. *Let $n > 6$ be an integer and $a_1, a_2, \ldots, a_k$ be all the natural numbers less than n and relatively prime to n. If*

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,$$

prove that n must be either a prime number or a power of 2.

I 28. *Show that if an infinite arithmetic progression of positive integers contains a square and a cube, it must contain a sixth power.*

I 29. *Let $a_1 = 11^{11}$, $a_2 = 12^{12}$, $a_3 = 13^{13}$, and*

$$a_n = |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|, n \geq 4.$$

Determine $a_{14^{14}}$.

I 30. *Prove that there exists two strictly increasing sequences a_n and b_n such that $a_n(a_n + 1)$ divides $b_n^2 + 1$ for every natural n.*

I 31. *Let k be a fixed positive integer. The infinite sequence a_n is defined by the formulae*

$$a_1 = k + 1, a_{n+1} = a_n^2 - ka_n + k \ \ (n \geq 1).$$

Show that if $m \neq n$, then the numbers a_m and a_n are relatively prime.

I 32. *The Fibonacci sequence $\{F_n\}$ is defined by*

$$F_1 = 1, \ F_2 = 1, \ F_{n+2} = F_{n+1} + F_n.$$

Show that $gcd(F_m, F_n) = F_{gcd(m,n)}$ for all $m, n \in \mathbf{N}$.

I 33. *The Fibonacci sequence $\{F_n\}$ is defined by*

$$F_1 = 1, \ F_2 = 1, \ F_{n+2} = F_{n+1} + F_n.$$

Show that $F_{mn-1} - F_{n-1}^m$ is divisible by F_n^2 for all $m \geq 1$ and $n > 1$.

I 34. *The Fibonacci sequence $\{F_n\}$ is defined by*

$$F_1 = 1, \ F_2 = 1, \ F_{n+2} = F_{n+1} + F_n.$$

Show that $F_{mn} - F_{n+1}^m + F_{n-1}^m$ is divisible by F_n^3 for all $m \geq 1$ and $n > 1$.

I 35. *The Fibonacci sequence $\{F_n\}$ is defined by*

$$F_1 = 1, \ F_2 = 1, \ F_{n+2} = F_{n+1} + F_n.$$

Show that $F_{2n-1}^2 + F_{2n+1}^2 + 1 = 3F_{2n-1}F_{2n+1}$ for all $n \geq 1$. [16]

---

[16]See **A5** and **B15**

I 36. *Prove that no Fibonacci number can be factored into a product of two smaller Fibonacci numbers, each greater than 1.*

I 37. *The sequence $\{x_n\}$ is defined by*

$$x_0 \in [0, 1], \ x_{n+1} = 1 - |1 - 2x_n|.$$

Prove that the sequence is periodic if and only if x_0 is irrational.

I 38. *Let x_1 and x_2 be relatively prime positive integers. For $n \geq 2$, define $x_{n+1} = x_n x_{n-1} + 1$.*

(a) Prove that for every $i > 1$, there exists $j > i$ such that $x_i{}^i$ divides $x_j{}^j$.

(b) Is it true that x_1 must divide $x_j{}^j$ for some $j > 1$?

I 39. *For a given positive integer k denote the square of the sum of its digits by $f_1(k)$ and let $f_{n+1}(k) = f_1(f_n(k))$. Determine the value of $f_{1991}(2^{1990})$.*

I 40. *Let $q_0, q_1, \cdots$ be a sequence of integers such that*

      (i) for any $m > n$, $m - n$ is a factor of $q_m - q_n$, and
      (ii) $|q_n| \leq n^{10}$ for all integers $n \geq 0$.

Show that there exists a polynomial $Q(x)$ satisfying $q_n = Q(n)$ for all n.

I 41. *Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, n_2, \ldots, n_k$ of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i $(1 \leq i < k)$.*

I 42. *Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?*

I 43. *The infinite sequence of 2's and 3's*

$$2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3,$$
$$3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, \ldots$$

has the property that, if one forms a second sequence that records the number of 3's between successive 2's, the result is identical to the given sequence. Show that there exists a real number r such that, for any n, the nth term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m. (Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.)

12. Combinatorial Number Theory

In great mathematics there is a very high degree of unexpectedness, combined with inevitability and economy.   Godfrey Harold Hardy

J 1. *Let p be a prime. Find all k such that the set $\{1, 2, \cdots, k\}$ can be partitioned into p subsets with equal sum of elements.*

J 2. *Prove that the set of integers of the form $2^k - 3 (k = 2, 3, ...)$ contains an infinite subset in which every two members are relatively prime.*

J 3. *The set of positive integers is parititioned into finitely many subsets. Show that some subset S has the following property : for every positive integer n, S contains infinitely many multiples of n.*

J 4. *Let M be a positive integer and consider the set*

$$S = \{n \in \mathbf{N} | M^2 \le n < (M+1)^2\}.$$

Prove that the products of the form ab with $a, b \in S$ are distinct.

J 5. *Let S be a set of integers such that*
- *there exist $a, b \in S$ with $gcd(a, b) = gcd(a - 2, b - 2) = 1$.*
- *if x and y are elements of S, then $x^2 - y$ also belongs to S. Prove that S is the set of all integers.*

J 6. *Show that for each $n \ge 2$, there is a set S of n integers such that $(a - b)^2$ divides ab for every distinct $a, b \in S$*

J 7. *Let a and b be positive integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, \cdots, n_k$ of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i $(1 \le i \le k)$.*

J 8. *Let n be an integer, and let X be a set of $n+2$ integers each of absolute value at most n. Show that there exist three distinct numbers $a, b, c \in X$ such that $c = a + b$.*

J 9. *Let $m \ge 2$ be an integer. Find the smallest integer $n > m$ such that for any partition of the set $\{m, m+1, \cdots, n\}$ into two subsets, at least one subset contains three numbers a, b, c such that $c = a^b$.*

J 10. *Let $S = \{1, 2, 3, \ldots, 280\}$. Find the smallest integer n such that each n-element subset of S contains five numbers which are pairwise relatively prime.*

J 11. *Let m and n be positive integers. If $x_1, x_2, \cdots, x_m$ are postive integers whose average is less than $n + 1$ and if $y_1, y_2, \cdots, y_n$ are postive integers whose average is less than $m + 1$, prove that some sum of one or more x's equals some sum of one or more y's.*

J 12. *For every natural number n, $Q(n)$ denote the sum of the digits in the decimal representation of n. Prove that there are infinitely many natural number k with $Q(3^k) > Q(3^{k+1})$.*

J 13. *Let n and k be given relatively prime natural numbers, $k < n$. Each number in the set $M = \{1, 2, ..., n-1\}$ is colored either blue or white. It is given that*

    ○ *for each $i \in M$, both i and $n - i$ have the same color;*
    ○ *for each $i \in M, i \neq k$, both i and $|i - k|$ have the same color.*

Prove that all numbers in M must have the same color.

