MODEL CATEGORIES OF DIAGRAM SPECTRA

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Abstract. Working in the category $\mathscr{T}$ of based spaces, we give the basic theory of diagram spaces and diagram spectra. These are functors $\mathscr{D} \longrightarrow \mathscr{T}$ for a suitable small topological category $\mathscr{D}$. When $\mathscr{D}$ is symmetric monoidal, there is a smash product that gives the category of $\mathscr{D}$-spaces a symmetric monoidal structure. Examples include

- Prespectra, as defined classically.
- Symmetric spectra, as defined by Jeff Smith.
- Orthogonal spectra, a coordinate free analogue of symmetric spectra with symmetric groups replaced by orthogonal groups in the domain category.
- Γ-spaces, as defined by Graeme Segal.
- $\mathscr{W}$-spaces, an analogue of Γ-spaces with finite sets replaced by finite CW complexes in the domain category.

We construct and compare model structures on these categories. With the caveat that Γ-spaces are always connective, these categories, and their simplicial analogues, are Quillen equivalent and their associated homotopy categories are equivalent to the classical stable homotopy category. Monoids in these categories are (strict) ring spectra. Often the subcategories of ring spectra, module spectra over a ring spectrum, and commutative ring spectra are also model categories. When this holds, the respective categories of ring and module spectra are Quillen equivalent and thus have equivalent homotopy categories. This allows interchangeable use of these categories in applications.

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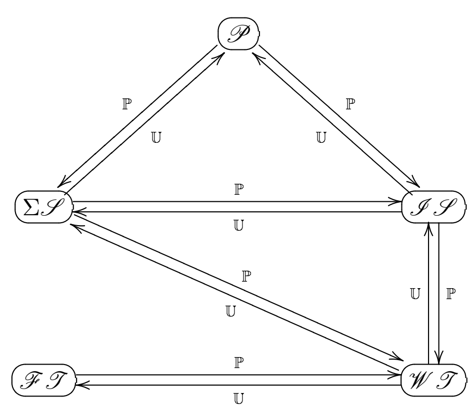
A few years ago there were no constructions of the stable homotopy category that began from a category of spectra with an associative and commutative smash product. Now there are several very different such constructions. These allow many new directions in stable homotopy theory, and they are being actively exploited by many people. We refer the reader to papers on particular categories [10, 11, 15, 21, 22, 24, 35, 39] and to [30] for discussions of the history, philosophy, advantages and disadvantages of the various approaches.

To avoid chaos, it is important to have comparison theorems relating the different constructions, so that the working mathematician can choose whichever category is most convenient for any particular application and can then transport the conclusions to any other such modern category of spectra. This is one of several papers that together show that all of the known approaches to highly structured ring and module spectra are essentially equivalent.

Several of the new categories are constructed from "diagram categories", by which we understand categories of functors from some fixed category $\mathscr{D}$ to some chosen ground category. We concentrate on such examples in this paper. In [36] and [24], the approach to stable homotopy theory based on diagram categories is compared to the approach based on classical spectra with additional structure of [11]. The categories of diagram spectra to be studied here are displayed in the following "Main Diagram":



We have the dictionary:

$\mathscr{P}$ is the category of $\mathscr{N}$-spectra, or prespectra.
$\Sigma\mathscr{S}$ is the category of Σ-spectra, or symmetric spectra.
$\mathscr{I}\mathscr{S}$ is the category of $\mathscr{I}$-spectra, or orthogonal spectra.
$\mathscr{F}\mathscr{T}$ is the category of $\mathscr{F}$-spaces, or Γ-spaces.
$\mathscr{W}\mathscr{T}$ is the category of $\mathscr{W}$-spaces.

As will be made precise, $\mathscr{N}$ is the category of non-negative integers, Σ is the category of symmetric groups, $\mathscr{I}$ is the category of orthogonal groups, $\mathscr{F}$ is the category of finite based sets, and $\mathscr{W}$ is the category of based spaces homeomorphic to finite CW complexes. We often use $\mathscr{D}$ generically to denote such a domain category for diagram spectra. When $\mathscr{D} = \mathscr{F}$ or $\mathscr{D} = \mathscr{W}$, there is no distinction between $\mathscr{D}$-spaces and $\mathscr{D}$-spectra, $\mathscr{D}\mathscr{T} = \mathscr{D}\mathscr{S}$. The functors $\mathbb{U}$ are forgetful functors, the functors $\mathbb{P}$ are prolongation functors, and in each case $\mathbb{P}$ is left adjoint to $\mathbb{U}$. All of these categories except $\mathscr{P}$ are symmetric monoidal. The functors $\mathbb{U}$ between symmetric monoidal categories are lax symmetric monoidal, the functors $\mathbb{P}$ between symmetric monoidal categories are strong symmetric monoidal, and these functors $\mathbb{P}$ and $\mathbb{U}$ restrict to adjoint pairs relating the various categories of rings, commutative rings, and modules over rings.

Symmetric spectra were introduced by Smith, and their homotopy theory was developed by Hovey, Shipley, and Smith [15]. Symmetric ring spectra were further studied in [39] and [37]. Under the name of $\mathscr{I}_*$-prespectra, orthogonal spectra were defined by May [27, §5], but their serious study begins here. They are further studied in [24]. Related but different notions defined in terms of $\mathscr{I}$ were introduced for use in infinite loop space theory by Boardman and Vogt [5]. Under the name of Γ-spaces, $\mathscr{F}$-spaces were introduced by Segal [38], and their homotopy theory was developed by Anderson [2] and Bousfield and Friedlander [7]. Under the name of Gamma-rings, Lydakis [21] and Schwede [35] introduced and studied $\mathscr{F}$-ring spaces. A version of $\mathscr{W}$-spaces was introduced by Anderson [3], and a simplicial analogue of $\mathscr{W}$-spaces has been studied by Lydakis [22]. There are few comparisons among these categories in the literature.

We develop the formal theory of diagram spectra in Part I, deferring categorical proofs and explanations to Part III. In particular, we explain the relationship between diagram ring spectra and diagram FSP's (functors with smash product) there. Our model theoretic work is in the central Part II. We define and compare model structures on categories of diagram spaces and on their categories of rings and modules. The most highly structured and satisfactory kind of comparison between model categories is specified by the notion of a Quillen equivalence, and most of our equivalences are of this form. The brief Appendix A records what we need about this notion. Each Part has its own introduction.

We define "stable model structures" simultaneously on the categories of $\mathscr{D}$-spectra for $\mathscr{D} = \mathscr{N}$, Σ, $\mathscr{I}$, and $\mathscr{W}$. In the case of symmetric spectra, our model structure is the same as that in the preprint version of [15]; the published version restricts attention to symmetric spectra of simplicial sets. Although that work inspired and provided a model for ours, our treatment of symmetric spectra is logically independent and makes no use of simplicial techniques. As one would expect, the categories of symmetric spectra of spaces and symmetric spectra of simplicial sets are Quillen equivalent; see §18.

Curiously, in all cases *except* that of symmetric spectra, whose homotopy theory is intrinsically more subtle, the stable equivalences are just the π_*-isomorphisms, namely the maps whose underlying maps of prespectra induce isomorphisms of homotopy groups. Using these stable model structures, we prove the following comparison theorem.

Theorem 0.1. *The categories of $\mathscr{N}$-spectra, symmetric spectra, orthogonal spectra, and $\mathscr{W}$-spaces are Quillen equivalent.*

In fact, we prove that the categories of $\mathscr{N}$-spectra and orthogonal spectra are Quillen equivalent and that the categories of symmetric spectra, orthogonal spectra, and $\mathscr{W}$-spaces are Quillen equivalent. These comparisons between $\mathscr{N}$-spectra and orthogonal spectra and between symmetric spectra and orthogonal spectra imply that the categories of $\mathscr{N}$-spectra and symmetric spectra are Quillen equivalent. This reproves a result of Hovey, Shipley, and Smith [15, 4.2.5]. The new proof leads to a new perspective on the stable equivalences of symmetric spectra.

Corollary 0.2. *A map f of cofibrant symmetric spectra is a stable equivalence if and only if $\mathbb{P}f$ is a π_*-isomorphism of orthogonal spectra.*

A similar characterization of the stable equivalences in terms of an interesting endofunctor $\mathbb{D}$ on the category of symmetric spectra is given in [39, 3.1.2]. Generalizations of that functor gave the starting point for a now obsolete approach to our comparison theorems; see [30].

Of course, the point of introducing categories of diagram spectra is to obtain point-set level models for the classical stable homotopy category that are symmetric monoidal under their smash product. On passage to homotopy categories, the derived smash product must agree with the classical (naive) smash product of prespectra. That is the content of the following addendum to Theorem 0.1.

Theorem 0.3. *The equivalences of homotopy categories induced by the Quillen equivalences of Theorem 0.1 preserve smash products.*

Here again, we compare $\mathscr{N}$-spectra and symmetric spectra to orthogonal spectra and then deduce the comparison between $\mathscr{N}$-spectra and symmetric spectra; a partial result in this direction was given in [15, 4.2.16].

Following the model of [37], we prove that, when $\mathscr{D} = \Sigma$, $\mathscr{I}$, or $\mathscr{W}$, the category of $\mathscr{D}$-ring spectra and the category of modules over a $\mathscr{D}$-ring spectrum inherit model structures from the underlying category of $\mathscr{D}$-spectra. Using these model structures, we obtain the following comparison theorems for categories of diagram ring and module spectra.

Theorem 0.4. *The categories of symmetric ring spectra, orthogonal ring spectra, and $\mathscr{W}$-ring spaces are Quillen equivalent model categories.*

Theorem 0.5. *For a cofibrant symmetric ring spectrum R, the categories of R-modules and of $\mathbb{P}R$-modules (of orthogonal spectra) are Quillen equivalent model categories. For a cofibrant orthogonal ring spectrum R, the categories of R-modules and of $\mathbb{P}R$-modules (of $\mathscr{W}$-spaces) are Quillen equivalent model categories.*

Here and in the analogous Theorems 0.8 and 0.12 below, the cofibrancy hypothesis results in no loss of generality (see Theorem 12.1).

Corollary 0.6. *For an orthogonal ring spectrum R, the categories of R-modules and of $\mathbb{U}R$-modules (of symmetric spectra) are Quillen equivalent model categories. For a $\mathscr{W}$-ring spectrum R, the categories of R-modules and of $\mathbb{U}R$-modules (of orthogonal spectra) are Quillen equivalent model categories.*

We would like the category of commutative $\mathscr{D}$-ring spectra to inherit a model structure from the underlying category of $\mathscr{D}$-spectra. However, because the sphere $\mathscr{D}$-spectrum is cofibrant in the stable model structure, a familiar argument due to Lewis [19] shows that this fails. In the context of symmetric spectra, Jeff Smith explained[1] the mechanism of this failure: if the zeroth term of a symmetric spectrum X is non-trivial, the symmetric powers of X do not behave well homotopically. As Smith saw, one can get around this by replacing the stable model structure by a Quillen equivalent "positive stable model structure".

In fact, we have such positive stable model categories of $\mathscr{D}$-spectra for all four of the categories considered so far, and all of the results above work equally well starting from these model structures. In the cases of symmetric and orthogonal spectra, we show that the categories of commutative ring spectra inherit positive stable model structures. The proof is closely analogous to the proof of the corresponding result in the context of the S-modules of Elmendorf, Kriz, Mandell, and May [11, VII§§3,5]. More generally, we show that the categories of modules, algebras and commutative algebras over a commutative S-algebra R are model categories. With these positive stable model structures, we prove the following comparison theorems.

Theorem 0.7. *The categories of commutative symmetric ring spectra and commutative orthogonal ring spectra are Quillen equivalent.*

Theorem 0.8. *Let R be a cofibrant commutative symmetric ring spectrum. The categories of R-modules, R-algebras, and commutative R-algebras are Quillen equivalent to the categories of $\mathbb{P}R$-modules, $\mathbb{P}R$-algebras, and commutative $\mathbb{P}R$-algebras (of orthogonal spectra).*

Corollary 0.9. *Let R be a commutative orthogonal ring spectrum. The categories of R-modules, R-algebras, and commutative R-algebras are Quillen equivalent to the categories of $\mathbb{U}R$-modules, $\mathbb{U}R$-algebras, and commutative $\mathbb{U}R$-algebras (of symmetric spectra).*

We do not know whether or not the category of commutative $\mathscr{W}$-ring spaces admits a model category structure; some of us suspect that it does not.

We now bring $\mathscr{F}$-spaces into the picture. Most of the previous work with them has been done simplicially. The category of $\mathscr{F}$-spaces has a stable model structure, and it is Quillen equivalent to the category of $\mathscr{F}$-simplicial sets; see §18. Since $\mathscr{F}$-spaces only model connective ($= (-1)$-connected) prespectra and the category of connective $\mathscr{W}$-spaces is not a model category (it fails to have limits), we cannot expect a Quillen equivalence between the categories of $\mathscr{F}$-spaces and connective $\mathscr{W}$-spaces. However, we have nearly that much. A Quillen equivalence is a Quillen adjoint pair that induces an equivalence of homotopy categories. We define a *connective Quillen equivalence* to be a Quillen adjoint pair that induces an equivalence between the respective homotopy categories of connective objects.

Theorem 0.10. *The functors $\mathbb{P}$ and $\mathbb{U}$ between $\mathscr{F}\mathscr{T}$ and $\mathscr{W}\mathscr{T}$ are a connective Quillen equivalence. The induced equivalence of homotopy categories preserves smash products.*

Theorem 0.11. *The categories of $\mathscr{F}$-ring spaces and $\mathscr{W}$-ring spaces are connectively Quillen equivalent.*

Theorem 0.12. *For a cofibrant $\mathscr{F}$-ring space R, the categories of R-modules and $\mathbb{P}R$-modules are connectively Quillen equivalent.*

Corollary 0.13. *For a connective $\mathscr{W}$-ring space R, the categories of R-modules and $\mathbb{U}R$-modules are connectively Quillen equivalent.*

[1] Private communication

The model structure on $\mathscr{W}$-spaces relevant to the last four results is not the stable model structure but rather a Quillen equivalent "absolute stable model structure". Lydakis [22] has studied a simplicial analogue of this model category, and we prove that $\mathscr{W}\mathscr{T}$ is Quillen equivalent to his category.

We do not know whether or not the homotopy categories of commutative $\mathscr{F}$-ring spaces and connective commutative $\mathscr{W}$-ring spaces are equivalent. The following remark provides a stopgap for the study of commutativity in these cases.

Remark 0.14. There is a definition of an action of an operad on a $\mathscr{D}$-spectrum. Restricting to an E_∞ operad, this gives the notion of an E_∞-$\mathscr{D}$-ring spectrum. See [30, §5]. It is an easy consequence of results in this paper (especially Lemma 15.5) that the homotopy categories of E_∞ symmetric ring spectra and commutative symmetric ring spectra are equivalent, as was first noted by Smith in the simplicial context, and that the homotopy categories of E_∞ orthogonal ring spectra and commutative orthogonal ring spectra are equivalent. We do not know whether or not the analogues for $\mathscr{W}$-spaces and $\mathscr{F}$-spaces hold, and here the homotopy theory of E_∞-rings seems more tractable than that of commutative rings. It is also an easy consequence of the methods of this paper that the homotopy categories of E_∞ symmetric ring spectra, E_∞ orthogonal ring spectra, and E_∞-$\mathscr{W}$-ring spaces are equivalent and that the homotopy categories of E_∞-$\mathscr{F}$-ring spaces and connective E_∞-$\mathscr{W}$-ring spaces are equivalent.

Part I. Diagram spaces and diagram spectra

We introduce functor categories $\mathscr{D}\mathscr{T}$ of $\mathscr{D}$-spaces in §1. When $\mathscr{D}$ is symmetric monoidal, so is $\mathscr{D}\mathscr{T}$. If R is a monoid in $\mathscr{D}\mathscr{T}$, we have a category $\mathscr{D}\mathscr{S}_R$ of R-modules, or "$\mathscr{D}$-spectra over R". It is symmetric monoidal if R is commutative. In §2, we define a new category $\mathscr{D}_R$ such that the categories of $\mathscr{D}_R$-spaces and $\mathscr{D}$-spectra over R are isomorphic. This reduces the study of diagram spectra to a special case of the conceptually simpler study of diagram spaces.

Our focus is on comparisons between such categories as $\mathscr{D}$ varies. In §3, we consider adjoint forgetful and prolongation functors $\mathbb{U} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}\mathscr{T}$ and $\mathbb{P} : \mathscr{C}\mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$ associated to a functor $\iota : \mathscr{C} \longrightarrow \mathscr{D}$. The main point is to understand the specialization of these functors to categories of diagram spectra.

Finally, in §4, we specialize to the examples that we are most interested in. For particular domain categories $\mathscr{D}$, we fix a canonical $\mathscr{D}$-monoid S that is related to spheres and obtain the category $\mathscr{D}\mathscr{S}$ of $\mathscr{D}$-spectra over S. It is symmetric monoidal when S is commutative. This fails for $\mathscr{N}$ but holds for Σ, $\mathscr{I}$, $\mathscr{F}$, and $\mathscr{W}$.

We have chosen to work with functors that take values in based spaces because some of our motivating examples make little sense simplicially. However, everything in Parts I and III can be adapted without difficulty to functors that take values in the category of based simplicial sets. The simplicially minded reader may understand "spaces" to mean "simplicial sets" and "continuous" to mean "simplicial". In fact, the categorical constructions apply verbatim to functors that take values in any symmetric monoidal category that is tensored and cotensored over either topological spaces or simplicial sets. Examples of such symmetric monoidal functor categories arise in other fields, such as algebraic geometry.

1. Categories of $\mathscr{D}$-spaces

Spaces will mean compactly generated spaces (= weak Hausdorff k-spaces). One reference is [32]; a thorough treatment is given in [18, App]. We let $\mathscr{T}$ denote the resulting category of based spaces. All of our categories are topological, meaning that they have spaces of morphisms and continuous composition. The category $\mathscr{T}$ is a closed symmetric monoidal topological category under the smash product and function space functors $A \wedge B$ and $F(A, B)$; its unit is S^0. We emphasize that the internal hom spaces $F(A, B)$ and the categorical hom spaces $\mathscr{T}(A, B)$ coincide.

Let $\mathscr{D}$ be a topological category. We assume that $\mathscr{D}$ is based, in the sense that it has a given initial and terminal object $*$. Thus the space $\mathscr{D}(d, e)$ of maps $d \to e$ is based with basepoint $d \to * \to e$. When $\mathscr{D}$ is given as an unbased category, we implicitly adjoin a base object $*$; in other words, we then understand $\mathscr{D}(d, e)$ to mean the union of the unbased space of maps $d \to e$ in $\mathscr{D}$ and a disjoint basepoint. The base object of $\mathscr{T}$ is a one-point space. By a functor between based categories, we always understand a functor that carries base objects to base objects; that is, we take this as part of our definition of "functor". A functor $F : \mathscr{D} \longrightarrow \mathscr{D}'$ between topological categories is continuous if $F : \mathscr{D}(d, e) \longrightarrow \mathscr{D}'(Fd, Fe)$ is a continuous map for all d and e.

Definition 1.1. A *$\mathscr{D}$-space* is a continuous functor $X : \mathscr{D} \longrightarrow \mathscr{T}$. Let $\mathscr{D}\mathscr{T}$ denote the category of $\mathscr{D}$-spaces and natural maps between them.

We think of a $\mathscr{D}$-space as a diagram of spaces whose shape is specified by $\mathscr{D}$. The category $\mathscr{D}\mathscr{T}$ is complete and cocomplete, with limits and colimits constructed levelwise (one object at a time). It is also tensored and cotensored. For a $\mathscr{D}$-space X and based space A, the tensor $X \wedge A$ is given by the levelwise smash product and the cotensor $F(A, X)$ is given by the levelwise function space. Thus

$$\mathscr{D}\mathscr{T}(X \wedge A, Y) \cong \mathscr{T}(A, \mathscr{D}\mathscr{T}(X, Y)) \cong \mathscr{D}\mathscr{T}(X, F(A, Y)). \tag{1.2}$$

We define homotopies between maps of $\mathscr{D}$-spaces by use of the cylinders $X \wedge I_+$.

Spaces and $\mathscr{D}$-spaces are related by a system of adjoint pairs of functors.

Definition 1.3. For an object d of $\mathscr{D}$, define the *evaluation functor* $Ev_d : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{T}$ by $Ev_d X = X(d)$ and define the *shift desuspension functor* $F_d : \mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$ by $(F_d A)(e) = \mathscr{D}(d, e) \wedge A$. The functors F_d and Ev_d are left and right adjoint,

$$\mathscr{D}\mathscr{T}(F_d A, X) \cong \mathscr{T}(A, Ev_d X). \tag{1.4}$$

Moreover, Ev_d is covariantly functorial in d and F_d is contravariantly functorial in d. We write $Ev_d^{\mathscr{D}}$ and $F_d^{\mathscr{D}}$ when necessary to avoid confusion.

Notation 1.5. We use the alternative notation $d^* = F_d S^0$. Thus $d^*(e) = \mathscr{D}(d, e)$ and $F_d A = d^* \wedge A$; d^* is the *$\mathscr{D}$-space represented by the object d*.

Recall that a skeleton $sk\mathscr{D}$ of a category $\mathscr{D}$ is a full subcategory with one object in each isomorphism class. The inclusion $sk\mathscr{D} \longrightarrow \mathscr{D}$ is an equivalence of categories. When $\mathscr{D}$ is topological and has a small skeleton $sk\mathscr{D}$, $\mathscr{D}\mathscr{T}$ is a topological category. The set $\mathscr{D}\mathscr{T}(X, Y)$ of maps $X \longrightarrow Y$ is the equalizer in the category of based spaces displayed in the diagram

$$\mathscr{D}\mathscr{T}(X, Y) \longrightarrow \prod_d F(X(d), Y(d)) \underset{\tilde{\nu}}{\overset{\tilde{\mu}}{\rightrightarrows}} \prod_{\alpha : d \to e} F(X(d), Y(e)),$$

where the products run over the objects and morphisms of $sk\mathscr{D}$. For $f = (f_d)$, the αth component of $\tilde{\mu}(f)$ is $Y(\alpha) \circ f_d$ and the αth component of $\tilde{\nu}(f)$ is $f_e \circ X(\alpha)$. By a comparison of represented functors, this implies that any $\mathscr{D}$-space X can be written as the coend of the contravariant functor d^* and the covariant functor X.

Lemma 1.6. *Let $\mathscr{D}$ have a small skeleton $sk\mathscr{D}$ and let X be a $\mathscr{D}$-space. Then the evaluation maps $\varepsilon : d^* \wedge X(d) \longrightarrow X$ induce a natural isomorphism*

$$\int^{d \in sk\mathscr{D}} d^* \wedge X(d) \longrightarrow X.$$

Explicitly, X is isomorphic to the coequalizer of the parallel arrows in the diagram

$$\bigvee_{d,e} e^* \wedge \mathscr{D}(d,e) \wedge X(d) \underset{\mathrm{id} \wedge \varepsilon}{\overset{\varepsilon \wedge \mathrm{id}}{\rightrightarrows}} \bigvee_d d^* \wedge X(d) \xrightarrow{\ \varepsilon\ } X,$$

where the wedges run over pairs of objects and objects of $sk\mathscr{D}$ and the parallel arrows are wedges of smash products of identity and evaluation maps.

We will explain the by now quite standard proof of the following fundamental result in §21, after fixing language about symmetric monoidal categories in §20. For the rest of this section, let $\mathscr{D}$ be a skeletally small symmetric monoidal category with unit u and product $\square$.

Theorem 1.7. *The category $\mathscr{D}\mathscr{T}$ has a smash product $\wedge$ and internal hom functor F under which it is a closed symmetric monoidal category with unit u^*.*

We often use the following addendum, which is also proven in §21.

Lemma 1.8. *For objects d and e of $\mathscr{D}$ and based spaces A and B, there is a natural isomorphism*

$$F_d A \wedge F_e B \longrightarrow F_{d \square e}(A \wedge B).$$

Monoids and commutative monoids are defined in any symmetric monoidal category, as are (right) R-modules M over monoids R: there is a map $M \wedge R \longrightarrow R$ such that the evident unit and associativity diagrams commute. The following definition and proposition give a more direct and explicit description of R-modules. The proof of the proposition is immediate from the definition of $\wedge$ in §21.

Definition 1.9. Let R be a monoid in $\mathscr{D}\mathscr{T}$ with unit λ and product ϕ. A $\mathscr{D}$-*spectrum over* R is a $\mathscr{D}$-space $X : \mathscr{D} \longrightarrow \mathscr{T}$ together with continuous maps $\sigma : X(d) \wedge R(e) \longrightarrow X(d \square e)$, natural in d and e, such that the composite

$$X(d) \cong X(d) \wedge S^0 \xrightarrow{\mathrm{id} \wedge \lambda} X(d) \wedge R(u) \xrightarrow{\ \sigma\ } X(d \square u) \cong X(d)$$

is the identity and the following diagram commutes:

$$\begin{array}{ccc} X(d) \wedge R(e) \wedge R(f) & \xrightarrow{\ \sigma \wedge \mathrm{id}\ } & X(d \square e) \wedge R(f) \\ {\scriptstyle \mathrm{id} \wedge \phi} \downarrow & & \downarrow {\scriptstyle \sigma} \\ X(d) \wedge R(e \square f) & \xrightarrow[\ \sigma\ ]{} & X(d \square e \square f). \end{array}$$

(Here and below, we suppress implicit use of the associativity isomorphisms for $\wedge$ and $\square$.) Let $\mathscr{D}\mathscr{S}_R$ denote the category of $\mathscr{D}$-spectra over R.

Proposition 1.10. *Let R be a monoid in $\mathscr{D}\mathscr{T}$. The categories of R-modules and of $\mathscr{D}$-spectra over R are isomorphic.*

We use R-modules and $\mathscr{D}$-spectra over R interchangeably throughout. As we will explain in §22, we can construct functors $\wedge_R$ and F_R exactly as in algebra, and we have the following extension of Theorem 1.7.

Theorem 1.11. *Let R be a commutative monoid in $\mathscr{D}\mathscr{T}$. Then the category $\mathscr{D}\mathscr{S}_R$ of R-modules has a smash product $\wedge_R$ and internal hom functor F_R under which it is a closed symmetric monoidal category with unit R.*

For a commutative monoid R in $\mathscr{D}\mathscr{T}$, we define a *(commutative) R-algebra* to be a (commutative) monoid in $\mathscr{D}\mathscr{S}_R$. As we will also explain in §22, this notion is equivalent to the more elementary notion of a (commutative) $\mathscr{D}$-FSP over R.

