

Lectures on Integer Partitions

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Preface

These lectures were delivered at the University of Victoria, Victoria, B.C., Canada, in June of 2000, under the auspices of the Pacific Institute for the Mathematical Sciences. My original intent was to describe the sequence of developments which began in the 1980's and has led to a unified and automated approach to finding partition bijections. These developments, embodied in the sequence [6, 17, 9, 20, 15, 21] of six papers, in fact form much of the content of these notes, but it seemed desirable to preface them with some general background on the theory of partitions, and I could not resist ending with the development in [3], which concerns integer partitions in a wholly different way.

The lecture notes were recorded by Joe Sawada, with such care that only a minimal buffing and polishing was necessary to get them into this form. My thanks go to Frank Ruskey, Florin Diacu and Irina Gavrilova for their hospitality in Victoria and for facilitating this work, and to Carla Savage for a number of helpful suggestions that improved the manuscript.

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July 12, 2000

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1 Overview

What I'd like to do in these lectures is to give, first, a review of the classical theory of integer partitions, and then to discuss some more recent developments. The latter will revolve around a chain of six papers, published since 1980, by Garsia-Milne, Jeff Remmel, Basil Gordon, Kathy O'Hara, and myself. In these papers what emerges is a unified and automated method for dealing with a large class of partition identities.

By a partition identity I will mean a theorem of the form "there are the same number of partitions of n such that . . . as there are such that" A great deal of human ingenuity has been expended on finding bijective and analytical proofs of such identities over the years, but, as with some other parts of mathematics, computers can now produce these bijections by themselves. What's more, it seems that what the computers discover are the very same bijections that we humans had so proudly been discovering for all of those years.

But before I get to those matters, let's discuss the introductory theory of integer partitions for a while. To do that effectively will require generating functions. Now I realize that many people, when they see a generating function coming in their direction, will cross to the other side of the street to avoid it. But I do hope that the extraordinary power of generating functions in the subject of integer partitions will help to make some converts.

These lectures are intended to be accessible to graduate students in mathematics and computer science.

2 Basic Generating Functions

Consider the identity $A = B$, where A and B count two different sets of objects. How can we prove such an identity? One approach is to count the elements in A and show that it is the same as number of elements in B . Another approach is to find a bijection between the two sets A and B . The traditional example that contrasts these two approaches is the one that considers the problem of showing that the number of people in an auditorium is the same as the number of seats. Following the first approach, we would count the people in the room and then count the seats in the room. But, following the second approach, we would only need to ask everyone to sit down, and see if there are any seats or people left over.

When dealing with integer partition identities, sometimes it is easier to use the first approach (generating functions), sometimes it is easier to use the second approach (bijective proofs), and sometimes both are equally easy or difficult. In the following pages we will see examples of all three situations.

What is an integer partition? If n is a positive integer, then a *partition* of n is a nonincreasing sequence of positive integers p_1, p_2, \dots, p_k whose sum is n . Each p_i is called a *part* of the partition. We let the function $p(n)$ denote the number of partitions of the integer n .

As an example, $p(5) = 7$, and here are all 7 of the partitions of the integer $n = 5$:

$$\begin{aligned}
 5 &= 5 \\
 &= 4 + 1 \\
 &= 3 + 2 \\
 &= 3 + 1 + 1 \\
 &= 2 + 2 + 1 \\
 &= 2 + 1 + 1 + 1 \\
 &= 1 + 1 + 1 + 1 + 1
 \end{aligned}$$

We take $p(n) = 0$ for all negative values of n and $p(0)$ is defined to be 1.

Integer partitions were first studied by Euler. For many years one of the most intriguing and difficult questions about them was determining the asymptotic properties of $p(n)$ as n got large. This question was finally answered quite completely by Hardy, Ramanujan, and Rademacher [11, 16] and their result will be discussed below (see p. 13). An example of a problem in the theory of integer partitions that remains unsolved, despite a good deal of effort having been expended on it, is to find a simple criterion for deciding whether $p(n)$ is even or odd. Though values of $p(n)$ have been computed for n into the billions, no pattern has been discovered to date. Many other interesting problems in the theory of partitions remain unsolved today. One of them, for instance, is to find a way to extend the scope of the bijective machinery that will be discussed below in sections 4-9.

The Ferrers diagram of an integer partition gives us a very useful tool for visualizing partitions, and sometimes for proving identities. It is constructed by stacking left-justified rows of cells, where the number of cells in each row corresponds to the size of a part. The first row corresponds to the largest part, the second row corresponds to the second largest part, and so on. As an illustration, the Ferrers diagram for the partition $26 = 10+7+3+2+2+1+1$ is shown in Figure 1.

We mention just briefly the closely related subject of Young tableaux. A Ferrers diagram can be turned into a Young tableau by filling each cell with a unique value from 1 through n such that the values across each row and down each column are increasing. Such a mapping of values to cells can be assigned by repeatedly placing the largest unassigned value into a corner position, i.e., a cell where there are no unassigned cells below or to the right. An introduction to the theory of Young tableaux can be found in [13].

As an example of the use of Ferrers diagrams in partition theory, we prove the following.

THEOREM 1 *The number of partitions of the integer n whose largest part is k is equal to the number of partitions of n with k parts.*

To prove this theorem we stare at a Ferrers diagram and notice that if we interchange the rows and columns we have a 1-1 correspondence between the two kinds of partitions. \square

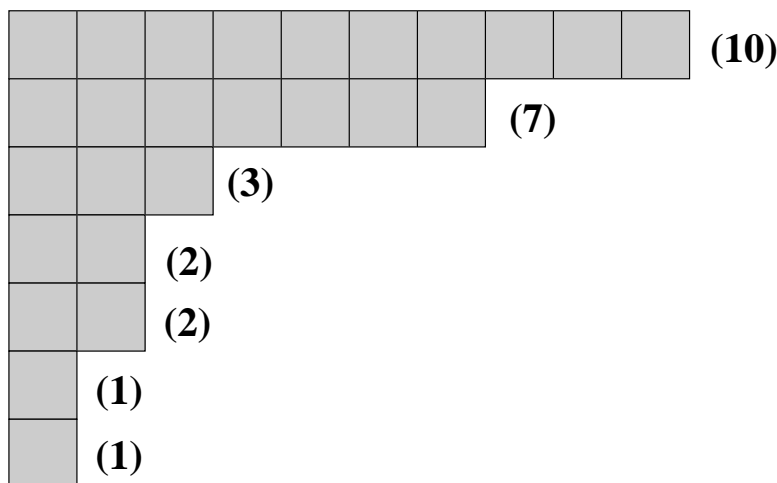


Figure 1: Ferrers diagram for $26 = 10 + 7 + 3 + 2 + 2 + 1 + 1$

We define the function $p(n, k)$ to be the number of partitions of n whose largest part is k (or equivalently, the number of partitions of n with k parts).

We will now derive Euler's generating function for the sequence $\{p(n)\}_{n=0}^{\infty}$. In other words, we are looking for some nice form for the function which gives us $\sum_{n=0}^{\infty} p(n)x^n$. Consider, (or as that word often implies "look out, here comes something from left field"):

$$(1 + x + x^2 + x^3 \cdots)(1 + x^2 + x^4 + x^6 \cdots)(1 + x^3 + x^6 \cdots)(1 + x^4 + x^8 \cdots) \cdots \quad (1)$$

We claim that by expanding this product, we obtain the desired result, namely $\sum_{n=0}^{\infty} p(n)x^n$. It is important to understand why this is true because when we look at several variations, they will be derived in a similar manner. To illustrate, consider the coefficient of x^3 . By choosing x from the first parenthesis, x^2 from the second, and 1 from the remaining parentheses, we obtain a contribution of 1 to the coefficient of x^3 . Similarly, if we choose x^3 from the third parenthesis, and 1 from all others, we will obtain another contribution of 1 to the coefficient of x^3 . So how does this relate to integer partitions?

Let the monomial chosen from the i -th parenthesis $1 + x^i + x^{2i} + x^{3i} \cdots$ in (1) represent the number of times the part i appears in the partition. In particular, if we choose the monomial $x^{c_i i}$ from the i -th parenthesis, then the value i will appear c_i times in the partition. Each selection of monomials makes one contribution to the coefficient of x^n and in general, each contribution must be of the form $x^{1c_1} \cdot x^{2c_2} \cdot x^{3c_3} \cdots = x^{c_1 + 2c_2 + 3c_3 \cdots}$. Thus the coefficient of x^n is the number of ways of writing $n = c_1 + 2c_2 + 3c_3 + \cdots$ where each $c_i \geq 0$. Notice that this is just another way to represent an integer partition. For example, the partition $25 = 6 + 4 + 4 + 3 + 2 + 2 + 2 + 1 + 1$ could be represented by $25 = 1(2) + 2(3) + 3(1) + 4(2) + 5(0) + 6(1)$. Thus, there is a 1-1 correspondence between choosing monomials whose product is x^n out of

the parentheses in (1) and the partitions of the integer n . \square

Now return to the original product in (1), and notice that each term is a geometric series. The product can be written as:

$$\frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots$$

Thinking as a combinatorialist, we are not concerned about whether these series converge, since we consider the powers of x to be merely placeholders. These previous observations lead to Euler's Theorem.

THEOREM 2 (EULER)

$$\mathcal{E}(x) \stackrel{\text{def}}{=} \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots = \sum_{n=0}^{\infty} p(n)x^n.$$

Now let's look at some variations.

