Supercategories

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1 Introduction

Motivated by the desire to better understand modular tensor categories (or whatever replaces them) in the context of (rational) *super*conformal 2-dimensional field theories, the following is an attempt to capture the basic axioms of superalgebra in a more "arrow-theoretic" way than commonly done, such that its generalization to less familiar contexts can proceed more systematically.

One possible way to conceive this endeavour is to think of the problem of finding the concept which is to a supergroup like a category is to a group.

	single object	**many objects**	**no invertibility**
ordinary	group	groupoid	category
super	supergroup	supergroupoid	supercategory

Table 1: We are looking for an "arrow theoretic" way to talk about supersymmetry. By this is meant an abstract diagrammatic formulation that may easily be internalized into various contexts which may look entirely different from the context of graded commutative algebra which is usually the starting point for the definition of superalgebra. One way to think of this problem is to think of completing the last two entries in the last row of the above table.

2 Flows on Categories

We would like to understand a supercategory as a category equipped with the "flow of an odd vector field" on it. From considerations in other contexts, one finds that the following is a good arrow-theoretic way to talk about vector fields and their flows:

Definition 1 (G-flow on a category)**.** *For G a group, a G-flow on a category C is*

- *a (strict) G-action on C*

$$R : \Sigma G \to \mathrm{Aut}(C)$$

- *for each $g \in G$ a natural transformation*

$$\begin{tikzcd} C \arrow[r, bend left=50, "\mathrm{Id}"{name=U}] \arrow[r, bend right=50, "R_g"'{name=D}] & C \arrow[Rightarrow, from=U, to=D, "f_g"] \end{tikzcd}$$

respecting the action, i.e. such that

$$\begin{tikzcd} C \arrow[r, bend left=50, "\mathrm{Id}"{name=U1}] \arrow[r, bend right=50, "R_{g_1}"'{name=D1}] & C \arrow[r, bend left=50, "\mathrm{Id}"{name=U2}] \arrow[r, bend right=50, "R_{g_2}"'{name=D2}] & C \arrow[Rightarrow, from=U1, to=D1, "f_{g_1}"] \arrow[Rightarrow, from=U2, to=D2, "f_{g_2}"] \end{tikzcd} = \begin{tikzcd} C \arrow[r, bend left=50, "\mathrm{Id}"{name=U}] \arrow[r, bend right=50, "R_{g_1 g_2}"'{name=D}] & C \arrow[Rightarrow, from=U, to=D, "f_{g_1 g_2}"] \end{tikzcd} .$$

In other words, a G-flow on C is a functor

$$F : \Sigma G \to \Sigma(\mathrm{Hom}_{\mathrm{Cat}}(\mathrm{Id}_C, -)) .$$

Remark. When G is Lie and C is a smooth category, we usually want to require everything in sight to be smooth to get the concept of a smooth flow on C.

Remark. One should think of the component maps of these transformations as providing the "flow lines" of the G-action. This is illustrated by the following example.

Example (ordinary vector fields and their flows) Let X be some manifold and let

$$\mathcal{P}_1(X)$$

be the smooth path groupoid of X. Its space of objects is X and its space of morphisms are thin-homotopy classes of smooth paths in X (compare [2]).

Every vector field $v \in \Gamma(TX)$ on X gives rise to smooth $\mathbb{R}$-flow on $\mathcal{P}_1(X)$

$$
t \;\mapsto\; \begin{tikzcd} \mathcal{P}_1(X) \arrow[r, bend left=50, "\mathrm{Id}"{name=U}] \arrow[r, bend right=50, "\exp(tv)"'{name=D}] & \mathcal{P}_1(X) \arrow[Rightarrow, from=U, to=D] \end{tikzcd}
$$

where the componet map of the transformation maps each point to the flow line of length t along v starting at that point.

I think that, conversely, every smooth $\mathbb{R}$-flow on $\mathcal{P}_1(X)$ defines a vector field on X this way. But I do not try to prove that here.

Example/Remark (ordinary Lie derivatives) A good way to understand the principle at work here is to compare this to the infinitesimal version, which may be more familiar.

Consider the 2-category whose objects are differential graded commutative algebras, whose morphisms are linear maps that are both chain maps as well as algebra derivations, and whose 2-morphisms are chain homotopies of these.

For X a smooth manifold as before, one object in this 2-category is $(\Omega^\bullet(X), d)$, the deRham complex of X. Every vector field v on X yields an inner derivation ι_v of this, which can be regarded as a chain map

$$
\begin{tikzcd} \Omega^\bullet(X) \arrow[r, bend left=50, "0"{name=U}] \arrow[r, bend right=50, "L_v := [d, \iota_v]"'{name=D}] & \Omega^\bullet(X) \arrow[Rightarrow, from=U, to=D, "\iota_v"] \end{tikzcd}
$$

which connects the 0-derivation with the Lie derivative $L_v := [d, \iota_v]$ induced by v. Compare [3, 4].