2. An interpretation of diagram spectra as diagram spaces

Let $\mathscr{D}$ be symmetric monoidal and fix a monoid $R : \mathscr{D} \longrightarrow \mathscr{T}$ in $\mathscr{D}\mathscr{T}$. We do not require R to be commutative, although that is the case of greatest interest. We reinterpret the category $\mathscr{D}\mathscr{S}_R$ of $\mathscr{D}$-spectra over R, alias the category of right R-modules, as the category $\mathscr{D}_R\mathscr{T}$ of $\mathscr{D}_R$-spaces, where $\mathscr{D}_R$ is a category constructed from $\mathscr{D}$ and R. If R is commutative, then $\mathscr{D}_R$ is a symmetric monoidal category. In this case, we can reinterpret the smash product $\wedge_R$ of R-modules as the smash product in the category of $\mathscr{D}_R$-spaces. This reduces the study of diagram spectra to the study of diagram spaces.

Just as in algebra, for a $\mathscr{D}$-space X, $X \wedge R$ is the free R-module generated by X. Recall the represented functors d^* from Notations 1.5 and remember that they behave contravariantly with respect to d.

Construction 2.1. We construct a category $\mathscr{D}_R$ and a functor $\delta : \mathscr{D} \longrightarrow \mathscr{D}_R$. When R is commutative, we construct a product $\Box_R$ on $\mathscr{D}_R$ such that $\mathscr{D}_R$ is a symmetric monoidal category and δ is strong symmetric monoidal functor. The objects of $\mathscr{D}_R$ are the objects of $\mathscr{D}$, and δ is the identity on objects. For objects d and e of $\mathscr{D}$, the space of morphisms $d \longrightarrow e$ in $\mathscr{D}_R$ is

$$\mathscr{D}_R(d,e) = \mathscr{D}\mathscr{S}_R(e^* \wedge R, d^* \wedge R),$$

and composition is inherited from composition in $\mathscr{D}\mathscr{S}_R$. Thus $\mathscr{D}_R$ may be identified with the full subcategory of $\mathscr{D}\mathscr{S}_R^{op}$ whose objects are the free R-modules $d^* \wedge R$. Observe that $\mathscr{D}(d,e) \cong \mathscr{D}\mathscr{T}(e^*, d^*)$. We specify δ on morphisms by smashing maps of $\mathscr{D}$-spaces with R. When R is commutative, $\Box_R$ is defined on objects as the product $\Box$ of $\mathscr{D}$. Its unit object is the unit object u of $\mathscr{D}$. The product $f \Box_R f'$ of morphisms $f : e^* \wedge R \longrightarrow d^* \wedge R$ and $f' : e'^* \wedge R \longrightarrow d'^* \wedge R$ is

$$f \wedge_R f' : (e \Box e')^* \wedge R \cong (e^* \wedge R) \wedge_R (e'^* \wedge R) \longrightarrow (d^* \wedge R) \wedge_R (d'^* \wedge R) \cong (d \Box d')^* \wedge R;$$

the isomorphisms are implied by the isomorphisms $(d \Box e)^* \cong d^* \wedge e^*$ of Lemma 1.8.

We shall prove the following result in §23.

Theorem 2.2. *Let R be a monoid in $\mathscr{D}\mathscr{T}$. Then the categories $\mathscr{D}\mathscr{S}_R$ of $\mathscr{D}$-spectra over R and $\mathscr{D}_R\mathscr{T}$ of $\mathscr{D}_R$-spaces are isomorphic. If R is commutative, then the isomorphism $\mathscr{D}\mathscr{S}_R \cong \mathscr{D}_R\mathscr{T}$ is an isomorphism of symmetric monoidal categories.*

Remark 2.3. If $R = (u_{\mathscr{D}})^*$, then $\delta : \mathscr{D} \longrightarrow \mathscr{D}_R$ is an identification. That is, as in any symmetric monoidal category, $\mathscr{D}$-spaces admit a unique structure of module over the unit for the smash product.

3. Forgetful and prolongation functors

We wish to compare the categories $\mathscr{D}\mathscr{T}$ as $\mathscr{D}$ varies. Thus let $\iota : \mathscr{C} \longrightarrow \mathscr{D}$ be a continuous functor between (based) topological categories. In practice, ι is faithful. We often regard it as an inclusion of categories and omit it from the notations.

Definition 3.1. Define the *forgetful functor* $\mathbb{U} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}\mathscr{T}$ on $\mathscr{D}$-spaces Y by letting $(\mathbb{U}Y)(c) = Y(\iota c)$.

The following result is standard category theory; we recall the proof in §23.

Proposition 3.2. *If $\mathscr{C}$ is skeletally small, then $\mathbb{U} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}\mathscr{T}$ has a left adjoint* prolongation functor *$\mathbb{P} : \mathscr{C}\mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$. For an object c of $\mathscr{C}$, $\mathbb{P}F_cX$ is naturally isomorphic to $F_{\iota c}X$. If $\iota : \mathscr{C} \longrightarrow \mathscr{D}$ is fully faithful, then the unit $\eta : \mathrm{Id} \longrightarrow \mathbb{U}\mathbb{P}$ of the adjunction is a natural isomorphism.*

The isomorphism $\mathbb{P}F_cX \cong F_{\iota c}X$ follows formally from the evident relation $Ev_c\mathbb{U}Y = Y(\iota c) = Ev_{\iota c}Y$. The last statement means that, when ι is fully faithful, $\mathbb{P}$ prolongs a $\mathscr{C}$-space X to a $\mathscr{D}$-space that restricts to X on $\mathscr{C}$.

When $\mathscr{D}$ is skeletally small, $\mathbb{U}$ also has a right adjoint, but we shall make no use of that fact. We are especially interested in the multiplicative properties of $\mathbb{U}$ and $\mathbb{P}$, and we prove the following basic result in §23. In the rest of this section, let $\iota : \mathscr{C} \longrightarrow \mathscr{D}$ be a strong symmetric monoidal functor between skeletally small symmetric monoidal categories.

Proposition 3.3. *The functor $\mathbb{P} : \mathscr{C}\mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$ is strong symmetric monoidal. The functor $\mathbb{U} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{C}\mathscr{T}$ is lax symmetric monoidal, but with $u^*_{\mathscr{C}} \cong \mathbb{U}u^*_{\mathscr{D}}$. The unit $\eta : \mathrm{Id} \longrightarrow \mathbb{U}\mathbb{P}$ and counit $\varepsilon : \mathbb{P}\mathbb{U} \longrightarrow \mathrm{Id}$ are monoidal natural transformations.*

The notion of a monoidal natural transformation is recalled in Definition 20.3.

We use the categories $\mathscr{D}_R$ to reduce comparisons of categories of diagram spectra to comparisons of categories of diagram spaces. By Proposition 3.3, if R is a monoid in $\mathscr{D}\mathscr{T}$, then $\mathbb{U}R$ is a monoid in $\mathscr{C}\mathscr{T}$, and $\mathbb{U}R$ is commutative if R is. We prove the first two statements of the following result in §23. The last two statements then follow from Propositions 3.2 and 3.3.

Proposition 3.4. *If R is a monoid in $\mathscr{D}\mathscr{T}$, then $\iota : \mathscr{C} \longrightarrow \mathscr{D}$ extends to a functor $\kappa : \mathscr{C}_{\mathbb{U}R} \longrightarrow \mathscr{D}_R$. If R is commutative, then κ is strong symmetric monoidal. Therefore, the forgetful functor $\mathbb{U} : \mathscr{D}_R\mathscr{T} \longrightarrow \mathscr{C}_{\mathbb{U}R}\mathscr{T}$ has a left adjoint prolongation functor $\mathbb{P} : \mathscr{C}_{\mathbb{U}R}\mathscr{T} \longrightarrow \mathscr{D}_R\mathscr{T}$. If R is commutative, then $\mathbb{U}$ is lax symmetric monoidal and $\mathbb{P}$ is strong symmetric monoidal.*

Using two observations of independent interest, we give an alternative description of $\mathbb{P}$ that makes no use of the categories $\mathscr{C}_{\mathbb{U}R}$ and $\mathscr{D}_R$.

Proposition 3.5. *Consider $\mathbb{P} : \mathscr{C}\mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$. Let Q be a monoid in $\mathscr{C}\mathscr{T}$. Then $\mathbb{P}Q$ is a monoid in $\mathscr{D}\mathscr{T}$, $\mathbb{P}$ restricts to a functor $\mathscr{C}\mathscr{S}_Q \longrightarrow \mathscr{D}\mathscr{S}_{\mathbb{P}Q}$, and the adjunction $(\mathbb{P}, \mathbb{U})$ restricts to an adjunction*

$$\mathscr{D}\mathscr{S}_{\mathbb{P}Q}(\mathbb{P}T, Y) \cong \mathscr{C}\mathscr{S}_Q(T, \mathbb{U}Y). \tag{3.6}$$

Proof. The first two statements are immediate from Proposition 3.3. For the last statement, we must show that if X is a Q-module and Y is a $\mathbb{P}Q$-module, then a map $f : \mathbb{P}X \longrightarrow Y$ of $\mathscr{D}$-spaces is a map of $\mathbb{P}Q$-modules if and only if its adjoint $\tilde{f} : X \longrightarrow \mathbb{U}Y$ is a map of Q-modules. The proof is a pair of diagram chases that boil down to use of the fact that η and ε are monoidal natural transformations. $\square$

Proposition 3.7. *Let $f : R \longrightarrow R'$ be a map of monoids in $\mathscr{D}\mathscr{T}$. By pullback of the action along f, an R'-module Y gives rise to an R-module f^*Y. By extension of scalars, an R-module X gives rise to an R'-module $X \wedge_R R'$. These functors give an adjunction*

$$\mathscr{D}\mathscr{S}_{R'}(X \wedge_R R', Y) \cong \mathscr{D}\mathscr{S}_R(X, f^*Y).$$

When R and R' are commutative, the functor f^* is lax symmetric monoidal and the functor $(-) \wedge_R R'$ is strong symmetric monoidal.

Proof. The proof is formally the same as for extension of scalars in algebra. □

Applying these results to $Q = \mathbb{U}R$ and the counit map $\varepsilon : \mathbb{P}\mathbb{U}R \longrightarrow R$, we obtain the following proposition by the uniqueness of adjoints.

Proposition 3.8. *Let R be a monoid in $\mathscr{D}\mathscr{T}$. Then $\mathbb{P} : \mathscr{C}_{\mathbb{U}R}\mathscr{T} \longrightarrow \mathscr{D}_R\mathscr{T}$ agrees under the isomorphisms of its source and target with the composite of the functor $\mathbb{P} : \mathscr{C}\mathscr{S}_{\mathbb{U}R} \longrightarrow \mathscr{D}\mathscr{S}_{\mathbb{P}\mathbb{U}R}$ and the extension of scalars functor $\mathscr{D}\mathscr{S}_{\mathbb{P}\mathbb{U}R} \longrightarrow \mathscr{D}\mathscr{S}_R$.*

4. Examples of diagram spectra

We now specialize the general abstract theory to the examples of interest in stable homotopy theory. Here we change our point of view. So far, we have considered general monoids R in $\mathscr{D}\mathscr{T}$, usually commutative. Now we focus on a particular, canonical, choice, which we denote by S, or $S_{\mathscr{D}}$ when necessary for clarity, to suggest spheres. It is a faithful functor in all of our examples. In this context, we call S-algebras $\mathscr{D}$-ring spectra. These diagram ring spectra and their modules are our main focus of interest.

We take S^n to be the one-point compactification of $\mathbb{R}^n$; the one-point compactification of $\{0\}$ is S^0, and it is convenient to let $S^n = *$ if $n < 0$. Similarly, for a finite dimensional real inner product space V, we take S^V to be the one-point compactification of V. Our first example is elementary, but crucial to the theory.

Example 4.1 (Prespectra)**.** Let $\mathscr{N}$ be the (unbased) category of non-negative integers, with only identity morphisms between them. The symmetric monoidal structure is given by addition, with 0 as unit. An $\mathscr{N}$-space is a sequence of based spaces. The canonical functor $S = S_{\mathscr{N}}$ sends n to S^n. It is strong monoidal, but it is *not* symmetric since permutations of spheres are not identity maps. This is the source of difficulty in defining the smash product in the stable homotopy category. A *prespectrum* is an $\mathscr{N}$-spectrum over S. Let $\mathscr{P}$, or alternatively $\mathscr{N}\mathscr{S}$, denote the category of prespectra. Since S^n is canonically isomorphic to the n-fold smash power of S^1, the category of prespectra defined in this way is isomorphic to the usual category of prespectra, whose objects are sequences of based spaces X_n and based maps $\Sigma X_n \longrightarrow X_{n+1}$.

The shift desuspension functors to $\mathscr{N}$-spectra are given by $(F_m A)_n = A \wedge S^{n-m}$. The smash product of $\mathscr{N}$-spaces (not $\mathscr{N}$-spectra) is given by

$$(X \wedge Y)_n = \bigvee_{p=0}^{n} X_p \wedge Y_{n-p}.$$

The category $\mathscr{N}_S$ such that an $\mathscr{N}$-spectrum is an $\mathscr{N}_S$-space has morphism spaces

$$\mathscr{N}_S(m, n) = S^{n-m}.$$

Because $S_{\mathscr{N}}$ is not symmetric, the category of $\mathscr{N}$-spectra does not have a smash product that makes it a symmetric monoidal category. For all other $\mathscr{D}$ that we consider, the functor $S_{\mathscr{D}}$ is a strong symmetric monoidal embedding $\mathscr{D} \longrightarrow \mathscr{T}$. Therefore the category of $\mathscr{D}$-spectra over S is symmetric monoidal.

Example 4.2 (Symmetric spectra)**.** Let Σ be the (unbased) category of finite sets $\mathbf{n} = \{1, \dots, n\}$, $n \geq 0$, and their permutations; thus there are no maps $\mathbf{m} \longrightarrow \mathbf{n}$ for $m \neq n$, and the set of maps $\mathbf{n} \to \mathbf{n}$ is the symmetric group Σ_n. The symmetric monoidal structure is given by concatenation of sets and block sum of permutations, with $\mathbf{0}$ as unit. The canonical functor $S = S_\Sigma$ sends $\mathbf{n}$ to S^n. A *symmetric spectrum* is a Σ-spectrum over S. Let $\Sigma\mathscr{S}$ denote the category of symmetric spectra. Define a strong symmetric monoidal faithful functor $\iota : \mathscr{N} \longrightarrow \Sigma$ by sending n to $\mathbf{n}$ and observe that $S_{\mathscr{N}} = S_\Sigma \circ \iota$. In effect, we have made S_Σ symmetric by adding permutations to the morphisms of $\mathscr{N}$. The idea of doing this is due to Jeff Smith.

The shift desuspension functors to symmetric spectra are given by

$$(F_m A)(\mathbf{n}) = \Sigma_{n+} \wedge_{\Sigma_{n-m}} (A \wedge S^{n-m}).$$

The smash product of Σ-spaces is given by

$$(X \wedge Y)(\mathbf{n}) \cong \bigvee_{p=0}^{n} \Sigma_{n+} \wedge_{\Sigma_p \times \Sigma_{n-p}} X(\mathbf{p}) \wedge Y(\mathbf{n} - \mathbf{p})$$

as a Σ_n-space. Implicitly, we are considering the set of partitions of the set $\mathbf{n}$. If we were considering the category of all finite sets k, we could rewrite this as

$$(X \wedge Y)(k) = \bigvee_{j \subset k} X(j) \wedge Y(k - j),$$

and this reinterpretation explains the associativity and commutativity of $\wedge$. The category Σ_S such that a Σ-spectrum is a Σ_S-space has morphism spaces

$$\Sigma_S(\mathbf{m}, \mathbf{n}) = \Sigma_{n+} \wedge_{\Sigma_{n-m}} S^{n-m}.$$

Example 4.3. The functor S_Σ is the case $A = S^1$ of the strong symmetric monoidal functor $S_A : \Sigma \longrightarrow \mathscr{T}$ that sends $\mathbf{n}$ to the n-fold smash power $A^{(n)}$ for a based space A. Moreover, the S_A give all strong symmetric monoidal functors $\Sigma \longrightarrow \mathscr{T}$. Applied to S_A, our theory constructs a symmetric monoidal category of "S_A-modules". The homotopy theory of these categories is relevant to localization theory.

Example 4.4 (Orthogonal spectra)**.** Let $\mathscr{I}$ be the (unbased) category of finite dimensional real inner product spaces and linear isometric isomorphisms; there are no maps $V \longrightarrow W$ unless $\dim V = \dim W = n$ for some $n \geq 0$, when the space of morphisms $V \longrightarrow W$ is homeomorphic to the orthogonal group $O(n)$. The symmetric monoidal structure is given by direct sums, with $\{0\}$ as unit. The canonical functor $S = S_{\mathscr{I}}$ sends V to S^V. An *orthogonal spectrum* is an $\mathscr{I}$-spectrum over S. Let $\mathscr{I}\mathscr{S}$ denote the category of orthogonal spectra. Define a strong symmetric monoidal faithful functor $\iota : \Sigma \longrightarrow \mathscr{I}$ by sending $\mathbf{n}$ to $\mathbb{R}^n$ and using the standard inclusions $\Sigma_n \longrightarrow O(n)$. Observe that $S_\Sigma = S_{\mathscr{I}} \circ \iota$.

The shift desuspension functors to orthogonal spectra are given on $W \supset V$ by

$$(F_V A)(W) = O(W)_+ \wedge_{O(W-V)} (A \wedge S^{W-V}),$$

where $W - V$ is the orthogonal complement of V in W; an analogous description applies whenever $\dim W \geq \dim V$, and $(F_V A)(W) = *$ if $\dim W < \dim V$. Note that we can restrict attention to the skeleton $\{\mathbb{R}^n\}$ of $\mathscr{I}$. For an inner product space V of dimension n, choose a subspace V_p of dimension p for each $p \leq n$. The smash product of $\mathscr{I}$-spaces is given by

$$(X \wedge Y)(V) \cong \bigvee_{p=0}^{n} O(V)_+ \wedge_{O(V_p)\times O(V-V_p)} X(V_p) \wedge Y(V - V_p)$$

as an $O(V)$-space. This describes the topology correctly, but to see the associativity and commutativity of $\wedge$, we can rewrite this set-theoretically as

$$(X \wedge Y)(V) = \bigvee_{W\subset V} X(W) \wedge Y(V - W).$$

The category $\mathscr{I}_S$ such that an $\mathscr{I}$-spectrum is an $\mathscr{I}_S$-space has morphism spaces

$$\mathscr{I}_S(V, W) = O(W)_+ \wedge_{O(W-V)} S^{W-V}$$

for $V \subset W$.

This example admits several variants. For instance, we can use real vector spaces and their isomorphisms, without insisting on inner product structures and isometries, or we can use complex vector spaces.

Example 4.5. Let V have dimension n and let $TO(V)$ be the Thom space of the tautological n-plane bundle over the Grassmannian of n-planes in $V \oplus V$. As observed in [28, §V.2], which gives many other examples, TO is a commutative $\mathscr{I}$-FSP (= $\mathscr{I}_*$-prefunctor there). Therefore TO is a commutative S-algebra by Proposition 22.6 below.

Our formal theory applies to examples like $R = TO$, but we focus on the canonical functors $S_{\mathscr{D}}$.

Example 4.6 ($\mathscr{W}$-spaces)**.** It is tempting to take $\mathscr{D} = \mathscr{T}$, but that does not have a small skeleton. Instead, we can take $\mathscr{D}$ to be the category $\mathscr{W}$ of based spaces homeomorphic to finite CW complexes. The theory works equally well if we redefine $\mathscr{W}$ in terms of countable rather than finite CW complexes or indeed in terms of any sufficiently large but skeletally small full subcategory of $\mathscr{T}$ that is closed under smash products. We have evident strong symmetric monoidal faithful functors $\Sigma \longrightarrow \mathscr{W}$ and $\mathscr{I} \longrightarrow \mathscr{W}$ under which $S_{\mathscr{W}}$ restricts to S_Σ and $S_{\mathscr{I}}$.

The shift desuspension functors to $\mathscr{W}$-spaces are given by

$$(F_A B)(C) = F(A, C) \wedge B.$$

This example suggests an alternative way of viewing Σ and $\mathscr{I}$.

Remark 4.7. It is sometimes convenient, and sometimes inconvenient, to change point of view and think of the objects of Σ and $\mathscr{I}$ as the spheres S^n and S^V, thus thinking of Σ and $\mathscr{I}$ as subcategories of $\mathscr{W}$. With this point of view, $\square$ is a subfunctor of $\wedge$ and S is the inclusion of a monoidal subcategory.

All of our examples so far are categories under $\mathscr{N}$. However, our last example is not of this type.

Example 4.8 ($\mathscr{F}$-spaces = Γ-spaces)**.** Let $\mathscr{F}$ be the category of finite based sets $\mathbf{n}^+ = \{0, 1, \ldots, n\}$ and all based maps, where 0 is the basepoint. This is the opposite of Segal's category Γ [38]. This category is based with base object the one point set $\mathbf{0}^+$. Take $\square$ to be the smash product of finite based sets; to be precise, we order the non-zero elements of $\mathbf{m}^+ \wedge \mathbf{n}^+$ lexicographically. The unit object is

$\mathbf{1}^+$. The canonical functor $S_{\mathscr{F}}$ sends $\mathbf{n}^+$ to $\mathbf{n}^+$ regarded as a discrete based space; it is the restriction to $\mathscr{F}$ of the functor $S_{\mathscr{W}}$.

In contrast to the cases of symmetric spectra and orthogonal spectra, the action of $S_{\mathscr{D}}$ required of $\mathscr{D}$-spectra gives no additional data when $\mathscr{D} = \mathscr{F}$ or $\mathscr{D} = \mathscr{W}$. Moreover, since the functor $\mathscr{F} \subset \mathscr{W}$ is fully faithful, $\mathbb{P} : \mathscr{F}\mathscr{T} \longrightarrow \mathscr{W}\mathscr{T}$ is a "prolongation" in the strong sense described in Proposition 3.2.

Lemma 4.9. *Let $S_{\mathscr{D}} : \mathscr{D} \longrightarrow \mathscr{T}$ be an embedding of $\mathscr{D}$ as a full symmetric monoidal subcategory of $\mathscr{T}$. Then a $\mathscr{D}$-space X admits a unique structure of $\mathscr{D}$-spectrum, and the categories of $\mathscr{D}$-spaces and $\mathscr{D}$-spectra are isomorphic. In particular, this applies to $\mathscr{D} = \mathscr{F}$ and $\mathscr{D} = \mathscr{W}$.*

Proof. This is an instance of Remark 2.3, but it is worthwhile to explain it explicitly. Omit the embedding $S_{\mathscr{D}}$ from the notation and write $\wedge$ for $\Box$. For spaces $A, B \in \mathscr{D}$, the action map $\sigma : X(A) \wedge B \longrightarrow X(A \wedge B)$ is the adjoint of the composite

$$B \xrightarrow{\alpha} \mathscr{T}(A, A \wedge B) = \mathscr{D}(A, A \wedge B) \xrightarrow{X} \mathscr{T}(X(A), X(A \wedge B)),$$

where $\alpha(b)(a) = a \wedge b$. The equality holds because $\mathscr{D}$ is a full subcategory of $\mathscr{T}$, and X is continuous by our definition of a $\mathscr{D}$-space. $\Box$

Part II. Model categories of diagram spectra and their comparison

We give some preliminaries about "compactly generated" topological model categories in §5. We show in §6 that, for any domain category $\mathscr{D}$, the category of $\mathscr{D}$-spaces has a "level model structure" in which the weak equivalences and fibrations are the maps that evaluate to weak equivalences or fibrations at each object of $\mathscr{D}$. This structure has been studied in more detail by Piacenza [33], [29, Ch. VI] and others. There is a relative variant in which we restrict attention to those objects in some subcategory $\mathscr{C}$ of $\mathscr{D}$.

In preparation for the study of stable model structures, we recall some homotopical facts about prespectra in §7; we use the terms "prespectrum" and "$\mathscr{N}$-spectrum" interchangeably, using the former when we are thinking in classical homotopical terms and using the latter when thinking about the relationship with other categories of diagram spectra.

We define and study "stable equivalences" in §8, and we give the categories of $\mathscr{N}$-spectra, symmetric spectra, orthogonal spectra, and $\mathscr{W}$-spaces a "stable model structure" in §9. The cofibrations are those of the level model structure relative to $\mathscr{N}$, and the weak equivalences are the stable equivalences. We give a single self-contained proof of the model axioms that applies to all four of these categories. We prove Theorem 0.1 and Corollary 0.2 in §10. We relate this theory to the classical theory of CW prespectra and handicrafted smash products and prove Theorem 0.3 in §11.

In §12, we prove that the categories of symmetric spectra, orthogonal spectra, and $\mathscr{W}$-spaces satisfy the pushout-product and monoid axioms of [37]. This answers the question of whether or not the monoid axiom holds for (topological) symmetric spectra, which was posed in the preprint version of [15]. This implies that the categories of $\mathscr{D}$-ring spectra and of modules over a $\mathscr{D}$-ring spectrum inherit model structures from the underlying category of $\mathscr{D}$-spectra in these cases. We prove Theorems 0.4–0.6 in §13.

We show in §14 that replacing the level model structure relative to $\mathscr{N}$ by the relative model structure relative to $\mathscr{N} - \{0\}$ leads to a "positive stable model structure" that is Quillen equivalent to the stable model structure but has fewer cofibrations. Its cofibrant objects have trivial zeroth terms. In §15, we use these model structures to construct model structures on the categories of commutative symmetric ring spectra and commutative orthogonal ring spectra. We prove Theorems 0.7 and 0.8 comparing these and related model categories in §16.

We return to $\mathscr{W}$-spaces in §17. We prove that the category of $\mathscr{W}$-spaces has a second, "absolute", stable model structure that also satisfies the pushout-product and monoid axioms. In the first stable model structure, we start from the level model structure *relative to* $\mathscr{N}$. In the second, we start from the *absolute* level model structure. The weak equivalences in both stable model structures are the π_*-isomorphisms. The cofibrations in the absolute stable structure are the same as those in the absolute level model structure, and there are more of them.

