On sparse sets hitting linear forms

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Abstract

Let $a_1 < a_2 < \ldots < a_s$ be gives integers and let $[n]$ denote the set $\{1, 2, \ldots, n\}$. In this note we investigate the size of minimum subsets of $[n]$ which intersect every set in $[n]$ of the form $\{a_1 x, a_2 x, \ldots, a_s x\}$ for some $x \in [n]$. For the most part, we can only estimate this extremal density although there is an interesting class of a_i's for which we can find the exact answer.

1 Introduction

This paper had its genesis in some work begun by the latter two authors some 25 years ago concerning the size and structure of the smallest subsets of $[n] := \{1, 2, \ldots, n\}$ which hit every set in some specified family of subsets of $[n]$. Unfortunately, although we came back to these questions from time to time, we never got around to writing up what we knew (and didn't know) when Paul was still alive. However, this meeting provided an ideal stimulus for pushing forward our knowledge boundaries even further, and summarizing in print the current state of affairs for what we feel is an attractive set of questions in combinatorial number theory.

2 Preliminaries

For a fixed $r \times s$ integer matrix $A = (a_{ij}), 1 \le i \le r, 1 \le j \le s$, let us call a subset $S = S_A(n) \subseteq [n] := \{1, 2, \ldots, n\}$ *A-hitting* if for every (non-trivial) vector $\bar{x} = (x_1, x_2, \ldots, x_s)^*$ with $x_i \in [n]$ for all i, and satisfying $Ax = \bar{0}$, we always have $x_j \in S$ for some j. (Here, non-trivial means all x_i are distinct, and y^* denotes the transpose of y). Further, define $s(n) = s_A(n)$ to be the minimize possible size of an A-hitting set $S_A(n)$. A classical problem in combinatorics is that of determining

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(or estimating) $s_A(n)$ for various choices of A. We mention several of these now, as an introduction to the cases we will be considering.

To begin with, take A to be the 1×3 matrix $A = (1\ 1\ -1)$. Thus, $A\bar{x} = \bar{0}$ implies $x_1 + x_2 - x_3 = 0$, i.e., $x_1 + x_2 = x_3$. In this case it is not hard to see that

$$s_A(n) = \lfloor \frac{n}{2} \rfloor,$$

with two somewhat different sets $s_A(n)$ achieving this bound. Namely, we can choose all *even* numbers in $[n]$, or we can choose all numbers $\leq n/2$ in $[n]$.

Next, consider the (even smaller) 1×2 matrix $A = (2\ \ -1)$. In this case, $A\bar{x} = \bar{0}$ implies that $x_2 = 2x_1$, so that any A-hitting set must intersect every set of the form $\{x, 2x\} \subseteq [n]$. What is $s_A(n)$ in this case? Motivated by the preceding example, there are two natural guesses one might make. One of these is $S_1 = \{x \in [n] \ : \ x = 2^{2k+1}t,\ k \geq 0,\ t \text{ odd}\}$. For this set we have $|S_1| \sim (\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots)n = \frac{n}{3}$. The other is $S_2 = \{x \in [n] \ : \ \frac{n}{2^{2k+2}} < x \leq \frac{n}{2^{2k+1}},\ k \geq 0\}$ which also turns out to have $|S_2| \sim \frac{n}{3}$. In fact, as we shall see later, it is not hard to show that $s_A(n) \sim \frac{n}{3}$ for this A.

For our next example, we mention the (very classical) case $A = (1\ 1\ \ -2)$. Here, $A\bar{x} = \bar{0}$ implies $x_1 + x_2 = 2x_3$, so the sets $\{x_1, x_2, x_3\} \subseteq [n]$ which any A-hitting set must intersect are just the 3-term arithmetic progressions (since $x_3 = \frac{x_1+x_2}{2}$). It is because of examples like this that we require solution vectors to be non-trivial, since (x, x, x) is a solution to $A\bar{x} = \bar{0}$ for *any* x. Erdős and Turán already conjectured in 1936 [7], that in this case $s_A(n) = (1 + o(1))n$, or phrased in another way, any set of integers with positive upper density must contain a 3-term arithmetic progression. This was finally first proved by Roth [16] in 1954, with subsequent striking extensions to this result by Szemerédi [19], Gowers [9], and others (we will discuss these further at the end of the paper). In general, all matrices A for which $A\ \bar{1} = 0$ (where $\bar{1}$ is the all 1's vector) have the special property that $s_A(n) = (1 + o(1))n$.

