INTRODUCTION TO SYMMETRIC SPECTRA. LECTURE 2

1. Model categories. Let $\mathcal{C}$ be a category with three distinguished classes of morphisms:

- **weak equivalences**, pictured as $\xrightarrow{\sim}$;
- **fibrations**, pictured as $\twoheadrightarrow$;
- **cofibrations**, pictured as $\hookrightarrow$;

each of which is closed under composition and contains all isomorphisms in $\mathcal{C}$. We say that these classes provide a **model category** structure on $\mathcal{C}$ if they satisfy the following axioms:

MC1 Finite limits and colimits exist in $\mathcal{C}$.

MC2 If f, g are morphisms in $\mathcal{C}$ such that $g \circ f$ is defined, and two of the three maps f, g, and $g \circ f$ are weak equivalences, then so is the third one.

MC3 Each of the classes of weak equivalences, fibrations, and cofibrations is closed under retracts.

MC4 Given a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array} \qquad (*)$$

there exists a lift $B \to X$ preserving commutativity if either (i) i is a cofibration and p is a fibration and a weak equivalence, or (ii) p is a fibration and i is a cofibration and a weak equivalence.

MC5 Any morphism f in $\mathcal{C}$ admits factorizations

$$f = p \circ i \qquad \text{and} \qquad f = q \circ j,$$

where p is a fibration, i is a cofibration and a weak equivalence, q is a fibration and a weak equivalence, and j is a cofibration.

Remarks: (1) In axiom MC3, recall that a morphism f is said to be a **retract** of a morphism g if there exists a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{r} & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{r'} & X' \end{array}$$

such that $r \circ i = \mathrm{id}_X$ and $r' \circ i' = \mathrm{id}_{X'}$. An easy exercise is to check that the class of isomorphisms in any category is closed under retracts.

(2) A map which is both a fibration and a weak equivalence is called an **acyclic fibration** by some people, and a **trivial fibration** by other people. We will adopt the first terminology, since it is less confusing (the second one may be thought to be related to the concept of a trivial fiber bundle, when in fact it is not). The definition of an **acyclic cofibration** is similar.

(3) Since $\mathcal{C}$ has finite limits and colimits, by abstract nonsense it has, in particular, an initial object $\varnothing$ and a final object $*$. An object X of $\mathcal{C}$ is said to be **fibrant** (resp., **cofibrant**) if the unique arrow $X \to *$ is a fibration (resp., the unique arrow $\varnothing \to X$ is a cofibration).

(4) In axiom MC5, one sometimes requires the existence of *functorial* factorizations (with respect to f). However, one does not need this assumption to develop most of the basic theory. My understanding is that in many of the interesting examples of model categories it is possible to prove the existence of functorial factorizations by some kind of set-theoretical argument.

(5) The following comment is important. If (i, p) is a pair of morphisms in $\mathcal{C}$ such that for *every* commutative square (*) in $\mathcal{C}$, there exists an arrow $B \to X$ which preserves the commutativity, we say that i satisfies the **left lifting property** (LLP) with respect to p, and that p satisfies the **right lifting property** (RLP) with respect to i. One can prove that

(i) The cofibrations are precisely the morphisms having the LLP w.r.t. the acyclic fibrations.

(ii) The acyclic cofibrations are precisely the morphisms having the LLP w.r.t. the fibrations.

(iii) The fibrations are precisely the morphisms having the RLP w.r.t. the acyclic cofibrations.

(iv) The acyclic fibrations are precisely the morphisms having the RLP w.r.t the cofibrations.

(6) In particular, in a model category, any two of the three classes of morphisms determine the third one. The case that may not be immediately obvious is that cofibrations and fibrations determine the weak equivalences; to see this, observe that cofibrations and fibrations determine the acyclic cofibrations and the acyclic fibrations by remark (5), (ii) and (iv), and then axioms MC2 and MC5 imply that the weak equivalences can be characterized as the maps that can be factored as an acyclic cofibration followed by an acyclic fibration.

2. Examples and applications.

First we list a few standard examples of model categories.

