PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 33, Fasc. 2 (2013), pp. 401–408

A FAMILY OF SEQUENCES OF BINOMIAL TYPE

BY

WOJCIECH MŁOTKOWSKI* (WROCŁAW) AND ANNA ROMANOWICZ** (WROCŁAW)

Abstract. For a delta operator $aD - bD^{p+1}$ we find the corresponding polynomial sequence of binomial type and relations with Fuss numbers. In the case of $D - \frac{1}{2}D^2$ we show that the corresponding Bessel–Carlitz polynomials are moments of the convolution semigroup of inverse Gaussian distributions. We also find probability distributions ν_t, $t > 0$, for which $\{y_n(t)\}$, the Bessel polynomials at t, is the moment sequence.

2000 AMS Mathematics Subject Classification: Primary: 05A40; Secondary: 60E07, 44A60.

Key words and phrases: Sequence of binomial type, Bessel polynomials, inverse Gaussian distribution.

1. INTRODUCTION

A sequence $\{w_n(t)\}_{n=0}^{\infty}$ of polynomials is said to be of *binomial type* (see [10]) if $\deg w_n(t) = n$ and for every $n \geqslant 0$ and $s, t \in \mathbb{R}$ we have

$$w_n(s+t) = \sum_{k=0}^{n} \binom{n}{k} w_k(s) w_{n-k}(t). \tag{1.1}$$

A linear operator Q of the form

$$Q = \frac{c_1}{1!}D + \frac{c_2}{2!}D^2 + \frac{c_3}{3!}D^3 + \ldots, \tag{1.2}$$

acting on the linear space $\mathbb{R}[x]$ of polynomials, is called a *delta operator* if $c_1 \neq 0$. Here D denotes the derivative operator: $D1 := 0$ and $Dt^n := n \cdot t^{n-1}$ for $n \geqslant 1$. We will write $Q = g(D)$, where

$$g(x) = \frac{c_1}{1!}x + \frac{c_2}{2!}x^2 + \frac{c_3}{3!}x^3 + \ldots \tag{1.3}$$

* Supported by the Polish National Science Centre grant No. 2012/05/B/ST1/00626.

** Supported by the Grant DEC-2011/02/A/ST1/00119 of the National Centre of Science.

There is a one-to-one correspondence between sequences of binomial type and delta operators, namely, if Q is a delta operator, then there is a unique sequence $\{w_n(t)\}_{n=0}^{\infty}$ of binomial type satisfying $Qw_0(t) = 0$ and $Qw_n(t) = n \cdot w_{n-1}(t)$ for $n \geqslant 1$. These w_n are called *basic polynomials* for Q.

A natural way of obtaining a sequence of binomial type is to start with a function f which is analytic in a neighborhood of zero:

$$f(x) = \frac{a_1}{1!}x + \frac{a_2}{2!}x^2 + \frac{a_3}{3!}x^3 + \ldots, \tag{1.4}$$

with $a_1 \neq 0$. Then a sequence of binomial type appears in the Taylor expansion of $\exp\big(t \cdot f(x)\big)$, namely

$$\exp\big(t \cdot f(x)\big) = \sum_{n=0}^{\infty} \frac{w_n(t)}{n!} x^n, \tag{1.5}$$

and the coefficients of w_n are partial Bell polynomials of $a_1, a_2, \ldots$ More generally, we can merely assume that f (as well as g) is a formal power series. Then f is the composition inverse of g: $f\big(g(x)\big) = g\big(f(x)\big) = x$.

The aim of the paper is to describe the sequences of polynomials of binomial type, which correspond to delta operators of the form $Q = aD - bD^{p+1}$. Then we discuss the special case $D - \frac{1}{2}D^2$ studied by Carlitz [3]. We find the related semigroup of probability measures and also the family of distributions corresponding to the Bessel polynomials.

2. THE RESULT

THEOREM 2.1. *For $a \neq 0, b \in \mathbb{R}$, and for $p \geqslant 1$, let $Q := aD - bD^{p+1}$. Then the basic polynomials are as follows*: $w_0(t) = 1$ *and for* $n \geqslant 1$

$$w_n(t) = \sum_{j=0}^{[(n-1)/p]} \frac{(n+j-1)!\, b^j}{j!\,(n-jp-1)!\, a^{n+j}} t^{n-jp}. \tag{2.1}$$

In particular, $w_1(t) = t/a$.