Finally, consider the somewhat larger example

$$\begin{bmatrix} 2 & -1 & 0 \\ 4 & 0 & -1 \end{bmatrix}.$$

Here, $A\bar{x} = \bar{0}$ implies $x_2 = 2x_1, x_3 = 4x_1$, so an A-hitting set $S_A(n)$ is one which intersects every subset of the form $\{x, 2x, 4x\} \subseteq [n]$.

As usual (it seems), there are two natural candidates for achieving the minimum size $s_A(n)$. They are

$$S_3 = \{x \in [n] \ : \ x = 2^{3k+2}t, \ k \geq 0, \ t \text{ odd}\}$$

and

$$S_4 = \{x \in [n] \ : \ \frac{n}{2^{3k+3}} < x \leq \frac{n}{2^{3k+2}}, \ k \geq 0\}.$$

And, as we might expect (by now),

$$|S_3| \sim \frac{n}{7} \sim |S_4|$$

which is the correct asymptotic value of $s_A(n)$.

The reason we have mentioned the two candidates in many of the preceding examples is that while they both gave the same asymptotic values for $s_A(n)$, one of the two makes perfect sense if we are considering subsets of $[0,1]$ (say), which hit all *real* solution vectors $\bar{x}$ to $A\bar{x} = \bar{0}$ (S_2 and S_4) whereas S_1 and S_3 don't. It is this difference which results in the rather different and very interesting behavior for matrices like $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$. We shall investigate this particular matrix in some detail in Section 4.

3 Asymptotic values

In general, we will be interested in the asymptotic behavior of $s_A(n)$. So, define

$$\sigma(A) \ := \ \liminf_{n\to\infty} \frac{s_A(n)}{n}$$

Thus, we have seen

A	$\sigma(A)$
$(1\ 1\ -1)$	$1/2$
$(2\ -1)$	$1/3$
$(1\ 1\ -2)$	1
$\begin{pmatrix} 2 & -1 & 0 \\ 4 & 0 & -1 \end{pmatrix}$	$1/7$

We could also ask the same questions for *real* solutions $\bar{x}$ to $A\bar{x} = \bar{0}$: How small can the measure of a set $R \subseteq [0,1]$ be which hits every (non-trivial) solutions $\bar{x}$ to $A\bar{x} = \bar{0}$?

Denote this value by $\rho(A)$. It is not hard to see that in general, $\sigma(A) \leq \rho(A)$.

For example, for $A = (2 \ \ -1)$,

$$\begin{aligned} \rho(A) &= \inf\{\mu(R) \ : \ R \subseteq [0,1] \text{ hits every set } \ \{x, 2x\} \subseteq [0,1]\} \\ &= 1/3 \qquad\qquad\qquad\qquad \text{where } \mu \text{ denotes Lebesgue measure} \end{aligned}$$

with the set $R = \{x \in [0,1] \ : \ \frac{1}{2^{2k+2}} < x \leq \frac{1}{2^{2k+1}}, \ k \geq 0\}$ being the only set (up to measure 0) achieving this value (as we will see shortly). As mentioned before, we can think of R as a "continuous" version of the set S_2 in the preceding section for the integer problem for A.

4 $\{x, 2x, 3x\}$

Much of the initial interest in the class of problems involved around a bet [1] that the second author made with the other two concerning hitting sets for solutions to $A\bar{x} = \bar{0}$ for the matrix $A = A_{1,2,3} = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \end{pmatrix}$. In this case, solution set to $A\bar{x} = \bar{0}$ are just sets of the form $\{x, 2x, 3x\} \subseteq [n]$ for some integer x. For the *real* case, Paul constructed the following candidate for a minimal set R hitting every set $\{x, 2x, 3x\} \subseteq [0,1]$:

$$R = \{x \in [0,1] \ : \ 6^{-k} \leq x < 2 \cdot 6^{-k}, k \geq 1\}$$

Thus R consists of all $x \in [0,1]$ which when expressed base 6 as $x = 0.x_1x_2x_3\ldots$ has its first nonzero "digit" x_i equal to 1.