1) *Chain complexes.* Let $\mathcal{A}$ be an abelian category with sufficiently many projective objects, and put $\mathcal{C} = \mathrm{Ch}^-(\mathcal{A})$, the category of complexes over $\mathcal{A}$ that are bounded on the right. Define

- a weak equivalence in $\mathcal{C}$ to be a quasi-isomorphism;
- a fibration in $\mathcal{C}$ to be an epimorphism (recall that a morphism $f : A^0 \to B^0$ of complexes is an epimorphism in $\mathcal{C}$ if and only if $f_k : A^k \to B^k$ is an epimorphism in $\mathcal{A}$ for every $k \in \mathbb{Z}$);

- a cofibration in $\mathcal{C}$ to be a monomorphism whose cokernel is a complex of projective objects.

It can be proved that this makes $\mathcal{C}$ into a model category. All objects are fibrant, and the cofibrant objects are the complexes of projective objects. There is a dual model structure on $\mathrm{Ch}^+(\mathcal{A}) \simeq \mathrm{Ch}^-(\mathcal{A}^{\mathrm{op}})$ for which all objects are cofibrant and the fibrant objects are the complexes of injective objects of $\mathcal{A}$; here we have to assume that $\mathcal{A}$ has enough injectives.

This example provides the bridge between homological algebra and "homotopical algebra" (as far as I know, this term was introduced by Quillen and refers to the study of model categories). Vague remark: as the reader knows, one can define the derived categories $D^{\pm}(\mathcal{A})$ by localizing $\mathrm{Ch}^{\pm}(\mathcal{A})$ with respect to the class of all quasi-isomorphisms. A similar definition can of course be made for a general model category $\mathcal{C}$: we define the **homotopy category** of $\mathcal{C}$ to be the localization

$$\mathcal{C} \to \mathrm{Ho}(\mathcal{C})$$

obtained by formally inverting all weak equivalences in $\mathcal{C}$. This localization is defined by an obvious universal property, and yields something rather unmanageable. On the other hand, the categories $D^{\pm}(\mathcal{A})$ (or, rather, morphisms in them) can be described in a different, much more concrete way. Moreover, if $\mathcal{A}$ has enough projectives (resp., injectives), it is well known that $D^-(\mathcal{A})$ (resp. $D^+(\mathcal{A})$) is equivalent to the homotopy category of complexes of projective (resp., injective) objects that are bounded on the right (resp., on the left). These statements have natural analogues in the context of general model categories, as we will see a little later.

In some sense, one could think of the theory of model categories as a "non-linear generalization" of homological algebra.

2) *Topological spaces*, take 1: Let $\mathcal{T}op$ be the category of topological spaces and continuous maps, and define a map in $\mathcal{T}op$ to be:

- a weak equivalence if it is a **weak homotopy equivalence**, i.e., induces isomorphisms of all the homotopy groups;
- a fibration if it is a **Serre fibration**, i.e., if, for each CW complex A, it has the RLP with respect to the natural inclusion $A = A \times \{0\} \hookrightarrow A \times [0,1]$;
- a cofibration if it has the LLP with respect to acyclic Serre fibrations.

[Remark: cofibrations can be characterized somewhat more concretely as the retracts of the closed inclusions $X \hookrightarrow Y$ where Y is obtained from X by attaching cells in some order, not depending on the dimension of the cells. Such an inclusion is sometimes called a **generalized relative CW inclusion**.]

The three classes of maps above define a model category structure on $\mathcal{T}op$. Each object is fibrant and the cofibrant objects are the retracts of generalized CW complexes.

3) *Topological spaces*, take 2: Define a map in $\mathcal{T}op$ to be:

- a weak equivalence if it is an actual homotopy equivalence;
- a fibration if it is a **Hurewicz fibration** (i.e., has the RLP similar to the one above, but where A is allowed to be any topological space);
- a cofibration if it is a **closed Hurewicz cofibration** (i.e., is an inclusion $i : A \hookrightarrow B$ as a closed subspace and has the Homotopy Extension Property).

This is also a model category structure on $\mathcal{T}op$ (rather different from the previous one).

4) *Simplicial sets.* Recall that $\mathcal{S}$ and $\mathcal{S}_*$ denote the categories of simplicial sets and pointed simplicial sets, respectively. Define a morphism in $\mathcal{S}$ to be a

- weak equivalence if it induces a (weak) homotopy equivalence of the geometric realizations (the reason the word weak is in parentheses is that geometric realizations of simplicial sets are CW complexes, and for CW complexes weak homotopy equivalences are known to be homotopy equivalences);
- cofibration if it is a monomorphism in $\mathcal{S}$;
- fibration if it satisfies the RLP with respect to the acyclic monomorphisms.