EXAMPLE 1

Let $f(n)$ be the number of partitions of n that have no part = 1. Recall that the monomial chosen from the factor $(1 + x + x^2 + x^3 + \cdots)$ indicates the number of 1's in the partition. Since we can only choose 1 from this term, we obtain the following generating function:

$$\begin{aligned} \sum_{n=0}^{\infty} f(n)x^n &= 1 \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots \\ &= \frac{1-x}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots \\ &= (1-x)\mathcal{E}(x). \end{aligned}$$

This generating function yields the following lemma, by matching the coefficients of like powers of x on both sides.

LEMMA 1 *The number of partitions of n with no parts equal to 1 is $p(n) - p(n-1)$.*

As a homework problem, try proving this identity bijectively. This is a general theme that will appear in some examples to come: we prove a partition identity through the use of generating functions, but to get a broader understanding, we attempt to find a bijective proof.

For another homework problem, suppose two sets of positive integers, S and T , are given. What is the generating function for the number of partitions of n whose parts all lie in S , and whose multiplicities of parts all lie in T ?

Let's look at two more examples.

EXAMPLE 2

Consider:

$$(1+x)(1+x^2)(1+x^4)(1+x^8) \cdots = \prod_{j=0}^{\infty} (1+x^{2^j}).$$

This is a bit (pun intended) like Euler's function $\mathcal{E}(x)$, or more like a shadow of it. Notice that $1+x$ is the start of $1+x+x^2 \cdots$, and $1+x^2$ is the start of $1+x^2+x^4 \cdots$, and $1+x^4$ is the start of $1+x^4+x^8 \cdots$, and so forth. Thus, we are counting the partitions of n all of whose parts are powers of 2, and furthermore, each power of 2 may occur at most once. Because each integer n has a unique binary expansion, there is exactly one solution for each n . Thus,

$$\prod_{j=0}^{\infty} (1+x^{2^j}) = 1+x+x^2+x^3 \cdots = \frac{1}{1-x}.$$

This equation can be thought of as the analytical expression of the fact that every positive integer can be uniquely written as a sum of distinct powers of 2.

EXAMPLE 3

Consider:

$$F(x) \stackrel{\text{def}}{=} \prod_{j=1}^{\infty} (1+x^j) = \sum_{n=0}^{\infty} f(n)x^n. \quad (2)$$

What is $f(n)$ counting in this case? More precisely, finish the statement ' $f(n)$ is the number of partitions of n such that ...' (This $f(n)$ counts the partitions of n into distinct parts.)

3 Identities and Asymptotics

Once we have a generating function for an object, the next question is often 'Can we find a recurrence formula?'. Many generating functions which are products, like the generating functions we've seen so far, can be converted to series by using logarithms. Then by differentiating these expressions with respect to x we can often find a recurrence (after some massaging of the resulting expressions, of course).

EXAMPLE 4

For the $F(x)$ defined in (2), write $F(x) = \sum_{n=0}^{\infty} f(n)x^n$ and consider:

$$\log \left\{ \prod_{j=1}^{\infty} (1+x^j) \right\} = \log \left\{ \sum_{n=0}^{\infty} f(n)x^n \right\}.$$

By differentiating we get:

$$\left(\sum_{j=1}^{\infty} \frac{jx^{j-1}}{1+x^j} \right) F = F'$$

To get a recurrence relation for $f(n)$ here, we insert the power series expansions for F , F' , and $1/(1+x^j)$, multiply out the product on the left, and equate coefficients of like powers of x . Try it!

In the same way we can obtain a recurrence relation for $p(n)$ from Euler's product (1). The result is that

$$p(n) = \frac{1}{n} \sum_{k \geq 1} \sigma(k)p(n-k),$$

where $\sigma(k)$ is the sum of the divisors of k . For example:

$$p(5) = \frac{1}{5}(p(4) + 3p(3) + 4p(2) + 7p(1) + 6p(0)) = 5.$$

Now consider the following:

$$\prod_{j=0}^{\infty} \frac{1}{(1-x^{2j+1})} = \sum_{n=0}^{\infty} f(n)x^n.$$

In this case, $f(n)$ counts the partitions of n into odd parts.

THEOREM 3 *The number of partitions of n into distinct parts equals the number of partitions of n into odd parts.*

Let's illustrate this theorem by looking at $n = 5$.

partition	odd	distinct
5	*	*
4+1		*
3+2		*
3+1+1	*	
2+2+1		
2+1+1+1		
1+1+1+1+1	*	

OK, so it works for $n = 5$; now for a proof by generating functions.

$$\begin{aligned} \text{DISTINCT} &= (1+x)(1+x^2)(1+x^3) \cdots \\ &= \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^6}{1-x^3} \cdot \frac{1-x^8}{1-x^4} \cdots \\ &= \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots \\ &= \text{ODD} \end{aligned}$$

That was an example of a very slick proof by generating functions. But there are many people who prefer bijective proofs. In this case what we would need for a bijective proof would be an explicit mapping that associates with every partition into odd parts a partition into distinct parts. The following argument gives such a mapping.

EULER'S BIJECTIVE PROOF: A partition into distinct parts can be written as

$$n = d_1 + d_2 + \cdots + d_k. \quad (3)$$

Each integer d_i can be uniquely expressed as a power of 2 times an odd number. Thus, $n = 2^{a_1}O_1 + 2^{a_2}O_2 + 2^{a_3}O_3 + \cdots + 2^{a_k}O_k$ where each O_i is an odd number. If we now group together the odd numbers we get an expression like:

$$\begin{aligned} n &= (2^{\alpha_1} + 2^{\alpha_2} + \cdots) \cdot 1 + (2^{\beta_1} + 2^{\beta_2} + \cdots) \cdot 3 + (2^{\gamma_1} + 2^{\gamma_2} + \cdots) \cdot 5 + \cdots \\ &= \mu_1 \cdot 1 + \mu_3 \cdot 3 + \mu_5 \cdot 5 \cdots \end{aligned}$$

In each series $(2^{\alpha_1} + 2^{\alpha_2} + \cdots)$, the α_i 's are distinct (why?). Thus the sum is the binary expansion of some μ_j . We now see the partition of n into odd parts that corresponds, under this bijection, to the given partition (3) into distinct parts. It is the partition that contains μ_1 1's, μ_3 3's, etc. \square

To illustrate this proof consider the partition of 5 into distinct parts $5 = 3 + 2$. What is its bijective mate, among the partitions of 5 into odd parts? To answer this we proceed as in the proof above,

$$\begin{aligned} 5 &= 3 + 2 \\ &= 2^0 \cdot 3 + 2^1 \cdot 1 \\ &= 2^1(1) + 2^0(3) \\ &= \text{two 1's and one 3} \\ &= 3 + 1 + 1. \end{aligned}$$

This bijective proof is a good example, because it turns out to be the same bijective proof that results from the more general "automated" method for finding bijections that will be discussed later, beginning in section 4.

EXAMPLE 5 THE CLASSIC MONEY CHANGING PROBLEM

Consider a country with only 9, 17, 31, and 1000 dollar bills. How many ways are there to change a 1000 dollar bill? In other words, how many ways can we partition the integer 1000, if the parts are restricted to being 9, 17, or 31? The solution is the coefficient of x^{1000} in the following:

$$\frac{1}{1 - x^9} \cdot \frac{1}{1 - x^{17}} \cdot \frac{1}{1 - x^{31}}$$

The problem of determining if even one solution exists, in general, is well known to be an NP-hard problem [1]

EXAMPLE 6

This example illustrates how partition problems pop up in other strange places. Consider:

$$F(x) = \prod_{j=0}^{\infty} (1 + x^{2^j} + x^{2^{j+1}}) = \sum_{n=0}^{\infty} f(n)x^n.$$

In this example $f(n)$ is evidently (?) the number of partitions of n into powers of 2, with the additional restriction that each power of 2 can appear at most twice. Interestingly enough, this function is also important in the theory of Stern-Brocot trees and continuants. A nice discussion of these topics is presented in Graham, Knuth and Patashnik [10]. A rather surprising fact is that the sequence $\{f(n)/f(n+1)\}_{n=0}^{\infty}$ consists [3] of exactly one occurrence of every positive rational number in reduced form! We discuss this fully in section 11 below. To obtain a simple recurrence, notice that $F(x^2) \cdot (1 + x + x^2) = F(x)$.

There are several Rogers-Ramanujan identities in the theory of integer partitions. We will give one here whose generating function proof [12] is 3-4 pages long, and whose bijective proof [7] is close to fifty pages long. More information on these identities can be found in George Andrews' book [2].

LEMMA 2 (A ROGERS-RAMANUJAN IDENTITY) *The number of partitions of n into parts congruent to 1 or 4 mod 5 is equal to the number of partitions into parts that are neither repeated nor consecutive.*

Note that it is not difficult to obtain a generating function for the first object, namely:

$$\prod_{j=0}^{\infty} \frac{1}{(1 - x^{5j+1})(1 - x^{5j+4})},$$

but to find a generating function for the second object is more complicated.

A partition is *self-conjugate* if it is equal to its conjugate, or in other words, if its Ferrers diagram is symmetric about the diagonal. For example, the Ferrers diagram for the partition $20 = 6 + 4 + 4 + 4 + 1 + 1$ is self-conjugate (see Figure 2) .