P r o o f. We have to show that if $n \geqslant 1$ then $Qw_n(t) = n \cdot w_{n-1}(t)$. It is obvious for $n = 1$, so assume that $n \geqslant 2$. Then we have

$$\begin{aligned} aDw_n(t) &= \sum_{j=0}^{[(n-1)/p]} \frac{(n+j-1)!\,(n-jp)b^j}{j!\,(n-jp-1)!\, a^{n-1+j}} t^{n-1-jp} \\ &= n\left(\frac{t}{a}\right)^{n-1} + \sum_{j=1}^{[(n-1)/p]} \frac{(n+j-1)!\,(n-jp)b^j}{j!\,(n-jp-1)!\, a^{n-1+j}} t^{n-1-jp} \end{aligned}$$

and

$$bD^{p+1}w_n(t) = \sum_{j=0}^{[(n-p-1)/p]} \frac{(n+j-1)!\,(n-jp)b^{j+1}}{j!\,(n-jp-p-1)!\,a^{n+j}} t^{n-1-jp-p}.$$

Now we substitute $j' := j+1$ and obtain

$$bD^{p+1}w_n(t) = \sum_{j'=1}^{[(n-1)/p]} \frac{(n+j'-2)!\,(n-j'p+p)b^{j'}}{(j'-1)!\,(n-j'p-1)!\,a^{n-1+j'}} t^{n-1-j'p}.$$

If $j \geqslant 1$ and $jp+1 < n$, then

$$\begin{aligned}\frac{(n+j-1)!\,(n-jp)}{j!\,(n-jp-1)!} - \frac{(n+j-2)!\,(n-jp+p)}{(j-1)!\,(n-jp-1)!}&\\ = \frac{(n+j-2)![(n+j-1)(n-jp)-j(n-jp+p)]}{j!(n-jp-1)!}&\\ = \frac{n(n+j-2)!\,(n-jp-1)}{j!\,(n-jp-1)!} = \frac{n(n+j-2)!}{j!\,(n-jp-2)!}&,\end{aligned}$$

and if $jp+1 = n$, then this difference is zero. Therefore we have

$$\begin{aligned}(aD-bD^{p+1})w_n(t) &= n\left(\frac{t}{a}\right)^{n-1} + \sum_{j=1}^{[(n-2)/p]} \frac{n(n+j-2)!\,b^j}{j!\,(n-jp-2)!\,a^{n-1+j}} t^{n-1-jp}\\ &= n \sum_{j=0}^{[(n-2)/p]} \frac{(n+j-2)!\,b^j}{j!\,(n-jp-2)!\,a^{n-1+j}} t^{n-1-jp} = n \cdot w_{n-1}(t),\end{aligned}$$

which concludes the proof. ■

Recall that *Fuss numbers of order* p are given by $\binom{np+1}{n}\frac{1}{np+1}$, and the corresponding generating function

$$\mathcal{B}_p(x) := \sum_{n=0}^{\infty} \binom{np+1}{n} \frac{x^n}{np+1} \tag{2.2}$$

is determined by the equation

$$\mathcal{B}_p(x) = 1 + x\cdot\mathcal{B}_p(x)^p. \tag{2.3}$$

In particular,

$$\mathcal{B}_2(x) = \frac{1-\sqrt{1-4x}}{2x}. \tag{2.4}$$

For more details, as well as combinatorial applications, we refer to [4].

Now we can exhibit the function f corresponding to the operator $aD - bD^{p+1}$ and to the polynomials (2.1).

COROLLARY 2.1. *For the polynomials* (2.1) *we have*

$$\exp\big(t\cdot f(x)\big)=\sum_{n=0}^{\infty}\frac{w_n(t)}{n!}x^n,$$

where

$$f(x)=\frac{x}{a}\cdot\mathcal{B}_{p+1}\left(\frac{bx^p}{a^{p+1}}\right). \tag{2.5}$$

P r o o f. Since f is the inverse function for $g(x):=ax-bx^{p+1}$, it satisfies $af(x)=x+bf(x)^{p+1}$, which is equivalent to

$$\left(\frac{af(x)}{x}\right)=1+\frac{bx^p}{a^{p+1}}\left(\frac{af(x)}{x}\right)^{p+1}.$$

Comparing with (2.3), we see that $af(x)/x=\mathcal{B}_{p+1}(bx^p/a^{p+1})$.

Alternatively, we could apply the formula $a_n=w_n'(0)$ for the coefficients in (1.4). ■

3. THE OPERATOR $D-\frac{1}{2}D^2$

One important source of sequences of binomial type are moments of convolution semigroups of probability measures on the real line. In this part we describe an example of such a semigroup, which corresponds to the delta operator $D-\frac{1}{2}D^2$. For more details we refer to [3], [5], [7], and [10] and to the entry A001497 in [11].