Theorem: These classes define a model category structure on $\mathcal{S}$ and also on $\mathcal{S}_*$, and there is an equivalence between $\mathrm{Ho}(\mathcal{S})$ and the localization of $\mathcal{T}op$ by the class of weak homotopy equivalences.

This result is one of the cornerstones of algebraic topology.

Remark: The description of the model category structure on $\mathcal{S}$ given above is probably the most economical one. However, the reader should know that there exists a less explicit, but purely algebraic, description of the same model structure that does not use geometric realization.

Definition: An object of $\mathcal{S}$ or $\mathcal{S}_*$ is said to be a **Kan complex** if it is fibrant with respect to the model structure defined above. See [GM], Ch. V.

Example: The object $\triangle[n] \in \mathcal{S}$ represented by $[n] \in \triangle$ is NOT a Kan complex for $n \geq 1$, contrary to what one might expect. The reason is that the maps

$$\begin{array}{ccc} [1] & \longrightarrow & [2] \\ \| & & \| \\ \{0,1\} & & \{0,1,2\} \end{array}, \qquad 0 \longmapsto 0, \qquad 1 \longmapsto 1$$

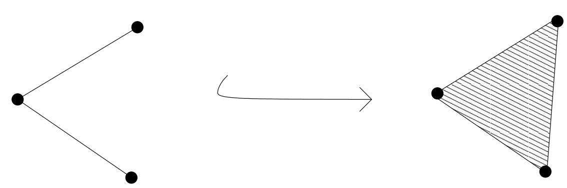
and

$$[1] \longrightarrow [2], \qquad 0 \longmapsto 0, \qquad 1 \longmapsto 2$$

define an inclusion

$$\triangle[1] \underset{\triangle[0]}{\cup} \triangle[1] \overset{i}{\hookrightarrow} \triangle[2]$$

which is certainly an acyclic cofibration; geometrically it looks as follows:



On the other hand, for every $n \geq 1$, let us consider the standard inclusion $[1] \hookrightarrow [n]$ $(0 \mapsto 0,\ 1 \mapsto 1)$ and the constant map $[1] \to [n]$ $(0 \mapsto 0,\ 1 \mapsto 0)$, and form the pushout

$$\varphi : \triangle[1] \underset{\triangle[0]}{\cup} \triangle[1] \longrightarrow \triangle[n].$$

Then it is easy to check that there exists no morphism $\triangle[2] \to \triangle[n]$ making the diagram

$$\begin{array}{ccc} \triangle[1] \cup_{\triangle[0]} \triangle[1] & \xrightarrow{\varphi} & \triangle[n] \\ {\scriptstyle i} \big\downarrow & \nearrow & \big\downarrow \\ \triangle[2] & \longrightarrow & * \end{array}$$

commute, which implies that $\triangle[n]$ is not a Kan complex.

On the other hand, if Z is any topological space, it is not hard to check that the total singular complex $\mathrm{Sing}(Z)$, introduced in the previous lecture, **is** a Kan complex. This allows us to construct what is called a **fibrant replacement functor**. Namely, for each object $X \in \mathcal{S}$, let

$$\eta_X : X \to \mathrm{Sing}(|X|)$$

denote the morphism*) obtained by adjunction from the identity map on $|X|$. Thus we get:

- a functor $\mathrm{Sing}(|\cdot|) : \mathcal{S} \to \mathcal{S}$ with the property that $\mathrm{Sing}(|X|)$ is both fibrant and cofibrant for all $X \in \mathcal{S}$;
- a natural transformation
$$\eta : \mathrm{Id}_{\mathcal{S}} \to \mathrm{Sing}(|\cdot|)$$
with the property that η_X is an acyclic cofibration for all $X \in \mathcal{S}$.

To finish our list of examples of model categories, we briefly mention another useful one.

5) *Differential graded algebras.* Let $\Bbbk$ be any commutative ring. The category $\text{dg-alg}_{\Bbbk}^{+}$ of nonnegatively graded DG algebras over $\Bbbk$ can be made into a model category by declaring the weak equivalences to be the quasi-isomorphisms, the fibrations to be surjections, and the cofibrations to be the homomorphisms satisfying the LLP with respect to the acyclic fibrations.