We prove Theorems 0.10–0.12 and Corollary 0.13 comparing $\mathscr{F}$-spaces and $\mathscr{W}$-spaces in §18. The stable model category of $\mathscr{F}$-spaces that we use is the one studied in [35]. Its cofibrations are those of the level model structure and its weak equivalences between cofibrant objects are the π_*-isomorphisms. It has fewer cofibrations and more fibrations than the simplicial analogue that was originally studied by Bousfield and Friedlander [7].

We compare diagram categories of spaces and diagram categories of simplicial sets in §19. The comparison between $\mathscr{F}$-spaces and $\mathscr{F}$-simplicial sets is used in the proofs of Theorems 0.10–0.12.

5. Preliminaries about topological model categories

We first construct model structures on categories of diagram spectra and then use a general procedure to lift them to model structures on categories of structured diagram spectra. The weak equivalences and fibrations in the lifted model structures are *created* in the underlying category of diagram spaces. That is, the underlying diagram spectrum functor preserves and reflects the weak equivalences and fibrations: a map of structured diagram spectra is a weak equivalence or fibration if and only if its underlying map of diagram spectra is a weak equivalence or fibration. We here describe the kind of model structures that we will encounter and explain the lifting procedure.

While we have the example of diagram spectra in mind, the considerations of this section apply more generally. Thus let $\mathscr{C}$ be any topologically complete and cocomplete category with tensors denoted $X \wedge A$ and homotopies defined in terms of $X \wedge I_+$. We let $\mathscr{A}$ be a topological category with a continuous and faithful forgetful functor $\mathscr{A} \longrightarrow \mathscr{C}$. We assume that $\mathscr{A}$ is topologically complete and cocomplete. This holds in all of the categories that occur in our work by the following pair of results. The first is [11, VII.2.10] and the second is [11, I.7.2].

Proposition 5.1. *Let $\mathscr{C}$ be a topologically complete and cocomplete category and let $\mathbb{T} : \mathscr{C} \longrightarrow \mathscr{C}$ be a continuous monad that preserves reflexive coequalizers. Then the category $\mathscr{C}[\mathbb{T}]$ of $\mathbb{T}$-algebras is topologically complete and cocomplete, with limits created in $\mathscr{C}$.*

The hypothesis on $\mathbb{T}$ holds trivially when $\mathscr{C}$ is closed symmetric monoidal with product $\wedge$ and $\mathbb{T}$ is the monad $\mathbb{T}X = R \wedge X$ that defines left modules over some monoid R in $\mathscr{C}$, since $\mathbb{T} : \mathscr{C} \longrightarrow \mathscr{C}$ is then a left adjoint. The following analogue is more substantial.

Proposition 5.2. *Let $\mathscr{C}$ be a cocomplete closed symmetric monoidal category. Then the monads that define monoids and commutative monoids in $\mathscr{C}$ preserve reflexive coequalizers.*

As in [11], we write q-cofibration and q-fibration for model cofibrations and fibrations, but we write cofibrant and fibrant rather than q-cofibrant and q-fibrant. The (usually weaker) notion of an h-cofibration plays an important role in model theory in topology. A map $i : A \longrightarrow X$ in $\mathscr{C}$ is an h-cofibration if it satisfies the Homotopy Extension Property (HEP) in $\mathscr{C}$. That is, for every map $f : X \longrightarrow Y$ and homotopy $h : A \wedge I_+ \longrightarrow Y$ such that $h_0 = f \circ i$, there is a homotopy $\tilde{h} : X \wedge I_+ \longrightarrow Y$ such that $\tilde{h}_0 = f$ and $\tilde{h} \circ (i \wedge \mathrm{id}) = h$. The universal test case is the mapping cylinder $Y = Mi = X \cup_i (A \wedge I_+)$, with the evident f and h, in which case $\tilde{h}$ is a retraction $X \wedge I_+ \longrightarrow Mi$.

In particular, an h-cofibration of $\mathscr{D}$-spaces is a level h-cofibration and therefore a level closed inclusion. For some purposes, we could just as well use level h-cofibrations where we use h-cofibrations, but the stronger condition plays a key role in some of our model theoretic work and is the most natural condition to verify. The theory of cofibration sequences works in exactly the same way for h-cofibrations of $\mathscr{D}$-spaces as for h-cofibrations of based spaces; we will be more explicit later. The various functors Ev_d, F_d, $\mathbb{U}$ and $\mathbb{P}$ defined in Part I all preserve colimits and smash products with spaces. By the retract of mapping cylinders criterion, they also preserve h-cofibrations. This elementary observation is crucial to our work, one point being that right adjoints, such as $\mathbb{U}$, do not preserve q-cofibrations.

Most work on model categories has been done simplicially rather than topologically. As observed in [11], it is convenient in topological contexts to require some form of "Cofibration Hypothesis". We shall incorporate this in our definition of what it means for $\mathscr{A}$ to be a "compactly generated model category".

Cofibration Hypothesis 5.3. Let I be a set of maps in $\mathscr{A}$. We say that I satisfies the Cofibration Hypothesis if it satisfies the following two conditions.

(i) Let $i : A \longrightarrow B$ be a coproduct of maps in I. In any pushout

$$\begin{array}{ccc} A & \longrightarrow & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ B & \longrightarrow & F \end{array}$$

in $\mathscr{A}$, the cobase change j is an h-cofibration in $\mathscr{C}$.

(ii) Viewed as an object of $\mathscr{C}$, the colimit of a sequence of maps in $\mathscr{A}$ that are h-cofibrations in $\mathscr{C}$ is their colimit as a sequence of maps in $\mathscr{C}$.

We can use the maps in such a set I as the analogues of (cell, sphere) pairs in the theory of cell complexes, and the following definition and result imply that q-cofibrations are h-cofibrations in compactly generated model categories.

Definition 5.4. Let I be a set of maps in $\mathscr{A}$. A map $f : X \longrightarrow Y$ is a *relative I-cell complex* if Y is the colimit of a sequence of maps $Y_n \longrightarrow Y_{n+1}$ such that $Y_0 = X$ and $Y_n \longrightarrow Y_{n+1}$ is obtained by cobase change from a coproduct of maps in I.

Lemma 5.5. *Let I satisfy the Cofibration Hypothesis. Then any retract of a relative I-cell complex is an h-cofibration in $\mathscr{C}$.*

We will define compactly generated model categories in terms of compact objects.

Definition 5.6. An object X of $\mathscr{A}$ is *compact* if

$$\mathscr{A}(X,Y) \cong \operatorname{colim} \mathscr{A}(X,Y_n)$$

whenever Y is the colimit of a sequence of maps $Y_n \longrightarrow Y_{n+1}$ in $\mathscr{A}$ that are h-cofibrations in $\mathscr{C}$.

Of course, for spaces, we understand compactness in the usual sense. Since points are closed in compactly generated spaces, an elementary argument shows that $\mathscr{T}(A,Y) \cong \operatorname{colim} \mathscr{T}(A,Y_n)$ if A is compact and Y is the union of a sequence of inclusions $Y_n \longrightarrow Y_{n+1}$.

Lemma 5.7. *If A is a compact space, then $F_d A$ is a compact $\mathscr{D}$-space. If X is a compact $\mathscr{D}$-space and A is a compact space, then $X \wedge A$ is a compact $\mathscr{D}$-space. If $Y \cup_X Z$ is the pushout of a level closed inclusion $i : X \longrightarrow Y$ and a map $f : X \longrightarrow Z$, where X, Y, and Z are compact $\mathscr{D}$-spaces, then $Y \cup_X Z$ is a compact $\mathscr{D}$-space.*

Proofs of the model axioms generally use some version of Quillen's small object argument [34, II, p.3.4]. The following version, details of a special case of which are given in [11, VII.5.2], suffices for most of our work. Abbreviate RLP and LLP for the right lifting property and left lifting property.

Lemma 5.8 (The small object argument)**.** *Let I be a set of maps of $\mathscr{A}$ such that each map in I has compact domain and I satisfies the Cofibration Hypothesis. Then maps $f : X \longrightarrow Y$ in $\mathscr{A}$ factor functorially as composites*

$$X \xrightarrow{\ i\ } X' \xrightarrow{\ p\ } Y$$

such that p satisfies the RLP with respect to any map in I and i satisfies the LLP with respect to any map that satisfies the RLP with respect to each map in I. Moreover, $i : X \longrightarrow X'$ is a relative I-cell complex.

This motivates the following definition.

Definition 5.9. Let $\mathscr{A}$ be a model category. We say that $\mathscr{A}$ is *compactly generated* if there are sets I and J of maps in $\mathscr{A}$ such that the domain of each map in I and each map in J is compact, I and J satisfy the Cofibration Hypothesis, the q-fibrations are the maps that satisfy the RLP with respect to the maps in J and the acyclic q-fibrations are the maps that satisfy the RLP with respect to the maps in I. Note that the maps in I must be q-cofibrations and h-cofibrations and the maps in J must be acyclic q-cofibrations and h-cofibrations. We call the maps in I the *generating q-cofibrations* and the maps in J the *generating acyclic q-cofibrations.*

Remark 5.10. There is a definition in terms of transfinite colimits of what it means for a set of maps to be small relative to a subcategory of $\mathscr{A}$. The more general notion of a *cofibrantly generated* model category $\mathscr{A}$ replaces the compactness condition with the requirement that I be small relative to the q-cofibrations and J be small relative to the acyclic q-cofibrations. See e.g. [13, §12.4] or [14, §2.1]. The Cofibration Hypothesis does not appear in the model theoretic literature, but it is almost always appropriate in topological settings.

All of our model categories are "topological", in the following sense. For maps $i : A \longrightarrow X$ and $p : E \longrightarrow B$ in $\mathscr{A}$, let

$$\mathscr{A}(i^*, p_*) : \mathscr{A}(X, E) \longrightarrow \mathscr{A}(A, E) \times_{\mathscr{A}(A,B)} \mathscr{A}(X, B) \tag{5.11}$$

be the map of spaces induced by $\mathscr{A}(i, \mathrm{id})$ and $\mathscr{A}(\mathrm{id}, p)$ by passage to pullbacks. Observe that the pair (i, p) has the lifting property if and only if $\mathscr{A}(i^*, p_*)$ is surjective.

Definition 5.12. A model category $\mathscr{A}$ is *topological* provided that $\mathscr{A}(i^*, p_*)$ is a Serre fibration if i is a q-cofibration and p is a q-fibration and is a weak equivalence if, in addition, either i or p is a weak equivalence.

The following result on lifting model structures is immediate by inspection of the proofs in [11, VII§5] or by combination of our version of the small object argument with the proof of [37, 2.3] or [13, 14.3.2]. Of course, viewed as a functor $\mathscr{C} \longrightarrow \mathscr{C}[\mathbb{T}]$, a monad $\mathbb{T}$ is the free functor left adjoint to the forgetful functor $\mathscr{C}[\mathbb{T}] \longrightarrow \mathscr{C}$. Since the forgetful functor preserves sequential colimits, $\mathbb{T}$ preserves compact objects.

Proposition 5.13. *Let $\mathscr{C}$ be a topologically complete and cocomplete category and let $\mathbb{T} : \mathscr{C} \longrightarrow \mathscr{C}$ be a continuous monad that preserves reflexive coequalizers. Assume that $\mathscr{C}$ is a compactly generated topological model category with generating sets I of cofibrations and J of acyclic cofibrations. Then $\mathscr{C}[\mathbb{T}]$ is a compactly generated topological model category with weak equivalences and fibrations created in $\mathscr{C}$ and generating sets $\mathbb{T}I$ of cofibrations and $\mathbb{T}J$ of acyclic cofibrations provided that*

(i) *$\mathbb{T}I$ and $\mathbb{T}J$ satisfy the Cofibration Hypothesis and*
(ii) *every relative $\mathbb{T}J$-cell complex is a weak equivalence.*

We need two pairs of analogues of the maps $\mathscr{A}(i^*, p_*)$. For a map $i : A \longrightarrow B$ of based spaces and a map $j : X \longrightarrow Y$ in $\mathscr{A}$, passage to pushouts gives a map

$$i \square j : (A \wedge Y) \cup_{A \wedge X} (B \wedge X) \longrightarrow B \wedge Y \tag{5.14}$$

and passage to pullbacks gives a map

$$F_{\square}(i, j) : F(B, X) \longrightarrow F(A, X) \times_{F(A,Y)} F(B, Y), \tag{5.15}$$

where $\wedge$ and F denote the tensor and cotensor in $\mathscr{A}$.

Inspection of definitions gives adjunctions relating (5.11), (5.14) and (5.15). Formally, these imply that the category of maps in $\mathscr{A}$ is tensored and cotensored over the category of maps in $\mathscr{T}$.

Lemma 5.16. *Let $i : A \longrightarrow B$ be a map of based spaces and let $j : X \longrightarrow Y$ and $p : E \longrightarrow F$ be maps in $\mathscr{A}$. Then there are natural isomorphisms of maps*

$$\mathscr{A}((i \square j)^*, p_*) \cong \mathscr{T}(i^*, \mathscr{A}(j^*, p_*)_*) \cong \mathscr{A}(j^*, F_{\square}(i, p)).$$

Therefore $(i \square j, p)$ has the lifting property in $\mathscr{A}$ if and only if $(i, \mathscr{A}(j^*, p_*))$ has the lifting property in $\mathscr{T}$.

Now assume that $\mathscr{A}$ is a closed symmetric monoidal category with product $\wedge_{\mathscr{A}}$ and internal function objects $F_{\mathscr{A}}$. For maps $i : X \longrightarrow Y$ and $j : W \longrightarrow Z$ in $\mathscr{A}$, passage to pushouts gives a map

$$i \square j : (Y \wedge_{\mathscr{A}} W) \cup_{X \wedge_{\mathscr{A}} W} (X \wedge_{\mathscr{A}} Z) \longrightarrow Y \wedge_{\mathscr{A}} Z, \tag{5.17}$$

and passage to pullbacks gives a map

$$F_{\square}(i, j) : F_{\mathscr{A}}(Y, W) \longrightarrow F_{\mathscr{A}}(X, W) \times_{F_{\mathscr{A}}(X,Z)} F_{\mathscr{A}}(Y, Z). \tag{5.18}$$

For maps i, j, and k in $\mathscr{A}$, these are related by a natural isomorphism of maps

$$\mathscr{A}((i\Box j)^*, k_*) \cong \mathscr{A}(i^*, F_\Box(j,k)_*). \tag{5.19}$$

6. The level model structure on $\mathscr{D}$-spaces

We give the category of $\mathscr{D}$-spaces a "level model structure". We shall be brief, since this material is well-known. An exposition that makes clear just how close this theory is to CW-theory in the category of spaces has been given by Piacenza [33], [29, Ch. VI]. Since the category $\mathscr{D}\mathscr{S}_R$ of $\mathscr{D}$-spectra over a monoid R is isomorphic to the category of $\mathscr{D}_R$-spaces, the category $\mathscr{D}\mathscr{S}_R$ obtains a level model structure by specialization. Recall that $\mathscr{D}_R$ has the same objects as $\mathscr{D}$. We assume that $\mathscr{D}$ is skeletally small and fix a skeleton $sk\mathscr{D}$.

Definition 6.1. We define five properties of maps $f : X \longrightarrow Y$ of $\mathscr{D}$-spaces.

(i) f is a *level equivalence* if each $f(d) : X(d) \longrightarrow Y(d)$ is a weak equivalence.
(ii) f is a *level fibration* if each $f(d) : X(d) \longrightarrow Y(d)$ is a Serre fibration.
(iii) f is a *level acyclic fibration* if it is both a level equivalence and a level fibration.
(iv) f is a *q-cofibration* if it satisfies the LLP with respect to the level acyclic fibrations.
(v) f is a *level acyclic q-cofibration* if it is both a level equivalence and a q-cofibration.

Of course, there is also a notion of a level cofibration, defined as in Definition 6.1(ii), but we shall make no use of it.

Definition 6.2. Let I be the set of h-cofibrations $S^{n-1}_+ \longrightarrow D^n_+$, $n \geq 0$ (interpreted as $* \longrightarrow S^0$ when $n = 0$). Let J be the set of h-cofibrations $i_0 : D^n_+ \longrightarrow (D^n \times I)_+$ and observe that each such map is the inclusion of a deformation retract. Define FI to be the set of all maps $F_d i$ with $d \in sk\mathscr{D}$ and $i \in I$. Define FJ to be the set of all maps $F_d j$ with $d \in sk\mathscr{D}$ and $j \in J$, and observe that each map in FJ is the inclusion of a deformation retract. Note that the domains and codomains of all maps in FI and FJ are compact.

We recall the following result of Quillen [34, II§3]; see also [14, Ch.2§2.4]. Recall that a model category is *left proper* if a pushout of a weak equivalence along a q-cofibration is a weak equivalence, *right proper* if a pullback of a weak equivalence along a q-fibration is a weak equivalence, and *proper* if it is left and right proper. All of our model categories are right proper, and many of them are proper.

Proposition 6.3. *$\mathscr{T}$ is a compactly generated proper topological model category with respect to the weak equivalences, Serre fibrations, and retracts of relative I-cell complexes. The sets I and J are the generating q-cofibrations and the generating acyclic q-cofibrations.*

Note that every space is fibrant. The model structure requires use of all based spaces, but weak equivalences only behave well with respect to standard constructions when we restrict to spaces with nondegenerate basepoints, meaning that the inclusion of the basepoint is an unbased h-cofibration. Recall that a based h-cofibration between nondegenerately based spaces is an unbased h-cofibration (satisfies the HEP in unbased spaces) [40, Prop. 9].

Definition 6.4. The category $\mathscr{D}$ is *nondegenerately based* if each of its morphism spaces is nondegenerately based. For any $\mathscr{D}$, a $\mathscr{D}$-space X is *nondegenerately based* if each $X(d)$ is nondegenerately based.

All of the categories $\mathscr{D}$ that we consider are nondegenerately based.

Theorem 6.5. *The category of $\mathscr{D}$-spaces is a compactly generated topological model category with respect to the level equivalences, level fibrations, and q-cofibrations. It is right proper, and it is left proper if $\mathscr{D}$ is nondegenerately based. The sets FI and FJ are the generating q-cofibrations and generating acyclic q-cofibrations, and the following identifications hold.*

(i) *The level fibrations are the maps that satisfy the RLP with respect to FJ or, equivalently, with respect to retracts of relative FJ-cell complexes, and all $\mathscr{D}$-spaces are level fibrant.*
(ii) *The level acyclic fibrations are the maps that satisfy the RLP with respect to FI or, equivalently, with respect to retracts of relative FI-cell complexes.*
(iii) *The q-cofibrations are the retracts of relative FI-cell complexes.*
(iv) *The level acyclic q-cofibrations are the retracts of relative FJ-cell complexes.*
(v) *If $\mathscr{D}$ is nondegenerately based, then any cofibrant $\mathscr{D}$-space is nondegenerately based.*

Proof. The only model axioms that are not obvious from the definitions are the lifting property that is not given by the definition of a q-cofibration and the two factorization properties. The latter are obtained by applying the small object argument of Lemma 5.8 to FJ and FI. The detailed statement of that lemma and adjunction arguments show that (i) through (iv) follow from their space level analogues; (ii) and (iii) give the remaining lifting property; see e.g. [14, 5.1.3].

To show that $\mathscr{D}\mathscr{T}$ is topological, we must show that if $i : A \longrightarrow X$ is a q-cofibration and $p : E \longrightarrow B$ is a level fibration, then the map $\mathscr{D}\mathscr{T}(i^*, p_*)$ of (5.11) is a Serre fibration which is a weak equivalence if i or p is a level equivalence. As in [34, SM7(a), p. II.2.3] or [14, 4.2.5], this reduces to showing that $\mathscr{D}\mathscr{T}(i^*, p_*)$ is a Serre fibration when i is in FI and an acyclic Serre fibration when i is in FJ. By adjunction, these conclusions follow from their space level analogues.

Right properness also follows directly from its space level analogue. To show that $\mathscr{D}\mathscr{T}$ is left proper, we must show that the pushout of a level equivalence along a q-cofibration is a level equivalence. The functors $F_d : \mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$ preserve h-cofibrations. Since $\mathscr{D}$ is nondegenerately based, $F_d A$ is nondegenerately based for any based CW complex A. Moreover, wedges of nondegenerately based spaces are nondegenerately based. Thus a relative FI-cell complex $i : X \longrightarrow Y$ is obtained by passage to pushouts and sequential colimits from based maps that are unbased h-cofibrations. Although X need not be nondegenerately based, i is a level unbased h-cofibration since pushouts and sequential colimits of unbased h-cofibrations are unbased h-cofibrations. Therefore any q-cofibration is a level unbased h-cofibration. The conclusion follows from the space level analogue that the pushout of a weak equivalence along an unbased h-cofibration is a weak equivalence. Part (v) also follows from this discussion. □

Now assume for a moment that $\mathscr{D}$ and therefore $\mathscr{D}\mathscr{T}$ are symmetric monoidal categories. We then have the following observation about the maps $i\Box j$ of (5.17).

Lemma 6.6. *If i and j are q-cofibrations, then $i\Box j$ is a q-cofibration which is level acyclic if either i or j is level acyclic. In particular, if Y is cofibrant, then*

$i \wedge \mathrm{id} : A \wedge Y \longrightarrow X \wedge Y$ is a q-cofibration, and the smash product of cofibrant $\mathscr{D}$-spaces is cofibrant.

Proof. Writing $i_n : S^{n-1}_+ \longrightarrow D^n_+$, Lemma 1.8 implies that

$$F_d i_m \Box F_e i_n \cong F_{d\Box e}(i_{m+n}).$$

From here, an easy formal argument using the adjunction (5.19) and the defining lifting property of q-cofibrations gives that $i\Box j$ is a q-cofibration; see [15, 5.3.4]. The acyclicity in the first statement follows by adjointness arguments from the fact that $\mathscr{D}\mathscr{T}$ is topological; compare [34, p. II.2.3]. □

Remark 6.7. The monoid axiom of [37] would require that any map obtained by cobase change and composition from maps of the form $i \wedge Y$, where i is a level acyclic q-cofibration and Y is arbitrary, be a level equivalence. Without nondegenerate basepoint hypotheses, this fails in general. Nevertheless, we shall later prove the monoid axiom for some of our stable model structures.

Let $\mathrm{Ho}_\ell \mathscr{D}\mathscr{T}$ denote the homotopy category obtained from the level model structure. Let $[X, Y]$ denote the set of maps $X \longrightarrow Y$ in $\mathrm{Ho}_\ell \mathscr{D}\mathscr{T}$ and $\pi(X, Y)$ denote the set of homotopy classes of maps $X \longrightarrow Y$. Then $[X, Y] \cong \pi(\Gamma X, Y)$, where $\Gamma X \longrightarrow X$ is a cofibrant approximation of X. Piacenza [33] has shown that we can refine the notion of an FI-cell complex to the notion of an FI-CW complex, just as for based spaces. The cellular approximation theorem holds and any cofibrant $\mathscr{D}$-space is homotopy equivalent to an FI-CW complex. Similarly, fiber and cofiber sequences of $\mathscr{D}$-spaces behave the same way as for based spaces, starting from the usual definitions of homotopy cofibers and fibers.

Definition 6.8. Let $f : X \longrightarrow Y$ be a map of $\mathscr{D}$-spaces. The *homotopy cofiber* $Cf = Y \cup_f CX$ of f is the pushout along f of the cone h-cofibration $i : X \longrightarrow CX$; here $CX = X \wedge I$, where I has basepoint 1. The *homotopy fiber* $Ff = X \times_f PY$ of f is the pullback along f of the path fibration $p : PY \longrightarrow Y$; here $PY = F(I, Y)$, where I has basepoint 0. Equivalently, these are the levelwise homotopy cofiber and fiber of f.

We record the following basic properties of the level homotopy category. They are elementary precursors of more sophisticated analogues that appear later.

Theorem 6.9. *Assume that $\mathscr{D}$ is nondegenerately based.*

(i) *Let A be a based CW complex. If X is a nondegenerately based $\mathscr{D}$-space, then $X \wedge A$ is nondegenerately based and*

$$[X \wedge A, Y] \cong [X, F(A, Y)]$$

for any Y. If $f : X \longrightarrow Y$ is a level equivalence of nondegenerately based $\mathscr{D}$-spaces, then $f \wedge \mathrm{id} : X \wedge A \longrightarrow Y \wedge A$ is a level equivalence.

(ii) *For nondegenerately based $\mathscr{D}$-spaces X_i, $\bigvee_i X_i$ is nondegenerately based and*

$$[\bigvee_i X_i, Y] \cong \prod_i [X_i, Y]$$

for any Y. A wedge of level equivalences of nondegenerately based $\mathscr{D}$-spaces is a level equivalence.

(iii) *If $i : A \longrightarrow X$ is an h-cofibration and $f : A \longrightarrow Y$ is any map of $\mathscr{D}$-spaces, where A, X, and Y are nondegenerately based, then $X \cup_A Y$ is nondegenerately based and the cobase change $j : Y \longrightarrow X \cup_A Y$ is an h-cofibration. If i is a level equivalence, then j is a level equivalence.*

(iv) *If i and i' are h-cofibrations and the vertical arrows are level equivalences in the following commutative diagram of nondegenerately based $\mathscr{D}$-spaces, then the induced map of pushouts is a level equivalence.*

$$\begin{array}{ccccc} X & \xleftarrow{\;i\;} & A & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X' & \xleftarrow[\;i'\;]{} & A' & \longrightarrow & Y' \end{array}$$

(v) *If X is the colimit of a sequence of h-cofibrations $i_n : X_n \longrightarrow X_{n+1}$ of nondegenerately based $\mathscr{D}$-spaces, then X is nondegenerately based and there is a $\lim^1$ exact sequence of pointed sets*

$$* \longrightarrow \lim{}^1[\Sigma X_n, Y] \longrightarrow [X, Y] \longrightarrow \lim[X_n, Y] \longrightarrow *$$

for any Y. If each i_n is a level equivalence, then the map from the initial term X_0 into X is a level equivalence.

(vi) *Let $f : X \longrightarrow Y$ be a map of nondegenerately based $\mathscr{D}$-spaces. Then Cf is nondegenerately based and, for any Z, there is a long exact sequence*

$$\cdots \longrightarrow [\Sigma^{n+1}X, Z] \longrightarrow [\Sigma^n Cf, Z] \longrightarrow [\Sigma^n Y, Z] \longrightarrow [\Sigma^n X, Z] \longrightarrow \cdots \longrightarrow [X, Z].$$

Proof. The statements about level equivalences are immediate from their analogues for weak equivalences of based spaces. Using Theorem 6.5(v), the statements about $[-, Y]$ follow by first passing to cofibrant approximations and then applying the analogue with $[-, Y]$ replaced by $\pi(-, Y)$. The latter results are proven exactly as on the space level. For example, by the naturality of the space level argument, cofiber sequences give rise to long exact sequences upon application of the functor $\pi(-, Y)$. The essential point is that if $i : A \longrightarrow X$ is an h-cofibration, then the canonical map $Ci \longrightarrow X/A$ is a homotopy equivalence. Again, in (v), X is homotopy equivalent to the telescope of the X_n, and there results a $\lim^1$ exact sequence for the computation of $\pi(X, Y)$. □

We shall need several relative variants of the absolute level model structure that we have been discussing.