In view of (2.1), the sequence of polynomials of binomial type corresponding to the delta operator $D-\frac{1}{2}D^2$ is

$$w_n(t)=\sum_{j=0}^{n-1}\frac{(n+j-1)!\,t^{n-j}}{j!\,(n-j-1)!\,2^j}=\sum_{k=1}^{n}\frac{(2n-k-1)!\,t^k}{(n-k)!\,(k-1)!\,2^{n-k}}, \tag{3.1}$$

with $w_0(t)=1$. They are related to the *Bessel polynomials*

$$y_n(t)=\sum_{j=0}^{n}\frac{(n+j)!}{j!\,(n-j)!}\left(\frac{t}{2}\right)^j=e^{1/t}\sqrt{\frac{2}{\pi t}}K_{-n-1/2}\,(1/t), \tag{3.2}$$

where $K_\nu(z)$ denotes the modified Bessel function of the second kind. Namely, for $n\geqslant 1$ we have

$$w_n(t)=t^n y_{n-1}\,(1/t)=t^n e^t\sqrt{\frac{2t}{\pi}}K_{1/2-n}(t). \tag{3.3}$$

Applying Corollary 2.1 and (2.4) we get

$$f(x) = x \cdot \mathcal{B}_2(x/2) = 1 - \sqrt{1-2x} = \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n, \tag{3.4}$$

where $a_n = (2n-3)!!$. The function $f(x\mathrm{i})$ admits the *Kolmogorov representation* (see formula (7.15) in [12]) as

$$1 - \sqrt{1-2x\mathrm{i}} = x\mathrm{i} + \int_0^{+\infty} \left(e^{ux\mathrm{i}} - 1 - ux\mathrm{i}\right) \frac{e^{-u/2}}{\sqrt{2\pi u^3}}\, du, \tag{3.5}$$

with the probability density function

$$\frac{\sqrt{u} e^{-u/2}}{\sqrt{2\pi}}$$

on $[0, +\infty)$. Therefore

$$\phi(x) = \exp\left(1 - \sqrt{1-2x\mathrm{i}}\right)$$

is the characteristic function of some infinitely divisible probability measure. It turns out that the corresponding convolution semigroup $\{\mu_t\}_{t>0}$ is contained in the family of *inverse Gaussian distributions*, see 24.3 in [1].

THEOREM 3.1. *For $t>0$ define probability distribution $\mu_t := \rho_t(u)\, du$, where*

$$\rho_t(u) := \frac{t \cdot \exp\bigl(-(u-t)^2/(2u)\bigr)}{\sqrt{2\pi u^3}} \tag{3.6}$$

for $u > 0$ and $\rho_t(u) = 0$ for $u \leqslant 0$. Then $\{\mu_t\}_{t>0}$ is a convolution semigroup, $\exp\left(t - t\sqrt{1-2x\mathrm{i}}\right)$ is the characteristic function of μ_t, and $\{w_n(t)\}_{n=0}^{\infty}$, defined by (3.1), *is the moment sequence of μ_t.*

Proof. It is sufficient to check moments of μ_t. Substituting $u := 2v$ and applying the formula

$$K_p(t) = \frac{1}{2} \left(\frac{t}{2}\right)^p \int_0^{\infty} \exp\left(-v - \frac{t^2}{4v}\right) \frac{dv}{v^{p+1}} \tag{3.7}$$

(see (10.32.10) in [9]), we obtain

$$\int_0^{\infty} u^n \frac{t \cdot \exp\bigl(-(u-t)^2/(2u)\bigr)}{\sqrt{2\pi u^3}}\, du = \frac{te^t}{\sqrt{2\pi}} \int_0^{\infty} \exp\left(\frac{-u}{2} - \frac{t^2}{2u}\right) u^{n-3/2}\, du$$

$$= \frac{te^t 2^{n-1}}{\sqrt{\pi}} \int_0^{\infty} \exp\left(-v - \frac{t^2}{4v}\right) v^{n-3/2}\, dv = \frac{te^t 2^{n-1}}{\sqrt{\pi}} 2 \left(\frac{t}{2}\right)^{n-1/2} K_{1/2-n}(t),$$

which, by (3.3), is equal to $w_n(t)$. ∎

4. PROBABILITY MEASURES CORRESPONDING TO THE BESSEL POLYNOMIALS

In this part we are going to give some remarks concerning Bessel polynomials (3.2). First we compute the exponential generating function (cf. formula (6.2) in [5]).

PROPOSITION 4.1. *For the exponential generating function of the sequence* $\{y_n(t)\}$ *we have*

$$\sum_{n=0}^{\infty} \frac{y_n(t)}{n!} x^n = \frac{\exp\left(\frac{1}{t} - \frac{1}{t}\sqrt{1-2tx}\right)}{\sqrt{1-2tx}}. \tag{4.1}$$

Proof. By (3.3) we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{y_n(t)}{n!} x^n &= \sum_{n=0}^{\infty} \frac{t^{n+1} w_{n+1}(1/t)}{n!} x^n = \sum_{n=1}^{\infty} \frac{t^n w_n(1/t)}{(n-1)!} x^{n-1} \\ &= \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{t^n w_n(1/t)}{n!} x^n\right) = \frac{d}{dx} \exp\left(\frac{1}{t} - \frac{1}{t}\sqrt{1-2tx}\right),\end{aligned}$$

which leads to (4.1). ■

Now we represent the Bessel polynomials (3.2) as a moment sequence.