J 14. *Let p be a prime number, $p \geq 5$, and k be a digit in the p-adic representation of positive integers. Find the maximal length of a non constant arithmetic progression whose terms do not contain the digit k in their p-adic representation.*

J 15. *Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression?*

J 16. *Is it possible to find 100 positive integers not exceeding 25000 such that all pairwise sums of them are different ?*

J 17. *Find the maximum number of pairwise disjoint sets of the form*

$$S_{a,b} = \{n^2 + an + b | n \in \mathbf{Z}\},$$

with $a, b \in \mathbf{Z}$.

J 18. *Let p be an odd prime number. How many p-element subsets A of $\{1, 2, \ldots 2p\}$ are there, the sum of whose elements is divisible by p?*

J 19. *Let $m, n \geq 2$ be positive integers, and let $a_1, a_2, \cdots, a_n$ be integers, none of which is a multiple of m^{n-1}. Show that there exist integers $e_1, e_2, \cdots, e_n$, not all zero, with $|e_i| < m$ for all i, such that $e_1 a_1 + e_2 a_2 + \cdots + e_n a_n$ is a multiple of m^n.*

J 20. *Determine the smallest integer $n \geq 4$ for which one can choose four different numbers a, b, c, and d from any n distinct integers such that $a + b - c - d$ is divisible by 20*

J 21. *A sequence of integers $a_1, a_2, a_3, \cdots$ is defined as follows : $a_1 = 1$, and for $n \geq 1$, a_{n+1} is the smallest integer greater than a_n such that $a_i + a_j \neq 3a_k$ for any i, j, and k in $\{1, 2, 3, \cdots, n+1\}$, not necessarily distinct. Determine a_{1998}.*

J 22. *Prove that for each positive integer n, there exists a positive integer with the following properties :*

    ○ *It has exactly n digits.*
    ○ *None of the digits is 0.*
    ○ *It is divisible by the sum of its digits.*

J 23. *Let k, m, n be integers such that $1 < n \leq m - 1 \leq k$. Determine the maximum size of a subset S of the set $\{1, 2, \cdots, k\}$ such that no n distinct elements of S add up to m.*

J 24. *Find the number of subsets of $\{1, 2, \cdots, 2000\}$, the sum of whose elements is divisible by 5.*

J 25. *Let A be a non-empty set of positive integers. Suppose that there are positive integers $b_1, \cdots, b_n$ and $c_1, \cdots, c_n$ such that*

> *(i) for each i the set $b_i A + c_i = \{b_i a + c_i | a \in A\}$ is a subset of A, and*
> *(ii) the sets $b_i A + c_i$ and $b_j A + c_j$ are disjoint whenever $i \neq j$.*

Prove that

$$\frac{1}{b_1} + \cdots + \frac{1}{b_n} \leq 1.$$

J 26. *A set of three nonnegative integers $\{x, y, z\}$ with $x < y < z$ is called historic if $\{z - y, y - x\} = \{1776, 2001\}$. Show that the set of all nonnegative integers can be written as the unions of pairwise disjoint historic sets.*

J 27. *Let p and q be relatively prime positive integers. A subset S of $\{0, 1, 2, \cdots\}$ is called ideal if $0 \in S$ and, for each element $n \in S$, the integers $n + p$ and $n + q$ belong to S. Determine the number of ideal subsets of $\{0, 1, 2, \cdots\}$.*

J 28. *Prove that the set of positive integers cannot be partitioned into three nonempty subsets such that, for any two integers x, y taken from two different subsets, the number $x^2 - xy + y^2$ belongs to the third subset.*

J 29. *Let A be a set of N residues (mod N^2). Prove that there exists a set B of N residues (mod N^2) such that the set $A + B = \{a + b | a \in A, b \in B\}$ contains at least half of all the residues (mod N^2).*

J 30. *Determine the largest positive integer n for which there exists a set S with exactly n numbers such that*

> *(i) each member in S is a positive integer not exceeding 2002,*
> *(ii) if a and b are two (not necessarily different) numbers in S, then there product ab does not belong to S.*

J 31. *Prove that, for any integer $a_1 > 1$ there exist an increasing sequence of positive integers $a_1, a_2, a_3, \cdots$ such that*

$$a_1 + a_2 + \cdots + a_n | a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2$$

for all $k \in \mathbf{N}$.

J 32. *An odd integer $n \geq 3$ is said to be "nice" if and only if there is at least one permution $a_1, \cdots, a_n$ of $1, \cdots, n$ such that the n sums $a_1 - a_2 + a_3 - \cdots - a_{n-1} + a_n$, $a_2 - a_3 + a_3 - \cdots - a_n + a_1$, $a_3 - a_4 + a_5 - \cdots - a_1 + a_2$, $\cdots$, $a_n - a_1 + a_2 - \cdots - a_{n-2} + a_{n-1}$ are all positive. Determine the set of all "nice" integers.*

J 33. *Assume that the set of all positive integers is decomposed into r distinct subsets $A_1 \cup A_2 \cup \cdots \cup A_r = \mathbf{N}$. Prove that one of them, say A_i, has the following property : There exist a positive integer m such that for any k one can find numbers $a_1, \cdots, a_k$ in A_i with $0 < a_{j+1} - a_j \leq m$ $(1 \leq j \leq k-1)$.*

J 34. *Determine for which positive integers k, the set*

$$X = \{1990, 1990 + 1, 1990 + 2, \cdots, 1990 + k\}$$

can be partitioned into two disjoint subsets A and B such that the sum of the elements of A is equal to the sum of the elements of B.