An easy calculation gives $\mu(R) = 1/5$, and so $\rho(1,2,3) := \rho(A_{1,2,3}) \leq 1/5$. In fact, this is the correct value of $\rho(1,2,3)$ as we now show.

Theorem 1

$$\rho(1,2,3) = 1/5 \tag{1}$$

Proof: Denote by Y the subinterval $(1/6, 1/3] \subseteq [0,1]$ and let X be any measurable set in $(1/6, 1]$

[1] Unfortunately, for just \$ 20.

which hits every set $\{y, 2y, 3y\}$ for some $y \in Y$. Define

$$\begin{aligned} X_1 &:= X \cup (1/6, 1/3], \\ X_2 &:= X \cup (1/3, 1/2], \\ X_3 &:= X \cup (1/2, 2/3], \\ X_4 &:= X \cup (2/3, 1] \end{aligned}$$

Note that

$$\begin{aligned} & 2y \in X \cap 2Y = X_2 \cup X_3 \\ \Longrightarrow \quad & y \in \frac{1}{2}(X \cap 2Y) = \frac{1}{2}X_2 \cup \frac{1}{2}X_3 \end{aligned}$$

Similarly,

$$\begin{aligned} & 3y \in X \cap 3Y = X_3 \cup X_4 \\ \Longrightarrow \quad & y \in \frac{1}{3}(X \cap 3Y) = \frac{1}{3}X_2 \cup \frac{1}{3}X_3 \end{aligned}$$

Thus,

$$\begin{aligned} Y &\subseteq X_1 \cup \frac{1}{2}(X \cap 2Y) \cup \frac{1}{3}(X \cap 3Y) \\ &= X_1 \cup \frac{1}{2}X_2 \cup \frac{1}{2}X_3 \frac{1}{3}X_3 \cup \frac{1}{3}X_4 \end{aligned}$$

and so

$$\begin{aligned} \frac{1}{6} &= \mu(Y) \leq \mu(X_1) + \frac{1}{2}\mu(X_2) + (\frac{1}{2} + \frac{1}{3})\mu(X_3) + \frac{1}{3}\mu(X_4) \\ &\leq \mu(X_1) + \mu(X_2) + \mu(X_3) + \mu(X_4) \\ &= \mu(X) \end{aligned}$$

with equality only if

$$\mu(X_1) = \frac{1}{6}, \quad \mu(X_2) = \mu(X_3) = \mu(X_4) = 0.$$

Now repeat the same argument for $Y' = (1/36, 1/6]$, concluding that $\mu(X') \leq 1/36$, etc. (No point in X can hit $\{y', 2y', 3y'\}$ for $y' \in Y'$). Thus, if R hits every set $\{x, 2x, 3x\} \subseteq [0, 1]$ then $\mu(R) \geq \sum_{n=1}^{\infty} 6^{-n} = 1/5$ with equality (up to a set of measure 0) only if

$$R = \cup_{k=1}^{\infty}(1/6^k, 2/6^k]$$

$\square$

Now, what about the *integer* case for $A_{1,2,3} = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \end{pmatrix}$. In light of all of the examples mentioned up to this point, it is natural conjecture that $\sigma(A) = 1/5$ as well (which is what Paul did. More precisely, he bet that we couldn't *find* sets $S(n) \subseteq [n]$ with $s(n) < (1/5 - \epsilon)n$ for $\epsilon > 0$ fixed and $n \to \infty$, which hit every integer set $\{x, 2x, 3x\} \subseteq [n]$).

Theorem 2

$$\sigma(1,2,3) \ := \ \sigma(A_{1,2,3}) \ < \ 0.1997$$

Proof: Define

$$\begin{aligned} D \ &:= \ \{2^i 3^j \ : \ i,j \geq 0\} = \{d_1 < d_2 < d_3 \ldots\} \\ &= \ \{1,2,3,4,6,8,9,12,16,18,24,27,\ldots\} \end{aligned}$$

and

$$D_k \ := \ \{d_1 < d_2 < \ldots < d_k\}$$

Write

$$[n] \ = \ \cup_{1 \leq t \leq n, (t,6)=1} C(t)$$

where $C(t) \ := \ \{2^i 3^j t \ : \ i,j \geq 0\} \cap [n]$.