3. Symmetric Ω-spectra. Let E be a spectrum in the sense of the definition given in the previous lecture. We say that E is an **Ω-spectrum** is each E_n is a Kan complex and the adjoint $E_n \to \mathrm{Hom}_{\mathcal{S}_*}(S^1, E_{n+1})$ of each of the structure maps $S^1 \wedge E_n \to E_{n+1}$ is a weak equivalence of simplicial sets.**) Note that another common term for "Ω-spectrum" is "quasi-fibrant spectrum".

A symmetric spectrum $E \in \mathcal{S}p^{\Sigma}$ is said to be a **symmetric Ω-spectrum** if $\mathcal{U}(E)$ is an Ω-spectrum, where we recall that

$$\mathcal{U} : \mathcal{S}p^{\Sigma} \to \mathcal{S}p^{\mathbb{N}}$$

denotes the forgetful functor. We let

$$\Omega\mathcal{S}p^{\mathbb{N}} \subset \mathcal{S}p^{\mathbb{N}} \quad \text{and} \quad \Omega\mathcal{S}p^{\Sigma} \subset \mathcal{S}p^{\Sigma}$$

be the full subcategories consisting of Ω-spectra, and by

$$\mathrm{Ho}(\Omega\mathcal{S}p^{\mathbb{N}}) \quad \text{and} \quad \mathrm{Ho}(\Omega\mathcal{S}p^{\Sigma})$$

the categories obtained from $\Omega\mathcal{S}p^{\mathbb{N}}$ and $\Omega\mathcal{S}p^{\Sigma}$ by formally inverting the **level equivalences** (i.e., morphisms $f : E \to F$ of spectra or symmetric spectra such that $f_n : E_n \to$

F_n is a weak equivalence of simplicial sets for all $n \geq 0$). Recall also that for our purposes, Boardman's stable homotopy category is defined as the localization of $\mathcal{S}p^{\mathbb{N}}$ by the class of stable homotopy equivalences; call this category $\mathrm{StHo}(\mathcal{S}p^{\mathbb{N}})$. It is obvious that a level equivalence of spectra is also a stable homotopy equivalence, and with this in mind we state the following

Theorem: (1) (Bousfield and Friedlander) The functor $\mathrm{Ho}(\Omega\mathcal{S}p^{\mathbb{N}}) \to \mathrm{StHo}(\mathcal{S}p^{\mathbb{N}})$ induced by the inclusion $\Omega\mathcal{S}p^{\mathbb{N}} \hookrightarrow \mathcal{S}p^{\mathbb{N}}$ is an equivalence of categories.
(2) (Hovey & Shipley & Smith) The functor $\mathrm{Ho}(\Omega\mathcal{S}p^{\Sigma}) \to \mathrm{Ho}(\mathcal{S}p^{\mathbb{N}})$ induced by the inclusion $\Omega\mathcal{S}p^{\Sigma} \hookrightarrow \Omega\mathcal{S}p^{\mathbb{N}}$ is an equivalence of categories.

However, $\Omega\mathcal{S}p^{\Sigma}$ is not the right category to look at, for many purposes. For instance, it is not complete or cocomplete, and it is also not closed under smash products. Thus we need to construct a model structure on the category of all symmetric spectra which is compatible with the smash product in the appropriate sense, and such that the associated homotopy category is equivalent to $\mathrm{Ho}(\Omega\mathcal{S}p^{\Sigma})$, and hence to Boardman's stable homotopy category. This will be explained later in the lecture.

4. K-theory of Waldhausen categories, revisited.

We briefly finish up the discussion of the K-theory spectra of Waldhausen categories that the last week's lecture was devoted to. Let $\mathcal{C}$ be a category with cofibrations (the definition was given a week ago). A **subcategory of weak equivalences** in $\mathcal{C}$ is a subcategory $w(\mathcal{C}) \subseteq \mathcal{C}$ whose morphisms are called **weak equivalences** and are pictured as $\xrightarrow{\sim}$, satisfying the following two axioms:

Weq 1 $w(\mathcal{C})$ contains all isomorphisms in $\mathcal{C}$; in particular, $\mathcal{O}b(w(\mathcal{C})) = \mathcal{O}b(\mathcal{C})$

Weq 2 *Gluing lemma*: given a diagram

$$\begin{array}{ccccc} B & \leftarrowtail & A & \longrightarrow & C \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ B' & \leftarrowtail & A' & \longrightarrow & C' \end{array}$$

in $\mathcal{C}$, the induced arrow

$$B \cup_A C \to B' \cup_{A'} C'$$

is also a weak equivalence.

