Algorithmic Solution to Problem 1
(and linear extensions of general one-level grid-like posets)

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Problem 1. (http://www.math.ucsd.edu/ r1pan/problems/p1.html)

We define a class of posets $\{P_n\}$ as follows.

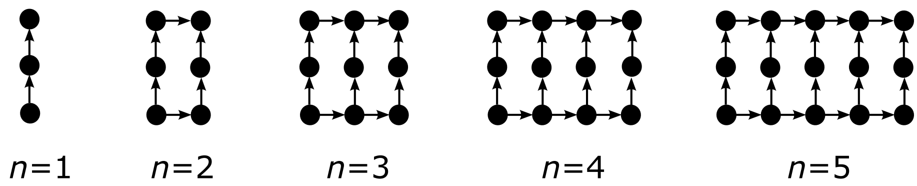

Figure 1: P_1, P_2, P_3, P_4 and P_5

$\mathcal{L}(P_n)$ is the number of linear extensions of poset P_n. One can easily figure out what the Hasse diagram of P_n is by observing posets in Figure 1.

Find $\mathcal{L}(P_n)$.

General problem.

We define a class of one-level grid-like posets $\{G[\mathbf{v}, \mathbf{t}, \mathbf{b}]\}$, where $\mathbf{v} = (v_1, v_2, \cdots, v_n)$, $\mathbf{t} = (t_1, t_2, \cdots, t_{n-1})$ and $\mathbf{b} = (b_1, b_2, \cdots, b_{n-1})$. v_i denotes the number of nodes in i-th vertical edge, t_i denotes the number of nodes in i-th top edge and b_i denotes the number of nodes in i-th base edge, endpoints not included. In all vertical edges, arrows are upward pointing and in all horizontal edges, arrows are right pointing.

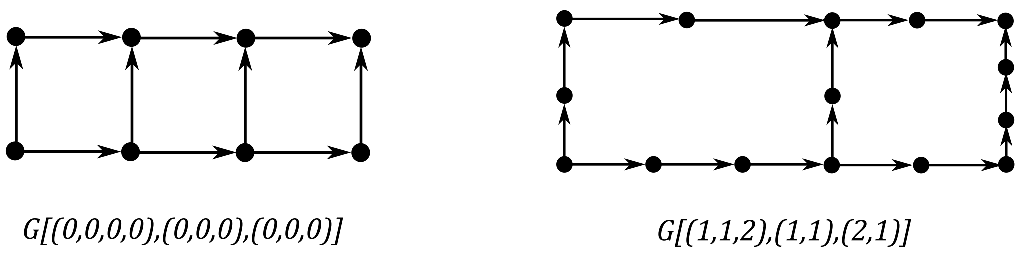

Figure 2: Two examples of one-level grid-like posets

For example, in Figure 1 above, $P_4 = G[(1,1,1,1),(0,0,0),(0,0,0)]$, and $G[(0,0,0,0),(0,0,0),(0,0,0)]$ and $G[(1,1,2),(1,1),(2,1)]$ are pictured in Figure 2.

For convenience, we rewrite P_n in Problem 1 as $G[(1^n), (0^{n-1}), (0^{n-1})]$.

In general, we want to find algorithm to compute the number of linear extensions of for arbitrary poset $\{G[\mathbf{v}, \mathbf{t}, \mathbf{b}]\}$.

Solution.

For any poset P, we use $\mathcal{L}(P)$ denote the number of linear extensions of P. The algorithm is somehow brute-force and recursive. The main idea is to break down boxes one-by-one from left to right.

The details are as follows, for any poset $G[(v_1, \cdots, v_n), (t_1, \cdots, t_{n-1}), (b_1, \cdots, b_{n-1})]$, we consider nodes in the first vertical edge and the first top edge and the node at the top-left corner, then we insert these $v_1 + t_1 + 1$ nodes into the second vertical edge and the first base edge. Suppose m is the number of nodes of the $v_1 + t_1 + 1$ nodes that are inserted into the second vertical edge, then apparently there are $v_1 + t_1 + 1 - m$ nodes inserted into the first base. Since at this moment, nodes in the first base edge are smallest in the entire poset and already linearly ordered, we can safely remove the first edge. After that, the new poset is $G[(v_2 + m, \cdots, v_n), (t_2, \cdots, t_{n-1}), (b_2, \cdots, b_{n-1})]$.

Be aware that there are $\binom{v_2+m}{m}$ ways to insert m nodes into the second vertical edge and there are $\binom{b_1+v_1+t_1+1-m}{v_1+t_1+1-m}$ ways to insert m nodes into the second vertical edge.

Then we are able to obtain a recursive formula for the number of linear extensions,

$$\begin{aligned}
&\mathcal{L}(G[(v_1, v_2, \cdots, v_n), (t_1, t_2, \cdots, t_{n-1}), (b_1, b_2, \cdots, b_{n-1})]) \\
= &\sum_{m=0}^{v_1+t_1+1} \binom{v_2+m}{v_2} \binom{b_1+v_1+t_1+1-m}{b_1} \mathcal{L}(G[(v_2+m, \cdots, v_n), (t_2, \cdots, t_{n-1}), (b_2, \cdots, b_{n-1})]).
\end{aligned}$$

Getting back to the original problem, suppose we want to compute $\mathcal{L}(P_4)$, the first step of recurrence is

$$\begin{aligned}
&\mathcal{L}(P_4) \\
= \;&\mathcal{L}(G[(1^4), (0^3), (0^3)]) \\
= \;&\binom{3}{1}\mathcal{L}(G[(3, 1^2), (0^2), (0^2)]) + \binom{2}{1}\mathcal{L}(G[(2, 1^2), (0^2), (0^2)]) + \binom{1}{1}\mathcal{L}(G[(1^3), (0^2), (0^2)]).
\end{aligned}$$

It's well known that counting linear extensions of an arbitrary poset is #P hard. Fortunately, we can show that, using the formula above, $\mathcal{L}(P_n)$ is computed dynamically with runtime $\mathcal{O}(n^3)$ and space $\mathcal{O}(n^2)$. Runtime and space is also obtained recursively.