Fact:

$$\begin{aligned} & \{x, 2x, 3x\} \subseteq [n] \\ \iff \ & \{x, 2x, 3x\} \subseteq C(t) = \{d_1 t, d_2 t, \ldots, d_k t\} \quad \text{for some } t \\ \iff \ & \{y, 2y, 3y\} \subseteq D(k) \end{aligned}$$

where $y = 2^i 3^j$ for some $i, j \geq 0$.

Question: How many $t \in [n], (t,6) = 1$, have $|C(t)| = k$?

Answer: We need $td_k \leq n < td_{k+1}$, i.e.,

$$\frac{n}{d_{k+1}} < t \leq \frac{n}{d_k}$$

Thus, there are $n(\frac{1}{d_k} - \frac{1}{d_{k+1}}) \cdot \frac{1}{3} + O(1)$ such t, the factor of 1/3 coming from the fact that $\phi(6) = 2$, i.e., we are requiring $(t, 6) = 1$.

Let $f(k)$ denote the size of the smallest set hitting all $\{x, 2x, 3x\} \subseteq D(k)$. Then it is clear that we have

$$\sigma(1,2,3) = \frac{1}{3}\sum_{k\geq 1} f(k)(\frac{1}{d_k} - \frac{1}{d_{k+1}}) \tag{2}$$

We point out that this derivation already appears in the earlier paper [11] of one of the authors. Of course, this leads to the next (non-obvious) question:

Question: What is $f(k)$?

Let us now identify the integers in $D = \{2^i 3^j \ : \ i, j \geq 0\} = \{d_1 < d_2 < d_3 < \dots\}$ with lattice points in the non-negative quadrant so that the integer $d_k = 2^i 3^j$ corresponds to the lattice point (i, j). With this representation, a set $\{y, 2y, 3y\}$ in D with $y = 2^i 3^j$ corresponds to the three lattice points (i, j), $(i + 1, j)$, $(i, j + 1)$ which we will call an "L" (see Figure 1). In order to avoid possible

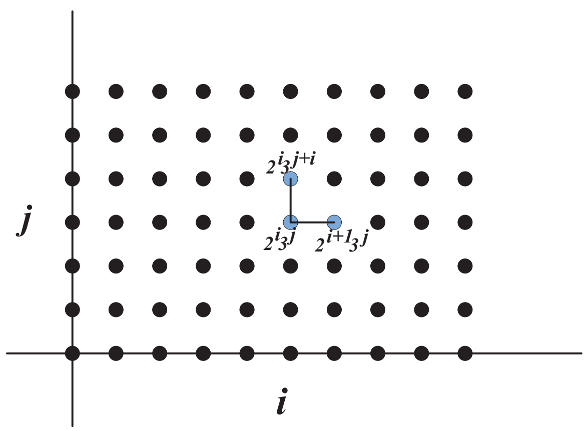

Figure 1: An L

confusion, we will let $\Delta(k)$ denote the lattice points corresponding to the integers in $D(k)$. In Table 1, we show two versions of $\Delta(25)$–the first with points labeled by the *value* of the integers in $D(25)$, and the second with points labeled by the *indices* of the corresponding integers in $D(25)$.

Note that $\Delta(k)$ in general consists exactly of those lattice points in the triangle bounded by the x and y axes (or in this case, the i and j axes), and the line $L(c) : i \log 2 + j \log 3 = c$ for an appropriate value c (actually, an interval of values). As c increases, the line $L(c)$ passes over one more lattice

81	162						
27	54	108					
9	18	36	72	144			
3	6	12	24	48	96	192	
1	2	4	8	16	32	64	128

(a) $\Delta(25)$ labelled by values

19	24						
12	16	21					
7	10	14	18	23			
3	5	8	11	15	20	25	
1	2	4	6	9	13	17	22

(b) $\Delta(25)$ labelled by indices

Table 1: Two labellings of $\Delta(25)$

point which is d_{k+1}, which is now also inside the growing triangle. The apparently chaotic behavior by which $\Delta(k)$ grows as $k \to \infty$ seems to us to be a major source of the difficulty in determining the exact value of $\rho(1,2,3)$. In fact, it would seem to us to be a minor miracle if $\rho(1,2,3)$ were to turn out to be rational.