THEOREM 4 *The number of partitions of n into parts that are both odd and distinct is equal to the number of self-conjugate partitions of n .*

Again, it is easy to find a generating function for the first object, namely:

$$\prod_{j=0}^{\infty} (1 + x^{2j+1}),$$

but a generating function for the latter object is not obvious. However, using Ferrers diagrams, a bijective proof is straightforward. The general idea is to 'bend' each odd, distinct

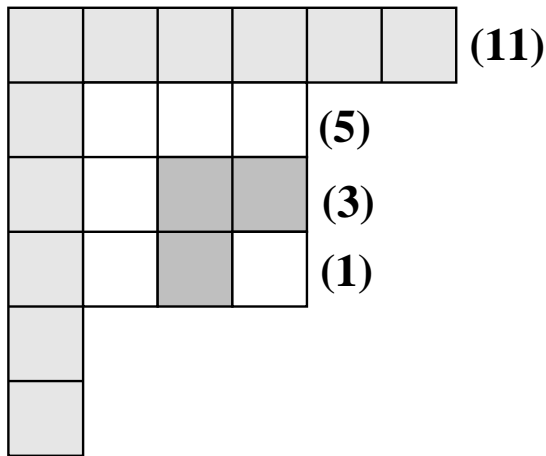


Figure 2: Converting the partition $20 = 11 + 5 + 3 + 1$ into one that is self-conjugate

part at the middle cell and then join the bent pieces together. This yields a self-conjugate partition, a process that is clearly reversible. As an example, the partition of 20 into the odd, distinct parts $11+5+3+1$ is illustrated in Figure 2.

Now recall the very hard problem of determining the parity of $p(n)$. Past attempts at solving this problem have involved throwing out a large, even number of partitions. Note, though, that the parity of $p(n)$ is unchanged if we throw out all pairs consisting of a non-self-conjugate partition and its conjugate. That leaves the self-conjugate partitions. Thus we see that the parity of $p(n)$ is the same as the parity of the number of partitions of n into parts that are odd and distinct, which is a much smaller number of partitions of n .

EXAMPLE 7 (A FIENDISH EXAMPLE)

Consider:

$$\prod_{j=1}^{\infty} (1 + x^j + y^j) = \sum_{m,n=0}^{\infty} f(m,n)x^m y^n. \tag{4}$$

In this case, each part j can contribute either to m or n , but not both. In addition each part can contribute at most once. Therefore the function $f(m,n)$ counts pairs of partitions of m and n respectively, such that each partition is composed of distinct parts and the pair of partitions have no part in common.

For a small variation, consider

$$\prod_{j=1}^{\infty} \left(1 + \frac{x^j}{1-x^j} + \frac{y^j}{1-y^j} \right) = \sum_{m,n} g(m,n)x^m y^n.$$

What does this g count? It counts the same thing as the f of (4), except that the parts of each of the partitions in the pair are no longer required to be distinct. That is, g counts

ordered pairs (π', π'') , where π' is a partition of m , π'' is a partition of n , and π', π'' have no common part.

Now what happens when we throw in one more term?

$$\prod_{j=1}^{\infty} (1 + x^j + y^j + z^j) = \sum_{m,n=0}^{\infty} f(m, n, r) x^m y^n z^r.$$

In this case, the function $f(m, n, r)$ counts triples of partitions of m, n and r , respectively, such that each partition is composed of distinct parts and pairwise they have no common part. It is important to note that the condition here is that *pairwise* they have no common part. This is a stronger condition than merely asking that the triple of partitions has no common part, which is a question that we will discuss on page 18. The distinction is rather like the difference between a collection of integers that is *pairwise relatively prime* vs. a *coprime* set.

Recall that $p(n, k)$ counts the partitions of n with largest part k . We can express such a partition as follows: $n = k + (\leq k) + (\leq k) + \dots$. If we now move the k to the other side of the equality we end up with a partition of $n - k$ into parts of size less than or equal to k . Using generating functions, the number of such partitions is given by the coefficient of x^{n-k} in:

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^k)},$$

i.e.,

$$\sum_{n=0}^{\infty} p(n, k) x^n = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

Now if we sum over k we get:

$$\mathcal{E}(x) = \prod_{j=1}^{\infty} \frac{1}{1-x^j} = \sum_{k \geq 1} \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

What we have here is an infinite product made into an infinite series, but not a power series. Further down this path lies the theory of q -series.

This type of development leads to a recurrence for $p(n, k)$. The partitions of n whose largest part is k come in two flavors: those that have exactly one part equal to k , and those that have more than one part equal to k . The former are counted by $p(n-1, k-1)$, and the latter by $p(n-k, k)$. The recurrence equation that results from this observation is $p(n, k) = p(n-1, k-1) + p(n-k, k)$. This recurrence is the starting point for most recursive programs used to tabulate $p(n)$, to list all partitions of n , to find a random partition, and for the ranking and unranking of partitions.

Now back to the question of finding an asymptotic series for $p(n)$. The following result by Hardy, Ramanujan, and Rademacher [11, 16] is the culmination of an intense research effort that took place in the first half of the twentieth century.

THEOREM 5 *We have*

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) \sqrt{k} \left[\frac{d \sinh\left(\frac{\pi}{k} \sqrt{\frac{2}{3}} \left(x - \frac{1}{24}\right)\right)}{dx \sqrt{\left(x - \frac{1}{24}\right)}} \right]_{x=n},$$

where

$$A_k(n) = \sum_{\substack{0 \leq h \leq k-1 \\ (h,k)=1}} \omega_{h,k} e^{-2\pi i n h/k}$$

and $\omega_{h,k}$ is a certain 24th root of unity.

A more complete account of this theorem can be found in [2]. To illustrate this formula, we steal an example from [2], for $n = 200$.

EXAMPLE 8

Feel free to verify this on your own: $p(200) = 3,972,999,029,388$. Using the previous theorem, the first 8 terms in the expansion of $p(200)$ are:

$$\begin{array}{r} + 3,972,998,993,185.896 \\ + 36,282.978 \\ - 87.584 \\ + 5.147 \\ + 1.424 \\ + 0.071 \\ + 0.000 \\ + 0.044 \\ \hline 3,972,999,029,387.975 \end{array}$$

This example illustrates the fact that the formula of Hardy, Ramanujan and Rademacher is not only an asymptotic series, it is a finite, exact formula for $p(n)$. It can be shown that if we sum the first $c\sqrt{n}$ terms in this expansion for some constant c , then the nearest integer to that sum will be the exact value of $p(n)$ [2]. The method that they used to find and to prove the validity of their formula is called *the circle method*, because the successive terms in the expansion arise from singularities of the generating function in a certain ordering of the rational points on the unit circle.

By taking only the first term of this expansion, we obtain the asymptotic behavior of $p(n)$,

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \quad (n \rightarrow \infty)$$

which shows that the growth of $p(n)$ is subexponential.

4 Pentagonal Numbers and Prefabs

Previously we have discussed the expression:

$$\prod_{j=1}^{\infty} \frac{1}{(1-x^j)} = \sum_{n=0}^{\infty} p(n)x^n.$$

But what about the expression $\prod_{j=1}^{\infty} (1-x^j)$?

THEOREM 6 (EULER'S PENTAGONAL NUMBER THEOREM)

$$\prod_{j=1}^{\infty} (1-x^j) = \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n+1)/2} = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \dots$$

The Euler pentagonal number theorem is a special case of the Jacobi triple product identity.

THEOREM 7 (THE JACOBI TRIPLE PRODUCT IDENTITY)

$$\prod_{n=1}^{\infty} (1-x^{2n})(1+x^{2n-1}z^2)(1+x^{2n-1}z^{-2}) = \sum_{n=-\infty}^{\infty} x^{n^2} z^{2n}.$$

The proof of this identity is somewhat lengthy, and will not be given here. A proof that contains several references to other proofs can be found in [19]

If we now replace x with x^k and z^2 with $-x^\ell$ we obtain the new expression:

$$\prod_{n=0}^{\infty} (1-x^{2kn-k-\ell})(1-x^{2kn-k+\ell})(1-x^{2kn}) = \sum_{n=-\infty}^{\infty} (-1)^n x^{kn^2+\ell n}.$$

Again, as combinatorialists, we consider the expression as a formal product and we do not worry about convergence. Now to get the Euler pentagonal number theorem we take $k = 3/2$ and $\ell = 1/2$ and then manipulate the resulting expression.

Another result of this kind is the following.

THEOREM 8

$$\{(1-x)(1-x^2)(1-x^3) \dots\}^3 = 1 - 3x + 5x^3 - 7x^6 + 9x^{10} - \dots,$$

where the coefficients are the odd numbers and the exponents are $\binom{n}{2}$.

How do we explain Theorem 6 combinatorially? Consider the expression $(1-x)(1-x^2)(1-x^3)\dots$. If we replaced the minus signs with plus signs, we would be counting the partitions of n into distinct parts. As it stands, however, the coefficient of x^n is the excess

of the number of partitions of n into an even number of distinct parts over the number with an odd number of distinct parts.

So what this theorem is saying combinatorially, is that there are the same number of partitions of n into an even number of distinct parts as there are partitions into an odd number of distinct parts unless n is a pentagonal number, $n = j(3j + 1)/2$, in which case the excess is $(-1)^j$. A combinatorial proof, due to Franklin, can be found in [12].

Now let's move on to a more general framework known as generalized partitions or prefabs, (or sometimes exponential structures). These generalized structures include, as special cases,

- integer partitions
- rooted unlabeled forests
- monic polynomials over a finite field
- plane partitions
- rooted unlabeled graphs
- ...

