On N consecutive integers in an arithmetic progression

By RONALD EVANS in Champaign (Illinois, U.S.A.)

Let $B_N(d)$ denote a block $\{c+d, c+2d, \dots, c+Nd\}$ of N consecutive integers in an arithmetic progression. It is known [1] that for each $N \geqq 17$ there exists a block $B_N(1)$ containing no integer relatively prime to each of the others. One might ask whether a similar result holds for blocks $B_N(2)$ of odd integers, or in general for blocks $B_N(d)$. We shall prove that in fact for any positive integers c and d and for all $N > N_0(d)$, there exists a block $B_N(d)$ whose integers are congruent to $c \pmod d$ which contains no integer relatively prime to each of the others. (This is of course trivial if $(c, d) > 1$.)

As the assertion is known for $d=1$, assume $d \geqq 2$ and let $t_1 < t_2 < \dots < t_k$ be the prime divisors of d. Let $r(N)$ be the number of integers $b = t_1^{a_1} t_2^{a_2} \dots t_k^{a_k}$ ($a_i = 0, 1, 2, \dots$) for which $b < N$. For a given i, the number of powers $t_i^{a_i}$ for which $t_i^{a_i} < N$ is $\leqq 1 + \frac{\log N}{\log t_i}$. Hence $r(N) \leqq \prod_{i=1}^{k} \left(1 + \frac{\log N}{\log t_i}\right) \leqq \left(1 + \frac{\log N}{\log 2}\right)^k$. Thus for all sufficiently large N, $r(N) < (\log N)^{k+1}$. By well-known theorems on distribution of primes, we conclude that for large N,

$$\pi(N/2) - \pi(N/4) > 2r(N), \tag{1}$$

$$\pi(3N/4) - \pi(N/2) > 4r(N). \tag{2}$$

There exists a prime t such that for all large N,

$$t_k < t < N/4. \tag{3}$$

Choose an integer $N_0(d)$ so large that (1), (2), and (3) hold for all $N > N_0(d)$. Fix $N > N_0(d)$ and let $r = r(N)$.

Let $b_1, \dots, b_r$ denote the integers $b = t_1^{a_1} t_2^{a_2} \dots t_k^{a_k}$ for which $b < N$. By (1), we can choose $2r$ distinct primes q_i such that

$$N/4 < q_i < [N/2] \qquad (i = 1, 2, \dots, 2r). \tag{4}$$

By (2), we can choose $4r$ distinct primes p_i such that

$$N/2 < p_i < [3N/4] \qquad (i = 1, 2, \dots, 4r). \tag{5}$$

Now let x be a solution of the system

(6) $$x \equiv c \pmod d$$

(7) $x \equiv 0 \pmod p$ for each prime $p \leqq N/2$ such that $p \notin \{t_1, \ldots, t_k, q_1, \ldots, q_{2r}\}$.

(8) $$x+db_i \equiv 0 \pmod{q_i} \qquad (i=1, 2, \ldots, r)$$

(9) $$x-db_i \equiv 0 \pmod{q_{r+i}} \qquad (i=1, 2, \ldots, r)$$

(10) $$x+dq_i \equiv 0 \pmod{p_i} \qquad (i=1, 2, \ldots, 2r)$$

(11) $$x-dq_i \equiv 0 \pmod{p_{2r+i}} \qquad (i=1, 2, \ldots, 2r).$$

(A solution exists as the moduli are relatively prime in view of (3), (4), and (5).)

We shall now show that the block $B_N(d) = \{x-d(N-[N/2]-1), \ldots, x+d[N/2]\}$ has the desired properties. That its integers are congruent to $c \pmod d$ follows from (6). To see that $B_N(d)$ contains no integer relatively prime to each of the others, we will produce, for each $u \in B_N(d)$, a corresponding $v \in B_N(d)$ such that $v \neq u$ and $(u, v) > 1$.

If $u=x$, we may choose $v = x+dt$ by (3) and (7). If $u = x+db_i$, we may choose $v = x+d(b_i-q_i)$ by (4) and (8). If $u = x-db_i$, we may choose $v = x+d(q_{r+i}-b_i)$ by (4) and (9). If $u = x+dq_i$, we may choose $v = x+d(q_i-p_i)$ by (4), (5) and (10). If $u = x-dq_i$, we may choose $v = x+d(p_{2r+i}-q_i)$ by (4), (5), and (11). Every other $u \in B_N(d)$ has the form $x \pm dm$, where m is divisible by a prime $p \leqq N/2$ such that $p \notin \{t_1, \ldots, t_k, q_1, \ldots, q_{2r}\}$. Hence by (7), we may choose $v=x$ for each of these u.

References

[1] R. J. Evans, On blocks of N consecutive integers, *Amer. Math. Monthly*, **76** (1969), 48—49.

UNIVERSITY OF ILLINOIS
CHAMPAIGN, ILLINOIS 61820

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