So we have converted our question about $f(k)$ to a geometrical (but equivalent) one:

What is the smallest set which hits all the "L's" in $\Delta(k)$?

Let us consider a possible candidate, which we call G_0. These are all the points (i,j) in $\Delta(k)$ with $i - j \equiv 0 \pmod 3$. In other words, G_0 consists of all the lattice points in $\Delta(k)$ which lie on a family of 45° parallel lines spaced 3 apart. It is easy to see that every "L" in $\Delta(k)$ must be hit by G_0. More generally, we could use either of the two translates $G_1 = \{(i,j) \in \Delta(k) \ : \ i-j \equiv 1(\mathrm{mod}3)\}$ or $G_2 = \{(i,j) \in \Delta(k) \ : \ i-j \equiv 2(\mathrm{mod}3)\}$ as well, and computation shows that occasionally, the sizes of these sets differ by 2 or more.

Note that it can happen that $(i,0) \in \Delta(r)$ but $(i-1,1) \notin \Delta(r)$ for some i and r (since the angle between the i-axis and the line $i\log 2 + j\log 3 = c$ is less than 45°). When this happens then even though $(i,0) \in G_{i\mathrm{mod}3}$, this point is not needed for $G_{i\mathrm{mod}3}$ to hit all "L's" in $\Delta(r)$. Hence, it should not be counted when computing the sizes of the minimum sets hitting all the "L's" in $\Delta(r)$. So, let us *define* $g(k)$ to be the size of the smallest of the three sets G_0, G_1 and G_2, where the size of G_j is decreased by 1 if the preceding situation applies. Then we have:

$$f(k) \leq g(k) \leq \lfloor \frac{k}{3} \rfloor \tag{3}$$

We should mention at this point that as far as we can tell, it is always true that $f(k) = g(k)$. However, we are unable to show that this is always the case. Now, for our estimate:

$$\begin{aligned}
\sigma(1,2,3) &= \frac{1}{3}\sum_{k\geq 1} f(k)(\frac{1}{d_k} - \frac{1}{d_{k+1}}) \\
&\leq \frac{1}{3}\sum_{k\geq 1} g(k)(\frac{1}{d_k} - \frac{1}{d_{k+1}}) \\
&\leq \frac{1}{3}\sum_{k<M} g(k)(\frac{1}{d_k} - \frac{1}{d_{k+1}}) + \frac{1}{3}\sum_{k\geq M} \frac{k}{3}(\frac{1}{d_k} - \frac{1}{d_{k+1}}) \\
&= \frac{1}{3}\sum_{k<M} g(k)(\frac{1}{d_k} - \frac{1}{d_{k+1}}) + \frac{1}{9}(\frac{M}{d_M} + \sum_{k>M} \frac{k}{3}\frac{1}{d_k}) \qquad (4)
\end{aligned}$$

For $M = 2000$, this gives

$$\sigma(1,2,3) < 0.1996805162$$

and the theorem is proved. □

The fact that M has to be rather large before we break the $1/5 = 0.2000$ barrier helps explain why it is difficult to actually display sets that are better than what the $\rho(1,2,3) = 1/5$ construction gives when converted to integers.

To establish a lower bound, we have

Theorem 3

$$\sigma(1,2,3) > 0.1990389$$

Proof: Since

$$\begin{aligned}
\sigma(1,2,3) &\geq \frac{1}{3}\sum_{k<M} f(k)(\frac{1}{d_k} - \frac{1}{d_{k+1}}) + \frac{1}{3}\sum_{k\geq M} f(M)(\frac{1}{d_k} - \frac{1}{d_{k+1}}) \\
&\geq \frac{1}{3}\sum_{k<M} f(k)(\frac{1}{d_k} - \frac{1}{d_{k+1}}) + \frac{1}{3}f(M)\frac{1}{d_k} \qquad (5)
\end{aligned}$$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$f(k)$	0	0	1	1	1	1	2	2	2	3	3	4	4	4	4	5	5	5	6	6	7	7	7	8	8

Table 2: Values of $f(k)$ for $k \leq 25$

then it is simply a matter of (patiently) determining $f(k)$ for some small values of k (we did this for $k \leq 50$; see Table 2 for values of $f(k), k \leq 25$).