Variant 6.10. Let $\mathscr{C}$ be a subcategory of $\mathscr{D}$. We define the *level model structure relative to $\mathscr{C}$* on the category of $\mathscr{D}$-spaces by restricting attention to those levels in $\mathscr{C}$. That is, we define the level equivalences and level fibrations relative to $\mathscr{C}$ to be those maps of $\mathscr{D}$-spaces that are level equivalences or level fibrations when regarded as maps of $\mathscr{C}$-spaces. We restrict to maps $F_c(-)$ with $c \in \mathscr{C}$ when defining the generating q-cofibrations and generating acyclic q-cofibrations. The proofs of the model axioms and of all other results in this section go through equally well in the relative context. Clearly, when $\mathscr{C}$ contains all objects of a skeleton of $\mathscr{D}$, the relative level model structure coincides with the absolute level model structure.

7. Preliminaries about π_*-isomorphisms of prespectra

We record some results about homotopy groups and π_*-isomorphisms of prespectra that are needed in our study of stable model structures. Recall that we are using the terms prespectrum and $\mathscr{N}$-spectrum interchangeably. We are following [4, 11, 20] in calling a sequence of spaces X_n and maps $\sigma : \Sigma X_n \longrightarrow X_{n+1}$ a "prespectrum", reserving the term "spectrum" for a prespectrum whose adjoint structure maps $\tilde{\sigma} : X_n \longrightarrow \Omega X_{n+1}$ are homeomorphisms. However, we make no use of such spectra in this paper. In fact, the following remark shows that, in a sense, the theory of such spectra is disjoint from the present theory of diagram spectra.

Remark 7.1. If the underlying prespectrum of a symmetric spectrum X is a spectrum, then X is trivial, and similarly for orthogonal spectra and $\mathscr{W}$-spaces. Indeed, the iterated adjoint structure map $X(\mathbf{n}) \longrightarrow \Omega^2 X(\mathbf{n}+\mathbf{2})$ takes image in the subspace of points fixed under the conjugation action of Σ_2, where Σ_2 acts on S^2 by permuting coordinates and acts on $X(\mathbf{n}+\mathbf{2})$ through the embedding of Σ_2 in Σ_{n+2} as the subgroup fixing the first n coordinates. This is a proper subspace unless $\Omega^2 X(\mathbf{n}+\mathbf{2})$ is a point.

Definition 7.2. The *homotopy groups* of a prespectrum X are defined by

$$\pi_q(X) = \text{colim}\ \pi_{q+n}(X_n).$$

A map of prespectra is called a *π_*-isomorphism* if it induces an isomorphism on homotopy groups. A prespectrum X is an *Ω-spectrum* (more logically, Ω-prespectrum) if its adjoint structure maps $\tilde{\sigma} : X_n \longrightarrow \Omega X_{n+1}$ are weak equivalences.

The following observation is trivial, but important.

Lemma 7.3. *A level equivalence of prespectra is a π_*-isomorphism. A π_*-isomorphism between Ω-spectra is a level equivalence.*

The following results are significantly stronger technically than their analogues in the previous section in that no hypotheses about nondegenerate basepoints are required. There is no contradiction since the suspension prespectrum functor does not convert weak equivalences of spaces to π_*-isomorphisms of prespectra in general.

Theorem 7.4. (i) *If $f : X \longrightarrow Y$ is a π_*-isomorphism of prespectra and A is a based CW complex, then $f \wedge \text{id} : X \wedge A \longrightarrow Y \wedge A$ is a π_*-isomorphism.*

(i′) *A map of prespectra is a π_*-isomorphism if and only if its suspension is a π_*-isomorphism, and the natural map $\eta : X \longrightarrow \Omega\Sigma X$ is a π_*-isomorphism for all prespectra X.*

(ii) *The homotopy groups of a wedge of prespectra are the direct sums of the homotopy groups of the wedge summands, hence a wedge of π_*-isomorphisms of prespectra is a π_*-isomorphism.*

(iii) *If $i : A \longrightarrow X$ is an h-cofibration and a π_*-isomorphism of prespectra and $f : A \longrightarrow Y$ is any map of prespectra, then the cobase change $j : Y \longrightarrow X \cup_A Y$ is a π_*-isomorphism.*

(iv) *If i and i' are h-cofibrations and the vertical arrows are π_*-isomorphisms in the comparison of pushouts diagram of Theorem 6.9(iv), then the induced map of pushouts is a π_*-isomorphism.*

(v) *If X is the colimit of a sequence of h-cofibrations $X_n \longrightarrow X_{n+1}$, each of which is a π_*-isomorphism, then the map from the initial term X_0 into X is a π_*-isomorphism.*

(vi) *For any map $f : X \longrightarrow Y$ of prespectra, there are natural long exact sequences*

$$\cdots \longrightarrow \pi_q(Ff) \longrightarrow \pi_q(X) \longrightarrow \pi_q(Y) \longrightarrow \pi_{q-1}(Ff) \longrightarrow \cdots ,$$

$$\cdots \longrightarrow \pi_q(X) \longrightarrow \pi_q(Y) \longrightarrow \pi_q(Cf) \longrightarrow \pi_{q-1}(X) \longrightarrow \cdots ,$$

and the natural map $\eta : Ff \longrightarrow \Omega Cf$ is a π_*-isomorphism.

Proof. This is standard but hard to find in the literature in this generality. We sketch the proofs. Part (i′) is clear since an inspection of colimits shows that $\pi_q(X)$ is naturally isomorphic to $\pi_{q+1}(\Sigma X)$, with the isomorphism realized by η_*. Part (v) is also clear. The first long exact sequence of (vi) results by passage to colimits from the level long exact sequences of homotopy groups. For the second, we see from (i′) that it suffices to prove the exactness of $\pi_q(X) \longrightarrow \pi_q(Y) \longrightarrow \pi_q(Cf)$, and this composite is clearly zero. For an element α in the kernel of $\pi_q(Y) \longrightarrow \pi_q(Cf)$, we may represent α by a map $g : S^{n+q} \longrightarrow Y_n$ with n large enough that there is a null homotopy $h : CS^{n+q} \longrightarrow Cf_n$. We then compare the cofiber sequences starting with the inclusions $S^{n+q} \longrightarrow CS^{n+q}$ and $Y_n \longrightarrow Cf_n$ to obtain a map $k : S^{q+n+1} \longrightarrow \Sigma X_n$ such that $\Sigma f_n \circ k$ is homotopic to Σg. The map k represents a preimage of α. See e.g. [20, III.2.1] or [11, I.3.4] for details of the spectrum level argument. The last statement in (vi) follows from the last statement of (i′) by comparing the two long exact sequences in (vi); see e.g. [20, p. 130]. For finite wedges, (ii) holds by inductive use of split cofiber sequences, and passage to colimits gives the general case. Part (iii) holds by a comparison of cofiber sequences, and part (iv) follows from (vi) and a diagram chase; see e.g. [11, I.3.5]. Part (i) follows from (ii), (iv), and (v). □

8. Stable equivalences of $\mathscr{D}$-spectra

In this section and the next, let $\mathscr{D}$ be a nondegenerately based symmetric monoidal domain category with a faithful strong symmetric monoidal functor $\iota : \mathscr{N} \longrightarrow \mathscr{D}$ and a sphere $\mathscr{D}$-monoid $S = S_{\mathscr{D}}$ that restricts along ι to the sphere prespectrum $S_{\mathscr{N}}$. We think of ι as an inclusion of categories. We let $\mathscr{D}\mathscr{S}$ be the category of $\mathscr{D}$-spectra over S, or right S-modules in $\mathscr{D}\mathscr{T}$. We are thinking of $\mathscr{N}\mathscr{S}$, $\Sigma\mathscr{S}$, $\mathscr{I}\mathscr{S}$, and $\mathscr{W}\mathscr{T}$, but there are surely other examples of interest. We have strong symmetric monoidal inclusions of categories

$$\mathscr{N} \subset \Sigma \subset \mathscr{I} \subset \mathscr{W} \tag{8.1}$$

that send n to $\mathbf{n}$, $\mathbf{n}$ to $\mathbb{R}^n$ and $\mathbb{R}^n$ to S^n. The sphere spectra for the smaller categories are the restrictions of the sphere spectra for the larger categories. To mesh notations, we write n for its image in any of the $\mathscr{D}$, and we let $F_n = F_n^{\mathscr{D}}$ denote the left adjoint to the nth space evaluation functor Ev_n; for a $\mathscr{D}$-spectrum X, we write $X(n) = Ev_n X = X_n$ interchangeably.

Convention 8.2. Until §17, we understand the level model structure on $\mathscr{D}$-spectra to mean the level model structure *relative to $\mathscr{N}$*, as defined in Variant 6.10. Since $\mathscr{N}$ contains all of the objects of a skeleton of Σ or $\mathscr{I}$, this is the same as the absolute level model structure in all cases above except the case of $\mathscr{W}$-spaces. We let $[X, Y]$ denote the set of maps $X \longrightarrow Y$ in the homotopy category with respect to the level model structure relative to $\mathscr{N}$. Recall that all of the results of §6 apply to this relative model structure.

Definition 8.3. Consider $\mathscr{D}$-spectra E and maps of $\mathscr{D}$-spectra $f : X \longrightarrow Y$.

(i) E is a *$\mathscr{D}$-Ω-spectrum* if $\mathbb{U}X$ is an Ω-spectrum.
(ii) f is a *π_*-isomorphism* if $\mathbb{U}f$ is a π_*-isomorphism.
(iii) f is a *stable equivalence* if $f^* : [Y, E] \longrightarrow [X, E]$ is a bijection for all $\mathscr{D}$-Ω-spectra E.

Observe that a level equivalence is a stable equivalence. Certain stable equivalences play a central role in the theory.

Definition 8.4. Define $\lambda_n : F_{n+1}S^1 \longrightarrow F_nS^0$ to be the map adjoint to the canonical inclusion $S^1 \longrightarrow (F_nS^0)_{n+1}$, namely $\eta : S^1 = (F_n^{\mathscr{N}} S^0)_{n+1} \longrightarrow (F_n^{\mathscr{D}} S^0)_{n+1}$.

Lemma 8.5. *For any $\mathscr{D}$-spectrum X,*

$$\lambda_n^* : \mathscr{D}\mathscr{S}(F_nS^0, X) \longrightarrow \mathscr{D}\mathscr{S}(F_{n+1}S^1, X)$$

coincides with $\tilde{\sigma} : X_n \longrightarrow \Omega X_{n+1}$ under the canonical homeomorphisms

$$X_n = \mathscr{T}(S^0, X_n) \cong \mathscr{D}\mathscr{S}(F_nS^0, X)$$

and

$$\Omega X_{n+1} = \mathscr{T}(S^1, X_{n+1}) \cong \mathscr{D}\mathscr{S}(F_{n+1}S^1, X).$$

Proof. With $X = F_nS^0$, $\tilde{\sigma}$ may be identified with a map

$$\tilde{\sigma} : \mathscr{D}\mathscr{S}(F_nS^0, F_nS^0) \longrightarrow \mathscr{D}\mathscr{S}(F_{n+1}S^1, F_nS^0),$$

and $\lambda_n : F_{n+1}S^1 \longrightarrow F_nS^0$ is the image of the identity map under $\tilde{\sigma}$. □

The following lemma is crucial. Because of it, the homotopy theory of symmetric spectra is significantly different, and considerably less intuitive at first sight, than the homotopy theories of $\mathscr{N}$-spectra, orthogonal spectra, and $\mathscr{W}$-spaces.

Lemma 8.6. *In all cases, the maps λ_n are stable equivalences. In $\mathscr{P}$, $\mathscr{I}\mathscr{S}$, and $\mathscr{W}\mathscr{T}$, the λ_n are π_*-isomorphisms. In $\Sigma\mathscr{S}$, the λ_n are* not *π_*-isomorphisms.*

Proof. The first statement is immediate from Lemma 8.5 and the definition of a stable equivalence. We prove that the λ_n are or are not π_*-isomorphisms separately in the four cases. Let $S^n = *$ if $n < 0$.

$\mathscr{N}$-spectra. Here $(F_nA)(q) = A \wedge S^{q-n}$. Thus F_nA is essentially a reindexing of the suspension $\mathscr{N}$-spectrum of A. The map $\lambda_n(q)$ is the identity unless $q = n$, when it is the inclusion $* \longrightarrow S^0$. Thus λ_n is a π_*-isomorphism.

Orthogonal spectra. We have

$$(F_nA)(q) = O(q)_+ \wedge_{O(q-n)} A \wedge S^{q-n}.$$

For $q \geq n+1$, $\lambda_n(q)$ is the canonical quotient map

$$O(q)_+ \wedge_{O(q-n-1)} S^1 \wedge S^{q-n-1} = O(q)_+ \wedge_{O(q-n-1)} S^{q-n} \longrightarrow O(q)_+ \wedge_{O(q-n)} S^{q-n}.$$

By Theorem 7.4(i′), it suffices to prove that the map $\Sigma^n\lambda_n$ is a π_*-isomorphism, and $(\Sigma^n\lambda_n)(q)$ takes the form

$$O(q)_+ \wedge_{O(q-n-1)} S^q \longrightarrow O(q)_+ \wedge_{O(q-n)} S^q.$$

Since $O(q)$ acts on S^q, this is isomorphic to the map

$$\pi \wedge \mathrm{id} : O(q)/O(q-n-1)_+ \wedge S^q \longrightarrow O(q)/O(q-n)_+ \wedge S^q,$$

where π is the evident quotient map. This map is $(2q-n-1)$-connected, hence $\Sigma^n\lambda_n$ is a π_*-isomorphism.

Symmetric spectra. The description of the maps λ_n is the same as for orthogonal spectra, except that orthogonal groups are replaced by symmetric groups. However, in contrast to the quotients $O(q)/O(q-n)$, the quotients Σ_q/Σ_{q-n} do not become highly connected as q increases. In fact, $\pi_*(F_nS^n)$, $n \geq 1$, is the sum of countably many copies of the stable homotopy groups of S^0; compare [15, 3.1.10].

$\mathscr{W}$-spaces. The qth map of λ_n can be identified with the evaluation map

$$\Sigma\Omega(\Omega^n S^q) \longrightarrow \Omega^n S^q.$$

Applying π_{q+r} and passing to colimits over q, these maps induce an isomorphism with target the stable homotopy groups of spheres, reindexed by n. □

We shall prove the following result at the end of the next section.

Proposition 8.7. *A map of $\mathscr{N}$-spectra, orthogonal spectra, or $\mathscr{W}$-spaces is a π_*-isomorphism if and only if it is a stable equivalence.*

For this reason, there is no need to mention stable equivalences when setting up the stable model structures in $\mathscr{P}$, $\mathscr{I}\mathscr{S}$ and $\mathscr{W}\mathscr{T}$: everything can be done more simply in terms of π_*-isomorphisms. At the price of introducing an unnecessary additional level of complexity in these cases, we have chosen to work with stable equivalences in order to give a uniform general treatment. As suggested by Lemma 8.6, the forward implication of Proposition 8.7 does hold in all cases.

Proposition 8.8. *A π_*-isomorphism in $\mathscr{D}\mathscr{S}$ is a stable equivalence.*

Proof. Following the analogous argument in [15], define $RX = F_S(F_1S^1, X)$, where F_S is the function $\mathscr{D}$-spectrum functor. Since F_nS^n is isomorphic to the nth smash power of F_1S^1, by Lemma 1.8, the n-fold iterate R^nX is isomorphic to $F_S(F_nS^n, X)$. The map $\lambda = \lambda_1 : F_1S^1 \longrightarrow F_0S^0 = S$ induces a map $\lambda^* : X \longrightarrow RX$ and thus a map $R^n\lambda^* : R^nX \longrightarrow R^{n+1}X$. Define QX to be the homotopy colimit (or telescope) of the R^nX and let $\iota : X \longrightarrow QX$ be the natural map. The defining adjunctions of the functors F_S and F_n, together with the isomorphism

$$F_mA \wedge_S F_nB \cong F_{m+n}(A \wedge B)$$

for based spaces A and B, imply that

$$\mathscr{T}(A, Ev_mF_S(F_nS^n, X)) \cong \mathscr{T}(A, \Omega^nX(m+n)).$$

Therefore $(R^nX)(m) \cong \Omega^nX(m+n)$. Since λ corresponds to $\tilde{\sigma}$ under adjunction, a quick inspection of colimits shows that

$$\pi_q((QX)(m)) \cong \pi_{q-m}(X).$$

Nevertheless, QX need not be a $\mathscr{D}$-Ω-spectrum in general. However, if E is a $\mathscr{D}$-Ω-spectrum, then $\lambda^* : E \longrightarrow RE$ is a level equivalence, hence so is $\iota : E \longrightarrow QE$, and QE is a $\mathscr{D}$-Ω-spectrum. Moreover, $\iota_* : [X, E] \longrightarrow [X, QE]$ is an isomorphism for any X. By the naturality of ι, ι_* is the composite of $Q : [X, E] \longrightarrow [QX, QE]$ and $\iota^* : [QX, QE] \longrightarrow [X, QE]$. Since $[QX, E] \cong [QX, QE]$, this shows that $[X, E]$ is naturally a retract of $[QX, E]$. If $f : X \longrightarrow Y$ is a π_*-isomorphism, then $Qf : QX \longrightarrow QY$ is a level equivalence. Thus $f^* : [Y, E] \longrightarrow [X, E]$ is a retract of the isomorphism $(Qf)^* : [QY, E] \longrightarrow [QX, E]$ and is therefore an isomorphism. □

The proof has the following useful corollary.

Corollary 8.9. *If E is a $\mathscr{D}$-Ω-spectrum, then $E' = F_S(F_1S^0, E)$ is a $\mathscr{D}$-Ω-spectrum such that E is level equivalent to $\Omega E'$ (which is isomorphic to RE).*

Colimits, h-fibrations, smash products with spaces, and fiber and cofiber sequences are preserved by $\mathbb{U}$, since they are specified in terms of levelwise constructions. This implies the following result about the π_*-isomorphisms of $\mathscr{D}$-spectra.

Proposition 8.10. *Lemma 7.3 and Theorem 7.4 hold with $\mathscr{P}$ replaced by $\mathscr{D}\mathscr{S}$.*

We have the following analogues of these results for stable equivalences.

Lemma 8.11. *A stable equivalence between $\mathscr{D}$-Ω-spectra is a level equivalence.*

Proof. This is formal. If $f : E \longrightarrow E'$ is a stable equivalence of $\mathscr{D}$-Ω-spectra, then $f^* : [E', E] \longrightarrow [E, E]$ is an isomorphism. A map $g : E' \longrightarrow E$ such that $g \circ f = f^*g = \mathrm{id}$ is an inverse isomorphism to f in the level homotopy category. □

Theorem 8.12. (i) *If $f : X \longrightarrow Y$ is a stable equivalence of $\mathscr{D}$-spectra and A is a based CW complex, then $f \wedge \mathrm{id} : X \wedge A \longrightarrow Y \wedge A$ is a stable equivalence.*

(i′) *A map of $\mathscr{D}$-spectra is a stable equivalence if and only if its suspension is a stable equivalence.*

(ii) *A wedge of stable equivalences of $\mathscr{D}$-spectra is a stable equivalence.*

(iii) *If $i : A \longrightarrow X$ is an h-cofibration and stable equivalence of $\mathscr{D}$-spectra and $f : A \longrightarrow Y$ is any map of $\mathscr{D}$-spectra, then the cobase change $j : Y \longrightarrow X \cup_A Y$ is a stable equivalence.*

(iv) *If i and i' are h-cofibrations and the vertical arrows are stable equivalences in the comparison of pushouts diagram of Theorem 6.9(iv), then the induced map of pushouts is a stable equivalence.*

(v) *If X is the colimit of a sequence of h-cofibrations $X_n \longrightarrow X_{n+1}$, each of which is a stable equivalence, then the map from the initial term X_0 into X is a stable equivalence.*

(vi) *If $f : X \longrightarrow Y$ is a map of $\mathscr{D}$-spectra and E is an Ω-spectrum, there are natural long exact sequences*

$$\cdots \longrightarrow [\Sigma X, E] \longrightarrow [Cf, E] \longrightarrow [Y, E] \longrightarrow [X, E] \longrightarrow [\Omega Cf, E] \longrightarrow \cdots,$$

$$\cdots \longrightarrow [\Sigma X, E] \longrightarrow [\Sigma Ff, E] \longrightarrow [Y, E] \longrightarrow [X, E] \longrightarrow [Ff, E] \longrightarrow \cdots.$$

Proof. Under nondegenerate basepoint hypotheses, most of these results follow directly from the elementary results about the level homotopy category in Theorem 6.9. To obtain them in full generality, we make use of Proposition 8.8 and the results on π_*-isomorphisms of Theorem 7.4. Cofibrant $\mathscr{D}$-spectra are nondegenerately based by Theorem 6.5(v), and cofibrant approximations of general $\mathscr{D}$-spectra are level equivalences, hence π_*-isomorphisms, hence stable equivalences. Thus we can first use cofibrant approximation and Theorem 7.4 to reduce each statement to a statement about cofibrant $\mathscr{D}$-spectra and then quote Theorem 6.9. The upshot is that statements about $[X, Y]$ that hold for nondegenerately based X and general Y also hold for general X and $\mathscr{D}$-Ω-spectra Y.

For (i), we see from Theorems 6.9(i) and 7.4(ii) that $[X \wedge A, E]$ is naturally isomorphic to $[X, F(A, E)]$ when E is a $\mathscr{D}$-Ω-spectrum, in which case $F(A, E)$ is also a $\mathscr{D}$-Ω-spectrum. Thus $f \wedge \mathrm{id}$ is a stable equivalence. For (i′), Theorems 6.9(i) and 7.4(i′) imply that $[\Sigma X, E] \cong [X, \Omega E]$ for all X when E is an Ω-spectrum, and (i′) follows in view of Corollary 8.9. Similarly, Theorems 6.9(ii) and 7.4(ii) imply that the functor $[-, E]$ converts wedges to products when E is an Ω-spectrum, and this implies (ii). Again, (vi) follows from Theorems 6.9(vi) and 7.4(vi). We use (vi) to prove (iii) and (iv).

For (iii), cofibrant approximation gives a commutative diagram

$$\begin{array}{ccccc} X' & \xleftarrow{i'} & A' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow & & \downarrow \\ X & \xleftarrow[i]{} & A & \xrightarrow[f]{} & Y \end{array}$$

in which X', A', and Y' are cofibrant, the vertical arrows are level acyclic fibrations, and the maps i' and f' are h-cofibrations. By Theorem 7.4(iv), the induced map of pushouts is a π_*-isomorphism and thus a stable equivalence. By the diagram, i' is a stable equivalence, and it suffices to prove that the cobase change $j' : Y' \longrightarrow X' \cup_{A'} Y'$ is a stable equivalence. Thus we may assume without loss of generality that the given A, X, and Y are cofibrant. We first deduce from the cofiber sequence $A \longrightarrow X \longrightarrow X/A$ that $[X/A, E] = 0$. Since $X \cup_A Y/Y \cong X/A$, we then deduce that $[Y, E] \longrightarrow [X \cup_A Y, E]$ is a bijection.

For (iv), we apply cofibrant approximation to the diagram of Theorem 6.9(iv) to see that we may assume without loss of generality that it is a diagram of cofibrant $\mathscr{D}$-spectra. A comparison of cofiber sequences gives that $X/A \longrightarrow X'/A'$ is a stable equivalence, and then another comparison of cofiber sequences gives that $X \cup_A Y \longrightarrow X' \cup_{A'} Y'$ is a stable equivalence.

For (v), we apply cofibrant approximation to obtain a sequence of h-cofibrations $j_n : Y_n \longrightarrow Y_{n+1}$ between cofibrant $\mathscr{D}$-spectra together with level acyclic fibrations $p_n : Y_n \longrightarrow X_n$ such that $p_{n+1} \circ j_n = i_n \circ p_n$. Since the i_n and p_n are stable equivalences, so are the j_n. Let $Y = \operatorname{colim} Y_n$. The map $p : Y \longrightarrow X$ induced by the p_n is a level equivalence, and the $\lim^1$-exact sequence of Theorem 6.9(v) implies that $[Y, E] \longrightarrow [Y_0, E]$ and thus $[X, E] \longrightarrow [X_0, E]$ are isomorphisms. □

9. The stable model structure on $\mathscr{D}$-spectra

We retain the hypotheses on $\mathscr{D}$ given at the start of §8. Definition 6.1 specifies the level equivalences, level fibrations, level acyclic fibrations, q-cofibrations, and level acyclic q-cofibrations in $\mathscr{D}\mathscr{S}$. Definition 8.3 specifies the stable equivalences. The class of stable equivalences is closed under retracts and is saturated (satisfies the two out of three property for composites).

Definition 9.1. Let $f : X \longrightarrow Y$ be a map of $\mathscr{D}$-spectra.

(i) f is an *acyclic q-cofibration* if it is a stable equivalence and a q-cofibration.
(ii) f is a *q-fibration* if it satisfies the RLP with respect to the acyclic q-cofibrations.
(iii) f is an *acyclic q-fibration* if it is a stable equivalence and a q-fibration.

We shall prove the following result. In outline, its proof follows that of Hovey, Shipley, and Smith [15] for symmetric spectra of simplicial sets, but there are significant differences of detail.

Theorem 9.2. *The category $\mathscr{D}\mathscr{S}$ is a compactly generated proper topological model category with respect to the stable equivalences, q-fibrations, and q-cofibrations.*

The set of generating q-cofibrations is the set FI specified in Definition 6.2. The set K of generating acyclic q-cofibrations properly contains the set FJ specified there. The idea is that level equivalences and stable equivalences coincide on $\mathscr{D}$-Ω-spectra, by Lemma 8.11, and the model structure is arranged so that the fibrant spectra turn out to be exactly the $\mathscr{D}$-Ω-spectra. We add enough generating acyclic q-cofibrations to FJ to ensure that the RLP with respect to the K-cell complexes forces the adjoint structure maps of fibrant spectra to be weak equivalences. Recall the maps λ_n from Definition 8.4 and the operation $\Box$ from (5.14).