J 35. *Prove that $n \geq 3$ be a prime number and $a_1 < a_2 < \cdots < a_n$ be integers. Prove that $a_1, \cdots, a_n$ is an arithmetic progression if and only if there exists a partition of $\{0, 1, 2, \cdots\}$ into classes $A_1, A_2, \cdots, A_n$ such that*

$$a_1 + A_1 = a_2 + A_2 = \cdots = a_n + A_n,$$

where $x + A$ denotes the set $\{x + a | a \in A\}$.

J 36. *Let a and b be non-negative integers such that $ab \geq c^2$ where c is an integer. Prove that there is a positive integer n and integers $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ such that*

$$x_1{}^2 + \cdots + x_n{}^2 = a, y_1{}^2 + \cdots + y_n{}^2 = b, x_1 y_1 + \cdots + x_n y_n = c$$

J 37. *Let n, k be positive integers such that n is not divisible by 3 and k is greater or equal to n. Prove that there exists a positive integer m which is divisible by n and the sum of its digits in the decimal representation is k.*

J 38. *Prove that for every real number M there exists an infinite arithmetical progression such that*

   ○ *each term is a positive integer and the common difference is not divisible by 10.*
   ○ *the sum of digits of each term exceeds M.*

J 39. *Find the smallest positive integer n, for which there exist n different positive integers $a_1, a_2, \cdots, a_n$ satifying the conditions :*

   a) the smallest common multiple of a_1, a_2, $\cdots$, a_n is 1985;
   b) for each $i, j \in \{1, 2, \cdots, n\}$, the numbers a_i and a_j have a common divisor;
   c) the product $a_1 a_2 \cdots a_n$ is a perfect square and is divisible by 243.

Find all n-tuples $(a_1, \cdots, a_n)$, satisfying a), b), and c).

J 40. *Let X be a non-empty set of positive integers which satisfies the following :*

   (a) If $x \in X$, then $4x \in X$.
   (b) If $x \in X$, then $[\sqrt{x}] \in X$.

Prove that $X = \mathbf{N}$.

J 41. *Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.*

J 42. *Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \ldots + 1/a_n = 1$. Determine whether N_{10} is even or odd.*

13. ADDITIVE NUMBER THEORY

On Ramanujan, G. H. Hardy Said : I remember once going to see him when he was lying ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. "No," he replied,

"it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."

K 1. *Show that any integer can be expressed as a sum of two squares and a cube.*

K 2. *Show that every integer greater than 1 can be written as a sum of two square-free integers.*

K 3. *Prove that every integer $n \geq 12$ is the sum of two composite numbers.*

K 4. *Prove that any positive integer can be represented as an aggregate of different powers of 3, the terms in the aggregate being combined by the signs $+$ and $-$ appropriately chosen.*

K 5. *The integer 9 can be written as a sum of two consecutive integers : 9=4+5 ; moreover it can be written as a sum of (more than one) consecutive positive integers in exactly two ways, namely 9=4+5= 2+3+4. Is there an integer which can be written as a sum of 1990 consecutive integers and which can be written as a sum of (more than one) consecutive integers in exactly 1990 ways ?*

K 6. *For each positive integer n, $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive squares.*

> *(a) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.*
> *(b) Find an integer n such that $S(n) = n^2 - 14$.*
> *(c) Prove that there are infintely many integers n such that $S(n) = n^2 - 14$.*

K 7. *For each positive integer n , let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways:*

$$4; 2 + 2; 2 + 1 + 1; 1 + 1 + 1 + 1.$$

Prove that, for any integer $n \geq 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$

K 8. *The positive function $p(n)$ is defined as the number of ways that the positive integer n can be written as a sum of positive integers.* [17] *Show that, for all $n > 1$,*

$$2^{[\sqrt{n}]} < p(n) < n^{3[\sqrt{n}]}.$$

K 9. *Let $a_1 = 1$, $a_2 = 2$, be the sequence of positive integers of the form $2^\alpha 3^\beta$, where α and β are nonnegative integers. Prove that every positive integer is exprssible in the form*

$$a_{i_1} + a_{i_2} + \cdots + a_{i_n},$$

where no summand is a multiple of any other.

K 10. *Let n be a non-negative integer. Find the non-negative integers a, b, c, d such that*

$$a^2 + b^2 + c^2 + d^2 = 7 \cdot 4^n.$$

K 11. *Find all integers $m > 1$ such that m^3 is a sum of m squares of consecutive integers.*

K 12. *A positive integer n is a square-free integer if there is no prime p such that $p^2 | n$. Show that every integer greater than 1 can be written as a sum of two square-free integers.*

K 13. *Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers.*

K 14. *(Jacobsthal) Let p be a prime number of the form $4k + 1$. Suppose that r is a quadratic residue of p and that s is a quadratic nonresidue of p. Show that $p = a^2 + b^2$, where*

$$a = \frac{1}{2} \sum_{i=1}^{p-1} \left(\frac{i(i^2 - r)}{p} \right), b = \frac{1}{2} \sum_{i=1}^{p-1} \left(\frac{i(i^2 - s)}{p} \right).$$

Here, $\left(\frac{k}{p} \right)$ denotes the Legendre Symbol.

K 15. *Let p be a prime with $p \equiv 1 \pmod 4$. Let a be the unique integer such that*

$$p = a^2 + b^2, a \equiv -1 \pmod 4, b \equiv 0 \pmod 2$$

Prove that

$$\sum_{i=0}^{p-1} \left(\frac{i^3 + 6i^2 + i}{p} \right) = 2 \left(\frac{2}{p} \right) a.$$

K 16. *Let n be an integer of the form $a^2 + b^2$, where a and b are relatively prime integers and such that if p is a prime, $p \leq \sqrt{n}$, then p divides ab. Determine all such n.*

---

[17] For example, $5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$, and so $p(5) = 7$.