Thus, taking $M = 50$ in (5) we find

$$\sigma(1, 2, 3) > 0.1990389$$

as claimed.

If $f(k) = g(k)$ for all k, as we believe, then in fact we would have

$$\sigma(1, 2, 3) = 0.199680516197\ldots$$

In any case, as one of the authors likes to say, every right-thinking person knows that $\sigma(1, 2, 3)$ is irrational.

We point out the quite similar arguments apply to more general sets like $\{x, ax, bx\}$ for example. However, when $\frac{\log a}{\log b}$ is irrational then the order in which lattice points enter the triangle (as $\Delta(k)$ grows) varies unpredictably, making it difficult to know when $f(k)$ increases, and preventing us from being able to evaluate the corresponding sum for $\sigma(1, a, b)$ exactly.

5 Other sets

Given the increase in difficulty involved in determining $\sigma(1, 2, 3)$, one might wonder if there is much hope in dealing with even larger examples. In particular, let us focus on matrices A of the form

$$\begin{pmatrix} a_1 & -1 & 0 & 0 & \ldots & 0 \\ a_2 & 0 & -1 & 0 & \ldots & 0 \\ & & & & & \\ a_s & 0 & 0 & 0 & \ldots & -1 \end{pmatrix}.$$

In this case we denote the asymptotic value $\rho(A)$ by $\rho(1, a_1, a_2, \ldots, a_s)$ where the sets to be hit have the form $\{x, a_1x, a_2x, \ldots, a_sx\}$. It turns out that there is an exceptional class of situations in which σ *can* be determined exactly. As an example, let us examine the family $\{x, 2x, 3x, 6x\}$.

Using the same methods employed for determining $\rho(1,2,3)$ in Theorem 1, it can be shown that $\rho(1,2,3,6) = 1/11$. What about $\sigma(1,2,3,6)$? This is answered by

Theorem 4

$$\sigma(1,2,3,6) = 1/12$$

Proof: Defining D and $D(k)$ as we did in Theorem 3 (since the only primes dividing $\prod_i a_i$ are still 2 and 3), we see that $\{y, 2y, 3y, 6y\}$ hits $D(k)$ if and only if $\{(i,j),(i+1,j),(i,j+1),(i+1,j+1)\}$. These four points correspond to the vertices of a unit square "S" (as opposed to the "L" in Theorem 3). Let us identify a special subset T of lattice points in the nonnegative quadrant by defining $T := \{(2i+1, 2j+1) \ : \ i,j \geq 0\}$, i.e., points with both coordinates odd. It is easy to see that every square "S" hits the set T. Furthermore, by considering the family F of the form $\{(2i,2j),(2i+1,2j),(2i,2j+1),(2i+1,2j+1)\}$, each containing a unique point of T as a Northeast corner, then we see $f(k)$, the size of the smallest set hitting every "S" in $\Delta(k)$, must be as large as $|T \cap \Delta(k)|$. However, it clearly does not have to be any larger than this since we can just use the set $T \cap \Delta(k)$ itself. Thus we have

$$f(k) = |T \cap \Delta(k)| \tag{6}$$

This implies that when the line $L(c) \ : \ i\log 2 + j \log 3 = c$ moves as c increases, the value of $f(k)$ exactly when a new lattice point from T is included in the triangle formed by the i and j axes and $L(c)$. Consequently, we have

$$\begin{aligned}
\sigma(1,2,3,6) &= \frac{1}{3}\sum_{k\geq 1} f(k)\left(\frac{1}{d_k} - \frac{1}{d_{k+1}}\right) \\
&= \frac{1}{3}\sum_{k\geq 1}(f(k)-f(k-1))\frac{1}{d_k} \\
&= \frac{1}{3}\sum_{f(k)>f(k-1)} \frac{1}{d_k} && \text{since } f(k)-f(k-1) \leq 1 \\
&= \frac{1}{3}\sum_{i,j\geq 0} \frac{1}{2^{2i+1}} \cdot \frac{1}{2^{2j+1}} && \text{since } f(k) \text{ jumps when } \ d_k \in T \\
&= \frac{1}{12}
\end{aligned}$$

This proves the theorem. $\square$

The same argument turns out to work whenever the configuration $C = C(1, a_1, a_2, \ldots, a_s)$ of lattice points corresponding to the set of coefficients $\{1, a_1, a_2, \ldots, a_s\}$ in A tiles the nonnegative quadrant (or more generally, the nonnegative orthant when $\prod_i a_i$ has more than two prime factors).