So what is a prefab? A *prefab* is a set \mathcal{P} of objects, together with an *order* function $|\cdots|$, which attaches to each object $x \in \mathcal{P}$ a nonnegative integer $|x|$ called its order. In addition there is a *synthesis* map \otimes which associates with each pair (x', x'') of objects in \mathcal{P} a new object $x' \otimes x''$, called their synthesis, in such a way that $|x' \otimes x''| = |x'| + |x''|$. Further, there is a distinguished subset of elements of \mathcal{P} called *primes* such that every object in the prefab can be obtained uniquely as a synthesis of the primes.

To illustrate the properties of a prefab, consider integer partitions. The order of a partition is the integer n that is being partitioned. In general, the synthesis of two objects is just the result obtained by writing the two objects down side by side. In the case of integer partitions, the synthesis of the partitions $5 = 3 + 2$ and $6 = 4 + 1 + 1$ is simply $11 = 4 + 3 + 2 + 1 + 1$. The primes are the partitions $1 = 1$, $2 = 2$, $3 = 3$, and so forth.

As another example, consider rooted forests. The order of a rooted forest is the number of nodes or vertices in the forest. The synthesis of two forests is the forest that you get when you write down the two given forests side by side, and the primes are the unlabeled rooted trees.

For these two examples, determining the primes was fairly straightforward, but in general it is not always so easy.

Let's now consider plane partitions. A *plane partition* is a partition of the integer n into the parts $p_{i,j}$ for $i, j \geq 0$ such that each $p_{i,j}$ is a nonnegative integer, $p_{i,j} \geq p_{i+1,j}$ and $p_{i,j} \geq p_{i,j+1}$. As an illustration, the following is an plane partition of 10:

$$\begin{array}{ccc} 2 & 1 & \\ 3 & 3 & 1 \end{array}$$

In other words as you go up a column, the parts are nonincreasing and as you go across a row the parts are nonincreasing. In this case the set of primes and the synthesis operation are not as obvious. The details can be found in chapter 12 of [14].

LEMMA 3 *In any prefab \mathcal{P} , let d_n be the number of prime objects of order n and let a_n be the total number of objects of order n , then*

$$\sum_{n=0}^{\infty} a_n x^n = \prod_{j=1}^{\infty} \frac{1}{(1-x^j)^{d_j}}.$$

The proof of this is similar to the proof of the validity of Euler's generating function for partitions. As each parenthesis in the product on the right passes by we can reach into it and take any nonnegative number of copies of the corresponding prime object that we please. The totality of these choices of primes, after synthesizing them, will produce a unique object whose order is the sum of the orders of the chosen primes.

Consider the generating function for plane partitions:

$$\prod_{j=1}^{\infty} \frac{1}{(1-x^j)^j}.$$

This generating function looks like one for a prefab with j primes of order j , for each $j \geq 0$. It is indeed that, and the identification was made by Bender and Knuth. A description of their work is in [14].

A related object is a solid partition, in which the parts are nonincreasing along each of 3 dimensions. It has been shown that solid partitions are not prefabs and it is currently an open problem to enumerate the solid partitions of the integer n .

Now let's use the previous lemma and consider rooted forests. Let t_n be the number of rooted trees of n (unlabeled) vertices - the primes. Using the lemma we have:

$$\prod_{j=1}^{\infty} \frac{1}{(1-x^j)^{t_j}} = \sum_{n=0}^{\infty} f_n x^n.$$

Pólya observed that there is a 1-1 correspondence between rooted trees of $n+1$ vertices and rooted forests of n vertices. Indeed we can obtain such a rooted tree from such a rooted forest by joining all the original roots in the forest to a new root node. This process is clearly reversible which implies that $t_{n+1} = f_n$ and thus

$$\prod_{j=1}^{\infty} \frac{1}{(1-x^j)^{t_j}} = \sum_{n=0}^{\infty} t_{n+1} x^n.$$

This equation determines the numbers $\{t_j\}_0^\infty$, and can be used to obtain a recurrence equation for them.

If we consider the prefab of all monic polynomials over $\text{GF}(q)$, we find that the primes are the irreducible monic polynomials and synthesis is ordinary multiplication of polynomials. We know that there are q^n such polynomials of degree n . Thus using the previous lemma, we have:

$$\prod_{j=1}^{\infty} \frac{1}{(1-x^j)^{i_j}} = \sum_{n=0}^{\infty} q^n x^n = \frac{1}{1-qx}$$

where i_j counts the irreducible polynomials of degree j . To find a formula for i_j , we take the logs of both sides and differentiate. The resulting equation involves a sum over divisors which can be solved using Möbius inversion. The resulting formula turns out to be the same formula that counts the q -ary aperiodic necklaces (Lyndon words) of length n .

We now move on to a theorem about prefabs that comes from [5].

THEOREM 9 *In a prefab \mathcal{P} , let $f_m(n)$ be the number of m -tuples of objects of order n in \mathcal{P} such that no prime object is a factor of every member of the m -tuple (a coprime set of objects), then*

$$\sum_{n \geq 0} f_m(n) x^n = \left(\sum_{n \geq 0} f(n)^m x^n \right) \left(\prod_{i \geq 1} (1-x^i)^{d_i} \right), \quad (5)$$

where d_i is the number of prime objects of order i in \mathcal{P} .

The proof is easy. Note that we can uniquely factor out the “gcd” of every m -tuple $(\omega_1, \dots, \omega_m)$ of objects of order n , by writing that m -tuple as a product of that gcd α and an m -tuple $(\omega'_1, \dots, \omega'_m)$ of coprime objects of orders $n - |\alpha|$. This fact is expressed in the language of generating functions by the identity

$$\sum_{n \geq 0} f(n)^m x^n = \frac{1}{\prod_{i \geq 1} (1-x^i)^{d_i}} \sum_{n \geq 0} f_m(n) x^n \quad (6)$$

from which the claimed result (5) follows at once.

We can apply this theorem to get the following consequences:

- The number of m -tuples of partitions of n with no common part is

$$p(n)^m - p(n-1)^m - p(n-2)^m + p(n-5)^m + p(n-7)^m - p(n-12)^m - \dots,$$

in which the decrements are the pentagonal numbers.

- In the prefab of rooted, unlabeled forests, for fixed n , the probability that if we choose two forests of n vertices i.u.a.r. then they will have no tree in common, is, according to (5) with $m = 2$,

$$1 + c_1 \left(\frac{f(n-1)}{f(n)} \right)^2 + c_2 \left(\frac{f(n-2)}{f(n)} \right)^2 + \dots,$$

in which $\prod_{i \geq 1} (1 - x^i)^{t_i} = \sum_i c_i x^i$ defines the c 's. Now it is well known that the number of rooted forests of n vertices is $f(n) \sim KC^n/n^{1.5}$, where $C = 2.95576\dots$. Hence each term $(f(n - k)/f(n))^2$ above approaches C^{-2k} , and in the limit as $n \rightarrow \infty$ we obtain the following.

PROPOSITION 1 *The probability that two rooted forests of n vertices have no tree in common approaches*

$$1 + \frac{c_1}{C^2} + \frac{c_2}{C^4} + \dots = \prod_{i \geq 1} \left(1 - \frac{1}{C^2}\right)^{t_i} = 0.8705\dots$$

as $n \rightarrow \infty$.

- The number of coprime m -tuples of monic polynomials of degree n over $GF[q]$ is $q^{nm} - q^{(n-1)m+1}$, i.e., the probability that m randomly chosen such polynomials will be coprime is $1 - 1/q^{m-1}$.

5 The Involution Principle

Now we're going to discuss a string of six papers in which a unified theory of partition bijections was developed. This research started in the late 1980's and continues today. In essence, these papers provide machinery that not only proves a large class of partition identities, but also produces in a very general way a bijection between the two sets of partitions that are involved. It turns out that in every case we know, the bijections found by this automated approach are the same as those that were previously found by humans.

The first of these six papers was Garsia and Milne's paper [6] on the Involution Principle. The initial motivation behind this paper was to prove a Rogers-Ramanujan identity bijectively, and indeed, the authors did prove the identity (and in the process claimed a \$100 prize that had been offered by George Andrews), however their general principle has since led to an approach that produces bijections in a large class of other partition identities.

The idea of the Involution Principle is as follows. Let \mathbf{A} and \mathbf{B} be sets with the same cardinality and let f be a bijection between the two sets. Now partition each set into positive and negative elements such that $f(\mathbf{A}^+) = \mathbf{B}^+$ and $f(\mathbf{A}^-) = \mathbf{B}^-$. Now define an involution¹ α on the set \mathbf{A} such that for all $a \in \mathbf{A}$ either

- (i) $\alpha(a) = a$ and $a \in \mathbf{A}^+$ or
- (ii) $\alpha(a) \neq a$ and if $a \in \mathbf{A}^+$ then $\alpha(a) \in \mathbf{A}^-$ and if $a \in \mathbf{A}^-$ then $\alpha(a) \in \mathbf{A}^+$.