A **Waldhausen category** is a triple $(\mathcal{C}, \mathrm{co}(\mathcal{C}), w(\mathcal{C}))$, where $(\mathcal{C}, \mathrm{co}(\mathcal{C}))$ is a category with cofibrations and $w(\mathcal{C}) \subseteq \mathcal{C}$ is a subcategory of weak equivalences.

Let us fix a Waldhausen category $\mathcal{C}$. A weak ago we have constructed, for each finite set Q, a Q-simplicial category $\mathcal{S}^Q_\bullet\mathcal{C}$ which roughly consists of objects of $\mathcal{C}$ equipped with $|Q|$ different filtrations and compatible choices of all multiple subquotients. The key observation is that if Q' is another finite set, and for each

$$\vec{n} = (n_q)_{q\in Q} \in \mathbb{Z}^Q_{\geq 0}$$

we define

$$\vec{n}' = ((n_q)_{q\in Q}, (1)_{q'\in Q'}) \in \mathbb{Z}^{Q\cup Q'}_{\geq 0},$$

then there is a natural equivalence

$$\mathcal{S}^Q_{\vec{n}}\mathcal{C} \xrightarrow{\sim} \mathcal{S}^{Q\cup Q'}_{\vec{n}'}\mathcal{C},$$

because a 1-step filtration amounts to no data whatsoever. In particular, for each fixed $\vec{n}$ we obtain a weak equivalence of simplicial sets

$$N_\bullet \left(w\mathcal{S}^{Q}_{\vec{n}} \mathcal{C} \right) \xrightarrow{\sim} N_\bullet \left(w\mathcal{S}^{Q \cup Q'}_{\vec{n}'} \mathcal{C} \right).$$

If we now let $\vec{n}$ vary, what we get is a morphism of $(Q \cup Q')$-simplicial objects in $\mathcal{S}_*$,

$$N_\bullet \left(w\mathcal{S}^{Q}_\bullet \mathcal{C} \right) \times \left(\prod_{Q'} \triangle[1] \right) \longrightarrow N_\bullet \left(w\mathcal{S}^{Q \cup Q'}_\bullet \mathcal{C} \right).$$

Finally, this isomorphism factors through

$$\mathrm{diag} \left(N_\bullet \left(w\mathcal{S}^{Q}_\bullet \mathcal{C} \right) \wedge S^{Q'} \right) \longrightarrow \mathrm{diag} \left(N_\bullet w\mathcal{S}^{Q \cup Q'}_\bullet \mathcal{C} \right).$$

The reason is that if $\vec{m} \in \mathbb{Z}^{Q \cup Q'}_{\geq 0}$ and any one of the components of $\vec{m}$ is zero, the corresponding category $\mathcal{S}^{Q \cup Q'}_{\vec{m}} \mathcal{C}$ is the trivial one consisting of only the zero object. Moreover, these maps are obviously $\mathrm{Aut}(Q) \times \mathrm{Aut}(Q')$-equivariant. These are the structure maps for the Waldhausen K-theory spectrum $K(\mathcal{C})$.

Recall that in the previous lecture we have defined the (stable) homotopy groups of any spectrum or symmetric spectrum E.

Definition: A spectrum or symmetric spectrum E is said to be **connective** if $\pi_k(E) = 0$ for $k < 0$.

Theorem (Waldhausen): If $\mathcal{C}$ is a Waldhausen category, the K-theory spectrum $K(\mathcal{C})$, constructed above, is a connective spectrum which is moreover an Ω-spectrum starting with the first term.

Note that $K(\mathcal{C})$ fails to be entirely a connective spectrum for a stupid reason: $K(\mathcal{C})_0$ is just the nerve of the category $w(\mathcal{C})$, and does not have much to do with the other terms in the sequence that defines $K(\mathcal{C})$.

We will next state one more result. As a preliminary, we make the following

Definitions: If $\mathcal{C}$ is a Waldhausen category, the **homotopy category** $\mathrm{Ho}(\mathcal{C})$ is obtained from $\mathcal{C}$ by formally inverting the weak equivalences. A functor $\Phi : \mathcal{C} \to \mathcal{C}'$ between Waldhausen categories is said to be **exact** if it preserves cofibrations, weak equivalences, and the pushout diagrams of the axiom Cof 3.