For example, for the set $\{1, 2, 12, 24\}$, the configuration C is the zigzag set $\{(i, j), (i + 1, j), (i + 2, j+1), (i+2, j+2)\}$. This tiles the positive quadrant, and the corresponding argument shows that

$$\sigma(1, 2, 12, 24) = \frac{1}{39}.$$

In general, for a set $A = \{a_1 < a_2 < \ldots < a_m\}$, define:

$\pi(A)$ = set of primes q dividing $\prod_i a_i$,

$D_A = \{d_1 < d_2 < \ldots\}$ = the set of numbers with only q_i as prime factors, and

$D_A(k) = \{d_1 < d_2 < \ldots < d_k\}$.

Let $f(k)$:= the size of a minimum set of $D_A(k)$ hitting every set of the form $\{a_1x, a_2x, \ldots, a_mx\}$, x an integer, and let $K(A) := \{k : f(k) > f(k - 1)\}$.

Theorem (Graham, Spencer, Witsenhausen [11])

$$\sigma(A) = \sigma(1, a_1, \ldots, a_m) = \prod_{q \in \pi(A)} (1 - q^{-1}) \sum_{k \in K(A)} d_k^{-1}$$

Although we have had this result for some time, it is only recently that we were able to use it to compute anything interesting.

We have also investigated several other infinite families of sets A. We list some of these results (without proof) below.

z	$\rho(A)$	$\sigma(a)$
3	1/5	0.199...
4	1/7	1/7
5	11/90	?
6	7/66	?
7	17/182	?
8	1/12	5/62

Table 3: $A = \{1, 2, z\}, z > 2$

The fact that we can evaluate $\sigma(1, 2, 8)$ arises from the fact that for this case, 2 is the only prime involved in the factors of the a_i, making this a 1-dimensional problem (which typically can be dealt with, since $f(k)$ is then essentially periodic).

For real z, $4 \le z \le 8$, we can show

$$\rho(1,2,z) = \frac{2}{z} - \frac{5}{4z-2}$$

(which probably wasn't easy to guess by just examining the values in Table 3 ! In general, we have techniques for evaluating $\rho(1,\alpha,\beta)$ for any α and β.

Conjecture: $\sigma(A)$ is rational if and only if $C(A)$ tiles the positive orthant.

We do not have a single example of an irrational $\sigma(A)$. Of course, $\sigma(1,2,3)$ is certainly irrational; all we lack is a proof! In this case, does $f(k) = g(k)$ for all k? In general, is there an efficient way to determine $f(k)$ for large k?

We remark here that in general, σ can be a lot smaller than ρ. For example, a recent result of Erdős and Spencer (1995) [6] deals with the set of coefficients $\{1,2,3,\ldots,s\}$. They show

$$\sigma(1,2,\ldots,s) = \Theta(\frac{1}{s\log s}).$$

On the other hand, it can be shown that

$$\rho(1,2,\ldots,s) = \frac{1}{2s-1},$$

which is a lot larger. Perhaps in some sense this is as large as the difference between σ and ρ can be.

Conjecture If $A = \{a_1, a_2, \ldots, a_m\}$, then $\rho(A)$ is rational whenever all the a_i are rational.

We conclude this section with one more class of matrices which have been investigated. These correspond to the matrices $A = (1\ 1\ \ -\alpha)$ for various values of α.

For the case $A = (1\ 1\ -3)$, our hitting sets $S_A(n)$ must intersect every non-trivial solution vector $\bar{x} = (x_1, x_2, x_3)$ to the equation $x_1 + x_2 = 3x_3$.

The second author had conjectured some years ago that $s_A(n) = n/2 + O(1)$. An early result of Lucht [14] shows that $\rho(A_3) = \sigma(A_3) = 1/2$. Subsequently, Chung and Goldwasser [4] showed that $s_{A_3}(n) = \lfloor n/2 \rfloor, n \ne 4$, and in fact, for $n \ge 23$, the set of even integers in $[n]$ is the unique minimum A_3-hitting set.