K 17. *If an integer n is such that $7n$ is the form $a^2 + 3b^2$, prove that n is also of that form.*

K 18. *Let A be the set of positive integers represented by the form $a^2 + 2b^2$, where a, b are integers and $b \neq 0$. Show that p is a prime number and $p^2 \in A$, then $p \in A$.*

K 19. *Show that an integer can be expressed as the difference of two squares if and only if it is not of the form $4k + 2 (k \in \mathbf{Z})$.*

K 20. *Show that there are infinitely many positive integers which cannot be expressed as the sum of squares.*

K 21. *Show that any integer can be expressed as the form $a^2 + b^2 - c^2$, where $a, b, c \in \mathbf{Z}$.*

K 22. *Let a and b be positive integers with $gcd(a, b) = 1$. Show that every integer greater than $ab - a - b$ can be expressed in the form $ax + by$, where $x, y \in \mathbf{N}_0$.*

K 23. *Let a, b and c be positive integers, no two of which have a common divisor greater than 1. Show that $2abc - ab - bc - ca$ is the largest integer which cannot be expressed in the form $xbc + yca + zab$, where $x, y, z \in \mathbf{N}_0$*

K 24. *Determine, with proof, the largest number which is the product of positive integers whose sum is 1976.*

K 25. *(Zeckendorf) Any positive integer can be represented as a sum of Fibonacci numbers, no two of which are consecutive.*

K 26. *Show that the set of positive integers which cannot be represented as a sum of distinct perfect squares is finite.*

K 27. *Let $a_1, a_2, a_3, \cdots$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_l$, where i, j, and k are not necessarily distinct. Determine a_{1998}.*

K 28. *A finite sequence of integers $a_0, a_1, \cdots, a_n$ is called quadratic if for each $i \in \{1, 2, \cdots, n\}$ we have the equality $|a_i - a_{i-1}| = i^2$.*

> *(a) Prove that for any two integers b and c, there exists a natural number n and a quadratic sequence with $a_0 = b$ and $a_n = c$.*
> *(b) Find the smallest natural number n for which there exists a quadratic sequence with $a_0 = 0$ and $a_n = 1996$.*

K 29. *A composite positive integer is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, z positive integers.*

14. The Geometry of Numbers

Srinivasa Aiyangar Ramanujan said   *"An equation means nothing to me unless it expresses a thought of God."*

L 1. *Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.*

L 2. *Prove no three lattice points in the plane form an equilateral triangle.*

L 3. *The sides of a polygon with 1994 sides are $a_i = \sqrt{i^2 + 4}$ $(i = 1, 2, \cdots, 1994)$. Prove that its vertices are not all on lattice points.*

L 4. *A triangle has lattice points as vertices and contains no other lattice points. Prove that its area is $\frac{1}{2}$.*

L 5. *Let R be a convex region[18] symmetrical about the origin with area greater than 4. Then R must contain a lattice point[19] different form the origin.*

L 6. *Show that the number $r(n)$ of representations of n as a sum of two squares has average value π, that is*

$$\frac{1}{n} \sum_{m=1}^{n} r(m) \to \pi \text{ as } n \to \infty.$$

L 7. *Prove that on a coordinate plane it is impossible to draw a closed broken line such that (i) coordinates of each vertex are ratoinal, (ii) the length of its every edge is equal to 1, (iii) the line has an odd number of vertices.*

---

[18]For any two points of R, their midpoint also lies in R.

[19]A point with integral coordinates

15. Miscellaneous Problems

Mathematics is not yet ready for such problems. Paul Erdös

M 1. *Let*

$$
\begin{array}{cccc}
a_{1,1} & a_{1,2} & a_{1,3} & \ldots \\
a_{2,1} & a_{2,2} & a_{2,3} & \ldots \\
a_{3,1} & a_{3,2} & a_{3,3} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m, n).

M 2. *The digital sum of a natural number n is denoted by $S(n)$. Prove that $S(8n) \geq \frac{1}{8} S(n)$ for each n.*

M 3. *Let p be an odd prime. Determine positive integers x and y for which $x \leq y$ and $\sqrt{2p} - \sqrt{x} - \sqrt{y}$ is nonnegative and as small as possible.*

M 4. *Let $\alpha(n)$ be the number of digits equal to one in the dyadic representation of a positive integer n. Prove that*

(a) the inequality $\alpha(n^2) \leq \frac{1}{2}\alpha(n)(1 + \alpha(n))$ holds,
(b) the above inequality is equality for infinitely many positive integers, and
(c) there exists a sequence $\{n_i\}$ such that $\frac{\alpha(n_i^2)}{\alpha(n_i)} \to 0$ as $i \to \infty$.

M 5. *Show that if a and b are positive integers, then*

$$
\left(a + \frac{1}{2} \right)^n + \left(b + \frac{1}{2} \right)^n
$$

is an integer for only finitely many positive integer n.

M 6. *If x is a real number such that $x^2 - x$ is an integer, and for some $n \geq 3$, $x^n - x$ is also an integer, prove that x is an integer.*

M 7. *Suppose that x and y are complex numbers such that*

$$
\frac{x^n - y^n}{x - y}
$$

are integers for some four consecutive positive integers n. Prove that it is an integer for all positive integers n.

M 8. *Determine the maximum value of $m^2 + n^2$, where m and n are integers satisfying $m, n \in \{1, 2, ..., 1981\}$ and $(n^2 - mn - m^2)^2 = 1$.*

M 9. *Denote by S the set of all primes p such that the decimal representation of $\frac{1}{p}$ has the fundamental period of divisible by 3. For every $p \in S$ such that $\frac{1}{p}$ has the fundamental period $3r$ one may write*

$$\frac{1}{p} = 0.a_1 a_2 \cdots a_{3r} a_1 a_2 \cdots a_{3r} \cdots ,$$

where $r = r(p)$; for every $p \in S$ and every integer $k \geq 1$ define $f(k, p)$ by

$$f(k, p) = a_k + a_{k+r(p)} + a_{k+2r(p)}.$$

a) Prove that S is finite.

b) Find the hightst value of $f(k, p)$ for $k \geq 1$ and $p \in S$.