¹Recall that an involution $\phi: A \rightarrow A$ is a map for which $\phi \circ \phi = id_A$

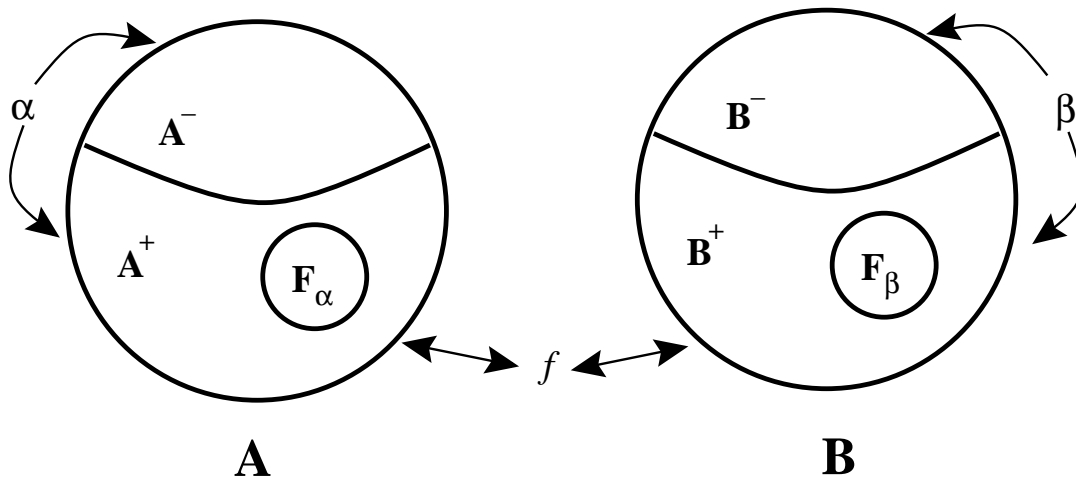


Figure 3: The setup for Garsia and Milne's Involution Principle

Define a similar involution β on the set \mathbf{B} . Note that by this definition, all fixed points are positive elements. The set \mathbf{F}_α denotes the fixed points of α and the set \mathbf{F}_β denotes the fixed points of β . This setup is illustrated in Figure 3.

The goal is to find a bijection between the sets \mathbf{F}_α and \mathbf{F}_β of fixed points of the involutions α, β . It is easy to see that these two sets are equinumerous (check this!) from the setup, so now we want to construct a bijection between \mathbf{F}_α and \mathbf{F}_β using the raw materials, namely the involutions α and β and the bijection f . Let $\alpha^* = f \circ \alpha$ which is a 1-1 function from \mathbf{A} to \mathbf{B} . Similarly, let $\beta^* = f^{-1} \circ \beta$ which is a 1-1 function from \mathbf{B} to \mathbf{A} . Now given a fixed point $a_1 \in \mathbf{F}_\alpha$, we want to find its image in \mathbf{F}_β . To do this we create a sequence $a_1, b_1, a_2, b_2, \dots$ by successively applying the functions $\alpha^*, \beta^*, \alpha^*, \beta^*, \dots$. In other words, we ping-pong back and forth between the sets \mathbf{A} and \mathbf{B} starting with the fixed point a_1 . This game stops when the ping-pong ball first lands in the set \mathbf{F}_β . Because the set \mathbf{B} is finite and since f is a bijection, there exists an n such that b_n is in the fixed point set \mathbf{F}_β .

THEOREM 10 (GARSIA-MILNE)

The map $a_1 \rightarrow b_n$ is a bijection between \mathbf{F}_α and \mathbf{F}_β .

6 Remmel's bijection machine

The paper [17] shows how to use the Garsia-Milne involution principle in general to develop a unified method that proves a large number of partition identities. This method not only is capable of showing that two sets of partitions are equinumerous, but it also supplies a bijection.

To understand this method we must get into a negative frame of mind. For example, instead of thinking of partitions into odd parts, we think of partitions with no even parts, i.e., partitions that do *not* contain any of the parts 2, 4, 6, 8, ... Similarly, rather than thinking of partitions into distinct parts, we think of partitions that do *not* have any of the following list of ‘diseases’: $\{1, 1\}, \{2, 2\}, \{3, 3\}, \dots$. To show that two sets of partitions are equinumerous, we try to match up their corresponding diseases.

THEOREM 11 (REMMEL)

Let $\mathcal{A} = \{\mathbf{A}_i\}_{i \in \omega}$, $\mathcal{B} = \{\mathbf{B}_i\}_{i \in \omega}$ be two lists of nonempty multisets such that the condition

$$\left| \bigcup_{i \in \mathbf{S}} \mathbf{A}_i \right| = \left| \bigcup_{i \in \mathbf{S}} \mathbf{B}_i \right| \quad (\forall \mathbf{S} \subseteq \omega) \quad (7)$$

holds. Then the number of partitions of n that contain no \mathbf{A}_i is equal to the number of partitions of n that contain no \mathbf{B}_i .

In this theorem, the expression $|\text{multiset}|$ represents the sum of the elements of the multiset. Also the number of occurrences of an element a in $\mathbf{A}_i \cup \mathbf{A}_j$ is given by the maximum number of occurrences of a in \mathbf{A}_i and \mathbf{A}_j . For example, if the integer 1 occurs twice in \mathbf{A}_i and three times in \mathbf{A}_j , then there will be three 1’s in $\mathbf{A}_i \cup \mathbf{A}_j$.

Let’s return to the example of the odd and distinct parts. The partitions of n into odd parts are the partitions of n that do not contain any of the multisets in the first column below, and the partitions into distinct parts are those that do not contain any of the multisets in the second column.

ODD (\mathcal{A})	DISTINCT (\mathcal{B})
$\mathbf{A}_1 = 2$	$\mathbf{B}_1 = \{1,1\}$
$\mathbf{A}_2 = 4$	$\mathbf{B}_2 = \{2,2\}$
$\mathbf{A}_3 = 6$	$\mathbf{B}_3 = \{3,3\}$
$\mathbf{A}_4 = 8$	$\mathbf{B}_4 = \{4,4\}$
$\mathbf{A}_5 = 10$	$\mathbf{B}_5 = \{5,5\}$
\vdots	\vdots

If the set \mathbf{S} of indices is, for example, $\{1, 3, 5\}$, then notice that $|\bigcup_{i \in \mathbf{S}} \mathbf{A}_i| = |\bigcup_{i \in \mathbf{S}} \mathbf{B}_i|$ since $2 + 6 + 10 = 1 + 1 + 3 + 3 + 5 + 5$. In fact this is true for every set \mathbf{S} , which means that the crucial hypothesis (7) is satisfied. Hence by the theorem, the number of partitions of n into odd parts equals the number of partitions of n into distinct parts. In this case, it should be obvious that (7) is satisfied since the multisets are pairwise disjoint. This situation occurs often enough that it is worth stating separately. Let’s write $P_n(\mathcal{A})$ for the set of all partitions of the integer n that do not contain any of the multisets in a list \mathcal{A} .

COROLLARY 1 (Remmel [17], Cohen [4]) *If \mathcal{A} and \mathcal{B} are sequences of pairwise disjoint multisets such that $|\mathbf{A}_i| = |\mathbf{B}_i|$ for all i then $|P_n(\mathcal{A})| = |P_n(\mathcal{B})|$.*

This corollary is much easier to work with, but is not as powerful as the theorem. Remmel's proof of Theorem 11 applies the Involution Principle. It not only proves the theorem, but it also gives a bijection.

PROOF OF THEOREM 11: Let \mathbf{A} be the collection of all ordered pairs (π, \mathbf{S}) such that π is a partition of n and \mathbf{S} is a set of indices such that all of the multisets A_s ($s \in \mathbf{S}$) are contained in π . In other words, the set \mathbf{S} indexes some, but not necessarily all, of the diseases in list \mathcal{A} that are found in the partition π . A similar construction is used for \mathbf{B} .

Following the prescription of the Involution Principle, we now attach a sign to each of these ordered pairs by decreeing that $\text{sign}((\pi, \mathbf{S})) = (-1)^{|\mathbf{S}|}$. The involution α is as follows. Given a partition π , let a_π equal the index of the largest multiset of \mathcal{A} that is contained in π . Then

$$\alpha((\pi, \mathbf{S})) = \begin{cases} (\pi, \mathbf{S} - \{a_\pi\}), & \text{if } a_\pi \in \mathbf{S}; \\ (\pi, \mathbf{S} \cup \{a_\pi\}), & \text{otherwise.} \end{cases}$$

Thus the involution α deletes the disease a_π from the set \mathbf{S} if it is contained in \mathbf{S} and adds it otherwise. This involution reverses the sign of the pair as long as the partition π contains at least one disease from \mathcal{A} . Thus the fixed points of this involution are the ordered pairs where the partition π contains no diseases from the list \mathcal{A} . The involution β is defined in a similar fashion.

To complete the setup of the Involution Principle in this application, it remains to describe the sign-preserving bijection $f : \mathbf{A} \rightarrow \mathbf{B}$. For a given $(\Pi, \mathbf{S}) \in \mathbf{A}$ we define

$$f(\Pi, \mathbf{S}) := (\lambda, \mathbf{S}), \text{ where } \lambda = [\Pi - (\cup_{i \in \mathbf{S}} A_i)] \cup [\cup_{i \in \mathbf{S}} B_i].$$

Having defined these pairs and involutions and the map f , the bijective proof of the theorem now follows from the Involution Principle. \square

Now for the fun part. We can manufacture our own theorems all day long simply by constructing pairs of lists of multisets that satisfy the condition (7). For each such theorem we will have a more-or-less transparent bijective proof.

Five examples are given to illustrate the power of this theorem.

EXAMPLE 9

Take

$$\begin{array}{l} \mathcal{A}: \{1, 1\}, \{2, 2\}, \{3, 3\}, \{4, 4\}, \{5, 5\}, \dots \\ \mathcal{B}: 2, 4, 6, 8, 10, \dots \end{array}$$

Thus the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts. In addition, we have a bijection between the two sets, although it takes a significant effort to track through the machinery to verbalize this bijection in a nice way. The resulting bijection turns out to be the same one found by Euler that we discussed above.