We conclude the discussion with the following

Theorem: Let $\mathcal{C}$ and $\mathcal{C}'$ be Waldhausen categories, each satisfying the following two properties:

(i) every arrow can be factored as $p \circ i$, where p is a weak equivalence and i is a cofibration;

(ii) an arrow is a weak equivalence if and only if it becomes an isomorphism in the homotopy category.

Then every exact functor $\Phi : \mathcal{C} \to \mathcal{C}'$ such that the induced functor

$$\mathrm{Ho}(\Phi) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C}')$$

is an equivalence of categories, induces a stable homotopy equivalence of spectra,

$$K(\Phi) : K(\mathcal{C}) \to K(\mathcal{C}').$$

(In fact, $K(\Phi)$ is a level equivalence starting with the first term, because $K(\mathcal{C})$ and $K(\mathcal{C}')$ are Ω-spectra starting with the first term.)

The result is due to

[Cis] D.-C. Cisinski, "Invariance de la K-théorie par équivalences dérivées", Preprint, 2004.

5. Why do we care about spectra? Here I will not attempt to explain why topologists care about spectra (due to the lack of competence). Rather, I will try to say a few words about why algebraists should care about categories of spectra (and related categories), and about model structures on them. All of this will probably be very vague, but hopefully explained in the future lectures. Moreover, many of the remarks I will make have already been mentioned by Beilinson in his talks last quarter, but since this minicourse is supposed to be independent of those talks, I will repeat them.

First of all, let us agree that for us spectra are objects that conveniently encode the homological information about the various categories that we are considering. On the other hand, spectra can be seen as machines for producing sequences of abelian groups, much like complexes — in fact, for our purpose spectra may be thought of as a nonlinear version of complexes of abelian groups, and the homotopy groups of spectra may be thought of as the analogues of the usual homology groups of complexes. Of course, the principal example to keep in mind is the K-theory spectrum $K(\mathcal{C})$ of a Waldhausen category $\mathcal{C}$, whose homotopy groups are the K-groups of $\mathcal{C}$:

$$K_i(\mathcal{C}) := \pi_i(K(\mathcal{C})) \qquad \forall i \geq 0.$$

As this example already shows, we can not avoid working with spectra even if we wanted to: they provide essentially the only way of encoding all the higher K-groups of $\mathcal{C}$ into one object with good functoriality properties.

Another aspect of Beilinson's work is that we have to work not only with spectra, but with sheaves of spectra as well. Recall from M. Abouzaid's lecture which started this minicourse that the main goal is to produce a formula for the determinant of cohomology of a perfect constructible complex on a real analytic manifold as a product of "local ε-factors". This makes the appearance of sheaf theory completely natural.

However, one has to be careful with sheaves of spectra for the following reason. The definition of a *presheaf* of spectra on any topological space, or even Grothendieck site $\mathcal{D}$, is the obvious one: it is just a contravariant functor $\mathcal{D}^{\mathrm{op}} \to \mathcal{S}p^{\mathbb{N}}$ (or $\mathcal{S}p^{\Sigma}$ if we wish to work with presheaves of symmetric spectra). For a long time many people have tried to define the category of sheaves of spectra as a full subcategory of the category of presheaves of spectra, but all these attempts have lead to rather clumsy theories. Jardine was the first person to realize that one has to adopt a "dual" point of view, explained below.

We first illustrate this point of view in an abelian situation. Let X be a topological space. In homological algebra (or algebraic geometry) one often has to deal with complexes of sheaves (say, of abelian groups) on X. Now a complex of abelian sheaves is the same thing as a sheaf of complexes of abelian groups; let us denote the category of such sheaves by

$$\mathrm{Sh}_X(\mathrm{Ch}(\mathcal{A}b)).$$

In the classical approach this category is a full subcategory of the category

$$\mathrm{PreSh}_X(\mathrm{Ch}(\mathcal{A}b))$$

of *presheaves* of complexes of abelian groups on X. Now consider the following class of morphisms in $\mathrm{PreSh}_X(\mathrm{Ch}(\mathcal{A}b))$:

$$S = \left\{ f : \mathcal{F}^0 \to \mathcal{G}^0 \;\middle|\; \begin{array}{c} \text{for each } x \in X, \text{ the induced morphism of stalks} \\ f_x : \mathcal{F}^0_x \to \mathcal{G}^0_x \text{ is an isomorphism of complexes} \end{array} \right\}.$$

Easy exercise: The composed functor

$$\begin{array}{ccc} \mathrm{Sh}_X(\mathrm{Ch}(\mathcal{A}b)) & \hookrightarrow & \mathrm{PreSh}_X(\mathrm{Ch}(\mathcal{A}b)) \\ & \swarrow & \\ & \mathrm{PreSh}_X(\mathrm{Ch}(\mathcal{A}b))[S^{-1}] & \end{array}$$

is an equivalence of categories.