M 10. *Determine all pairs (a, b) of real numbers such that $a[bn] = b[an]$ for all positive integer n. (Note that $[x]$ denotes the greatest integer less than or equal to x.)*

M 11. *Let n be a positive integer that is not a perfect cube. Define real numbers a, b, c by*

$$a = n^{\frac{1}{3}}, b = \frac{1}{a - [a]}, c = \frac{1}{b - [b]},$$

where $[x]$ denotes the integer part of x. Prove that there are infinitely many such integers n with the property that there exist integers r, s, t, not all zero, such that $ra + sb + tc = 0$.

M 12. *Find, with proof, the number of positive integers whose base-n representation consists of distinct digits with the property that, except for the leftmost digit, every digit differs by ± 1 from some digit further to the left.*

16. Sources

The only way to learn Mathematics is to do Mathematics. Paul Halmos

Divisibility Theory I

A 1 (Ksk). *http://www-math.mit.edu/∼kedlaya/papers*

A 2. *Poland 2001*

A 3. *1969 Eötvös-Kürschák Mathematics Competition*

A 4. *IMO 1988/6*

A 5.

A 6 (IHH, pp. 211).

A 7. *Belarus 1994*

A 8 (UmDz pp.13). [20] *Unused Problem for the Balkan Mathematical Olympiad*

A 9. *Putnam 1972*

A 10. *IMO Long List 1985 P (RO2)*

A 11. *IMO 2000/5*

A 12. *Bulgaria 1998*

A 13. *Slovenia 1994*

A 14. *Japan 1999*

A 15. *IMO Short List 1998*

A 16.

A 17. *IMO 1974/3*

A 18 (GhEw pp.104).

A 19. *Putnam 1996*

A 20.

A 21. *IMO 1972/3*

A 22.

A 23. *Putnam 2000*

A 24. *Amer. Math. Monthly, Problem E2623, Proposed by Ivan Niven*

A 25.

---

[20]See the References

A 26. *Kazakhstan 1998*

A 27. *IMO 1979/1*

A 28. *IMO Short List 1996*

A 29. *IMO Short List 2002 N3*

A 30. *IMO Short List 2001 N4*

A 31. *Australia 2002*

A 32. *Poland 2002*

A 33. *Bosnia and Herzegovina 2002*

A 34. *Math. Magazine, Problem 1438, Proposed by David M. Bloom*

A 35.

A 36 (PJ pp.110)**.** *UC Berkeley Preliminary Exam 1990*

A 37 (Ae pp.137)**.**

A 38.

A 39. *Iran 1994*

A 40. *Germany 1982*

A 41. *IMO Short List 1997*

A 42. *Romania 1995, Proposed by I. Cucurezeanu*

A 43 (IMV, pp. 15)**.**

A 44 (IMV, pp. 15)**.**

A 45. *IMO Short List 2001 N1*

A 46. *Germany 2000*

A 47. *IMO 1984/6*

A 48. *IMO 1986/1*

A 49. *Iran 2001*

A 50. *IMO ShortList 1993 IND5*

A 51. *Czech and Slovak Mathematical Olympiad 1999*

A 52. *Romania 1987, Proposed by L. Panaitopol*

A 53. *Bulgaria 1995*

A 54. *APMO 2003/3*

A 55 (AaJc, pp. 250)**.**

A 56. *Putnam 1988/B6*

Divisibility Theory II

B 1. *IMO 1990/3 (ROM5)*

B 2. *IMO 1999/4*

B 3. *Ha Duy Hung : 2003/09/14* [21]

B 4. *APMO 1997/2*

B 5. *APMC 2002*

B 6. *IMO 1992/1*

B 7. *Russia 2001*

B 8. *IMO 1994/4*

B 9. *IMO 2003/2*

B 10. *IMO 2002/3*

B 11. *IMO Short List 2000 N4*

B 12 (Tma pp. 73)**.**

B 13 (ElCr pp. 11)**.**

B 14. *Romania 1998*

B 15.

B 16.

B 17. *Turkey 1994*

B 18. *Mediterranean Mathematics Competition 2002*

B 19. *IMO 1998/4*

B 20. *IMO 1977/5*

B 21. *APMO 1998*

B 22.

B 23. *Iran 1998*

B 24. *Balkan Mathemaitical Olympiad for Juniors 2002*

B 25. *Ireland 1998*

B 26. *Iran 1999*

B 27. *Belarus 1999, Proposed by I. Voronovich*

B 28. *IMO 2002/4*

B 29. *Singapore 1997*

---

[21]Contributor, Date

B 30. *Belarus 1998, Proposed by E. Barabanov, I. Voronovich*

B 31. *IMO 1960/1*

B 32. *IMO 1975/4*

B 33. *IMO Short List 1994 N7*

B 34. *IMO ShortList 1990 ROM1*

B 35. *IMO Long List 1985 (TR5)*

B 36. *IMO Long List 1987*

B 37. *IMO Short List 1986 P10 (NL1)*

B 38. *British Mathematical Olympiad 2003, 2-1*

B 39. *APMO 2002/2*

B 40. *Putnam 1997/B3*

Arithmetic in Z_n
1. Primitive Roots

C 1.

C 2 (IMV, pp. 90)**.**

C 3. *Math. Magazine, Problem 1419, Proposed by William P. Wardlaw*

C 4 (Km, Problems Sheet 3-9)**.**

C 5 (Km, Problems Sheet 3-11)**.**

C 6 (AaJc, pp. 178)**.**

C 7 (AaJc, pp. 181)**.**

2. Qudratic Residues

C 8. *IMO 1996/4*

C 9 (IHH pp.147)**.**

C 10 (IMV, pp. 72)**.**

C 11. *Putnam 1991/B5*

3. Congruences

C 12 (Tma, pp. 127)**.**

C 13. *Putnam 1991/B4*

C 14.