Definition 9.3. Let $M\lambda_n$ be the mapping cylinder of λ_n. Then λ_n factors as the composite of a q-cofibration $k_n : F_{n+1}S^1 \longrightarrow M\lambda_n$ and a deformation retraction $r_n : M\lambda_n \longrightarrow F_nS^0$. For $n \geq 0$, let K_n be the set of maps of the form $k_n \Box i$, $i \in I$. Let K be the union of FJ and the sets K_n for $n \geq 0$.

We need a characterization of the maps that satisfy the RLP with respect to K. The following definition is not quite standard, but is convenient for our purposes.

Definition 9.4. A commutative diagram of based spaces

$$\begin{array}{ccc} D & \xrightarrow{g} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow[f]{} & B \end{array}$$

in which p and q are Serre fibrations is a *homotopy pullback* if the induced map $D \longrightarrow A \times_B E$ is a weak equivalence or, equivalently, if $g : p^{-1}(a) \longrightarrow q^{-1}(f(a))$ is a weak equivalence for all $a \in A$.

Proposition 9.5. *A map $p : E \longrightarrow B$ satisfies the RLP with respect to K if and only if p is a level fibration and the diagram*

$$\begin{array}{ccc} E_n & \xrightarrow{\tilde{\sigma}} & \Omega E_{n+1} \\ {\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle \Omega p_{n+1}} \\ B_n & \xrightarrow[\tilde{\sigma}]{} & \Omega B_{n+1} \end{array} \tag{9.6}$$

is a homotopy pullback for each $n \geq 0$.

Proof. Clearly p satisfies the RLP with respect to K if and only if p satisfies the RLP with respect to FJ and the K_n for $n \geq 0$. The maps that satisfy the RLP with respect to FJ are the level fibrations. Thus assume that p is a level fibration in the rest of the proof. By the definition of K_n, p has the RLP with respect to K_n if and only if p has the RLP with respect to $k_n \Box I$. By Lemma 5.16, this holds if and only if $\mathscr{D}\mathscr{S}(k_n^*, p_*)$ has the RLP with respect to I, which means that $\mathscr{D}\mathscr{S}(k_n^*, p_*)$ is an acyclic Serre fibration. Since k_n is a q-cofibration and p is a level fibration, $\mathscr{D}\mathscr{S}(k_n^*, p_*)$ is a Serre fibration because the level model structure is topological. We conclude that p satisfies the RLP with respect to K if and only if p is a level fibration and $\mathscr{D}\mathscr{S}(k_n^*, p_*)$ is a weak equivalence for $n \geq 0$. Let $j_n : F_nS^0 \longrightarrow M\lambda_n$ be the evident homotopy inverse of $r_n : M\lambda_n \longrightarrow F_nS^0$. Then $\mathscr{D}\mathscr{S}(k_n^*, p_*) \simeq \mathscr{D}\mathscr{S}((j_n\lambda_n)^*, p_*)$. This is a weak equivalence if and only if

$$\mathscr{D}\mathscr{S}(\lambda_n^*, p_*) : \mathscr{D}\mathscr{S}(F_nS^0, E) \longrightarrow \mathscr{D}\mathscr{S}(F_nS^0, B) \times_{\mathscr{D}\mathscr{S}(F_{n+1}S^1, B)} \mathscr{D}\mathscr{S}(F_{n+1}S^1, E))$$

is a weak equivalence. But this is isomorphic to the map

$$E_n \longrightarrow B_n \times_{\Omega B_{n+1}} \Omega E_{n+1}$$

and is thus a weak equivalence if and only if (9.6) is a homotopy pullback. □

Corollary 9.7. *The trivial map $F \longrightarrow *$ satisfies the RLP with respect to K if and only if F is a $\mathscr{D}$-Ω-spectrum.*

Corollary 9.8. *If $p : E \longrightarrow B$ is a stable equivalence that satisfies the RLP with respect to K, then p is a level acyclic fibration.*

Proof. Certainly $p : E \longrightarrow B$ is a level fibration. We must prove that p is a level equivalence. Let $F = p^{-1}(*)$ be the fiber (defined levelwise) over the basepoint. Since $F \longrightarrow *$ is a pullback of p, it satisfies the RLP with respect to K and is therefore a $\mathscr{D}$-Ω-spectrum. Since p is acyclic, so is $F \longrightarrow *$. Therefore, by Lemma 8.11, F is level acyclic. By the level long exact sequences, each $p_n : E_n \longrightarrow B_n$ induces an isomorphism of homotopy groups in positive degrees. To deal with π_0, observe that, in the homotopy pullback (9.6), the map Ωp_{n+1} depends only on basepoint components and is a weak equivalence. Therefore p_n is a weak equivalence as required. □

The q-cofibrations are the same for the stable as for the level model structure. The essential part of the proof of the model axioms for the stable model structure is to characterize the acyclic q-cofibrations, the q-fibrations, and the acyclic q-fibrations. Observe that the small object argument applies to K since the domains of the maps in K are compact by Lemma 5.7.

Proposition 9.9. *Let $f : X \longrightarrow Y$ be a map in $\mathscr{D}\mathscr{S}$.*

(i) *f is an acyclic q-cofibration if and only if it is a retract of a relative K-cell complex.*
(iii) *f is a q-fibration if and only if it satisfies the RLP with respect to K, and X is fibrant if and only if it is a $\mathscr{D}$-Ω-spectrum.*
(iii) *f is an acyclic q-fibration if and only if it is a level acyclic fibration.*

Proof. (i) Let f be a retract of a relative K-cell complex. Since the maps in K are acyclic q-cofibrations, f is an acyclic q-cofibration by the closure properties of the class of q-cofibrations given by the level model structure and the closure properties of the class of stable equivalences given by Theorem 8.12. Conversely, let $f : X \longrightarrow Y$ be an acyclic q-cofibration. Using the small object argument, factor f as the composite of a relative K-cell complex $i : X \longrightarrow X'$ and a map $p : X' \longrightarrow Y$ that satisfies the RLP with respect to K. We have just seen that i is a stable equivalence. Since f is a stable equivalence, so is p. By Corollary 9.8, p is a level acyclic fibration. Since f is a q-cofibration, it has the LLP with respect to p. Now a standard retract argument applies. There is a map $g : Y \longrightarrow X'$ such that $g \circ f = i$ and $p \circ g = \mathrm{id}$. Thus g and p are maps under X and f is a retract of the relative K-cell complex i.
(ii) Since f satisfies the RLP with respect to K if and only if it satisfies the RLP with respect to all retracts of relative K-cell complexes, this follows from (i) and the definition of a q-fibration.
(iii). By the level model structure, a map is a level acyclic fibration if and only if it satisfies the RLP with respect to the q-cofibrations, and this implies trivially that it is a q-fibration. Thus, since a level equivalence is a stable equivalence, a level acyclic fibration is an acyclic q-fibration. Conversely, an acyclic q-fibration satisfies the RLP with respect to K, by (ii), and is therefore a level acyclic fibration by Corollary 9.8. □

The proof that $\mathscr{DS}$ is a model category. The definition of a q-fibration gives one of the lifting axioms. The identification of the acyclic q-fibrations as the level acyclic fibrations gives the other lifting axiom and the factorization of a map as a composite of a q-cofibration and an acyclic q-fibration, via the level model structure. It remains to prove that a map $f : X \longrightarrow Y$ factors as the composite of an acyclic q-cofibration and a q-fibration. Applying the small object argument to K, we obtain a factorization of f as the composite of a relative K-cell complex $i : X \longrightarrow X'$ and a map $p : X' \longrightarrow Y$ that satisfies the RLP with respect to K. By the previous proposition, i is an acyclic q-cofibration and p is a q-fibration. $\square$

The proof that $\mathscr{DS}$ is topological. Let $i : A \longrightarrow X$ be a q-cofibration and $p : E \longrightarrow B$ be a q-fibration. Since p is a level fibration, the map

$$\mathscr{DS}(i^*, p_*) : \mathscr{DS}(X, E) \longrightarrow \mathscr{DS}(A, E) \times_{\mathscr{DS}(A,B)} \mathscr{DS}(X, B)$$

is a Serre fibration because the level model structure is topological. Similarly, if p is acyclic, then p is level acyclic and $\mathscr{DS}(i^*, p_*)$ is a weak equivalence. We must show that $\mathscr{DS}(i^*, p_*)$ is a weak equivalence if i is acyclic, and it suffices to show this when $i \in K$. If $i \in FJ$, this again holds by the result for the level model structure. Thus suppose that $i \in k_n \Box I$, say $i = k_n \Box j$. We have seen in the proof of Proposition 9.5 that $\mathscr{DS}(k_n^*, p_*)$ is a weak equivalence. Thus, since $\mathscr{T}$ is a topological model category, $\mathscr{T}(j^*, \mathscr{DS}(k_n^*, p_*)_*)$ is a weak equivalence. By Lemma 5.16, this implies that $\mathscr{DS}(i^*, p_*)$ is a weak equivalence. $\square$

The proof that $\mathscr{DS}$ is proper. Since q-cofibrations are h-cofibrations and q-fibrations are level fibrations, the following lemma generalizes the claim. $\square$

Lemma 9.10. *Consider the following commutative diagram:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow[g]{} & Y. \end{array}$$

(i) *If the diagram is a pushout in which i is an h-cofibration and f is a stable equivalence, then g is a stable equivalence.*
(ii) *If the diagram is a pullback in which j is a level fibration and g is a stable equivalence, then f is a stable equivalence.*

Proof. (i) The induced map $X/A \longrightarrow Y/B$ is an isomorphism. We compare the cofibration sequences $[-, E]$ of Theorem 8.12(vi) for the cofibration sequences $A \longrightarrow X \longrightarrow X/A$ and $B \longrightarrow Y \longrightarrow Y/B$ and apply the five lemma. (ii) Dually, the induced map from the fiber of i to the fiber of j is an isomorphism. We compare fibration sequences using Theorem 8.12(vi). $\square$

We have left one unfinished piece of business from the previous section.

The proof of Proposition 8.7. By Proposition 8.8, we need only show that a stable equivalence f in $\mathscr{DS}$ is a π_*-isomorphism when $\mathscr{DS}$ is $\mathscr{P}$, $\mathscr{IS}$, or $\mathscr{WT}$. Factor f as the composite of an acyclic q-cofibration and an acyclic q-fibration. Since an acyclic q-fibration is a level acyclic fibration, it is a level equivalence and therefore a π_*-isomorphim. We must show that an acyclic q-cofibration is a π_*-isomorphism. We first show that the maps in K are π_*-isomorphisms. The maps in FJ are inclusions of deformation retracts and are therefore π_*-isomorphisms. The maps $k_n \square i$ with $i \in FI$ specified in Definition 9.3 are also π_*-isomorphisms. Indeed, by Lemma 8.6, the maps λ_n and therefore the maps k_n are π_*-isomorphisms. By Theorem 7.4(i), so are their smash products with based CW complexes. By passage to pushouts and a little diagram chase, this implies that the maps $k_n \square i$ are π_*-isomorphisms. By Theorem 7.4, it follows that any relative K-cell complex is a π_*-isomorphism. Since the acyclic q-cofibrations are the retracts of the relative K-cell complexes, the conclusion follows. □

In fact, we now see that, in our development of the stable model structure on $\mathscr{D}\mathscr{S}$ in these three cases, we can start out by defining the weak equivalences to be either the stable equivalences or the π_*-isomorphisms. We arrive at the same acyclic q-cofibrations and acyclic q-fibrations either way.

10. Comparisons among $\mathscr{P}$, $\Sigma\mathscr{S}$, $\mathscr{I}\mathscr{S}$, and $\mathscr{W}\mathscr{T}$

We now turn to the proofs that our various adjoint pairs are Quillen equivalences. Write $\mathbb{U} : \mathscr{D}\mathscr{S} \longrightarrow \mathscr{C}\mathscr{S}$ generically for the forgetful functor associated to any of the inclusions $\mathscr{C} \subset \mathscr{D}$ of (8.1); the alert reader will notice that the arguments apply more generally. As noted in Proposition 3.2, for each of the inclusions $\mathscr{C} \subset \mathscr{D}$, we have $\mathbb{P} \circ F_n^{\mathscr{C}} \cong F_n^{\mathscr{D}}$.

The characterizations of the q-fibrations and acyclic q-fibrations given by Propositions 9.5 and 9.9 directly imply the following lemma. Recall Definition A.1.

Lemma 10.1. *Each forgetful functor $\mathbb{U} : \mathscr{D}\mathscr{S} \longrightarrow \mathscr{C}\mathscr{S}$ preserves q-fibrations and acyclic q-fibrations. Therefore each $(\mathbb{P}, \mathbb{U})$ is a Quillen adjoint pair.*

We wish to apply Lemma A.2(iii) to demonstrate that these pairs are Quillen equivalences. For that, we need to know that $\mathbb{U}$ creates the stable equivalences in its domain category. This is false for $\mathbb{U} : \Sigma\mathscr{S} \longrightarrow \mathscr{P}$ because the λ_n are stable equivalences of symmetric spectra but the $\mathbb{U}\lambda_n$ are not stable equivalences (= π_*-isomorphisms) of $\mathscr{N}$-spectra. This makes a direct proof of the Quillen equivalence between $\mathscr{N}$-spectra and symmetric spectra fairly difficult; compare [15, §4]. However, this is the only case in which the condition fails.

Lemma 10.2. *The forgetful functors*

$$\mathbb{U} : \mathscr{I}\mathscr{S} \longrightarrow \mathscr{P}, \quad \mathbb{U} : \mathscr{I}\mathscr{S} \longrightarrow \Sigma\mathscr{S}, \quad \textit{and} \quad \mathbb{U} : \mathscr{W}\mathscr{T} \longrightarrow \mathscr{I}\mathscr{S}$$

and their composites create the stable equivalences in their domain categories.

Proof. This is immediate in the first and third case, since there the stable equivalences coincide with the π_*-isomorphisms in both the domain and codomain categories. To prove that $\mathbb{U} : \mathscr{I}\mathscr{S} \longrightarrow \Sigma\mathscr{S}$ creates the weak equivalences of orthogonal spectra, let $f : X \longrightarrow Y$ be a map of orthogonal spectra such that $\mathbb{U}f$ is a stable equivalence and let $f' : X' \longrightarrow Y'$ be a fibrant approximation of f. Then $\mathbb{U}f'$ is a stable equivalence between symmetric Ω-spectra and thus a π_*-isomorphism, and it follows that f is a π_*-isomorphism. □

Thus $\mathbb{U} : \mathscr{D}\mathscr{S} \longrightarrow \mathscr{C}\mathscr{S}$ creates the stable equivalences in $\mathscr{D}\mathscr{S}$ whenever the stable equivalences and π_*-isomorphisms coincide in $\mathscr{D}\mathscr{S}$. In these cases, we also have the following result about the unit $\eta : \mathrm{Id} \longrightarrow \mathbb{U}\mathbb{P}$ of the adjunction.

Lemma 10.3. *Consider $\mathbb{U} : \mathscr{D}\mathscr{S} \longrightarrow \mathscr{C}\mathscr{S}$ and $\mathbb{P} : \mathscr{C}\mathscr{S} \longrightarrow \mathscr{D}\mathscr{S}$. If the stable equivalences and π_*-isomorphisms coincide in $\mathscr{D}\mathscr{S}$, then $\eta : X \longrightarrow \mathbb{U}\mathbb{P}X$ is a stable equivalence for all cofibrant $\mathscr{C}$-spectra X.*

Proof. Since the functors $\mathbb{P}$ and $\mathbb{U}$ preserve colimits, h-cofibrations, and smash products with based spaces and since cofibrant $\mathscr{C}$-spectra are retracts of FI-cell $\mathscr{C}$-spectra, we see from Theorem 8.12 that it suffices to prove the result when $X = F_nS^n$, $n \geq 0$. Let $\gamma_n^{\mathscr{C}} : F_n^{\mathscr{C}}S^n \longrightarrow F_0^{\mathscr{C}}S^0$ be adjoint to the identity map $S^n \longrightarrow S^n = (F_0S^0)(n)$. Then $\gamma_n^{\mathscr{C}}$ is the composite of the maps $\Sigma^m\lambda_m$, $0 \leq m < n$. These maps are stable equivalences by Lemma 8.6; moreover, with $\mathscr{C}$ replaced by $\mathscr{D}$, they are π_*-isomorphisms. Since $\mathbb{U}$ preserves π_*-isomorphisms and π_*-isomorphisms in $\mathscr{C}\mathscr{S}$ are stable equivalences, the conclusion follows from the commutative diagram

$$\begin{array}{ccc} F_n^{\mathscr{C}}S^n & \xrightarrow{\gamma_n^{\mathscr{C}}} & F_0^{\mathscr{C}}S^0 = S_{\mathscr{C}} \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ \mathbb{U}F_n^{\mathscr{D}}S^n & \xrightarrow[\mathbb{U}\gamma_n^{\mathscr{D}}]{} & \mathbb{U}F_0^{\mathscr{D}}S^0 = S_{\mathscr{C}}, \end{array}$$

in which the right arrow η is an isomorphism. □

Theorem 10.4. *The categories of $\mathscr{N}$-spectra and orthogonal spectra, of symmetric spectra and orthogonal spectra, and of orthogonal spectra and $\mathscr{W}$-spaces are Quillen equivalent.*

Proof. This is immediate from Lemmas A.2(iii), 10.1, 10.2, and 10.3. □

Corollary 10.5. *The categories of $\mathscr{N}$-spectra and symmetric spectra are Quillen equivalent.*

Proof. We have the following pair of adjoint pairs:

$$\mathscr{P} \underset{\mathbb{U}}{\overset{\mathbb{P}}{\rightleftarrows}} \Sigma\mathscr{S} \underset{\mathbb{U}}{\overset{\mathbb{P}}{\rightleftarrows}} \mathscr{I}\mathscr{S}.$$

The composite pair $(\mathbb{P}\mathbb{P}, \mathbb{U}\mathbb{U})$ and the second pair $(\mathbb{P}, \mathbb{U})$ are Quillen equivalences. By Lemma A.2, so is the first pair $(\mathbb{P}, \mathbb{U})$. □

Proof of Corollary 0.2. The result asserts that a map $f : X \longrightarrow Y$ of cofibrant symmetric spectra is a stable equivalence if and only if $\mathbb{P}f$ is a π_*-isomorphism of orthogonal spectra. By the naturality of η, Lemma 10.3 implies that f is a stable equivalence if and only if $\mathbb{U}\mathbb{P}f$ is a stable equivalence. Since $\mathbb{U}$ creates the stable equivalences of orthogonal spectra, this gives the conclusion. □

We now turn to the proof of Theorem 0.3, which asserts that our induced equivalences of homotopy categories preserve smash products. In the comparisons that do not involve $\mathscr{P}$, $\mathbb{P}$ is strong symmetric monoidal and the conclusion is formal (see Lemma A.3). Of course, since the equivalence of homotopy categories induced by $\mathbb{P}$ preserves smash products, so does the inverse equivalence induced by $\mathbb{U}$. We bring prespectra into the picture and complete the proof in the next section.

11. CW prespectra and handicrafted smash products

For historical continuity, we bring the abstract theory down to earth by relating it to the classical theory of CW prespectra and handicrafted smash products, due to Boardman [4] and Adams [1].

Classically, a *CW prespectrum* is a sequence of based CW complexes X_n and isomorphisms from ΣX_n onto a subcomplex of X_{n+1}; we may regard these isomorphisms as inclusions of subcomplexes. We have another such notion, which actually applies equally well to $\mathscr{D}$-spectra for $\mathscr{D} = \mathscr{N}$, Σ, $\mathscr{I}$, or $\mathscr{W}$. We define a *CW $\mathscr{D}$-spectrum* to be an FI-cell complex whose cells are attached only to cells of lower dimension, where we define the dimension of a cell $F_n D^m_+$ to be $m - n$. Of course, a CW $\mathscr{D}$-spectrum is cofibrant. The following description of $\mathscr{N}$-spectra, which is implied by Lemma 1.6, makes it easy to compare these two notions. Recall that, for a based space A, $(F_n A)_q = A \wedge S^{q-n}$, where $S^m = *$ if $m < 0$. The map $\lambda_n : F_{n+1}\Sigma A \longrightarrow F_n A$ is the adjoint of the identity map $\Sigma A \longrightarrow (F_n A)_{n+1}$. For an $\mathscr{N}$-spectrum X, let $X\langle n\rangle$ be the evident $\mathscr{N}$-spectrum such that

$$X\langle n\rangle_q = \begin{cases} X_q & \text{if } q \le n \\ \Sigma^{q-n} X_n & \text{if } q > n \end{cases}$$

and observe that $X\langle 0\rangle = F_0 X_0$.

Lemma 11.1. *An $\mathscr{N}$-spectrum X is isomorphic to the colimit of the right vertical arrows in the inductively constructed pushout diagrams*

(11.2)
$$\begin{array}{ccccc} F_{n+1}\Sigma X_n & \xrightarrow{\lambda_n} & F_n X_n & \longrightarrow & X\langle n\rangle \\ {\scriptstyle F_{n+1}\sigma_n}\downarrow & & & & \downarrow \\ F_{n+1}X_{n+1} & & \longrightarrow & & X\langle n+1\rangle . \end{array}$$

Lemma 11.3. *A CW prespectrum X is a CW $\mathscr{N}$-spectrum and is thus cofibrant.*

Proof. For a CW complex A, $F_n A$ is easily checked to be a CW $\mathscr{N}$-spectrum, naturally in cellular maps of A; moreover, $\lambda_n : F_{n+1}\Sigma A \longrightarrow F_n A$ is cellular. Just as for spaces, a base change of a cellular inclusion of CW $\mathscr{N}$-spectra along a cellular map is a cellular inclusion of CW $\mathscr{N}$-spectra, and a colimit of cellular inclusions of CW $\mathscr{N}$-spectra is a CW $\mathscr{N}$-spectrum. □

As is made precise in the following lemma, the converse holds up to homotopy.

Lemma 11.4. *If X is a cofibrant $\mathscr{N}$-spectrum, then the X_n have the homotopy types of CW complexes and the $\sigma_n : \Sigma X_n \longrightarrow X_{n+1}$ are h-cofibrations. If X is any prespectrum such that the X_n have the homotopy types of CW complexes and the σ_n are h-cofibrations, then X has the homotopy type of a CW prespectrum.*

Proof. The first statement is a direct levelwise inspection of definitions when X is an FI-cell $\mathscr{N}$-spectrum, and the general case follows. The second statement is classical, but we give a proof in our context. Since the maps $F_{n+1}\sigma_n$ in (11.2) are h-cofibrations, so are the right vertical arrows in (11.2). Therefore the colimit X is homotopy equivalent to the corresponding telescope. We can construct based CW complexes Y_n, homotopy equivalences $f_n : Y_n \longrightarrow X_n$, and isomorphisms onto subcomplexes $\tau_n : \Sigma Y_n \longrightarrow Y_{n+1}$ such that $\sigma_n \Sigma \circ f_n \simeq f_{n+1} \circ \tau_n$. Then $Y \cong \operatorname{colim} Y\langle n\rangle$ is a CW prespectrum, and $Y \simeq \operatorname{tel} Y\langle n\rangle \simeq \operatorname{tel} X\langle n\rangle$. □

This implies the following observation, which is unexpected from a model theoretic point of view.

Proposition 11.5. *Let X be a cofibrant $\mathscr{D}$-spectrum, where $\mathscr{D} = \Sigma$, $\mathscr{I}$, or $\mathscr{W}$. Then the underlying prespectrum $\mathbb{U}X$ has the homotopy type of a CW prespectrum and thus of a cofibrant $\mathscr{N}$-spectrum.*

Proof. For a finite CW complex A, the spaces $(F_m^{\mathscr{D}} A)_n$ have the homotopy types of CW complexes. Therefore, for an FI-cell spectrum X and thus for any cofibrant $\mathscr{D}$-spectrum X, each X_n has the homotopy type of a CW complex. The conclusion follows from Lemma 11.4. □

We now fix a choice of a naive or "handicrafted" smash product of prespectra.

Definition 11.6. Define the (naive) smash product of prespectra X and Y by

$$(X \wedge Y)_{2n} = X_n \wedge Y_n \quad \text{and} \quad (X \wedge Y)_{2n+1} = \Sigma(X_n \wedge Y_n),$$

with the evident structure maps.

Proposition 11.7. *For any cofibrant prespectrum X, the functor $X \wedge Y$ of Y preserves π_*-isomorphisms.*

Proof. Each X_n has the homotopy type of a CW complex, hence each functor $X_n \wedge Y$ preserves π_*-isomorphisms by Theorem 7.4(i). The groups $\pi_*(X \wedge Y)$ are

$$\begin{aligned}
\pi_q(X \wedge Y) &\cong \operatorname{colim}_n \pi_{2n+q}(X_n \wedge Y_n) \\
&\cong \operatorname{colim}_{m,n} \pi_{m+n+q}(X_m \wedge Y_n) \\
&\cong \operatorname{colim}_m \operatorname{colim}_n \pi_{m+n+q}(X_m \wedge Y_n) \\
&= \operatorname{colim}_m \pi_{m+q}(X_m \wedge Y),
\end{aligned}$$

and the conclusion follows. □

We must explain the relationship between the naive smash product and the smash product of $\mathscr{D}$-spectra for $\mathscr{D} = \Sigma$, $\mathscr{I}$, and $\mathscr{W}$. The definition of the latter given in §21 implies that there are canonical maps $X_m \wedge Y_n \longrightarrow (X \wedge_S Y)_{m+n}$. These maps for $m = n$ and the structure maps $\Sigma(X \wedge_S Y)_{2n} \longrightarrow (X \wedge_S Y)_{2n+1}$ of the prespectrum $\mathbb{U}(X \wedge_S Y)$ specify maps

$$\phi_q : (\mathbb{U}X \wedge \mathbb{U}Y)_q \longrightarrow \mathbb{U}(X \wedge_S Y)_q. \tag{11.8}$$

As would also be true for any other choice of handicrafted smash product of prespectra, these maps do not form a map of prespectra, due to permutations of spheres. However, there are natural homotopies $\phi_{q+1}\sigma_q \simeq \sigma_q \Sigma \phi_q$. That is, ϕ is a "weak map" of prespectra. This is the kind of map that appears in the classical representation of homology and cohomology theories on spaces. The homotopy groups of prespectra are functorial with respect to weak maps, and the ϕ behave as follows.