This implies that we can construct the derived category $D(\mathrm{Sh}_X(\mathcal{A}b))$ as the localization of the category $\mathrm{PreSh}_X(\mathrm{Ch}(\mathcal{A}b))$ by the class of morphisms that induce *stalkwise quasi-isomorphisms* of complexes.

Note that objects of $D(\mathrm{Sh}_X(\mathcal{A}b))$ are "machines for producing sequences of abelian sheaves" on X, and thus $D(\mathrm{Sh}_X(\mathcal{A}b))$ can be thought of as the linear version of the stable category of sheaves of spectra on X.

At least from the point of view of homological algebra, this is the main observation that underlies the modern theories of sheaves of simplicial sets and spectra. Namely, the category of sheaves is always constructed from the category of presheaves by formally inverting the morphisms that induce stalkwise weak equivalences (in a suitable sense).

This will be explained much more explicitly in the future lectures, but at this point I should say why model categories turn out to be useful in this theory. The reason is that in general localization of categories is a mysterious process which is hard to control. Probably one of the reasons Quillen has invented homotopical algebra is that if the class of morphisms one wants to invert is the class of weak equivalences for some model structure on the category, then computations in the localized category become much more manageable.

This is especially so if the model structure satisfies the stronger property of being a *simplicial* model structure (this notion will be discussed some other time). All model structures that we encounter in practice will satisfy this condition.

Finally, it is of course clear that in order to speak about model structures on categories of presheaves of spectra one should first understand model structures on the categories of spectra themselves, and this is what we will conclude this lecture with.

6. Model categories of spectra. This final section will probably appear very dry, since it mostly consists of statements of results taken from [HSS]. The reader should also be warned that the results on symmetric spectra stated in this lecture and the previous one constitute only a very small portion of the results in [HSS].

Theorem 1 (Bousfield & Friedlander): $\mathcal{S}p^{\mathbb{N}}$ is a proper simplicial model category with respect to the following three classes of maps:

- **strict weak equivalences**, defined to be the level equivalences (§ 3);
- **strict fibrations**, defined to be the level fibrations, i.e., morphisms $f : E \to F$ in $\mathcal{S}p^{\mathbb{N}}$ such that each $f_n : E_n \to F_n$ is a fibration in $\mathcal{S}_*$;

- **strict cofibrations**, defined as the morphisms $f : E \to F$ in $\mathcal{S}p^{\mathbb{N}}$ such that the induced maps

$$E_0 \to F_0 \quad \text{and}$$
$$E_{n+1} \bigcup_{S^1 \wedge E_n} (S^1 \wedge F_n) \to F_{n+1}$$

are cofibrations (i.e., monomorphisms) in $\mathcal{S}_*$ for all $n \geq 0$.

This is called the **strict model structure*****) on $\mathcal{S}p^{\mathbb{N}}$.

Theorem 2 (Bousfield & Friedlander): $\mathcal{S}p^{\mathbb{N}}$ is a proper simplicial model category with respect to:

- **stable weak equivalences**, defined as the stable homotopy equivalences;
- **stable cofibrations**, defined to be the same as the strict cofibrations above;
- **stable fibrations**, defined by the RLP with respect to the acyclic cofibrations.

This is called the **stable model structure*****) on $\mathcal{S}p^{\mathbb{N}}$.

Definitions: A **projective cofibration** or a **stable cofibration** of symmetric spectra is a map which has the LLP with respect to every level acyclic fibration (i.e., a morphism $f : E \to F$ in $\mathcal{S}p^{\Sigma}$ such that $f_n : E_n \to F_n$ is an acyclic fibration for all $n \geq 0$).

A **stable acyclic cofibration** of symmetric spectra is a map in $\mathcal{S}p^{\Sigma}$ which is both a stable cofibration and a stable homotopy equivalence.