C 15. *Math. Magazine, Problem 1494, Proposed by Emeric Deutsch and Ira M. Gessel*

C 16. *Putnam 1997/B5*

C 17. *USA 1991*

C 18.

C 19.

C 20. *APMO 1991/4*

C 21 (AaJc, pp. 139)**.**

C 22. *Putnam 1986/B3*

C 23. *Purdue POW, Spring 2003 Series/5*

C 24. *Turkey 1993*

C 25. *IMO Short List 2000 N1*

C 26. *Amer. Math. Monthly, Problem ???, Proposed by M. S. Klamkin and A. Liu*

C 27. *Turkey 2000*

Primes and Composite Numbers

D 1 (DfAk, pp. 50)**.** *Leningrad Mathematical Olympiad 1991, Here is the hint : $2z^2 = 3xy \implies x^3 + y^3 + z^3 = x^3 + y^3 + z^3 - 3xyz$!! In fact, It's known that $512^3 + 675^3 + 720^3 = 229 \cdot 720 \cdot 7621$.*

D 2.

D 3. *IMO Short List 1990 USS1*

D 4 (Tma, pp. 101)**.**

D 5. *Math. Magazine, Problem Q789, Proposed by Norman Schaumberger*

D 6. *IMO 2001/6*

D 7.

D 8. *Balkan Mathematical Olympiad 1989*

D 9. *IMO 1987/6*

D 10. *Math. Magazine, Problem 1392, Proposed by George Andrews*

D 11 (AaJc pp.212)**.**

D 12.

D 13 (AaJc pp.176)**.**

D 14 (GjJj pp.140)**.**

D 15 (Ns pp.176)**.**

D 16. *IMO 2003/6*

D 17. *Yugoslavia 2001*

D 18.

D 19 (Er pp.10). *Eötvös Competition 1896*

D 20. *Math. Magazine, Problem 1404, Proposed by H. Gauchmen and I. Rosenholtz*

D 21. *Math. Magazine, Problem Q614, Proposed by Rod Cooper*

D 22.

D 23. *IMO Short List 1992 P16*

D 24. *Math. Magazine, Problem Q684, Proposed by Noam Elkies*

D 25. *IMO 1989/5*

D 26. *USA 1982*

D 27 (GjJj pp.36).

D 28. *IMO Short List 1996 N1*

D 29 (Tma, pp. 102).

D 30 (Tma, pp. 128).

D 31 (DfAk, pp. 9). *Leningrad Mathematical Olympiad 1987*

Rational and Irrational Numbers

E 1 (Vvp, pp. 40).

E 2 (AI, pp. 106). *For a proof, See [Km].*

E 3 (AI, pp. 116). *For a proof, See [KaMr].*

E 4. *The Grosman Meomorial Mathematical Olympiad 1999*

E 5. *Belarus 2002*

E 6. *Belarus 2001*

E 7. *Putnam 1980*

E 8.

E 9.

E 10.

E 11.

E 12.

E 13.

E 14. *Austria 2002*

E 15. *Romania 2001, Proposed by Marius Ghergu*

E 16. *USA 1973*

E 17. *IMO 1987/5*

E 18. *IMO ShortList 1991 P19 (IRE 5)*

E 19. *IMO Long List 1967 P20 (DDR)*

E 20. *IMO Long List 1967 (GB)*

E 21. *Poland 2002*

E 22.

E 23. *IMO Short List 1999*

E 24. *Berkeley Math Circle Monthly Contest 1999-2000*

E 25. *Iran 1998*

E 26. *APMO 1994/5*

E 27.

E 28. *Putnam 1995*

Diophantine Equations I

F 1. *AIME 1989/9*

F 2. *UC Berkeley Preliminary Exam 1983*

F 3. *Taiwan 1998*

F 4.

F 5.

F 6.

F 7. *Poland 2002*

F 8. *Bulgaria 1999*

F 9. *Ireland 1995*

F 10. *Putnam 2001*

F 11. *India 1998*

F 12.

F 13. *Italy 1994*

F 14. *Canada 1991*

F 15. *Balkan Mathematical Olympaid 1998*

F 16. *Poland 2003*

F 17. *Vietnam 2002*

F 18. *Romania 1995, Proposed by M. Becheanu*

F 19 (UmDz pp.14)**.** *Unused Problem for the Balkan Mathematical Olympiad*

F 20. *APMO 1993/4*

F 21. *APMO 1989/2*

F 22. *Poland 1998*

F 23.

F 24. *IMO 1982/4*

F 25. *IMO Short List 2002 N1*

F 26. *Ukraine 2002*

F 27. *IMO Short List 2000 N5*

F 28. *IMO Short List 1997 N6*

F 29. *Belarus 2000*

F 30. *IMO Short List 1993 GEO3*

F 31 (Eb1, pp. 19)**.** *Amer. Math. Monthly 61(1954), 126; 62(1955), 263*

F 32. *IMO Long List 1987 (Romaina)*

F 33. *IMO Long List 1967 P (PL)*

F 34. *IMO Long List 1985 (SE1)*

F 35. *IMO Long List 1985 (TR3)*

F 36. *Italy 1996*

Diophantine Equations II

G 1. *British Mathematical Olympiad 2002/2003, 1-1*

G 2. *Latvia 1995*

G 3. *Belarus 1999, Proposed by S. Shikh*

G 4. *IMO Long List 1987 (Greece)*

G 5 (MaGz pp.13-16)**.**

G 6. *India 1995*

G 7.

G 8 (Rdc pp.51)**.**

G 9. *Austria 2002*

G 10. *Austria 2002*

G 11. *IMO Short List 2002 N4*

G 12. *IMO Short List 2001 N2*

G 13. *Hong Kong 2000*

G 14. *Belarus 1999*

G 15. *Baltic Way 2002*

G 16.