Proposition 11.9. *Let $\mathscr{D} = \mathscr{I}$ or $\mathscr{D} = \mathscr{W}$. For a cofibrant $\mathscr{D}$-spectrum X and any $\mathscr{D}$-spectrum Y, $\phi : \mathbb{U}X \wedge \mathbb{U}Y \longrightarrow \mathbb{U}(X \wedge_S Y)$ is a π_*-isomorphism. The analogue for symmetric spectra is false.*

Proof. We shall prove in Proposition 12.3 below that the functor $X \wedge_S Y$ of Y preserves π_*-isomorphisms. Applying this to a cofibrant approximation of Y and using Propositions 11.7 and 11.5, we see that we may assume that both X and Y are cofibrant. Passing to retracts, we see that we may assume that X and Y are FI-cell complexes. By double induction and passage to suspensions, wedges, pushouts, and colimits, it suffices to prove the result when $X = F_m^{\mathscr{D}} S^0$ and $Y = F_n^{\mathscr{D}} S^0$. Here $X \wedge_S Y \cong F_{m+n}^{\mathscr{D}} S^0$ by Lemma 1.8 and $F^{\mathscr{D}} \cong \mathbb{P}F^{\mathscr{P}}$. We have an evident weak map

$$\phi : F_m^{\mathscr{P}} S^0 \wedge F_n^{\mathscr{P}} S^0 \longrightarrow F_{m+n}^{\mathscr{P}} S^0$$

that sends $S^{q-m} \wedge S^{q-n}$ to S^{2q-m-n} at level $2q$ and is a π_*-isomorphism. Again, it is due to permutations of spheres that this is only a weak map. The following diagram of weak maps of prespectra commutes:

$$\begin{array}{ccc}
F_m^{\mathscr{P}} S^0 \wedge F_n^{\mathscr{P}} S^0 & \xrightarrow{\ \phi\ } & F_{m+n}^{\mathscr{P}} S^0 \\
{\scriptstyle \eta\wedge\eta}\downarrow & & \downarrow{\scriptstyle \eta} \\
\mathbb{UP}F_m^{\mathscr{P}} S^0 \wedge \mathbb{UP}F_n^{\mathscr{P}} S^0 & \xrightarrow[\ \phi\ ]{} & \mathbb{UP}F_{m+n}^{\mathscr{P}} S^0.
\end{array}$$

The maps η and therefore $\eta \wedge \eta$ are π_*-isomorphisms, by Lemma 10.3 and Proposition 11.7, hence the bottom map ϕ is a π_*-isomorphism. In the case of symmetric spectra, this argument does not apply and, by inspection of definitions as in Lemma 8.6, the source and target of the bottom map ϕ have different homotopy groups. □

Proof of Theorem 0.3. The maps ϕ of (11.8) together with the natural homotopies $\phi_{q+1}\sigma_q \simeq \sigma_q \Sigma \phi_q$ prescribe what May and Thomason call a "preternatural transformation" [31, A.1]. They observe [31, A.2] (see also [20, I.7.6]) that use of the "cylinder construction" K gives a natural commutative diagram of weak maps

$$\begin{array}{ccc}
K(\mathbb{U}(X) \wedge \mathbb{U}(Y)) & \xrightarrow{\ K\phi\ } & K\mathbb{U}(X \wedge_S Y) \\
{\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \psi} \\
\mathbb{U}(X) \wedge \mathbb{U}(Y) & \xrightarrow[\ \phi\ ]{} & \mathbb{U}(X \wedge_S Y)
\end{array}$$

in which the ψ are natural π_*-isomorphisms of prespectra and $K\phi$ is a natural map of prespectra. When $\mathscr{D} = \mathscr{I}$ or $\mathscr{D} = \mathscr{W}$, ϕ is a π_*-isomorphism, hence so is $K\phi$. On passage to homotopy categories, we can invert ψ and conclude that the equivalence induced by $\mathbb{U} : \mathscr{D}\mathscr{S} \longrightarrow \mathscr{P}$ preserves smash products. Because the equivalence induced by $\mathbb{U} : \mathscr{I}\mathscr{S} \longrightarrow \Sigma\mathscr{S}$ also preserves smash products, it follows formally that the equivalence induced by $\mathbb{U} : \Sigma\mathscr{S} \longrightarrow \mathscr{P}$ preserves smash products. Proposition 11.9 shows that the equivalence is not given in the most naive way. □

12. Model categories of ring and module spectra

So far in our work, we have largely ignored the main point of the introduction of categories of diagram spectra, namely the fact that the category of $\mathscr{D}$-spectra is symmetric monoidal under its smash product $\wedge_S$ when the sphere $\mathscr{D}$-space S is a commutative $\mathscr{D}$-monoid. This holds for all of the categories except $\mathscr{P}$ displayed in the Main Diagram in the introduction. We are writing $\wedge_S$ to avoid confusion with smash products with spaces and as a reminder that the category $\mathscr{D}\mathscr{S}$ of $\mathscr{D}$-spectra coincides with the category of S-modules.

It is now an easy matter to obtain (stable) model structures on categories of $\mathscr{D}$-ring and module spectra when $\mathscr{D}$ is Σ, $\mathscr{I}$, or $\mathscr{W}$; we write $\mathscr{D}$ generically for any of these three categories. As we indicate at the end of the section, most of the proof of the following theorem can be quoted from the axiomatic treatment of Schwede and Shipley [37].

Theorem 12.1. *Let R be a $\mathscr{D}$-ring spectrum, where $\mathscr{D} = \Sigma$, $\mathscr{I}$, or $\mathscr{W}$.*

(i) *The category of left R-modules is a compactly generated proper topological model category with weak equivalences and q-fibrations created in $\mathscr{D}\mathscr{S}$.*
(ii) *If R is cofibrant as a $\mathscr{D}$-spectrum, then the forgetful functor from R-modules to $\mathscr{D}$-spectra preserves q-cofibrations, hence every cofibrant R-module is cofibrant as a $\mathscr{D}$-spectrum.*
(iii) *If R is commutative, the symmetric monoidal category $\mathscr{D}\mathscr{S}_R$ of R-modules also satisfies the pushout-product and monoid axioms.*
(iv) *If R is commutative, the category of R-algebras is a compactly generated right proper topological model category with weak equivalences and q-fibrations created in $\mathscr{D}\mathscr{S}$.*
(v) *If R is commutative, every q-cofibration of R-algebras whose source is cofibrant as an R-module is a q-cofibration of R-modules, hence every cofibrant R-algebra is cofibrant as an R-module.*
(vi) *If $f : Q \longrightarrow R$ is a weak equivalence of $\mathscr{D}$-ring spectra, then restriction and extension of scalars define a Quillen equivalence between the categories of Q-modules and of R-modules.*
(vii) *If $f : Q \longrightarrow R$ is a weak equivalence of commutative $\mathscr{D}$-ring spectra, then restriction and extension of scalars define a Quillen equivalence between the categories of Q-algebras and of R-algebras.*

In the language of [37], we shall prove that $\mathscr{D}\mathscr{S}$ satisfies the monoid and pushout-product axioms. We shall make repeated use of the following observation. Recall that, by Lemma 5.5, a q-cofibration is an h-cofibration.

Lemma 12.2. *If $i : X \longrightarrow Y$ is an h-cofibration of $\mathscr{D}$-spectra and Z is any $\mathscr{D}$-spectrum, then $i \wedge_S \mathrm{id} : X \wedge_S Z \longrightarrow Y \wedge_S Z$ is an h-cofibration.*

Proof. Smashing with Z preserves colimits and smash products with spaces and so preserves the relevant retraction. □

The following result is the heart of the proof of the monoid and pushout product axioms and thus of the proof of Theorem 12.1.

Proposition 12.3. *For any cofibrant $\mathscr{D}$-spectrum X, the functor $X \wedge_S (-)$ preserves π_*-isomorphisms and stable equivalences.*

Proof. Of course, when $\mathscr{D} = \mathscr{I}$ or $\mathscr{D} = \mathscr{W}$, π_*-isomorphisms are the same as stable equivalences. We shall prove the result when $X = F_n S^n$ shortly. Using the fact that $F_n A \cong (F_n S^0) \wedge A$ together with Theorems 7.4 and 8.12, we deduce first that the conclusion holds when $X = F_n S^0$, next that it holds when $X = F_n A$ for a finite CW-complex A, and then that it holds when X is any FI-cell complex. Passage to retracts gives the general case. We treat the case $X = F_n S^n$ separately for symmetric spectra and for orthogonal spectra and $\mathscr{W}$-spaces.

Symmetric spectra. Using Example 4.2, Lemma 1.8, and (22.2) to write out the relevant smash product, we find that, for $q \geq n$,

$$\begin{aligned} (F_n S^n \wedge_S Y)(q) &\cong \Sigma_{q+} \wedge_{\Sigma_{q-n}} (S^n \wedge Y(q-n)) \\ &\cong (\Sigma_n/\Sigma_{q-n})_+ \wedge (S^q \wedge Y(q-n)). \end{aligned}$$

The second isomorphism is obtained by writing the free right Σ_{q-n}-set Σ_q as a disjoint union of orbits Σ_{q-n} and is only an isomorphism of spaces, not of Σ_q-spaces.

Even this much depends heavily on the fact that the Σ_q are discrete. We choose orbit representatives one q at a time, using the chosen representatives for copies of Σ_{q-n} in Σ_q as representatives for some of the copies of Σ_{q+1-n} in Σ_{q+1}. We find by passage to colimits over q that $\pi_*(F_nS^n \wedge_S Y)$ is naturally the sum of countably many copies of $\pi_*(Y)$. Thus the functor $F_nS^n \wedge_S Y$ preserves π_*-isomorphisms. To show that it preserves stable equivalences, we now see by application of functorial cofibrant approximation in the level model structure that the conclusion holds for stable equivalences in general if it holds for stable equivalences between cofibrant symmetric spectra. For cofibrant Y and any E, we have

$$[F_nS^n \wedge_S Y, E] \cong [Y, F_S(F_nS^n, E)],$$

naturally in Y. Since $F_S(F_nS^n, E) = R^nE$ is a symmetric Ω-spectrum by the proof of Proposition 8.8, the conclusion follows.

Orthogonal spectra and $\mathscr{W}$-spaces. Let Y be either an orthogonal spectrum or a $\mathscr{W}$-space. Here we give a proof that does not require an explicit description of the smash product $F_nS^n \wedge_S Y$. By Theorem 7.4(vi) and Lemma 12.2, it suffices to prove that $\pi_*(F_nS^n \wedge_S Y) = 0$ if $\pi_*(Y) = 0$. Let $\gamma_n : F_nS^n \longrightarrow F_0S^0 = S$ be the canonical π_*-isomorphism (as in the proof of Lemma 10.3). Let $\alpha \in \pi_q(F_nS^n \wedge_S Y)$ and choose a representative map $f\colon F_rS^{q+r} \to F_nS^n \wedge_S Y$. Since $\pi_*(Y) = 0$, we can choose r large enough that the composite

$$(\gamma_n \wedge_S \mathrm{id}) \circ f\colon F_rS^{q+r} \longrightarrow F_nS^n \wedge_S Y \longrightarrow S \wedge_S Y \cong Y$$

is null homotopic. Let $g = (\gamma_n \wedge_S \mathrm{id}) \circ f$ and let g' be the map

$$\mathrm{id} \wedge_S g\colon F_{n+r}S^{n+q+r} \cong F_nS^n \wedge_S F_rS^{q+r} \longrightarrow F_nS^n \wedge_S Y$$

obtained from g by smashing with F_nS^n. Then g' is also null homotopic. Now let f' be the composite

$$f \circ (\gamma_n \wedge_S \mathrm{id})\colon F_{n+r}S^{n+q+r} \cong F_nS^n \wedge_S F_rS^{q+r} \longrightarrow F_rS^{q+r} \longrightarrow F_nS^n \wedge_S Y.$$

Then f' also represents α. We show that $\alpha = 0$ by showing that the maps f' and g' are homotopic. We can rewrite f' and g' as the composites of the map

$$\mathrm{id} \wedge_S f\colon F_{n+r}S^{n+q+r} \cong F_nS^n \wedge_S F_rS^{q+r} \longrightarrow F_nS^n \wedge_S F_nS^n \wedge_S Y$$

and the maps $F_nS^n \wedge_S F_nS^n \wedge_S Y \longrightarrow F_nS^n \wedge_S Y$ obtained by applying γ_n to the first or second factor F_nS^n. Thus, it suffices to show that the maps $\mathrm{id} \wedge \gamma_n$ and $\gamma_n \wedge \mathrm{id}$ from $F_nS^n \wedge_S F_nS^n$ to F_nS^n are homotopic. So far the argument has been identical for orthogonal spectra and for $\mathscr{W}$-spaces. We prove this last step for orthogonal spectra. The conclusion for $\mathscr{W}$-spaces follows upon application of the functor $\mathbb{P}$. For orthogonal spectra, the adjoints

$$S^{2n} \longrightarrow (F_nS^n)_{2n} = O(2n)_+ \wedge_{O(n)} S^{2n} \cong O(2n)/O(n)_+ \wedge S^{2n}$$

of the two maps send s to $1 \wedge s$ and to $\tau \wedge s$, where $\tau \in O(2n)$ is the evident transposition on $\mathbb{R}^n \times \mathbb{R}^n$. These maps are homotopic since $O(2n)/O(n)$ is connected. □

We shall later need the following consequence of this result.

Corollary 12.4. *When $\mathscr{D} = \mathscr{I}$ or $\mathscr{D} = \mathscr{W}$, $\gamma_k \wedge_S \mathrm{id} : F_kS^k \wedge_S Y \longrightarrow Y$ is a π_*-isomorphism for any $\mathscr{D}$-spectrum Y.*

Proof. Let $q : X \longrightarrow Y$ be a π_*-isomorphism, where X is cofibrant. By Proposition 12.3, $\gamma_k \wedge_S \mathrm{id}_X$ and $\mathrm{id} \wedge_S q : F_kS^k \wedge_S X \longrightarrow F_kS^k \wedge_S Y$ are π_*-isomorphisms. Since $q \circ (\gamma_k \wedge_S \mathrm{id}_X) = (\gamma_k \wedge_S \mathrm{id}_Y) \circ (\mathrm{id} \wedge_S q)$, this gives the conclusion. □

Proposition 12.5 (Monoid axiom)**.** *For any acyclic q-cofibration $i : A \longrightarrow X$ of $\mathscr{D}$-spectra and any $\mathscr{D}$-spectrum Y, the map $i \wedge_S \mathrm{id} : A \wedge_S Y \longrightarrow X \wedge_S Y$ is a stable equivalence and an h-cofibration. Moreover, cobase changes and sequential colimits of such maps are also weak equivalences and h-cofibrations.*

Proof. Let $Z = X/A$ and note that Z is homotopy equivalent to the cofiber Ci. Then Z is an acyclic cofibrant $\mathscr{D}$-space. Since the functor $- \wedge_S Y$ preserves cofiber sequences, Theorem 8.12(vi) implies that it suffices to prove that $Z \wedge_S Y$ is acyclic. Let $j : Y' \to Y$ be a cofibrant approximation of Y. By Proposition 12.3, $\mathrm{id}_Z \wedge_S j$ is a stable equivalence. Thus, we may assume that Y as well as Z is cofibrant. Here Proposition 12.3 gives the conclusion since $* \longrightarrow Z$ is a stable equivalence and $* \wedge_S Y = *$. The last statement holds since cobase changes and sequential colimits of maps that are h-cofibrations and stable equivalences are h-cofibrations and stable equivalences, by Theorem 8.12. □

For maps $i : X \longrightarrow Y$ and $j : W \longrightarrow Z$ of $\mathscr{D}$-spectra, we have the map

$$i \Box j : (Y \wedge_S W) \cup_{X \wedge_S W} (X \wedge_S Z) \longrightarrow Y \wedge_S Z$$

of (5.17). By Lemma 6.6, if i and j are q-cofibrations, then so is $i \Box j$.

Proposition 12.6 (Pushout-product axiom)**.** *If $i : X \longrightarrow Y$ and $j : W \longrightarrow Z$ are q-cofibrations of $\mathscr{D}$-spectra and i is a stable equivalence, then the q-cofibration $i \Box j$ is a stable equivalence.*

Proof. By the monoid axiom, $i \wedge_S \mathrm{id} : X \wedge_S Z \longrightarrow Y \wedge_S Z$ is a stable equivalence for any Z. By Theorem 8.12(iii) (or 7.4(iii)), any cobase change of an h-cofibration that is a stable equivalence is a stable equivalence. It is immediate from the definition of $i \Box j$ that its composite with the cobase change of $i \wedge_S \mathrm{id}_W$ along $\mathrm{id}_X \wedge_S j$ is $i \wedge_S \mathrm{id}_Z$. Therefore $i \Box j$ is a stable equivalence. □

Observe that the unit S of the smash product of $\mathscr{D}$-spectra is cofibrant.

Proof of Theorem 12.1. Most of this is given by the general theory of Schwede and Shipley [37], and we focus on (i) and (iv). For these model structures, we are thinking of a variant of the theory of [37] that is based on Proposition 5.13. The generating q-cofibrations and acyclic q-cofibrations are obtained by applying the free R-module functor $R \wedge_S (-)$ or the free R-algebra functor $\mathbb{T}$ to the generating q-cofibrations and acyclic q-cofibrations of $\mathscr{D}$-spectra. Here $\mathbb{T}X = \bigvee_{i \geq 0} R \wedge_S X^{(i)}$. The defining adjunctions for the functors $R \wedge_S (-)$ and $\mathbb{T}$ imply that, if A is a compact $\mathscr{D}$-spectrum, then $R \wedge_S A$ is a compact R-module and $\mathbb{T}A$ is a compact R-algebra, in the sense of Definition 5.6.

The pushout-product and monoid axioms allow verification of (i) and (ii) of Proposition 5.13. That is, the sets of generating q-cofibrations and generating acyclic q-cofibrations satisfy the Cofibration Hypothesis 5.3, and the relative cell complexes generated by the latter are stable equivalences. As in Lemma 5.5, a relative $(R \wedge_S FI)$-cell or $(R \wedge_S K)$-cell R-module is an h-cofibration of R-modules and thus an h-cofibration of $\mathscr{D}$-spectra. Arguing as in [11, VII.3.9 and 3.10], with a slight elaboration to deal with maps of K not in FJ, the same is true for relative $\mathbb{T}FI$-cell or $\mathbb{T}K$-cell R-algebras. This gives the Cofibration Hypothesis 5.3.

The monoid axiom implies directly that a relative $R \wedge_S K$-cell complex is a stable equivalence. This gives the model structure in (i), and it is proper and topological by the same proofs as for the stable model structure on $\mathscr{D}\mathscr{S}$. The second statement of (i) holds since the right adjoint $F_S(R,-)$ to the forgetful functor preserves acyclic q-cofibrations by the adjoint form of the pushout-product axiom.

The proof that relative $\mathbb{T}K$-cell complexes are stable equivalences and the proof of (v) require the combinatorial analysis of pushouts (= amalgamated free products) in the category of R-algebras that is given in [11, VII.6.1] and [37, 6.2]. That the model structure is right proper and topological is inherited from $\mathscr{D}\mathscr{S}$. The role of iterated smash products in the specification of $\mathbb{T}$ makes it clear that this category cannot be expected to be left proper.

For (vi) and (vii), the following generalization of Proposition 12.3 verifies a hypothesis that allows us to quote the general results of [37]. □

Proposition 12.7. *For a cofibrant right R-module M, the functor $M \wedge_R N$ of N preserves π_*-isomorphisms and stable equivalences.*

Proof. It suffices to prove the result when M is an $(FI \wedge_S R)$-cell R-module. As in the proof of Proposition 12.3, we see by induction up the cell filtration that it suffices to prove the result when $M = F_n A \wedge_S R$ for a based CW-complex A. Then $M \wedge_R N \cong F_n A \wedge_S N$ and the result holds by Proposition 12.3. □

13. Comparisons of ring and module spectra

We here prove Theorems 0.4 and 0.5 and Corollary 0.6, which compare various categories of ring and module diagram spectra. We treat the comparisons between structured symmetric and orthogonal spectra; the comparisons between structured orthogonal spectra and $\mathscr{W}$-spaces are proven in exactly the same way.

Proof of Theorem 0.4. The functors $\mathbb{P}$ and $\mathbb{U}$ between symmetric and orthogonal spectra preserve ring spectra, and they restrict to an adjoint pair relating the categories of symmetric and orthogonal ring spectra. This is a Quillen adjoint pair since, in both cases, the forgetful functor to $\mathscr{D}$-spaces creates the weak equivalences and q-fibrations. Since the underlying symmetric spectrum of a cofibrant symmetric ring spectrum is cofibrant, by Theorem 12.1(v), the restricted pair is a Quillen equivalence by Lemma A.2(iii). □

Proof of Theorem 0.5. For a symmetric ring spectrum R, $(\mathbb{P},\mathbb{U})$ induces a Quillen adjoint pair between the categories of R-modules and $\mathbb{P}R$-modules. If R is cofibrant, then R and all cofibrant R-modules are cofibrant as symmetric spectra, by Theorem 12.1, and the restricted pair is a Quillen equivalence by Lemma A.2(iii). □

Proof of Corollary 0.6. For an orthogonal ring spectrum R, the functor $\mathbb{U}$ from R-modules to $\mathbb{U}R$-modules has left adjoint the functor $\mathbb{P}(-) \wedge_{\mathbb{P}\mathbb{U}R} R$. Again, this is a Quillen adjoint pair. Let $\gamma : Q \longrightarrow \mathbb{U}R$ be a cofibrant approximation. Since $\mathbb{U}$ creates stable equivalences, the adjoint $\tilde{\gamma} : \mathbb{P}Q \longrightarrow R$ is a stable equivalence. We have the following commutative diagram of right adjoints in Quillen adjoint pairs relating categories of modules:

(13.1)
$$\begin{array}{ccc} \mathscr{M}_{\mathbb{P}Q} & \xleftarrow{\tilde{\gamma}^*} & \mathscr{M}_R \\ {\scriptstyle \mathbb{U}}\downarrow & & \downarrow{\scriptstyle \mathbb{U}} \\ \mathscr{M}_Q & \xleftarrow[\gamma^*]{} & \mathscr{M}_{\mathbb{U}R} \end{array}$$

The left arrow $\mathbb{U}$ and the arrows induced by the stable equivalences γ and $\tilde{\gamma}$ are the right adjoints of Quillen equivalences, by Theorems 0.5 and 12.1(vi), hence so is the right arrow $\mathbb{U}$. □

14. The positive stable model structure on $\mathscr{D}$-spectra

We return to the context of §8, letting $\mathscr{D}$ be any of $\mathscr{N}$, Σ, $\mathscr{I}$, or $\mathscr{W}$. In the last three cases, we seek a model category of commutative $\mathscr{D}$-ring spectra. However, because the sphere $\mathscr{D}$-spectrum is cofibrant, the stable model structure cannot create a model structure on the category of commutative $\mathscr{D}$-ring spectra. A fibrant approximation of S as a commutative $\mathscr{D}$-ring spectrum would be an Ω-spectrum with zeroth space a commutative topological monoid weakly equivalent to QS^0. That would imply that QS^0 is weakly equivalent to a product of Eilenberg-Mac Lane spaces. This is a manifestation of Lewis's observation [19] that one cannot have an ideal category of spectra that is ideally related to the category of spaces.

Thus, following an idea of Jeff Smith, we modify the stable model structure in such a way that $S_{\mathscr{D}}$ is no longer cofibrant. This is very easy to do. Basically, we just modify the arguments of §§6, 8, 9 by starting with the level model structure relative to $\mathscr{N} - \{0\}$ rather than relative to $\mathscr{N}$.

We define *positive* classes of maps from the classes of maps specified in Definition 6.1 by restricting to levels $n > 0$ in (i) and (ii) there. We obtain further positive classes defined in terms of these positive classes exactly as in Definition 9.1. We obtain sets of maps F^+I, F^+J, and K^+ by omitting the maps with $n = 0$ from the sets FI, FJ, and K specified in Definitions 6.2 and 9.3. We say that a $\mathscr{D}$-spectrum X is a *positive $\mathscr{D}$-Ω-spectrum* if the structure maps $\tilde{\sigma} : X_n \longrightarrow \Omega X_{n+1}$ of its underlying prespectrum are weak equivalences for $n > 0$. With these definitions, we have the following results.

Theorem 14.1. *The category $\mathscr{D}\mathscr{S}$ is a compactly generated proper topological model category with respect to the positive level equivalences, positive level fibrations, and positive level q-cofibrations. The sets F^+I and F^+J are the generating sets of positive q-cofibrations and positive level acyclic q-cofibrations. The positive q-cofibrations are those q-cofibrations that are homeomorphisms at level* 0.

Proof. Since the model structure we have specified is the level model structure relative to $\mathscr{N} - \{0\}$, only the last statement is not part of the relative version of Theorem 3.4. The last statement follows from the fact that a map is a positive q-cofibration if and only if it is a retract of a relative F^+I-cell complex and the observation that a relative FI-cell complex is a homeomorphism at level 0 if and only if no standard cells $F_0 i$ occur in its construction. □

Theorem 14.2. *The category $\mathscr{D}\mathscr{S}$ is a compactly generated proper topological model category with respect to the stable equivalences, positive q-fibrations, and positive q-cofibrations. The sets F^+I and K^+ are the generating positive q-cofibrations and generating positive acyclic q-cofibrations. The positive fibrant $\mathscr{D}$-spectra are the positive $\mathscr{D}$-Ω-spectra. When $\mathscr{D} = \Sigma$, $\mathscr{I}$, or $\mathscr{W}$, the pushout-product and monoid axioms are satisfied.*

For the proof, we need a characterization of the stable equivalences in terms of the positive level model structure. Let $[X, Y]^+$ denote the set of maps $X \longrightarrow Y$ in the homotopy category associated to the positive level model structure.