Some results are given in the following table. Some of them are new to OEIS and some sequences are already known. We will not bother to write down the proofs that the sequences we find matches formulas on OEIS, although some proofs are nontrivial.

Structure of posets	Sequence of numbers of linear extensions ($n \geq 1$)	OEIS
$G[(0^n),(0^{n-1}),(0^{n-1})]$	1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796,58786, 208012, 742900, 2674440, $\cdots$	A000108
$G[(1,0^{n-1}),(0^{n-1}),(0^{n-1})]$	1, 3, 9, 28, 90, 297, 1001, 3432, 11934, 41990, 149226, 534888, 1931540, 7020405, $\cdots$	A000245
$G[(2,0^{n-1}),(0^{n-1}),(0^{n-1})]$	1, 4, 14, 48, 165, 572, 2002, 7072, 25194, 90440, 326876, 1188640, $\cdots$	A002057
$G[(3,0^{n-1}),(0^{n-1}),(0^{n-1})]$	1, 5, 20, 75, 275, 1001, 3640, 13260, 48450, 177650, 653752, $\cdots$	A000344
$G[(4,0^{n-1}),(0^{n-1}),(0^{n-1})]$	1, 6, 27, 110, 429, 1638, 6188, 23256, 87210, 326876, 1225785, $\cdots$	A003517
$G[(5,0^{n-1}),(0^{n-1}),(0^{n-1})]$	1, 7, 35, 154, 637, 2548, 9996, 38760, 149226, 572033, 2187185, $\cdots$	A000588
$G[(6,0^{n-1}),(0^{n-1}),(0^{n-1})]$	1, 8, 44, 208, 910, 3808, 15504, 62016, 245157, 961400, 3749460, $\cdots$	A003518
$G[(1^n),(0^{n-1}),(0^{n-1})]$	1, 6, 71, 1266, 30206, 902796, 32420011, 1359292626, 65164480466, 3515569641156, $\cdots$	New
$G[(2^n),(0^{n-1}),(0^{n-1})]$	1, 20, 1301, 177260, 41385102, 14760468600, 7465847167005, 5083351577582300, $\cdots$	New
$G[(3^n),(0^{n-1}),(0^{n-1})]$	1, 70, 26599, 29609650, 72574079902, 332014782982540, 2545213373338499072, $\cdots$	New
$G[(4^n),(0^{n-1}),(0^{n-1})]$	1, 252, 578005, 5442949764, 145145279070542, 8831078509305669632, $\cdots$	New
$G[(0^n),(1^{n-1}),(0^{n-1})]$	1, 3, 12, 55, 273, 1428, 7752, 43263, 246675, 1430715, 8414640, $\cdots$	A001764
$G[(0^n),(2^{n-1}),(0^{n-1})]$	1, 4, 22, 140, 969, 7084, 53820, 420732, 3362260, 27343888, 225568798, $\cdots$	A002293
$G[(0^n),(3^{n-1}),(0^{n-1})]$	1, 5, 35, 285, 2530, 23751, 231880, 2330445, 23950355, 250543370, $\cdots$	A002294
$G[(0^n),(4^{n-1}),(0^{n-1})]$	1, 6, 51, 506, 5481, 62832, 749398, 9203634, 115607310, 1478314266, $\cdots$	A002295
$G[(0^n),(1^{n-1}),(1^{n-1})]$	1, 6, 53, 554, 6362, 77580, 986253, 12927170, 173452334, 2370742868, $\cdots$	A066357
$G[(1^n),(1^{n-1}),(1^{n-1})]$	1, 20, 962, 75080, 8133732, 1127589120, 190416834360, 37902843124640, $\cdots$	New
$G[(1^n),(1^{n-1}),(0^{n-1})]$	1, 10, 215, 7200, 328090, 18914190, 1318595475, 107813147200, $\cdots$	New
$G[(1^n),(2^{n-1}),(1^{n-1})]$	1, 35, 3164, 475391, 100270569, 27235367376, 9047105899944, 3551983608942083, $\cdots$	New
$G[(1^n),(2^{n-1}),(0^{n-1})]$	1, 15, 510, 27525, 2040219, 192349620, 22005761490, 2959779740625, $\cdots$	New
$G[(2^n),(1^{n-1}),(0^{n-1})]$	1, 35, 4382, 1193423, 568123143, 418198953480, 440137511828322, $\cdots$	New
$G[(0,1,0,1,\cdots),(0^{n-1}),(0^{n-1})]$	1, 3, 11, 87, 431, 5127, 29851, 457347, 2969687, 54616335, 385008243, $\cdots$	New
$G[(1,0,1,0,\cdots),(0^{n-1}),(0^{n-1})]$	1, 3, 19, 87, 923, 5127, 72815, 457347, 7949099, 54616335, 1107735543, $\cdots$	New