THEOREM 4.1. *For* $n \geqslant 0$ *and* $t > 0$ *we have*

$$y_n(t) = \int_0^{\infty} u^n \frac{\exp\bigl(-(u-1)^2/(2tu)\bigr)}{\sqrt{2\pi t u}}\, du. \tag{4.2}$$

Proof. Substituting $u := 2tv$, applying (3.7) and (3.2) we get

$$\begin{aligned}\int_0^{\infty} u^n \frac{\exp\bigl(-(u-1)^2/(2tu)\bigr)}{\sqrt{2\pi t u}}\, du &= \frac{e^{1/t}}{\sqrt{2\pi t}} \int_0^{\infty} u^{n-1/2} \exp\left(\frac{-u}{2t} - \frac{1}{2tu}\right) du \\ &= \frac{e^{1/t}(2t)^{n+1/2}}{\sqrt{2\pi t}} \int_0^{\infty} v^{n-1/2} \exp\left(-v - \frac{1}{4t^2 v}\right) dv \\ &= e^{1/t}\sqrt{\frac{2}{\pi t}} K_{-n-1/2}(1/t) = y_n(t),\end{aligned}$$

which concludes the proof.

Alternatively, we could apply Theorem 3.1 and the relation (3.3). ■

Denote by ν_t the corresponding probability measure, i.e.

$$\nu_t := \frac{\exp\bigl(-(u-1)^2/(2tu)\bigr)}{\sqrt{2\pi t u}} \chi_{(0,+\infty)}(u)\, du. \tag{4.3}$$

Although the family $\{\nu_t\}_{t>0}$ is not a convolution semigroup, we will see that every ν_t is infinitely divisible.

THEOREM 4.2. *For $t > 0$ we have the convolution formula*

$$\nu_t = \gamma_t * \mathbf{D}_t \mu_{1/t}, \tag{4.4}$$

where γ_t denotes the gamma distribution with shape $1/2$ and scale $2t$,

$$\gamma_t = \frac{\exp\bigl(-u/(2t)\bigr)}{\sqrt{2\pi t u}} \chi_{(0,+\infty)}(u)\, du, \tag{4.5}$$

and $\mathbf{D}_t \mu_{1/t}$ is the dilation of $\mu_{1/t}$ by t,

$$\mathbf{D}_t \mu_{1/t} = \frac{\exp\bigl(-(u-1)^2/(2tu)\bigr)}{\sqrt{2\pi t u^3}} \chi_{(0,+\infty)}(u)\, du. \tag{4.6}$$

In particular, ν_t is infinitely divisible.

P r o o f. From Theorem 4.1 we see that the characteristic function of ν_t, i.e.

$$\psi_t(x) = \frac{\exp\left(\frac{1}{t} - \frac{1}{t}\sqrt{1 - 2tx\mathrm{i}}\right)}{\sqrt{1 - 2tx\mathrm{i}}}, \tag{4.7}$$

is the product of

$$\frac{1}{\sqrt{1 - 2tx\mathrm{i}}},$$

the characteristic function of γ_t, and

$$\exp\left(\frac{1}{t} - \frac{1}{t}\sqrt{1 - 2tx\mathrm{i}}\right),$$

the characteristic function of $\mathbf{D}_t \mu_{1/t}$, which proves (4.4). Since both γ_t and $\mathbf{D}_t \mu_{1/t}$ are infinitely divisible, so is their convolution ν_t. ■

Let us list some interesting integer sequences which arise from the polynomials (3.1) and (3.2), together with their numbers in the *On-Line Encyclopedia of Integer Sequences* [11] and the corresponding probability distribution. For their combinatorial applications we refer to [11]:

1. A144301: $w_n(1)$, moments of μ_1;
2. A107104: $w_n(2)$, moments of μ_2;
3. A043301: $w_{n+1}(2)/2$, moments of the density function $u \cdot \rho_2(u)/2$;
4. A080893: $2^n \cdot w_n(1/2)$, moments of the density function $\rho_{1/2}(u/2)/2$;
5. A001515: $y_n(1)$, moments of ν_1;
6. A001517: $y_n(2)$, moments of ν_2;
7. A001518: $y_n(3)$, moments of ν_3;
8. A065919: $y_n(4)$, moments of ν_4.

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Wojciech Młotkowski
Institute of Mathematics
University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: mlotkow@math.uni.wroc.pl

Anna Romanowicz
E-mail: annaromanowicz85@gmail.com

Received on 28.4.2013;
revised version on 25.9.2013