For the case of $x_1+x_2 = kx_3$ for integer $k \geq 4$, the corresponding value of $\sigma(A_k)$ was determined by Lucht [14] who showed that $\sigma(A_k) = \frac{2k-2}{k^2-2}$. For real $\alpha \geq 4$, it was shown in [5] that $\rho(A_\alpha) \leq \frac{2\alpha-2}{\alpha^2-2}$. Presumably, this is the correct value of $\rho(A_\alpha)$.

6 Some final remarks

As promised early, we will mention a few more remarks concerning the case $A_3 = (1 \ \ -2 \ 1)$ (corresponding to 3-term arithmetic progressions), and the more general case

$$A_k = \begin{pmatrix} 1 & -2 & 1 & 0 & \dots & & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \\ 0 & 0 & \dots & & 1 & -2 & 1 \end{pmatrix}$$

for k-term arithmetic progressions. Since $\sigma_{A_k}(n) = (1+o(1))n$ for all k (by the classic result of Szemerédi [19]), people have traditionally focused on the complementary statement of the problem, defining $r_k(n) \ := \ n - \sigma_{A_k}(n)$ as the size of the largest subset of $[n]$ which contains no k-term arithmetic progression. The first significant results here were the bounds of Salem/Spencer [17] and Behrend [1], which were of the form

$$r_3(n) > n\exp(-c\sqrt{\log n})$$

for a suitable constant $c >$. This of course shows that $r_3(n) > n^{1-\epsilon}$ for any $\epsilon > 0$ when n is large. Roth [16] in 1954 then showed that

$$r_3(n) = O(\frac{n}{(\log\log n)^c}) = o(n),$$

which was followed 15 years later by Szemerédi's result [18] $r_4(n) = o(n)$, and finally by the celebrated bound in 1974 [19]

$$r_k(n) = o(n) \qquad \text{for all } k.$$

(In this paper he also introduced the fundamental "regularity lemma"; see [13]).

Further improvements were:

$$
\begin{aligned}
r_3(n) &= O(\frac{n}{(\log n)^{1/3}} && \text{by Heath-Brown (1987) [12]} \\
& && \text{and Szemerédi (1990) [19]} \\
r_3(n) &= O(n(\frac{\log\log n}{\log n})^{1/3}) && \text{by Bourgain (1999) [3]} \\
r_4(n) &= O(\frac{n}{(\log\log n)^c}) && \text{by Gowers (1999)[8]}
\end{aligned}
$$

Very recently (in fact, not yet published at the time of this writing), Gowers has proved the striking bound

$$r_k(n) = O(\frac{n}{(\log\log n)^{c_k}})$$

for all k where $c_k > 0$ is a constant depending on k. This may well represent the truth here, although almost no progress has been made on the lower bounds for the past 50 years.

As an application of Gowers' result, define $W(k)$ to be the least integer such that any 2-coloring of the integers $\{1, 2, \ldots, W(k)\}$ must always contain a k-term arithmetic progression in a single color. The existence of $W(k)$ is guaranteed by a classic result of van der Waerden (1927) [21] (also see [10]). The original bounds for $W(n)$ grew like the Ackermann function (because of the double induction used in the proof) an so were not even primitive recursive. In 1988, Shelah [15] found a different proof which gave the bound

$$W(n) \leq T(n)$$

where $T(n)$ is defined recursively by:

$$T(1) = 2, T(k+1) = \overbrace{2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}}^{T(k)}$$

However, Gowers' bound on $r_k(n)$ implies

$$W(n) < 2^{2^{2^{2^{2^{n+9}}}}} \qquad \text{for all } n$$

(This result earned a reward of \$ 1000 offered by the 3rd author, parallel to the \$1000 reward given to Szemerédi by the second author for his estimate $r_k(n) = o(n)$. This problem is turning out to be rather expensive!) Undeterred, however, the 3rd author conjectures (for \$1000):

$$W(n) \leq 2^{n^2} \qquad \text{for all } n.$$

The best lower bound currently available is still that of Berlekamp [2] from 1968:

$$W(n+1) \geq n2^n, \qquad n \text{ prime.}$$

Clearly there is still a lot of room for improvement here.

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