The definition of a G-flow on a category is exactly modelled after this example, supposed to capture the integrated as opposed to infinitesimal version.

Example (inner automorphisms). Among all automorphisms of a group G, precisely the inner automorphisms are those which gives rise to flows on G, in the above sense, in that the action of inner automorphisms

$$
g \;\mapsto\; \begin{tikzcd} \Sigma G \arrow[r, bend left=50, "\mathrm{Id}"{name=U}] \arrow[r, bend right=50, "\mathrm{Ad}_g"'{name=D}] & \Sigma G \arrow[Rightarrow, from=U, to=D, "g"] \end{tikzcd}
$$

is connected to the identity automorphism. That's in fact the very definition of inner automorphism, once one looks at the naturality square for the above natural transformation:

$$
\begin{tikzcd}
\bullet \arrow[r, "g"] \arrow[d, "h"'] & \bullet \arrow[d, "\mathrm{Ad}_g h"] \\
\bullet \arrow[r, "g"] & \bullet
\end{tikzcd}
$$

for all $h \in G$.

Definition 2 (Morphisms of categories with G-flow)**.** *For (C, R) and (C', R') two categories equipped with a G-flow, a morphism*

$$F : (C, R) \longrightarrow (C', R')$$

between them is a functor

$$F : \; C \longrightarrow C'$$

such that

$$
\begin{tikzcd}
C \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "R_g"'] \arrow[r, phantom, "\Downarrow"] & C \arrow[d, "f"] \\
 & C'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
C \arrow[d, "F"] & \\
C' \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "R'_g"'] \arrow[r, phantom, "\Downarrow"] & C'
\end{tikzcd}
$$

Example. In [4] it is discussed that the condition on an Ehresmann connection on the total space of a G-bundle is precisely of this form, albeit in the infinitesimal context. The two categories with G-flow which appear in that context are $\mathcal{P}_1(P)$ (paths in the total space of the G-bundle with the obvious G-action on them) and ΣG itself, with the adjoint G-action.

3 Odd flows and supercategories

Definition 3 (supercategory)**.** *A supercategory is a category equipped with a $\mathbb{Z}_2$-flow.*

Hence a supercategory is a diagram

$$
\begin{tikzcd}
C \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\Downarrow"] & C
\end{tikzcd}
$$

in Cat, such that

$$\begin{tikzcd} C \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\Downarrow"] & C \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\Downarrow"] & C \end{tikzcd} = \begin{tikzcd} C \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow \mathrm{Id}"] & C \end{tikzcd} .$$

Here we agree, for notational convenience, that unlabeled arrows from Id to s denote the specified transformation belonging to s.

Remark. In light of our interpretation of flows on categories as something generalizing the notion of ordinary flows of ordinary vector fields (as described in the examples in 2), we should think of a supercategory as being a category equipped with an "odd vector field".

Example (modules over graded commutative algebras). Let A be a $\mathbb{Z}_2$-graded commutative algebra and let Mod_A be the category of right A-modules. Any A-module V is naturally $\mathbb{Z}_2$-graded itself

$$V = V_0 \oplus V_1 .$$

The functor

$$s : \mathrm{Mod}_A \to \mathrm{Mod}_A$$

exchanges the degree of these modules

$$(sV)_0 := V_1$$

$$(sV)_1 := V_0$$

and there are canonical isomorphisms

$$s_V : V \xrightarrow{\sim} sV .$$

The action of s on morphisms is given by conjugations with these isomorphisms

$$s : (V \xrightarrow{f} W) \mapsto \begin{tikzcd} V \arrow[r, "f"] & W \arrow[d, "s_W"] \\ sV \arrow[u, "s_V^{-1}"] & sW \end{tikzcd} .$$

This manifestly makes the s_V the components of the natural isomorphism

$$\begin{tikzcd} \mathrm{Mod}_A \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{Mod}_A \end{tikzcd} .$$

Notice that s is often denoted "Π". It is called the *parity change functor*.

In fact, we can understand s here as coming from tensoring with the module sA, which is the algebra itself, but with the degrees reversed. This will be important in 4, when we interpret braided monoidal supercategories as certain one-object 3-categories with extra structure.