Lemma 14.3. *For $\mathscr{D}$-Ω-spectra E, $[X,E]^+$ is naturally isomorphic to $[X,E]$.*

Proof. Let $q : X' \longrightarrow X$ be a cofibrant approximation to X in the positive level model structure. Then $q^* : [X,E]^+ \longrightarrow [X',E]^+$ is an isomorphism. Since q is a π_*-isomorphism and thus a stable equivalence by Proposition 8.8, $q^* : [X,E] \longrightarrow [X',E]$ is also an isomorphism. However, since X' is cofibrant in both model structures, $[X',E] = \pi(X',E) = [X',E]^+$. □

Proposition 14.4. *A map $f : X \longrightarrow Y$ is a stable equivalence if and only if $f^* : [Y,E]^+ \longrightarrow [X,E]^+$ is a bijection for all positive $\mathscr{D}$-Ω-spectra E.*

Proof. First, let f be a stable equivalence and E be a positive $\mathscr{D}$-Ω-spectrum. Construct RE as in the proof of Proposition 8.8. Then RE is a $\mathscr{D}$-Ω-spectrum and the natural map $E \longrightarrow RE$ is a positive level equivalence. By application of Lemma 14.3 to RE, $f^* : [Y,E]^+ \longrightarrow [X,E]^+$ is a bijection since $f^* : [Y,RE] \longrightarrow [X,RE]$ is a bijection. Since a $\mathscr{D}$-Ω-spectrum is a positive $\mathscr{D}$-Ω-spectrum, the converse implication is immediate from Lemma 14.3. □

From here, Theorem 14.2 is proven by the same arguments as for the stable model structure, with everything restricted to positive levels. Its last statement implies the following analogue of Theorem 12.1 for the positive stable model structure.

Theorem 14.5. *Parts (i), (iii), (iv), (vi), and (vii) of Theorem 12.1 are also valid for the positive stable model structure on $\mathscr{D}\mathscr{S}$ for $\mathscr{D} = \Sigma$, $\mathscr{I}$, or $\mathscr{W}$.*

Parts (ii) and (v) of Theorem 12.1, concerning q-cofibrations, are not valid here since S is not cofibrant. However, since we have both model structures on hand, this is not a serious defect. For example, parts (vi) and (vii) in the previous theorem no longer follow directly from [37]. Rather, they follow from parts (vi) and (vii) of Theorem 12.1 and the following comparison result, whose proof is immediate.

Proposition 14.6. *The identity functor from $\mathscr{D}\mathscr{S}$ with its positive stable model structure to $\mathscr{D}\mathscr{S}$ with its stable model structure is the left adjoint of a Quillen equivalence. It restricts to a Quillen equivalence on the category of $\mathscr{D}$-ring spectra, on the category of left modules over a $\mathscr{D}$-ring spectrum, and on the category of algebras over a commutative $\mathscr{D}$-ring spectrum.*

Remark 14.7. The proofs in the previous section show that Theorems 0.1, 0.4, and 0.5 remain valid when reinterpreted in terms of the positive stable model structures. The essential point is that, since these structures have fewer cofibrant objects, verification of the hypothesis of Lemma A.2(iii) for the stable model structures is more than enough to verify the hypothesis for the positive stable model structures.

15. The model structure on commutative $\mathscr{D}$-ring spectra

We prove the following two theorems. Let $\mathscr{D} = \Sigma$ or $\mathscr{D} = \mathscr{I}$ throughout this section. To clarify algebraic ideas, we refer to $\mathscr{D}$-ring spectra as "S-algebras". Let $\mathbb{C}$ be the monad on $\mathscr{D}$-spectra that defines commutative S-algebras. Thus $\mathbb{C}X = \bigvee_{i\geq 0} X^{(i)}/\Sigma_i$, where $X^{(i)}$ denotes the ith smash power, with $X^{(0)} = S$.

Theorem 15.1. *The category of commutative S-algebras is a compactly generated proper topological model category with q-fibrations and weak equivalences created in the positive stable model category of $\mathscr{D}$-spectra. The sets $\mathbb{C}F^+I$ and $\mathbb{C}K^+$ are the generating sets of q-cofibrations and acyclic q-cofibrations.*

Theorem 15.2. *Let R be a commutative S-algebra.*

(i) *The category of commutative R-algebras is a compactly generated proper topological model category whose weak equivalences, q-fibrations, and q-cofibrations are the maps whose underlying maps of commutative S-algebras are weak equivalences, q-cofibrations, or q-cofibrations.*

(ii) *If $f : Q \longrightarrow R$ is a weak equivalence of commutative S-algebras, then restriction and extension of scalars define a Quillen equivalence between the categories of commutative Q-algebras and commutative R-algebras.*

Exactly as in algebra, the category of commutative R-algebras is isomorphic to the category of commutative S-algebras under R. Therefore the model structure in part (i) is immediate from the model structure in the category of commutative S-algebras [9, 3.10]. The sets $R \wedge_S \mathbb{C}F^+I$ and $R \wedge_S \mathbb{C}K^+$ are the generating sets of q-cofibrations and acyclic q-cofibrations. As in algebra, the smash product $\wedge_S$ is the coproduct in the category of commutative S-algebras. Thus the maps in these sets are q-cofibrations of commutative S-algebras because they are coproducts of q-cofibrations of commutative S-algebras with the identity map of R. In both theorems, evident adjunctions show that the domains of the maps in our generating sets are compact. By Propositions 5.1, 5.2, and 5.13, the following two lemmas give the model structure in Theorem 15.1.

Lemma 15.3. *The sets $\mathbb{C}F^+I$ and $\mathbb{C}K^+$ satisfy the Cofibration Hypothesis 5.3.*

Lemma 15.3 directly implies that the sets $R \wedge_S \mathbb{C}F^+I$ and $R \wedge_S \mathbb{C}K^+$ satisfy the Cofibration Hypothesis in the category of commutative R-algebras. Indeed, the right vertical arrow in a pushout diagram

$$\begin{array}{ccc} R \wedge_S \mathbb{C}X & \longrightarrow & A \\ \downarrow & & \downarrow \\ R \wedge_S \mathbb{C}Y & \longrightarrow & B \end{array}$$

of commutative R-algebras can be identified with the right vertical arrow in the pushout diagram

$$\begin{array}{ccc} \mathbb{C}X & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathbb{C}Y & \longrightarrow & B \end{array}$$

of commutative S-algebras. The point is that, as for commutative monoids in any symmetric monoidal category, the pushout of a diagram $R' \longleftarrow R \longrightarrow R''$ of commutative R-algebras is the smash product $R' \wedge_R R''$.

Lemma 15.4. *Every relative $\mathbb{C}K^+$-cell complex is a stable equivalence.*

We single out for emphasis the key step of the proof of Lemma 15.4. It is the analogue for symmetric and orthogonal spectra of [11, III.5.1] for the S-modules of Elmendorf, Kriz, Mandell, and May. We do not know whether or not the analogue for $\mathscr{W}$-spaces or $\mathscr{F}$-spaces holds, and it is for this reason that we do not have results for commutative rings in those cases. It is an insight of Smith that restriction to positive cofibrant symmetric spectra suffices to obtain the following conclusion.

Lemma 15.5. *Let K be a based CW complex, X be a $\mathscr{D}$-spectrum, and $n > 0$. Then the quotient map*

$$q : (E\Sigma_{i+} \wedge_{\Sigma_i} (F_nK)^{(i)}) \wedge_S X \longrightarrow ((F_nK)^{(i)}/\Sigma_i) \wedge_S X$$

is a level homotopy equivalence. For any positive cofibrant $\mathscr{D}$-spectrum X,

$$q : E\Sigma_{i+} \wedge_{\Sigma_i} X^{(i)} \longrightarrow X^{(i)}/\Sigma_i$$

is a π_*-isomorphism.

Proof. We give the details for $\mathscr{D} = \mathscr{I}$. The result for $\mathscr{D} = \Sigma$ is proven by the same argument, but with orthogonal groups replaced by symmetric groups. By Example 4.4, Lemma 1.8, and inspection of coequalizers,

$$((F_nK)^{(i)} \wedge_S X)(q) \cong O(q)_+ \wedge_{O(q-ni)} (K^{(i)} \wedge X(q-ni)).$$

The action of $\sigma \in \Sigma_i$ is to permute the factors in $K^{(i)}$ and to act through $\sigma \oplus \mathrm{id}_{q-ni}$ on $O(q)$, where $\sigma \in O(ni)$ permutes the summands of $\mathbb{R}^{ni} = (\mathbb{R}^n)^i$. Since Σ_i acts on $O(q)$ as a subgroup of $O(ni)$, the action commutes with the action of $O(q-ni)$. Therefore, passing to orbits over Σ_i,

$$((F_nK)^{(i)}/\Sigma_i \wedge_S X)(q) \cong O(q)_+ \wedge_{\Sigma_i \times O(q-ni)} (K^{(i)} \wedge X(q-ni)).$$

Similarly,

$$((E\Sigma_{i+} \wedge_{\Sigma_i} (F_nK)^{(i)}) \wedge_S X)(q) \cong (E\Sigma_i \times O(q))_+ \wedge_{\Sigma_i \times O(q-ni)} (K^{(i)} \wedge X(q-ni)).$$

The quotient map $E\Sigma_i \times O(q) \longrightarrow O(q)$ is a $(\Sigma_i \times O(q-ni))$-equivariant homotopy equivalence since $O(q)$ is a free $(\Sigma_i \times O(q-ni))$-space that can be triangulated as a finite $(\Sigma_i \times O(q-ni))$-CW complex. The first statement follows. For the second statement, we may assume that X is an F^+I-cell spectrum, and the proof then is the same induction up the cellular filtration as in the proof of [11, III.5.1]. □

The first statement has the following consequence.

Lemma 15.6. *Let K be a based CW complex and let $n > 0$. Then the functor $\mathbb{C}F_nK \wedge_S (-)$ of $\mathscr{D}$-spectra preserves stable equivalences.*

Proof. By induction up the cellular filtration of $E\Sigma_{i+}$, the successive subquotients of which are wedges of copies of $\Sigma_{i+} \wedge S^n$, and use of results in §8, the functor $E\Sigma_{i+} \wedge_{\Sigma_i} (-)$ preserves stable equivalences. □

Similarly, the second statement implies the following result.

Lemma 15.7. *The functor $\mathbb{C}$ preserves stable equivalences between positive cofibrant $\mathscr{D}$-spectra. In particular, each map in $\mathbb{C}K^+$ is a stable equivalence.*

From here, the proofs of Lemmas 15.3 and 15.4 are analogous to the proofs of corresponding results about S-modules in [11]. We shall not give details of arguments that are essentially identical. For the Cofibration Hypothesis 5.3, we record the following result, whose proof is the same as in [11, XII.2.3].

Lemma 15.8. *The functor $\mathbb{C} : \mathscr{D}\mathscr{S} \longrightarrow \mathscr{D}\mathscr{S}$ preserves h-cofibrations.*

Since the functor $\mathbb{C}$ commutes with colimits, Cofibration Hypothesis 5.3(i) for the set $\mathbb{C}F^+I$ is equivalent to the following lemma.

Lemma 15.9. *Let $X \longrightarrow Y$ be a wedge of maps in F^+I and let $f : \mathbb{C}X \longrightarrow R$ be a map of commutative R-algebras. Then the cobase change $j : R \longrightarrow R \wedge_{\mathbb{C}X} \mathbb{C}Y$ is an h-cofibration.*

Proof. The proof is similar to that of the analogous result for commutative S-algebras in [11, VII§3]. We use the geometric realization of simplicial $\mathscr{D}$-spectra. This is constructed levelwise and has properties just like the geometric realization of simplicial spaces and of simplicial spectra; see [26, §11] and [11, X§1]. We also use the two-sided bar construction; see [26, §9] and [11, XII].

We first give a convenient, although rather baroque, model for the inclusion $i : S^{q-1}_+ \longrightarrow D^q_+$. Think of the unit interval I as the geometric realization of the standard simplicial 1-simplex $\Delta[1]$. For any space A, $(A \times I)_+ \cong A_+ \wedge I_+$ is homeomorphic to the geometric realization of the simplicial space $A_+ \wedge \Delta[1]_+$. Since $\Delta[1]$ is discrete, the space of q-simplices of $A_+ \wedge \Delta[1]_+$ is the wedge of one copy of A_+ for each simplex of $\Delta[1]$. An explicit examination of the faces and degeneracies of $\Delta[1]$ [25, p.14] shows that $A_+ \wedge \Delta[1]_+$ can be identified with the simplicial bar construction $B_*(A_+, A_+, A_+)$, whose space of q-simplices is the wedge of $q+2$ copies of A_+. The faces and degeneracies are given by successive applications of the folding map $\nabla : A_+ \vee A_+ \longrightarrow A_+$ and inclusions of wedge summands, and all q-simplices with $q > 1$ are degenerate. The inclusion of the zeroth and last wedge summands A_+ in each simplicial degree induce the inclusions i_0 and i_1 of A_+ in $A_+ \wedge I_+$ on passage to realization. Write $B(-)$ for the geometric realization of simplicial bar constructions $B_*(-)$ and let CA be the unreduced cone on A. The quotient map $A_+ \wedge I_+ \longrightarrow (CA)_+$ is isomorphic to the map

$$B(A_+, A_+, A_+) \longrightarrow B(A_+, A_+, S^0)$$

induced by the evident map $A_+ \longrightarrow pt_+ = S^0$, and the inclusion $i_0 : A_+ \longrightarrow (CA)_+$ is isomorphic to the map $\iota : A_+ \longrightarrow B(A_+, A_+, S^0)$ induced from the inclusion of A_+ in the space of zero simplices. Taking $A = S^{q-1}$ and identifying $i : S^{q-1}_+ \longrightarrow D^q_+$ with $i_0 : S^{q-1}_+ \longrightarrow (CS^{q-1})_+$, we can identify i with $\iota : S^{q-1}_+ \longrightarrow B(S^{q-1}_+, S^{q-1}_+, S^0)$.

The functor F_n commutes with colimits and with smash products with based spaces, hence commutes with geometric realization and the bar construction. We can apply wedges to the construction to obtain a similar description of a wedge of a set of standard cells. Explicitly, if $X = \vee_i F_{n_i} S^{q_i - 1}_+$ and $Y = \vee_i F_{n_i} D^{q_i}_+$, then $Y \cong B(X, X, T)$ under X, where $T = \vee_i F_{n_i} S^0$. Here $B(X, X, T)$ is the geometric realization of the evident simplicial $\mathscr{D}$-spectrum whose $\mathscr{D}$-spectrum of q-simplices is the wedge of $q + 1$ copies of X and a copy of T.

By Proposition 5.1, the category of commutative S-algebras is tensored over the category of unbased spaces; an explicit construction of tensors is given in [11, VII.2.10]. The functor $\mathbb{C}$ from $\mathscr{D}$-spectra to commutative S-algebras commutes with colimits and converts smash products $X \wedge A_+$ to tensors $\mathbb{C}X \otimes A$, where X is a $\mathscr{D}$-spectrum and A is an unbased space. As is discussed in an analogous situation in [11, VII§3], it follows that $\mathbb{C}$ converts geometric realizations and bar constructions to similar constructions defined in terms of the category of simplicial commutative S-algebras. Exactly as in [11, VII.3.3], the geometric realization of a simplicial commutative S-algebra R_* can be computed by forgetting the ring structure on each R_q, taking the geometric realization as a simplicial $\mathscr{D}$-spectrum, and giving this geometric realization the structure of commutative S-algebra that it inherits from R_*. With the notation above, we have the identification

$$R \wedge_{\mathbb{C}X} \mathbb{C}Y \cong R \wedge_{\mathbb{C}X} B(\mathbb{C}X, \mathbb{C}X, \mathbb{C}T) \cong B(R, \mathbb{C}X, \mathbb{C}T) \tag{15.10}$$

under R. It follows as in [11, VII.3.9] that $j : R \longrightarrow R \wedge_{\mathbb{C}X} \mathbb{C}Y$ is an h-cofibration. In summary, the degeneracy operators of the simplicial $\mathscr{D}$-spectrum $B_*(R, \mathbb{C}X, \mathbb{C}T)$

are inclusions of wedge summands, hence $B_*(R, \mathbb{C}X, \mathbb{C}T)$ is *proper*, in the sense that its degenerate q-simplices map by an h-cofibration into its q-simplices; compare [11, X.2.2]. This implies that the map from the $\mathscr{D}$-spectrum of zero simplices into the realization is an h-cofibration, and the map from R into the $\mathscr{D}$-spectrum $R \wedge_S \mathbb{C}T$ is the inclusion of a wedge summand and thus also an h-cofibration. □

Since the maps in $\mathbb{C}K^+$ are relative $\mathbb{C}F^+I$-cell complexes, the previous lemma and Lemma 1.2 imply Cofibration Hypothesis 5.3(i) for $\mathbb{C}K^+$. Cofibration Hypothesis 5.3(ii) for both $\mathbb{C}F^+I$ and $\mathbb{C}K^+$ is implied by the following analogue of [11, VII.3.10], which admits the same easy proof.

Lemma 15.11. *Let $\{R_i \longrightarrow R_{i+1}\}$ be a sequence of maps of commutative S-algebras that are h-cofibrations of $\mathscr{D}$-spectra. Then the underlying $\mathscr{D}$-spectrum of the colimit of the sequence computed in the category of commutative S-algebras is the colimit of the sequence computed in the category of $\mathscr{D}$-spectra.*

Using Lemma 15.6, the proof of Lemma 15.9 leads to the following analogue of the monoid axiom.

Proposition 15.12. *Let $i : R \longrightarrow R'$ be a q-cofibration of commutative S-algebras. Then the functor $(-)\wedge_R R'$ on commutative R-algebras preserves stable equivalences.*

Proof. We may assume that i is a relative $\mathbb{C}F^+I$-cell complex. First let i be the map $\mathbb{C}X \longrightarrow \mathbb{C}Y$ obtained by applying $\mathbb{C}$ to a wedge $X \longrightarrow Y$ of maps in F^+I. By (15.10), the functor $(-) \wedge_{\mathbb{C}X} \mathbb{C}Y$ is isomorphic to the bar construction $B(-, \mathbb{C}X, \mathbb{C}T)$. In each simplicial degree, the functor $B_q(-, \mathbb{C}X, \mathbb{C}T)$ preserves stable equivalences by inductive use of Lemma 15.6. By the $\mathscr{D}$-spectrum analogue of [11, X.2.4], it follows that the functor $B(-, \mathbb{C}X, \mathbb{C}T)$ preserves stable equivalences. Given a pushout diagram of commutative $\mathscr{D}$-ring spectra

$$\begin{array}{ccc} \mathbb{C}X & \longrightarrow & R \\ \downarrow & & \downarrow \\ \mathbb{C}Y & \longrightarrow & R', \end{array}$$

we have $R' \cong R \wedge_{\mathbb{C}X} \mathbb{C}Y$ and thus $(-) \wedge_R R' \cong (-) \wedge_{\mathbb{C}X} \mathbb{C}Y$. Therefore the conclusion holds in this case, and the general case follows by passage to colimits, using Lemma 15.11. □

Proof of Lemma 15.4. By passage to pushouts and colimits, it suffices to prove that if $i : X \longrightarrow Y$ is a wedge of maps in K^+ and $f : \mathbb{C}X \longrightarrow R$ is a map of commutative S-algebras, then the cobase change $j : R \longrightarrow R \wedge_{\mathbb{C}X} \mathbb{C}Y$ is a stable equivalence. Applying the small object argument, factor f as the composite of a relative $\mathbb{C}F^+I$-cell complex $f' : \mathbb{C}X \longrightarrow R'$ and a map $p : R' \longrightarrow R$ that satisfies the RLP with respect to $\mathbb{C}F^+I$. By adjunction, p regarded as a map of $\mathscr{D}$-spectra satisfies the RLP with respect to F^+I. Thus p is an acyclic positive q-fibration of $\mathscr{D}$-spectra. Consider the commutative diagram

$$\begin{array}{ccc} R' & \xrightarrow{j'} & R' \wedge_{\mathbb{C}X} \mathbb{C}Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p\wedge \mathrm{id}} \\ R & \xrightarrow[j]{} & R \wedge_{\mathbb{C}X} \mathbb{C}Y. \end{array}$$

Since p is a stable equivalence, $p \wedge \mathrm{id}$ is a stable equivalence by Proposition 15.12. Using $R' \cong R' \wedge_{\mathbb{C}X} \mathbb{C}X$, Proposition 15.12 also gives that the cobase change j' is a stable equivalence. Therefore j is a stable equivalence. □

Formal arguments show that the model structures in Theorems 15.1 and 15.2 are right proper and topological. Since the pushout of a diagram $A' \leftarrow A \longrightarrow A''$ of commutative R-algebras is $A' \wedge_A A''$ and a q-cofibration of commutative R-alegbras is a q-cofibration of commutative S-algebras, Proposition 15.12 implies that the category of commutative R-algebras is left proper. In turn, via Lemma A.2, this implies Theorem 15.2(ii).

16. Comparisons of modules, algebras, and commutative algebras

We prove Theorems 0.7 and 0.8 and Corollary 0.9 here.

Proof of Theorem 0.7. The functors $\mathbb{P} : \Sigma\mathscr{S} \longrightarrow \mathscr{I}\mathscr{S}$ and $\mathbb{U} : \mathscr{I}\mathscr{S} \longrightarrow \Sigma\mathscr{S}$ restrict to an adjoint pair between the category of commutative symmetric ring spectra and the category of commutative orthogonal ring spectra. We must prove that $(\mathbb{P}, \mathbb{U})$ is a Quillen equivalence. Since weak equivalences and q-fibrations of commutative ring spectra are created in the positive stable model categories of underlying spectra, $\mathbb{U}$ creates weak equivalences and preserves q-fibrations. Thus we have a Quillen adjoint pair. By Lemma A.2, it suffices to prove that the unit map $\eta : R \longrightarrow \mathbb{U}\mathbb{P}R$ is a stable equivalence for every cofibrant commutative symmetric ring spectrum R.

We may assume that R is a $\mathbb{C}F^+I$-cell complex. We claim first that η is a stable equivalence when $R = \mathbb{C}X$ for a positive cofibrant symmetric spectrum X, and it suffices to prove that $\eta : X^{(i)}/\Sigma_i \longrightarrow \mathbb{U}\mathbb{P}(X^{(i)}/\Sigma_i)$ is a stable equivalence for $i \geq 1$. On the right, $\mathbb{P}(X^{(i)}/\Sigma_i) \cong (\mathbb{P}X)^{(i)}/\Sigma_i$, and $\mathbb{P}X$ is a positive cofibrant orthogonal spectrum. Applying the second statement of Lemma 15.5 to X and to $\mathbb{P}X$, a quick diagram chase shows that the claim holds if and only if

$$\eta : E\Sigma_{i+} \wedge_{\Sigma_i} X^{(i)} \longrightarrow \mathbb{U}\mathbb{P}(E\Sigma_{i+} \wedge_{\Sigma_i} X^{(i)})$$

is a stable equivalence. Using Lemma 10.3 and the fact that suspensions of $X^{(i)}$ are positive cofibrant, this holds by induction up the skeletal filtration of $E\Sigma_i$. By passage to colimits, the result for general R follows from the result for an $\mathbb{C}F^+I$-cell complex that is constructed in finitely many stages. We have proven the result when R requires only a single stage, and we assume the result when R is constructed in n stages. Thus suppose that R is constructed in $n+1$ stages. Then R is a pushout $R_n \wedge_{\mathbb{C}X} \mathbb{C}Y$, where R_n is constructed in n-stages and $X \longrightarrow Y$ is a wedge of maps in F^+I. By (15.10), $R \cong B(R_n, \mathbb{C}X, \mathbb{C}T)$. Since the simplicial bar construction is proper and since $\mathbb{U}$ and $\mathbb{P}$ commute with colimits and smash products with spaces and thus with geometric realization, the analogue of [11, X.2.4] shows that it suffices to prove that η is a stable equivalence on the $\mathscr{D}$-spectrum

$$R_n \wedge_S (\mathbb{C}X)^{(q)} \wedge \mathbb{C}T \cong R_n \wedge_S \mathbb{C}(X \vee \cdots \vee X \vee T)$$

of q-simplices for each q. By the definition of $\mathbb{C}F^+I$-cell complexes, we see that this smash product (= pushout) of commutative $\mathscr{D}$-ring spectra can be constructed in n-stages, hence the conclusion follows from the induction hypothesis. □

Proof of Theorem 0.8. Let R be a cofibrant commutative symmetric ring spectrum. Theorems 12.1 and 14.5 give the stable and positive stable model structures on the categories of R-modules and R-algebras and Theorem 15.2 gives the positive stable model structure on the category of commutative R-algebras. The pair $(\mathbb{P}, \mathbb{U})$ induces adjoint pairs between the categories of R-modules, R-algebras, and commutative R-algebras and the categories of $\mathbb{P}R$-modules, $\mathbb{P}R$-algebras, and commutative $\mathbb{P}R$-algebras. We must show that these pairs are Quillen equivalences. For the module and algebra case, the conclusion holds for both the stable model structures and the positive stable model structures. Since $\mathbb{U} : \mathscr{I}\mathscr{S} \longrightarrow \Sigma\mathscr{S}$ preserves (positive) q-fibrations and creates weak equivalences, the same is true of the induced forgetful functors. Thus $(\mathbb{P}, \mathbb{U})$ is a Quillen adjoint pair in all cases and, by Lemma A.2, we need only prove that the unit of the adjunction is a stable equivalence when applied to a cofibrant object. For modules and algebras, a cofibrant object in the positive stable model structure is also cofibrant in the stable model structure, so we need only consider the latter case.

Thus consider $\eta : X \longrightarrow \mathbb{U}\mathbb{P}X$. Theorem 0.7 gives that η is a stable equivalence when X is a cofibrant commutative symmetric ring spectrum, such as $X = R$. If X is a cofibrant commutative R-algebra, then the unit $R \longrightarrow X$ and therefore its composite with the unit $S \longrightarrow R$ are q-cofibrations of commutative symmetric ring spectra, so that X is a cofibrant commutative symmetric ring spectrum and η is a stable equivalence. If X is a cofibrant R-algebra, then X is also cofibrant as an R-module by Theorem 12.1(iii). Thus it remains to prove that η is a stable equivalence when X is a cofibrant R-module. Arguing as in the proof of Lemma 10.3, it suffices to prove this when $X = R \wedge_S F_n^{\Sigma} S^n$. We have a canonical π_*-isomorphism $\gamma_n : F_n^{\mathscr{I}} S^n \longrightarrow S$ of orthogonal spectra. Using the mapping cylinder construction, we can factor γ_n as the composite of an acyclic q-cofibration and a homotopy equivalence. Thus, by Proposition 12.5, γ_n induces a π_*-isomorphism

$$\mathbb{P}(R \wedge_S F_n^{\Sigma} S^n) \cong \mathbb{P}R \wedge_S F_n^{\mathscr{I}} S^n \longrightarrow \mathbb{P}R \wedge_S S \cong \mathbb{P}R.$$

Applying $\mathbb{U}$ and using a naturality diagram, we see that η is a stable equivalence when $X = R \wedge_S F_n^{\Sigma} S^n$ since η is a stable equivalence when $X = R$. □

Proof of Corollary 0.9. As in the proof of Corollary 0.6 in §12, this follows from Theorems 12.1, 14.5, 15.2, and 0.8. □

17. The absolute stable model structure on $\mathscr{W}$-spaces

The stable model structure on $\mathscr{W}$-spaces studied so far was based on the level model structure relative to $\mathscr{N}$. That is, the level equivalences and level fibrations of $\mathscr{W}$-spaces were only required to be weak equivalences or fibrations when evaluated at S^n for $n \geq 0$. The objects of $\mathscr{F} \subset \mathscr{W}$ are the discrete based spaces $\mathbf{n}^+ = \{0, 1, \cdots, n\}$, and these are not spheres. We need a stable model structure based on the absolute level model structure in order to make a comparison.