An **injective fibration** of symmetric spectra is a map in $\mathcal{S}p^{\Sigma}$ which has the RLP with respect to every level acyclic cofibration. Note that the adjective "injective" refers to the lifting properties of the map and not to the property of being a monomorphism.

A **stable fibration** in $\mathcal{S}p^{\Sigma}$ is a map which has the RLP with respect to every stable acyclic cofibration. Unlike the case of cofibrations, a stable fibration is not the same as an injective fibration!

The **projective level structure** on $\mathcal{S}p^{\Sigma}$ consists of

- weak equivalences = level equivalences;
- projective (or stable) cofibrations;
- level fibrations.

The **injective level structure** on $\mathcal{S}p^{\Sigma}$ consists of

- weak equivalences = level equivalences;
- level cofibrations (= monomorphisms);
- injective fibrations.

Theorem 3 (Hovey & Shipley & Smith): The projective level structure and the injective****) level structure are proper simplicial model structures on $\mathcal{S}p^{\Sigma}$.

It remains to discuss the stable model structure on $\mathcal{S}p^{\Sigma}$. First we make the following

Definition: An **injective symmetric spectrum** is a fibrant object for the injective level structure on $\mathcal{S}p^\Sigma$. A morphism $f : X \to Y$ of symmetric spectra is a **stable equivalence** if the induced map

$$\pi_0 \operatorname{Map}_{\mathcal{S}p^\Sigma}(Y, E) \to \pi_0 \operatorname{Map}_{\mathcal{S}p^\Sigma}(X, E)$$

is a bijection for every injective symmetric Ω-spectrum E.

Theorem 4 (Hovey & Shipley & Smith): $\mathcal{S}p^\Sigma$ is a proper simplicial model category with respect to:

- weak equivalences = stable equivalences;
- stable (or projective) cofibrations;
- stable fibrations.

This is called the stable model structure on $\mathcal{S}p^\Sigma$. We have equivalences of categories

$$\operatorname{Ho}(\mathcal{S}p^\Sigma_{\text{stable}}) \simeq \operatorname{Ho}(\mathcal{S}p^{\mathbb{N}}_{\text{stable}}) \simeq \text{Boardman's stable homotopy category.}$$

Post-lecture footnotes

*) Observe that X is a Kan complex if and only if X is a retract of $\operatorname{Sing}(|X|)$. Indeed, the "if" direction follows from the definition of a model category (namely, the class of fibrations is closed under retracts). For the converse, apply the lifting axiom to the diagram

$$\begin{array}{ccc} X & \xrightarrow{\operatorname{id}_X} & X \\ {\scriptstyle \eta_X}\downarrow & & \downarrow \\ \operatorname{Sing}(|X|) & \longrightarrow & * \end{array}$$

Since η_X is an acyclic cofibration and $X \to *$ is a fibration (because we are assuming that X is a Kan complex), we see that there exists a morphism $f : \operatorname{Sing}(|X|) \to X$ such that $f \circ \eta_X = \operatorname{id}_X$, which proves the claim.

This argument is in fact a special case of a standard trick in the theory of model categories, known as the "retract argument":

Proposition: Let $\mathcal{C}$ be a category and $f = p \circ i$ a factorization of an arrow in $\mathcal{C}$. Then:

(i) if p has the RLP with respect to f, then f is a retract of i;

(ii) if i has the LLP with respect to f, then f is a retract of p.

Exercise: Prove this proposition and use it to deduce statements (i) – (iv) in Remark (5) after the definition of a model category.

**) Note that if E_{n+1} is a Kan complex, then so is $\operatorname{Hom}_{\mathcal{S}_*}(S^1, E_{n+1})$. This is just a formal statement: the functor $S^1 \wedge \cdot$ obviously preserves acyclic cofibrations in $\mathcal{S}_*$, and therefore, by adjunction, the functor $\operatorname{Hom}_{\mathcal{S}_*}(S^1, \cdot)$ preserves fibrations.

***) The strict model structure on $\mathcal{S}p^{\mathbb{N}}$ only plays an auxiliary role in the theory of spectra: it is needed to define the stable model structure, but has no applications by itself. The same is true for the projective and injective model structures on $\mathcal{S}p^\Sigma$.

****) The injective model structure is in fact just an artifact of working with symmetric spectra of simplicial sets rather than topological spaces. The whole theory becomes significantly simpler when one works in the topological context.