EXAMPLE 10

Take

$$\begin{array}{l} \mathcal{A}: \quad d, \quad 2d, \quad 3d, \quad 4d, \quad \dots \\ \mathcal{B}: \quad \underbrace{\{1, 1, \dots, 1\}}_d, \quad \underbrace{\{2, 2, \dots, 2\}}_d, \quad \underbrace{\{3, 3, \dots, 3\}}_d, \quad \underbrace{\{4, 4, \dots, 4\}}_d, \quad \dots \end{array}$$

Thus, the number of partitions of n with no part divisible by d is equal to the number of partitions n with no part repeated d or more times. The bijection computed from Garsia-Milne is the same as the one found by Glaisher [8].

EXAMPLE 11

Take

$$\begin{array}{l} \mathcal{A}: \quad 2, \quad 3, \quad 4, \quad 6, \quad 8, \quad 9, \quad \dots \\ \mathcal{B}: \quad \{1,1\}, \quad 3, \quad \{2,2\}, \quad 6, \quad \{4,4\}, \quad 9, \quad \dots \end{array}$$

In this case, the method shows that the number of partitions of n into parts congruent to $\pm 1 \pmod 6$ is equal to the number of partitions of n into distinct parts congruent to $\pm 1 \pmod 3$. Again, we also obtain a bijection. This theorem is attributed to Schur [18].

Finally, one can create new theorems about partition identities quite easily with this apparatus. The following, taken from [17], shows an extreme example of this sort of activity.

EXAMPLE 12 (A TOUR DE FORCE)

The number of partitions of n of each of the following types are all equal:

- (i) the parts congruent to 1 or 4 mod 5 do not differ by 8
- (ii) the parts congruent to 2 or 3 mod 5 do not differ by 6
- (iii) parts congruent to 2 or 3 mod 5 do not differ by 4
- (iv) parts congruent to 1 or 4 mod 5 do not differ by 2
- (v) no repeated multiples of 5 among the parts
- (vi) no multiples of 10

The proof of this tour de force is given by constructing the following:

$$\begin{aligned}
\mathcal{A}_1 &: \{1,9\}, & \{6,14\}, & \{11,19\}, & \dots \\
\mathcal{A}_2 &: \{2,8\}, & \{7,13\}, & \{12,18\}, & \dots \\
\mathcal{A}_3 &: \{3,7\}, & \{8,12\}, & \{13,17\}, & \dots \\
\mathcal{A}_4 &: \{4,6\}, & \{9,11\}, & \{14,16\}, & \dots \\
\mathcal{A}_5 &: \{5,5\}, & \{10,10\}, & \{15,15\}, & \dots \\
\mathcal{A}_6 &: 10, & 20, & 30, & \dots
\end{aligned}$$

EXAMPLE 13 (A NON-DISJOINT CASE)

$$\begin{aligned}
\mathcal{A} &: \{2,4\}, & \{4,6\}, & \{6,8\}, & \dots \\
\mathcal{B} &: \{1,1,2,2\}, & \{2,2,3,3\}, & \{3,3,4,4\}, & \dots
\end{aligned}$$

This example is slightly trickier to verify because of the multiset union. Once it is verified that the condition (7) is satisfied, we find that the number of partitions of n with no consecutive even parts is equal to the number of partitions of n with no consecutive repeated parts.

6.1 Bibliographic notes

Many of the results of this section were obtained by Daniel I. A. Cohen [4], using his method of “PIE-sums,” at just about the same time as Remmel’s paper [17] appeared. In fact Corollary 1 above was obtained by him in exactly the form in which we state it here, and Cohen also derived a multitude of special cases of this disjoint multiset situation. However there is no bijection in [4], and indeed the one due to Remmel that we have given here required the prior development of the machinery of the Involution Principle.

7 Sieve equivalence

The third paper [20] in the string of six papers that we are discussing focuses on the hypothesis that is crucial to Remmel’s theorem. This condition (7) states that for every subset \mathbf{S} :

$$|\cup_{i \in \mathbf{S}} \mathbf{A}_i| = |\cup_{i \in \mathbf{S}} \mathbf{B}_i|.$$

If we study this condition we shall see that it is none other than a condition that ensures that two calculations that use the sieve method (a.k.a. the principle of inclusion-exclusion, or PIE) will get the same answers.

To use the sieve method in any combinatorial problem, we start with a set of objects (in this case the partitions of n) Ω and a list of properties (multisets or diseases) of those objects, \mathcal{P} . The inputs into any sieve method computation are the numbers $N(\supseteq \mathbf{S})$, for all subsets $S \subseteq \mathcal{P}$, which denote the number of objects in Ω that have at least the list \mathbf{S} of properties. The sieve method can return outputs such as the number N_0 of objects with no properties; the number $N(= T)$ of objects that have exactly the properties in a given set \mathbf{T} ;



Figure 4: The sieve method

or the number N_j of objects that have exactly j properties. Conceptually, the sieve method is as shown in Figure 4.

If we now consider two different lists of properties, say \mathcal{A} and \mathcal{B} such that for all sets \mathbf{S} of indices of the properties we have $N(\supseteq \mathbf{S}; \mathcal{A}) = N(\supseteq \mathbf{S}; \mathcal{B})$. Then it should be obvious that the outputs generated by the sieve method will be the same since the inputs are all the same. Two such sets of properties are said to be *sieve-equivalent*. If we let Ω be the partitions of n and if the properties \mathcal{A} are lists of multisets, then

$$N(\supseteq \mathbf{S}; \mathcal{A}) = p(n - |\cup_{i \in \mathbf{S}} \mathbf{A}_i|).$$

As an example consider the two multisets $\{1, 1, 2\}$ and $\{1, 3\}$. Clearly any partition with these two properties must contain the parts $\{1, 1, 2, 3\}$ or in other words it must contain the multiset union of the two properties. To see that the number of such partitions is equal to $p(n - 7)$, notice that we can take any partition of $n - 7$ and adjoin the parts 1,1,2,3 to get a partition of n . Thus for two lists of multisets, the following equation holds:

$$p(n - |\cup_{i \in \mathbf{S}} \mathbf{A}_i|) = p(n - |\cup_{i \in \mathbf{S}} \mathbf{B}_i|).$$

To summarize, the hypotheses in Remmel's Theorem are simply saying the following: If we do a PIE calculation on the partitions of n using the list \mathcal{A} of properties, and then we do another one using the list \mathcal{B} of properties, then these two sieve calculations will yield the same outputs, because all of their inputs are the same. Of course, Remmel's Theorem also finds a bijection, and we will deal with that matter shortly. See section 10 for further consequences of this point of view.

Using the notion of prefabs, we can generalize the results for partitions to other objects.

THEOREM 12 *The number of rooted forests of n vertices such that the trees are all different (distinct parts) equals the number of rooted forests with no even tree (odd parts).*

So what is an even tree? If we take two copies of the same rooted tree and join their two roots together, with the new root being one of the original roots, then the resulting tree is an even tree. This is illustrated in Figure 5. The lists of multisets of trees are:

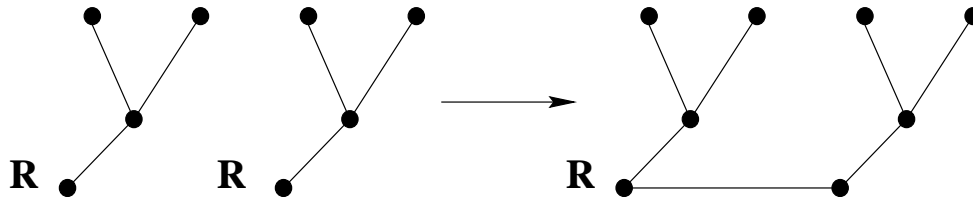


Figure 5: An even tree

- \mathcal{A} : 2 copies of every rooted tree
 \mathcal{B} : 1 copy of every even tree

8 Gordon's algorithm

We now turn our attention to the most important contribution of Remmel's Theorem, the bijection. However, now we want to formulate the problem in the sieve context.

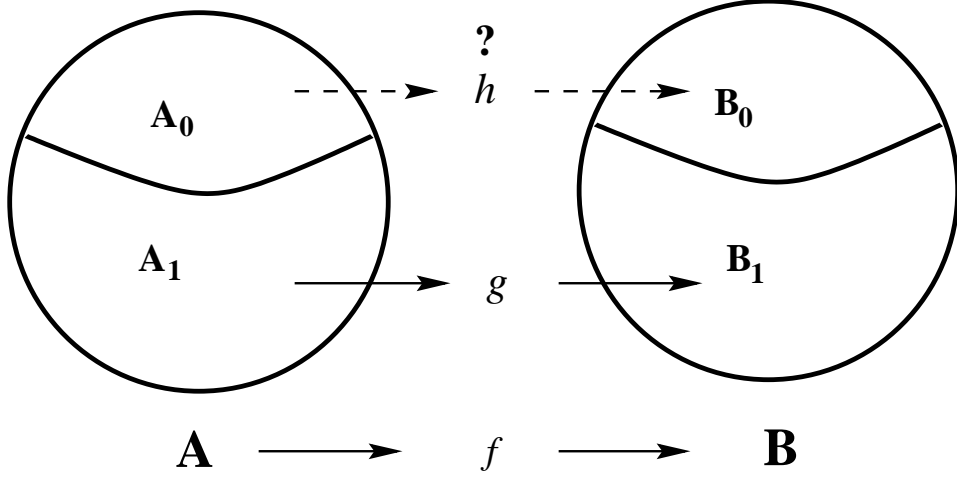
Imagine that two sets Ω, Ω' of objects are given, along with a list \mathcal{A} of properties of objects in Ω , and a list \mathcal{B} of properties of objects in Ω' . We are given also that for every index set S of positive integers, the number of objects in Ω that have at least all of the properties indexed by S in the list \mathcal{A} is equal to the number that have at least all of the properties indexed by S in the list \mathcal{B} .