G 17. *IMO 1997/5*

G 18. *Belarus 2000*

G 19. *Iran 1998*

G 20. *Belarus 1996*

G 21. *Korea 1998*

G 22. *Hungary 1998*

G 23 (Eb2, pp. 20). *Q657, Math. Magazine 52(1979), 47, 55*

G 24. *IMO Short List 1980 (GB)*

G 25 (DfAk, pp. 18). *Leningrad Mathematical Olympiad 1988*

G 26. *British Mathematical Olympiad 2002/2003, 1-5*

G 27. *British Mathematical Olympiad 2000, 2-3*

Functions in Number Theory

H 1. *IMO ShortList 1991 P20 (IRE 3)*

H 2. *APMO 1993/2*

H 3.

H 4 (Tma, pp.175).

H 5 (Rh pp.104). *Quantum, Problem M59, Contributed by B. Martynov*

H 6.

H 7 (Rdc pp.36).

H 8. *Russia 1998*

H 9. *Canada 1999*

H 10. *IMO 1998/3*

H 11. *Taiwan 1998*

H 12. *Amer. Math. Monthly, Problem 10346, Proposed by David Doster*

H 13. *Korea 2000*

H 14 (IHH pp.142)**.**

H 15.

H 16.

H 17.

H 18. *Amer. Math. Monthly, Problem 10837, Proposed by Hojoo Lee*

H 19. *IMO Short List 2000 N2*

H 20. *IMO Short List 1983 (Belgium)*

H 21. *Singapore 1996*

H 22. *Turkey 1995*

H 23. *Belarus 1999, Proposed by D. Bazylev*

H 24 (Km, Problems Sheet 1-11)**.**

Sequences of Integers

I 1. *Slovenia 1995*

I 2.

I 3. *USA 1993*

I 4. *IMO Long List 1987 (GB)*

I 5. *IMO Long List 1985 (RO3)*

I 6. *IMO Short List 1991 P14 (USS 2)*

I 7. *Canada 1998*

I 8. *Vietnam 1999*

I 9.

I 10. *IMO Short List 1988*

I 11. *Amer. Math. Monthly, Problem E2619, Proposed by Thomas C. Brown*

I 12. *Putnam 1983*

I 13. *IMO Short List 1989*

I 14. *IMO 1967/4*

I 15. *Amer. Math. Monthly, Problem E2510, Proposed by Saul Singer*

I 16. *Putnam 1999*

I 17. *United Kingdom 1998*

I 18. *Serbia 1998*

I 19. *United Kingdom 2002*

I 20 (KiMr pp. 233)**.**

I 21.

I 22.

I 23. *Putnam 1983*

I 24. *Pi Mu Epsilon Journal, Problem 339, Proposed by Paul Erdös*

I 25. *Iran 1998*

I 26. *APMC 1999*

I 27. *IMO 1991/2*

I 28. *IMO Short List 1993*

I 29. *IMO Short List 2001 N3*

I 30. *IMO Short List 1999 N3*

I 31. *Poland 2002*

I 32 (Nv pp.58)**.**

I 33 (Nv pp.74)**.**

I 34 (Nv pp.75)**.**

I 35 (Eb1 pp.21)**.**

I 36. *Math. Magazine, Problem 1390, Proposed by J. F. Stephany*

I 37 (Ae pp.228)**.**

I 38. *IMO Short List 1994 N6*

I 39. *IMO Short List 1990 HUN1*

I 40. *Taiwan 1996*

I 41. *USA 2002*

I 42. *Putnam 1985/A4*

I 43. *Putnam 1993/A6*

Combinatorial Number Theory

J 1. *IMO Long List 1985 (PL2)*

J 2. *IMO 1971/3*

J 3. *Berkeley Math Circle Monthly Contest 1999-2000*

J 4. *India 1998*

J 5. *USA 2001*

J 6. *USA 1998*

J 7. *Romania 1998*

J 8. *India 1998*

J 9. *Romania 1998*

J 10. *IMO 1991/3*

J 11. *Math. Magazine, Problem 1466, Proposed by David M. Bloom*

J 12. *Germany 1996*

J 13. *IMO 1985/2*

J 14. *Romania 1997, Proposed by Marian Andronache and Ion Savu*

J 15. *IMO 1983/5*

J 16. *IMO Short List 2001*

J 17. *Turkey 1996*

J 18. *IMO 1995/6*

J 19. *IMO Short List 2002 N5*

J 20. *IMO Short List 1998 P16*

J 21. *IMO Short List 1998 P17*

J 22. *IMO ShortList 1998 P20*

J 23. *IMO Short List 1996*

J 24 (TaZf pp.10)**.** *High-School Mathematics (China) 1994/1*

J 25. *IMO Short List 2002 A6*

J 26. *IMO Short List 2001 C4*

J 27. *IMO Short List 2000 C6*

J 28. *IMO Short List 1999 A4*

J 29. *IMO Short List 1999 C4*

J 30. *Australia 2002*

J 31 (Ae pp.228)**.**

J 32. *IMO ShortList 1991 P24 (IND 2)*

J 33. *IMO Short List 1990 CZE3*

J 34. *IMO Short List 1990 MEX2*

J 35. *USA 2002*

J 36. *IMO Short List 1995*

J 37. *IMO Short List 1999*

J 38. *IMO Short List 1999*

J 39. *Romania 1995*

J 40. *Japan 1990*

J 41. *USA 2003*

J 42. *Putnam 1997/A5*

Additive Number Theory

K 1. *Amer. Math. Monthly, Problem 10426, Proposed by Noam Elkies and Irving Kaplanky*

K 2 (IHH, pp. 474)**.**

K 3 (Tma, pp. 22)**.**

K 4 (Rdc pp.24)**.**

K 5. *IMO Short List 1990 AUS3*

K 6. *IMO 1992/6*

K 7. *IMO 1997/6*

K 8 (Hua pp.199)**.**

K 9. *Math. Magazine, Problem Q814, Proposed by Paul Erdös*

K 10. *Romania 2001, Proposed by Laurentiu Panaitopol*

K 11. *Amer. Math. Monthly, Problem E3064, Proposed by Ion Cucurezeanu*

K 12.