Definition 4 (Grading operator)**.** *Given any supercategory C, we say a grading on C is a subcategory*

- $$C_0 \hookrightarrow C$$

  closed under s;

- *together with a nontrivial transformation*

  $$\begin{tikzcd} C_0 \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow b"] & C_0 \end{tikzcd}$$

  such that

  $$\begin{tikzcd} C_0 \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow b"] & C_0 \arrow[r, "s"'] & C_0 \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\mathrm{Id}"'] \arrow[r, phantom, "\Downarrow b"] & C_0 \arrow[r, "s"'] & C_0 \end{tikzcd}$$

  is involutary and central with respect to horizontal composition of transformations.

Remark. This condition should be thought of as saysing that "the fermion number operator $(-1)^{\text{fermion number}}$ anticommutes with the parity change operator".

Remark. The maximal subcategory C_0 with the above property has the same objects as C and all even-graded morphisms of C, but no odd-graded morphisms. In applications we are mostly interested in working with such a maximal C_0. Usually this still remembers the odd-graded morphisms as the corresponding internal Hom-spaces. But the existence of C is actually crucial for the notion of supersymmetry: there are categories C_0 as above, which do not come from any supercategory C, namely if there is no bijection between even and odd objects of C_0.

Example. For $C = \mathrm{Mod}_A$ the supercategory of modules for the graded commutative algebra A, let
$$C_0 = C_{\mathrm{even}} \oplus C_{\mathrm{odd}}$$
be the subcategory of all even graded morphisms. Let the component map b_V of the grading operator
$$b_V : V \to V$$
be the identity if V is even graded and be multiplication with -1 if V is odd graded. Then the above condition says that the natural transformation whose components are
$$sV \xrightarrow{sb_V} sV \xrightarrow{b_{sV}} sV$$
has to square to the identity and commute with all other natural transformations. Indeed, we have
$$\begin{tikzcd} sV \arrow[r, "sb_V"] \arrow[rr, bend right, "\cdot -1"'] & sV \arrow[r, "b_{sV}"] & sV \end{tikzcd} .$$

4 Braided monoidal supercategories

We formulate the concept of a braided monoidal supercategory by making use of the considerations in [1].

Definition 5. *A braided monoidal supercategory is a supercategory C such that the graded subcategory C_0 comes from a $(\mathbb{Z}_2 \xrightarrow{\mathrm{Id}} \mathbb{Z}_2)$ -stabilized 3-category K:*
$$C_0 = \mathrm{Hom}_K(\mathrm{Id}_\bullet, --)\,.$$

Let us unwrap this definition. That K is a $(\mathbb{Z}_2 \xrightarrow{\mathrm{Id}} \mathbb{Z}_2)$ -stabilized 3-category means that there are precisely two 1-morphisms
$$\bullet \xrightarrow{\mathrm{Id}} \bullet$$
and
$$\bullet \xrightarrow{\sigma} \bullet\ ,$$
which form the group $\mathbb{Z}_2$. In addition, there is a singled-out 2-morphism
$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "\mathrm{Id}", ""{name=U, below}] \arrow[r, bend right=50, "\sigma"', ""{name=D, above}] & \bullet \arrow[Rightarrow, from=U, to=D, "J", "\sim"'] \end{tikzcd}$$
between the identity 1-morphism and the nontrivial 1-morphism, together with its inverse.

The presence of this 2-morphism J makes $\mathrm{Hom}_K(\mathrm{Id}_\bullet, --)$ naturally the C_0-part of a supercategory C:

the parity operator is simply postcomposition with J

$$\begin{tikzcd} s : \bullet \arrow[rr, bend left=50, "\mathrm{Id}"] \arrow[rr, bend right=50, "\mathrm{Id}"'] \arrow[rr, phantom, "\Downarrow U"] && \bullet & \mapsto & \bullet \arrow[rr, bend left=50, "\mathrm{Id}"] \arrow[rr, bend right=40, "\mathrm{Id}" description] \arrow[rr, bend right=90, "\sigma"'] \arrow[rr, phantom, shift left=2, "\Downarrow U"] \arrow[rr, phantom, shift right=6, "\Downarrow J"] && \bullet . \end{tikzcd}$$

Remark. Notice that not every $\mathbb{Z}_2$-equivariant category C is automatically a supercategory: a necessary condition for C to be super is that there is a bijection beween its odd and its even graded objects. It is this bijection (the very super*symmetry*) which makes a $\mathbb{Z}_2$-graded category a supercategory.

Example. Our previous example, the supercategory $C = \mathrm{Mod}_A$ of modules for a graded commutative algebra A, is in fact a braided monoidal supercategory in the obvious way. The object J here is the module sA, namely the algebra A itself, but in odd grade.

References

[1] U.S. *On G-equivariant fusion categories*

[2] U. S., K. Waldorf, *Parallel transport and functors*

[3] U.S., J. Stasheff, *Structure of Lie n-algebras*

[4] U.S., J. Stasheff, *Connections for Lie n-algebras*