But now we require more. Not only will we assume that these *numbers* of objects are equal, we will assume also that we are given, for every S , a bijection f_S between these two equicardinal sets of objects. In other words, we will assume that for every set S of positive integers, we are given a map

$$f_S : \cap_{i \in S} A_i \rightarrow \cap_{i \in S} B_i.$$

Since we are given not only that these pairs of sets are equicardinal but also we have maps between them, it is reasonable to ask how we can construct a map between the sets of objects in Ω, Ω' that have no properties at all, on their respective lists \mathcal{A}, \mathcal{B} . In asking such a question we are seeking to expand the scope of the sieve method at a very general level, one which is by no means restricted to partitions and their cousins, the prefabs. The original sieve gives equality of numbers of objects with no properties. We want a bijection between those two sets of objects with no properties. So, how can we construct a bijection between the two sets counted by $N_{\mathcal{A}}(=\emptyset)$ and $N_{\mathcal{B}}(=\emptyset)$?

This question was solved by Basil Gordon in [9], the fourth paper in the series. In this paper, Gordon gives an algorithm to find the bijection. To get a complete understanding of this algorithm, the reader should attempt to write a program that carries it out. But here is a brief summary.

Figure 6: The case $n = 1$

First, we formulate the problem as follows. $\mathcal{A} = \{\mathbf{A}_i\}_{i=1}^n$ such that all $\mathbf{A}_i \subseteq \mathbf{A}$ and $\mathcal{B} = \{\mathbf{B}_i\}_{i=1}^n$ such that all $\mathbf{B}_i \subseteq \mathbf{B}$. Let $\mathbf{A}_{\mathbf{S}} = \bigcap_{i \in \mathbf{S}} \mathbf{A}_i$ and $\mathbf{B}_{\mathbf{S}} = \bigcap_{i \in \mathbf{S}} \mathbf{B}_i$ where $\mathbf{S} \subseteq [n]$, and let $\mathbf{A}_0 = \mathbf{A} - \bigcup_{i=1}^n \mathbf{A}_i$ and $\mathbf{B}_0 = \mathbf{B} - \bigcup_{i=1}^n \mathbf{B}_i$. Given that the two sets of properties are sieve-equivalent, then for all \mathbf{S} we have $|\mathbf{A}_{\mathbf{S}}| = |\mathbf{B}_{\mathbf{S}}|$. If we also have bijections $f_{\mathbf{S}} : \mathbf{A}_{\mathbf{S}} \rightarrow \mathbf{B}_{\mathbf{S}}$ for all \mathbf{S} , then the goal is to construct a bijection $h : \mathbf{A}_0 \rightarrow \mathbf{B}_0$. The case when $n = 1$ is illustrated in Figure 6.

In this special case ($n = 1$), we follow a technique equivalent to the involution principle. In other words, given an x from \mathbf{A}_0 we want to find its image in \mathbf{B}_0 . As before, we play a game of ping-pong. If $f(x) \in \mathbf{B}_0$ then take $h(x) = f(x)$. Otherwise we form $h_1(x) = f \circ g^{-1} \circ f(x)$. If $h_1(x) \in \mathbf{B}_0$ then take $h(x) = h_1(x)$. Otherwise we repeat the procedure creating $h_2(x) = f \circ g^{-1} \circ h_1(x)$. Again, because these sets are finite, the game must end, i.e., $h_n(x)$ lands in \mathbf{B}_0 for some n .

To find the bijection for arbitrary n requires a little more work. Inductively, suppose the bijection $h' : \mathbf{A}'_0 \rightarrow \mathbf{B}'_0$ has been constructed for all sets \mathbf{A}' , \mathbf{B}' and families \mathcal{A}' , \mathcal{B}' composed of less than n subsets of \mathbf{A}' , \mathbf{B}' , and for given bijections $f'_{\mathbf{S}} : \mathbf{A}'_{\mathbf{S}} \rightarrow \mathbf{B}'_{\mathbf{S}}$. Then, for all non-empty subsets \mathbf{T} of positive integers, set

$$\mathbf{A}_{\mathbf{T},0} = \mathbf{A}_{\mathbf{T}} - \bigcup_{i \notin \mathbf{T}} \mathbf{A}_i \text{ and } \mathbf{B}_{\mathbf{T},0} = \mathbf{B}_{\mathbf{T}} - \bigcup_{i \notin \mathbf{T}} \mathbf{B}_i.$$

The sets $\{\mathbf{A}_{\mathbf{T},0}\}$ form a partition of the set $\bigcup_i \mathbf{A}_i$ and similarly the sets $\{\mathbf{B}_{\mathbf{T},0}\}$ form a partition of the set $\bigcup_i \mathbf{B}_i$. Now fix a non-empty subset \mathbf{T} and use induction with

$$\mathbf{A}' = \mathbf{A}_{\mathbf{T}}, \mathbf{B}' = \mathbf{B}_{\mathbf{T}}, \mathbf{N}' = \mathbf{N} - \mathbf{T},$$

$$\mathcal{A} = \{\mathbf{A}_i \cap \mathbf{A}_{\mathbf{T}} | i \in \mathbf{N}'\},$$

$$\mathcal{B} = \{\mathbf{B}_i \cap \mathbf{B}_T \mid i \in \mathbf{N}'\}.$$

Let $\mathbf{S} \subseteq \mathbf{N}'$ and let

$$\mathbf{A}'_{\mathbf{S}} = \mathbf{A}_{\mathbf{S}} \cap \mathbf{A}_{\mathbf{T}} = \mathbf{A}_{\mathbf{S} \cup \mathbf{T}},$$

$$\mathbf{B}'_{\mathbf{S}} = \mathbf{B}_{\mathbf{S} \cup \mathbf{T}}.$$

For all such \mathbf{S} set $f'_{\mathbf{S}} = f_{\mathbf{S} \cup \mathbf{T}}$. The conditions of the inductive assumption are all satisfied and we get a bijection $h_{\mathbf{T}} : \mathbf{A}_{\mathbf{T},0} \rightarrow \mathbf{B}_{\mathbf{T},0}$.

Since $\mathbf{T} \neq \mathbf{U}$ implies $\mathbf{A}_{\mathbf{T},0} \cap \mathbf{A}_{\mathbf{U},0} = \mathbf{B}_{\mathbf{T},0} \cap \mathbf{B}_{\mathbf{U},0} = \emptyset$, we can combine the $2^n - 1$ maps $h_{\mathbf{T}}$ to make a bijection

$$g : \cup_{i=1}^n \mathbf{A}_i \rightarrow \cup_{i=1}^n \mathbf{B}_i.$$

Then apply the case $n = 1$ to $f = f_{\emptyset}$, g to get a bijection $h : \mathbf{A}_0 \rightarrow \mathbf{B}_0$.

9 The accelerated algorithm of Kathy O'Hara

We have now seen approaches by Garsia-Milne-Remmel and Gordon to produce bijections. In all cases studied, the bijections produced are the same, as mappings, as the classical ones; however, the classical maps are easier to state, since there is no ping-pong-ing back and forth between the two sets. At least this was the case until Kathy O'Hara came up with a speedup process for the special case of disjoint multisets.

O'Hara's algorithm, which eliminates the ping-pong effect, works as follows. We are given two lists of pairwise disjoint multisets $\{\mathbf{A}_i\}$, $\{\mathbf{B}_i\}$ such that $|\mathbf{A}_i| = |\mathbf{B}_i|$ for all i . Now given the partition π which contains none of the \mathbf{A}_i 's, repeat the following until no \mathbf{B}_i multiset is contained in π :

- (a) Replace some \mathbf{B}_i in π by its matched \mathbf{A}_i .

The key to this algorithm is that the mapping that it produces is independent of the order in which the \mathbf{B}_i 's are chosen - but this requires a good deal of effort to prove.

EXAMPLE 14

Consider the following partition of 28 into odd parts:

$$\begin{aligned} 28 &= \underbrace{7+7}_{14} + 7 + 3 + 3 + 1 \\ &= 14 + 7 + \underbrace{3+3}_{6} + 1 \\ &= 14 + 7 + 6 + 1. \end{aligned}$$

The result is a partition into distinct parts.

10 Identically distributed partition statistics

We have already noted that in a sieve-equivalence situation, the two problems under consideration will have, for each j , the same numbers of objects that have exactly j properties. In the most recent paper [21] of the six that we are discussing, some consequences of this fact were studied.

EXAMPLE 15 AN EXTENSION OF EULER'S RESULT

The number of partitions of n with exactly j repeated part sizes is equal to the number of partitions of n with exactly j even part sizes. Euler's original 'odd-distinct' theorem is the special case where $j = 0$.

Notice what a simple consequence this is of the sieve point of view. We can state it more clearly, perhaps, as follows. Let's say that a *partition statistic* is a nonnegative integer valued function defined on the partitions of integers. The number of parts, is an example, as are the number of even parts, the number of repeated parts that are multiples of 6, and so forth.

We can associate with every partition statistic X a probability distribution by defining

$$\text{Prob}_n(X = j) \stackrel{\text{def}}{=} \frac{|\{\pi \in \mathcal{P}(n) : X(\pi) = j\}|}{p(n)},$$

where $\mathcal{P}(n)$ is the set of partitions of n . If \mathcal{A}, \mathcal{B} are two sieve-equivalent lists of multisets, then we obtain, from Remmel's theorem, two *identically distributed* partition statistics X, Y . The value of the statistic $X(\pi)$ (resp. $Y(\pi)$) on a partition π is the number of i such that π contains the multiset \mathcal{A}_i (resp. \mathcal{B}_i).