K 13. *Putnam 2000*

K 14.

K 15. *Amer. Math. Monthly, Problem 2760, Proposed by Kenneth S. Williams*

K 16. *APMO 1994/3*

K 17. *India 1998*

K 18. *Romania 1997, Proposed by Marcel Tena*

K 19.

K 20.

K 21.

K 22.

K 23. *IMO 1983/3*

K 24. *IMO 1976/4*

K 25.

K 26. *IMO Short List 2000 N6*

K 27. *IMO Short List 1998 P21*

K 28. *IMO Short List 1996 N3*

K 29. *Putnam 1988/B1*

The Geometry of Numbers

L 1. *Putnam 1993/B5*

L 2.

L 3. *Israel 1994*

L 4.

L 5 (Hua pp.535)**.**

L 6 (GjJj pp.215)**.**

L 7. *IMO Short List 1990 USS3*

Miscellaneous Problems

M 1. *Putnam 1985/B3*

M 2. *Latvia 1995*

M 3. *IMO Short List 1992 P17*

M 4.

M 5 (Ns pp.4)**.**

M 6. *Ireland 1998*

M 7. *Amer. Math. Monthly, Problem E2998, Proposed by Clark Kimberling*

M 8. *IMO 1981/3*

M 9. *IMO Short List 1999 N4*

M 10. *IMO Short List 1998 P15*

M 11. *IMO Short List 2002 A5*

M 12. *USA 1990*

17. References

AaJc Andrew Adler, John E. Coury, *The Theory of Numbers - A Text and Source Book of Problems*, John and Bartlet Publishers

Ae Arthur Engel, *Problem-Solving Strategies*, Springer-Verlag

AI A. N. Parshin, I. R. Shafarevich, *Number Theory IV - Encyclopaedia of Mathematical Sciences, Volume 44*, Spinger-Verlag

DfAk Dmitry Fomin, Alexey Kirichenko, *Leningrad Mathematical Olympiads 1987-1991*, MathPro Press

Eb1 Edward J. Barbeau, *Pell's Equation*, Springer-Verlag

Eb2 Edward J. Barbeau, *Power Play*, MAA

ElCr Edward Lozansky, Cecil Rousseau, *Winning Solutions*, Springer-Verlag

Er Elvira Rapaport, *Hungarian Problem Book I*, MAA

GhEw G. H. Hardy, E. M. Wright, *An Introduction to the theory of numbers*, Fifth Edition, Oxford University Press

GjJj Gareth A. Jones, J. Mary Jones, *Elementary Number Theory*, Springer-Verlag

Hs *The MacTutor History of Mathematics Archive*, http://www-gap.dcs.st-and.ac.uk/~history/index.html

Hua Hua Loo Keng, *Introduction to Number Theory*, Springer-Verlag

IHH Ivan Niven, Herbert S. Zuckerman, Hugh L. Montogomery, *An Introduction to the Theory of Numbers*, Fifth Edition, John Wiley and Sons, Inc.

IMV I. M. Vinogradov, *An Introduction to The Theory of Numbers*, Pergamon Press

KaMr K. Alladi, M. Robinson, *On certain irrational values of the logarithm.* Lect. Notes Math. 751, 1-9

KiMr Kenneth Ireland, Michael Rosen, *A Classical Introduction to Modern Number Theory*, Springer-Verlag

Km K. Mahler, *On the approximation of π*, Proc. Kon. Akad. Wetensch. A., Vol 56, 30-42

Km Keith Matthews, *MP313 Number Theory III, Semester 2, 1999*, http://www.numbertheory.org/courses/MP313/index.html

Ksk Kiran S. Kedlaya, *When is (xy+1)(yz+1)(zx+1) a square?*, Math. Magazine, Vol 71 (1998), 61-63

MaGz Martin Aigner, Günter M. Ziegler, *Proofs from THE BOOK*, Springer-Verlag

Nv Nicolai N. Vorobiev, *Fibonacci Numbers*, Birkhäuser

PJ Paulo Ney de Souza, Jorge-Nuno Silva, *Berkeley Problems in Mathematics*, Second Edition, Springer-Verlag

Pp *Purdue Univ. POW*, http://www.math.purdue.edu/academics/pow

Rc Robin Chapman, *A Polynomial Taking Integer Values*, Math. Magazine, Vol 69 (1996), 121

Rdc Robert D. Carmichael, *The Theory of Numbers*

Rh   R. Honsberger, *Mathematical Chestnuts from Around the World*, MAA

Rs   Rizos Sakellariou, *On Equal Sums of Like Powers (Euler's Conjecture)*, http://www.cs.man.ac.uk/cnc/EqualSums/equalsums.html

TaZf Titu Andreescu, Zuming Feng, *102 Combinatorial Problems From the Training of the USA IMO Team*, Birkhäuser

Tma Tom M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag

UmDz Uroš Milutinović, Darko Z̃ubrinić, *Balkanian Mathematical Olmpiades 1984-1991*

Vvp V. V. Prasolov, *Problems and Theorems in Linear Algebra*, AMS

Wlp Kiran S. Kedlaya, *William Lowell Putnam Mathematics Competition Archive*, http://www.unl.edu/amc/a-activities/a7-problems/putnam/-pindex.html

Appendix

How Many Problems Are In This Book ?

```
Divisibility Theory I
```
: 56 problems
```
Divisibility Theory II
```
: 40 problems
```
Arithmetic in
```
Z_n : 27 problems
```
Primes and Composite Numbers
```
: 31 problems
```
Rational and Irrational Numbers
```
: 28 problems
```
Diophantine Equations I
```
: 36 problems
```
Diophantine Equations II
```
: 27 problems
```
Functions in Number Theory
```
: 24 problems
```
Sequences of Integers
```
: 43 problems
```
Combinatorial Number Theory
```
: 42 problems
```
Additive Number Theory
```
: 29 problems
```
The Geometry of Numbers
```
: 7 problems
```
Miscellaneous Problems
```
: 12 problems

402 PROBLEMS