Several examples of pairs of identically distributed partition statistics, taken from [21], follow, and many others are easy to construct from the sieve-equivalence machinery.

- $X =$ the number of even part sizes, $Y =$ the number of repeated part sizes. (This is Example 15 again.)
- $X =$ the number of consecutive even part sizes, and $Y =$ the number of consecutive repeated part sizes.
- $X =$ the number of part sizes that are perfect squares; $Y =$ the number of part sizes i whose multiplicity is $\geq i$.
- $X =$ the number of part sizes that are $\equiv 2,3,4 \pmod{6}$; $Y =$ the number of part sizes that are either an odd multiple of 3 or else repeated and not a multiple of 3.
- Fix an integer $d > 1$. Let $X =$ the number of part sizes that are multiples of d ; $Y =$ the number of part sizes whose multiplicity is $\geq d$.

- Let M_1, M_2 be two sets of positive integers. Define $2M_1 = \{j : (j/2) \in M_1\}$. Suppose that $2M_1 \subseteq M_1$ and $M_2 = M_1 - 2M_1$. Then define the statistic $X =$ the number of part sizes that are not in M_2 ; $Y =$ the number of part sizes i such that either $i \notin M_1$ or $i \in M_1$ and is repeated.

For an example in the other direction, consider this.

EXAMPLE 16 (ROGERS-RAMANUJAN IDENTITY)

Recall Lemma 2 which states that the number of partitions of n into parts congruent to 1 or 4 mod 5 is equal to the number of partitions whose parts are neither repeated nor consecutive. This is an example where Remmel's theorem does not apply. To illustrate we write down the lists of properties (diseases):

<i>gaps = 0 or 1</i>	<i>parts $\equiv 1$ or 4 mod 5</i>
1 1	2
2 1	3
2 2	5
3 2	7
3 3	8
4 3	10

It should be quite clear that there is no way to order the properties so that Remmel's theorem will apply. To see that this does not work by the sieve method notice that the partitions of 4 with exactly one gap of size 0 or 1 are:

$$22, 1111$$

and the partitions of 4 with exactly one part size congruent to 0, 2 or 3 mod 5 are:

$$31, 22, 211$$

Thus, these two sets of properties are not sieve-equivalent since the numbers of partitions are different.

11 Counting the rational numbers

We promised one more section that illustrates the use of integer partition counting functions in a rather unusual role: counting the rationals. The discussion follows [3].

The list that we are about to construct, of the positive rational numbers, will begin like this:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{3}{4}, \frac{4}{1}, \frac{5}{4}, \frac{4}{7}, \frac{7}{3}, \frac{8}{5}, \frac{5}{7}, \frac{7}{2}, \frac{7}{5}, \dots$$

Some of the interesting features of this list are

1. The denominator of each fraction is the numerator of the next one. That means that the n th rational number in the list looks like $b(n)/b(n+1)$ ($n = 0, 1, 2, \dots$), where b is a certain function of the nonnegative integers whose values are

$$\{b(n)\}_{n \geq 0} = \{1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, \dots\}.$$

2. The function values $b(n)$ actually count something nice. In fact, $b(n)$ is the number of ways of writing the integer n as a sum of powers of 2, each power being used at most twice. For instance, we can write $5 = 4 + 1 = 2 + 2 + 1$, so there are two such ways to write 5, and therefore $b(5) = 2$. Let's say that $b(n)$ is the number of *hyperbinary* representations of the integer n .
3. Consecutive values of this function b are always relatively prime, so that each rational occurs in reduced form when it occurs.
4. Every positive rational occurs once and only once in this list.

11.1 The tree of fractions

For the moment, let's forget about enumeration, and just imagine that fractions grow on the tree that is completely described, inductively, by the following two rules:

- $\frac{1}{1}$ is at the top of the tree, and
- Each vertex $\frac{i}{j}$ has two children: its left child is $\frac{i}{i+j}$ and its right child is $\frac{i+j}{j}$.

We show the following properties of this tree.

1. *The numerator and denominator at each vertex are relatively prime.* This is certainly true at the top vertex. Otherwise, suppose r/s is a vertex on the highest possible level of the tree for which this is false. If r/s is a left child, then its parent is $r/(s-r)$, which would clearly also not be a reduced fraction, and would be on a higher level, a contradiction. If r/s is a right child, then its parent is $(r-s)/s$, which leads to the same contradiction. \square
2. *Every reduced positive rational number occurs at some vertex.* The rational number 1 certainly occurs. Otherwise, by induction on the levels of the tree, as above. \square
3. *No reduced positive rational number occurs at more than one vertex.* First, the rational number 1 occurs only at the top vertex of the tree, for if not, it would be a child of some vertex r/s . But the children of r/s are $r/(r+s)$ and $(r+s)/s$, neither of which can be 1. Otherwise, use induction on the levels of the tree, as before. \square

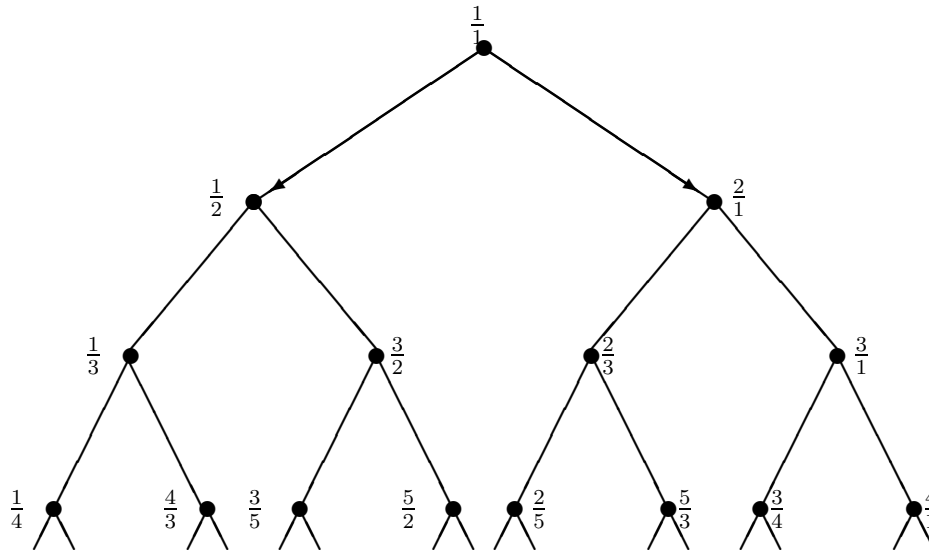


Figure 7: The tree of fractions

It follows that a list of all positive rational numbers, each appearing once and only once, can be made by writing down $1/1$, then the fractions on the level just below the top of the tree, reading from left to right, then the fractions on the next level down, reading from left to right, etc.

We claim that if that be done, then the denominator of each fraction is the numerator of its successor. This is clear if the fraction is a left child and its successor is the right child of the same parent. If the fraction is a right child then its denominator is the same as the denominator of its parent and the numerator of its successor is the same as the numerator of the parent of its successor, hence the result follows by downward induction on the levels of the tree. Finally, the rightmost vertex of each row has denominator 1, and the leftmost vertex of the next row has numerator 1, proving the claim.

Thus, after we make a single sequence of the rationals by reading the successive rows of the tree as described above, the list will be in the form $\{f(n)/f(n+1)\}_{n \geq 0}$, for some f .

Now, as the fractions sit in the tree, the two children of $f(n)/f(n+1)$ are $f(2n+1)/f(2n+2)$ and $f(2n+2)/f(2n+3)$. Hence from the rule of construction of the children of a parent, it must be that

$$f(2n+1) = f(n) \quad \text{and} \quad f(2n+2) = f(n) + f(n+1) \quad (n = 0, 1, 2, \dots).$$

These recurrences, together with $f(0) = 1$, evidently determine our function f on all non-negative integers.

We claim that $f(n) = b(n)$, the number of hyperbinary representations of n , for all $n \geq 0$.

This is true for $n = 0$, and suppose it is true for all integers $0, 1, \dots, 2n$. Now $b(2n+1) = b(n)$, because if we are given a hyperbinary expansion of $2n+1$, the “1” must appear, hence by subtracting 1 from both sides and dividing by 2, we’ll get a hyperbinary representation of n . Conversely, given such an expansion of n , double each part and add a 1 to obtain a representation of $2n+1$.

Furthermore, $b(2n+2) = b(n) + b(n+1)$, for a hyperbinary expansion of $2n+2$ might have either two 1’s or no 1’s in it. If it has two 1’s, then by deleting them and dividing by 2 we obtain an expansion of n . If it has no 1’s, then we just divide by 2 to get an expansion of $n+1$. These maps are reversible, proving the claim.

It follows that $b(n)$ and $f(n)$ satisfy the same recurrence formulas and take the same initial values, hence they agree for all nonnegative integers. We state the final result as follows.

THEOREM 13 *The n th rational number, in reduced form, can be taken to be $b(n)/b(n+1)$, where $b(n)$ is the number of hyperbinary representations of the integer n , for $n = 0, 1, 2, \dots$. That is, $b(n)$ and $b(n+1)$ are relatively prime, and each positive reduced rational number occurs once and only once in the list $b(0)/b(1), b(1)/b(2), \dots$*

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