Definition 6.1 specifies the absolute level equivalences, absolute level fibrations, absolute level acyclic fibrations, absolute q-cofibrations, and absolute level acyclic q-cofibrations of $\mathscr{W}$-spaces. Replacing stable equivalences by π_*-isomorphisms in Definition 9.1, we define absolute acyclic q-cofibrations, absolute q-fibrations, and absolute acyclic q-fibrations in terms of these absolute level classes of maps.

For a finite based CW-complex A, let $F_A : \mathscr{T} \longrightarrow \mathscr{W}\mathscr{T}$ denote the left adjoint to evaluation at A. We restrict attention to objects A in a skeleton of $\mathscr{W}$. All of these functors F_A are used in Definition 6.2, which specifies the sets FI and

FJ of generating absolute q-cofibrations and generating absolute level acyclic q-cofibrations of the absolute level model structure.

As in Definition 8.4 and and Lemma 8.5, define $\lambda_A : F_{\Sigma A}S^1 \longrightarrow F_A S^0$ to be that map of $\mathscr{W}$-spaces such that

$$\lambda_A^* : \mathscr{W}\mathscr{T}(F_A S^0, X) \longrightarrow \mathscr{W}\mathscr{T}(F_{\Sigma A}S^1, X)$$

corresponds under adjunction to $\tilde{\sigma} : X(A) \longrightarrow \Omega X(\Sigma A)$ for all $\mathscr{W}$-spaces X. The following lemma generalizes part of Lemma 8.6.

Lemma 17.1. *The maps λ_A are π_*-isomorphisms.*

Proof. Using Example 4.6, we identify $\lambda_A(S^q)$ as the evaluation map

$$\Sigma\Omega F(A, S^q) \longrightarrow F(A, S^q).$$

This gives a π_*-isomorphism by Lemma 8.6 when A is a sphere. Using (i′) and (vi) of Theorem 7.4 to obtain long exact sequences, it follows in general by induction on the number of cells of A. □

Define K to be the union of the set FJ with the sets $k_A \square I$ defined as in Definition 9.3, where $k_A : F_{\Sigma A}S^1 \longrightarrow M\lambda_A$ is the absolute acyclic q-cofibration given in terms of the mapping cylinder of λ_A.

Theorem 17.2. *The category of $\mathscr{W}$-spaces is a compactly generated proper topological model category with respect to the π_*-isomorphisms, absolute q-fibrations, and absolute q-cofibrations (of the absolute level model structure). The sets FI and K are the generating sets of absolute q-cofibrations and absolute acyclic q-cofibrations.*

The comparison of our two stable model structures takes the following form.

Proposition 17.3. *The identity functor from $\mathscr{W}\mathscr{T}$ with its original stable model structure to $\mathscr{W}\mathscr{T}$ with its absolute stable model structure is the left adjoint of a Quillen equivalence.*

We insert several preliminary results about $\mathscr{W}$-spaces before turning to the proof of Theorem 17.2. Recall that $\mathscr{W}$-spaces and $\mathscr{W}$-spectra coincide, so that a $\mathscr{W}$-space X has a natural pairing

$$\sigma : X(A) \wedge B \longrightarrow X(A \wedge B).$$

With B fixed, these define a map of $\mathscr{W}$-spaces $X \wedge B \longrightarrow X(- \wedge B)$.

Remark 17.4. In view of $\sigma : X(A)\wedge I_+ \longrightarrow X(A\wedge I_+)$, we see that any $\mathscr{W}$-space X is a homotopy-preserving functor. Of course, a weak equivalence in $\mathscr{W}$ is a homotopy equivalence, by Whitehead's theorem. Thus any X is a "homotopy functor", in the sense that it preserves weak equivalences.

Definition 17.5. Let X be a $\mathscr{W}$-space and A be a finite based CW-complex. Define a prespectrum $X[A]$ by setting $X[A]_n = X(S^n \wedge A)$, with structure maps given by instances of σ. Note that $X[S^0] = \mathbb{U}X$, $\mathbb{U} : \mathscr{W}\mathscr{T} \longrightarrow \mathscr{P}$. We also have the prespectrum $X[S^0] \wedge A$. The maps

$$\sigma : X(S^n) \wedge A \longrightarrow X(S^n \wedge A)$$

specify a map of prespectra

$$\sigma[A] : X[S^0] \wedge A \longrightarrow X[A].$$

The homotopy groups $\pi_*(X[S^0] \wedge A)$ are the homology groups of A with respect to the homology theory represented by the prespectrum $X[S^0]$. The insight that the following result should be true is due to Lydakis, who proved an analogue in the simplicial setting [22, 11.7].

Proposition 17.6. *For every $\mathscr{W}$-space X and finite based CW-complex A, $\sigma[A]$ is a π_*-isomorphism. Therefore, if $f : X \longrightarrow Y$ is a π_*-isomorphism, in the sense that $f[S^0] : X[S^0] \longrightarrow Y[S^0]$ is a π_*-isomorphism, then $f[A] : X[A] \longrightarrow Y[A]$ is a π_*-isomorphism for every A.*

Proof. The second statement follows directly from the first and Theorem 7.4(i). We prove the first statement in stages. First suppose that $X = F_B S^0$, where B is a finite based CW-complex. Then, on nth-spaces, $\sigma[A]$ is the canonical map

$$F(B, S^n) \wedge A \longrightarrow F(B, S^n \wedge A).$$

It is easy to check directly that this map is a π_*-isomorphism. This is just an explicit prespectrum level precursor of a standard result about Spanier-Whitehead duality. Since $F_B C \cong (F_B S^0) \wedge C$, Theorem 7.4(i) implies that $\sigma[A]$ is a π_*-isomorphism when $X = F_B C$ for any based CW-complex C. Using Theorem 7.4, it follows that $\sigma[A]$ is a π_*-isomorphism when X is a cell FI-complex. For a general X, we factor the trivial map $* \longrightarrow X$ as the composite of a cell FI-complex $* \longrightarrow X'$ and a level acyclic fibration $p : X' \longrightarrow X$. Since $\sigma[A]$ is a π_*-isomorphism for X', it is a π_*-isomorphism for X. □

The following definitions and lemma turn out to describe the fibrant $\mathscr{W}$-spaces in the absolute stable model structure.

Definition 17.7. Consider a commutative diagram of based spaces

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow[g]{} & Y. \end{array}$$

The diagram is a *homotopy cocartesian* square if the induced map from the homotopy pushout $M(i, f)$ to Y is a weak equivalence. It is a *homotopy cartesian* square if the induced map from A to the homotopy pullback $P(g, j)$ is a weak equivalence. (The homotopy pullback diagrams of Definition 9.4 are special cases).

Definition 17.8. A $\mathscr{W}$-space E is *linear* if it converts homotopy cocartesian squares to homotopy cartesian squares.

Lemma 17.9. *The following properties of a $\mathscr{W}$-space are equivalent.*

(i) *E is linear.*
(ii) *$E[A]$ is an Ω-spectrum for all $A \in \mathscr{W}$.*
(iii) *$\tilde{\sigma} : E(A) \longrightarrow \Omega E(\Sigma A)$ is a weak equivalence for all $A \in \mathscr{W}$.*

Proof. Recall that our functors are assumed to be based, so that $E(*) = *$. If E is linear, then $E(A)$ is weakly equivalent to the homotopy pullback $\Omega E(\Sigma A)$ of the diagram $* \longrightarrow E(\Sigma A) \longleftarrow *$. This weak equivalence is homotopic to the adjoint structure map $\tilde{\sigma}$, hence E satisfies (iii). Conversely, if E satisfies (iii), then the map $\pi_q(E(A)) \longrightarrow \pi_q(E[A]) = \operatorname{colim} \pi_{q+n}(E(S^n \wedge A))$ is an isomorphism for $q \geq 0$, and these $\pi_q(E(A))$ form part of a homology theory. By the five lemma, this implies that, for a cofiber sequence $A \longrightarrow B \xrightarrow{f} C$, the induced map from $E(A)$ to the homotopy fiber of $E(f)$ is a weak equivalence. In turn, this implies that E is linear. The equivalence of (ii) and (iii) is elementary. □

From here, the proof of Theorem 17.2 is exactly the same as the proof of Theorem 9.2, but with the stable equivalences there replaced by the π_*-isomorphisms here; see also the proof of Proposition 8.7 in §10. We use Proposition 17.6 repeatedly, and we apply the results on π_*-isomorphisms of §7 to the restricted maps $f[A]$ of prespectra associated to maps f of $\mathscr{W}$-spaces. We record the main steps of the proof since they give useful characterizations of the classes of maps that enter into the model structure.

Proposition 17.10. *A map $p : E \longrightarrow B$ satisfies the RLP with respect to K if and only if p is an absolute level fibration and the diagram*

$$\begin{array}{ccc} E(A) & \xrightarrow{\tilde{\sigma}} & \Omega E(\Sigma A) \\ {\scriptstyle p(A)}\downarrow & & \downarrow{\scriptstyle \Omega p(\Sigma A)} \\ B(A) & \xrightarrow[\tilde{\sigma}]{} & \Omega B(\Sigma A) \end{array} \tag{17.11}$$

is a homotopy pullback for each finite based CW-complex A.

Using the third criterion in Lemma 17.9, this gives the following result.

Corollary 17.12. *The trivial map $F \longrightarrow *$ satisfies the RLP with respect to K if and only if F is linear.*

Corollary 17.13. *If $p : E \longrightarrow B$ is a π_*-isomorphism that satisfies the RLP with respect to K, then p is an absolute level acyclic fibration.*

Proposition 17.14. *Let $f : X \longrightarrow Y$ be a map of $\mathscr{W}$-spaces.*

(i) *f is an absolute acyclic q-cofibration if and only if it is a retract of a relative K-cell complex.*
(iii) *f is an absolute q-fibration if and only if it satisfies the RLP with respect to K, and X is fibrant if and only if it is linear.*
(iii) *f is an absolute acyclic q-fibration if and only if it is an absolute level acyclic fibration.*

For the study of $\mathscr{W}$-ring and module spaces, we have the following result, which implies that Theorem 12.1 applies to $\mathscr{W}$-spaces under the absolute as well as the original stable model structure.

Proposition 17.15. *Under the absolute stable model structure, the category of $\mathscr{W}$-spaces satisfies the pushout-product and monoid axioms.*

Exactly as in the proofs of Propositions 12.6 and 12.5, this is a consequence of the following analogue of Proposition 12.3.

Proposition 17.16. *For any cofibrant $\mathscr{W}$-space X, the functor $X \wedge_S (-)$ preserves π_*-isomorphisms.*

Proof. As in the proof of Proposition 12.3, but taking into account that there are more cofibrant objects to deal with, it suffices to prove that $\pi_*(F_A S^0 \wedge_S Y) = 0$

if $\pi_*(Y) = 0$, where A is any finite based CW complex. Let Z be a Spanier-Whitehead k-dual to A, with duality maps $\eta : S^k \longrightarrow A \wedge Z$ and $\varepsilon : Z \wedge A \longrightarrow S^k$. By adjunction, η gives rise to a map $\tilde{\eta} : F_A S^k \longrightarrow F_0 Z$ and the adjoint

$$Z \longrightarrow F(A, S^k) = (F_A S^0)(S^k)$$

of ε gives rise to a map $\tilde{\varepsilon} : F_k Z \longrightarrow F_A S^0$. Consider the composites

$$\alpha : F_{\Sigma^k A} S^k \cong F_k S^0 \wedge_S F_A S^k \xrightarrow{\mathrm{id} \wedge \tilde{\eta}} F_k S^0 \wedge_S F_0 Z \cong F_k Z \xrightarrow{\tilde{\varepsilon}} F_A S^0$$

and

$$\beta : F_{\Sigma^k A} S^k \cong F_k S^k \wedge F_A S^0 \xrightarrow{\gamma_k \wedge \mathrm{id}} F_0 S^0 \wedge_S F_A S^0 \cong F_A S^0.$$

These maps have adjoints $S^k \longrightarrow (F_A S^0)(\Sigma^k A) = F(A, \Sigma^k A)$, which in turn have adjoints $\overline{\alpha} : \Sigma^k A \longrightarrow \Sigma^k A$ and $\overline{\beta} : \Sigma^k A \longrightarrow \Sigma^k A$. Inspecting definitions, we see that $\overline{\alpha}$ is the composite

$$\Sigma^k A \cong S^k \wedge A \xrightarrow{\eta \wedge \mathrm{id}} A \wedge Z \wedge A \xrightarrow{\mathrm{id} \wedge \varepsilon} A \wedge S^k = \Sigma^k A,$$

which is homotopic to the identity by the definition of a k-duality, and $\overline{\beta}$ is the identity map. Thus $\alpha \simeq \beta$. Since $\pi_*(Y) = 0$, $\pi_*(F_k Z \wedge_S Y) = 0$ by Theorem 7.4(i) and Proposition 12.3. Therefore $\alpha \wedge_S \mathrm{id}_Y$ induces the zero map on π_*. By Corollary 12.4, $\beta \wedge_S \mathrm{id}_Y$ induces an isomorphism on π_*. Therefore $\pi_*(F_A S^0 \wedge_S Y) = 0$. □

We add some observations about connectivity for use in the next section.

Definition 17.17. A prespectrum X is *n-connected* if $\pi_q(X) = 0$ for $q \leq n$; X is *connective* if it is (-1)-connected. A $\mathscr{W}$-space X is connective if its underlying prespectrum $X[S^0]$ is connective; X is *strictly connective* if $X(A)$ is n-connected when A is n-connected.

Observe that, on passage to the homotopy groups $\pi_q(X(A))$ of its spaces, a connective linear $\mathscr{W}$-space X defines a homology theory in all degrees.

Lemma 17.18. *A connective linear $\mathscr{W}$-space is strictly connective. The following conditions on a map $f : X \longrightarrow Y$ between connective linear $\mathscr{W}$-spaces are equivalent.*

(i) *f is a π_*-isomorphism.*
(ii) *$f : X(S^0) \longrightarrow Y(S^0)$ is a weak equivalence.*
(iii) *f is a level equivalence.*

Proof. If T is an n-connected Ω-spectrum, then its zeroth space is n-connected. If X is connective and linear and A is n-connected, then $X[A]$ is n-connected because its homotopy groups are the homology groups of A with respect to a connective homology theory. Since $X[A]$ is an Ω-spectrum with zeroth space $X(A)$, $X(A)$ is n-connected and X is strictly connective. In the second statement, (i) and (ii) are clearly equivalent and (ii) and (iii) are equivalent since a map of homology theories is an isomorphism if and only if it is an isomorphism on coefficients. □

18. The comparison between $\mathscr{F}$-spaces and $\mathscr{W}$-spaces

It remains to relate $\mathscr{F}$-spaces to $\mathscr{W}$-spaces. It is important to keep in mind the two quite different forgetful functors defined on $\mathscr{W}$-spaces, namely

$$\mathbb{U}_{\mathscr{F}} : \mathscr{W}\mathscr{T} \longrightarrow \mathscr{F}\mathscr{T} \quad \text{and} \quad \mathbb{U}_{\mathscr{P}} : \mathscr{W}\mathscr{T} \longrightarrow \mathscr{P}.$$

We write $\mathbb{U}$ for the former and write $\mathbb{P}$ for its left adjoint $\mathscr{F}\mathscr{T} \longrightarrow \mathscr{W}\mathscr{T}$.

We have the level model structure on the category of $\mathscr{F}$-spaces given by the level equivalences, level fibrations, and q-cofibrations. We recall what we need about the stable model structure from [35, App B].

Definition 18.1. Let $f : X \longrightarrow Y$ be a map of $\mathscr{F}$-spaces.

(i) f is a *π_*-isomorphism* if $\mathbb{U}_{\mathscr{P}}\mathbb{P}f$ is a π_*-isomorphism of prespectra.
(ii) f is a *stable equivalence* if a cofibrant approximation $f' : X' \longrightarrow Y'$ of f (in the level model structure) is a π_*-isomorphism.
(iii) f is an *acyclic q-cofibration* if it is a stable equivalence and a q-cofibration.
(iv) f is a *q-fibration* if it satisfies the RLP with respect to the acyclic q-cofibrations.
(iv) f is an *acyclic q-fibration* if it is a stable equivalence and a q-fibration.

One reason for the distinction between π_*-isomorphisms and stable equivalences is that we have not proven that $\mathbb{P}$ preserves level equivalences or even carries level equivalences to π_*-isomorphisms in general. Another is that this definition of a stable equivalence agrees with the one given in [35, App B]; see Remark 19.9 below.

For an $\mathscr{F}$-space X, we write $X_n = X(\mathbf{n}^+)$; recall that $X_0 = *$. Let $\delta_i : \mathbf{n}^+ \longrightarrow \mathbf{1}^+$ be the projection given by $\delta_i(i) = 1$ and $\delta_i(j) = 0$ for $j \neq i$. Let $\phi : \mathbf{2}^+ \longrightarrow \mathbf{1}^+$ be the based map such that $\phi(1) = 1 = \phi(2)$.

Definition 18.2. An $\mathscr{F}$-space X is *special* if the map $X_n \longrightarrow X_1^n$ induced by the n projections $\delta_i : \mathbf{n}^+ \longrightarrow \mathbf{1}^+$ is a weak equivalence. If X is special, then $\pi_0(X_1)$ is an abelian monoid with product $\pi_0(X_1) \times \pi_0(X_1) \cong \pi_0(X_2) \longrightarrow \pi_0(X_1)$ induced by ϕ. A special $\mathscr{F}$-space X is *very special* if $\pi_0(X_1)$ is an abelian group.

Theorem 18.3. *The category $\mathscr{F}\mathscr{T}$ is a cofibrantly generated model category with respect to the stable equivalences, q-fibrations, and q-cofibrations. An $\mathscr{F}$-space is fibrant if and only if it is very special.*

We refer the reader to [35] for the proof. While the result is deduced there from its simplicial analogue, a topological argument works just as well. However, it is not known and, as explained in [35, A.6], seems unlikely to be true that the stable model structure on $\mathscr{F}\mathscr{T}$ is compactly generated, so that a more general version of the small object argument than Lemma 5.8 is needed. The set of generating q-cofibrations is FI, and of course its elements have compact domains. However, there does not seem to be a canonical choice of a set of generating acyclic q-cofibrations, and the elements of the set chosen in [35, App B] do not all have compact domains. All elements of the set are π_*-isomorphisms, and this has the following consequences.

Lemma 18.4. *All acyclic q-cofibrations are π_*-isomorphisms.*

Lemma 18.5. *The pair $(\mathbb{P}, \mathbb{U})$ is a Quillen adjoint pair.*

Proof. Since $\mathbb{U} : \mathscr{W}\mathscr{T} \longrightarrow \mathscr{F}\mathscr{T}$ carries absolute level equivalences and absolute level fibrations to level equivalences and level fibrations, $(\mathbb{P}, \mathbb{U})$ is a Quillen adjoint pair with respect to these level model structures and thus $\mathbb{P}$ preserves q-cofibrations (and level acyclic q-cofibrations). Now the previous lemma gives that $\mathbb{P}$ preserves acyclic q-cofibrations since it obviously preserves π_*-isomorphisms. □

In particular, $\mathbb{U}$ preserves fibrant objects, as could easily be checked directly.

Lemma 18.6. *If Y is a linear $\mathscr{W}$-space, then $\mathbb{U}Y$ is a very special $\mathscr{F}$-space.*

The following result, which was left open in [35], is a consequence of its counterpart, Proposition 17.15, for $\mathscr{W}$-spaces. It implies that Theorem 12.1 applies to $\mathscr{F}$-spaces.

Proposition 18.7. *The stable model structure on the category of $\mathscr{F}$-spaces satisfies the pushout-product and monoid axioms.*

Proof. This is an exercise in the use of cofibrant approximation of maps. The essential points are that smash products and pushouts of cofibrant approximations are cofibrant approximations and that the functor $\mathbb{P}$ preserves colimits and smash products and creates the stable equivalences between cofibrant objects. □

Because the topological prolongation functor $\mathbb{P}$ is harder to analyze than its simplicial counterpart, we shall derive the following result from its known simplicial analogue in the next section. In essence, this result goes back to Segal [38] and is at the heart of his infinite loop space machine.

Proposition 18.8. *Let X be a cofibrant $\mathscr{F}$-space. Then $\mathbb{P}X$ is a strictly connective $\mathscr{W}$-space. If X is very special, then $\mathbb{P}X$ is a cofibrant linear $\mathscr{W}$-space. That is, the functor $\mathbb{P}$ preserves cofibrant-fibrant objects.*

Granting these results, Lemma 17.18 and the fact that $\mathbb{UP} \cong \mathrm{Id}$ immediately give the following consequences.

Lemma 18.9. *The following conditions on a map $f : X \longrightarrow X'$ between cofibrant very special $\mathscr{F}$-spaces are equivalent.*
(i) *f is a π_*-isomorphism.*
(ii) *$f : X_1 \longrightarrow X'_1$ is a weak equivalence.*
(iii) *f is a level equivalence.*
(iv) *$\mathbb{P}f$ is an absolute level equivalence of $\mathscr{W}$-spaces.*

Lemma 18.10. *If Y is a connective linear $\mathscr{W}$-space and $f : X \longrightarrow \mathbb{U}Y$ is a cofibrant-fibrant approximation of the $\mathscr{F}$-space $\mathbb{U}Y$, then the composite*

$$\varepsilon \circ \mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{PU}Y \longrightarrow Y$$

is an absolute level equivalence of $\mathscr{W}$-spaces.

The results above directly imply Theorems 0.10, 0.11, and 0.12. Corollary 0.13 follows as in the proof of Corollary 0.6 in §13.

19. Simplicial and topological diagram spectra

Let $\mathscr{S}_*$ denote the category of pointed simplicial sets, abbreviated ssets, and let $\mathbb{T} : \mathscr{S}_* \longrightarrow \mathscr{T}$ and $\mathbb{S} : \mathscr{T} \longrightarrow \mathscr{S}_*$ be the geometric realization and total singular complex functors. Both are strong symmetric monoidal. Let $\nu : \mathrm{Id} \longrightarrow \mathbb{ST}$ and $\rho : \mathbb{TS} \longrightarrow \mathrm{Id}$ be the unit and counit of the $(\mathbb{S}, \mathbb{T})$ adjunction. Both are monoidal natural weak equivalences. Recall that a map f of spaces is a weak equivalence or Serre fibration if and only if $\mathbb{S}f$ is a weak equivalence or Kan fibration of ssets.

For a discrete category $\mathscr{D}$, a $\mathscr{D}$-sset is a functor $Y : \mathscr{D} \longrightarrow \mathscr{S}_*$, and we have the category $\mathscr{D}\mathscr{S}_*$ of $\mathscr{D}$-ssets. When we are given a canonical symmetric monoidal functor $S_{\mathscr{D}} : \mathscr{D} \longrightarrow \mathscr{S}_*$, we define $\mathscr{D}$-spectra over $S_{\mathscr{D}}$ in the evident fashion. Let us write $\mathscr{D}\mathscr{S}[\mathscr{S}_*]$ and $\mathscr{D}\mathscr{S}[\mathscr{T}]$ for the categories of $\mathscr{D}$-spectra of ssets over $S_{\mathscr{D}}$ and $\mathscr{D}$-spectra of spaces over $\mathbb{T}S_{\mathscr{D}}$. Both are symmetric monoidal categories. Levelwise application of $\mathbb{S}$ gives a lax symmetric monoidal functor $\mathbb{S} : \mathscr{D}\mathscr{S}[\mathscr{T}] \longrightarrow \mathscr{D}\mathscr{S}[\mathscr{S}_*]$

with unit map $\nu : S_{\mathscr{D}} \longrightarrow \mathbb{S}\mathbb{T}S_{\mathscr{D}}$. Levelwise application of $\mathbb{T}$ gives a strong symmetric monoidal functor $\mathbb{T} : \mathscr{D}\mathscr{S}[\mathscr{S}_*] \longrightarrow \mathscr{D}\mathscr{S}[\mathscr{T}]$. These functors are right and left adjoint, and they induce adjoint functors when restricted to categories of rings, commutative rings, and modules over rings.

Warning 19.1. The functor $\mathbb{T}\mathbb{S} : \mathscr{T} \longrightarrow \mathscr{T}$ is *not* continuous. Therefore we do not have a functor $\mathbb{T}\mathbb{S} : \mathscr{D}\mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$ when the topological category $\mathscr{D}$ is not discrete.

When $\mathscr{D} = \mathscr{W}$, we shall see how to get around this problem in Theorem 19.11.

As far as the relevant homotopy categories go, we can work interchangeably with $\mathscr{D}$-spectra of ssets and $\mathscr{D}$-spectra of spaces.

Proposition 19.2. *Let $\mathscr{D}$ be discrete and suppose that the category of $\mathscr{D}$-spectra of ssets is a model category such that every level equivalence is a weak equivalence. Define a weak equivalence of $\mathscr{D}$-spectra of spaces to be a map f such that $\mathbb{S}f$ is a weak equivalence. Then $\mathbb{S}$ and $\mathbb{T}$ induce adjoint equivalences of homotopy categories that induce adjoint equivalences between the respective homotopy categories of rings, commutative rings, and modules over rings.*

Proof. Since $\eta : Y \longrightarrow \mathbb{S}\mathbb{T}Y$ is a level equivalence for all $\mathscr{D}$-spectra of ssets, an argument much like the proof of Lemma A.2 applies. □

The proposition applies to symmetric spectra [15] and to $\mathscr{F}$-spectra [35]. In the latter case, just as for $\mathscr{F}$-spaces, $\mathscr{F}$-spectra of ssets are the same as $\mathscr{F}$-ssets. As noted in the preprint version of [15] and in [35], Lemma A.2 applies to give the following stronger conclusion in these cases.

Theorem 19.3. *Let $\mathscr{D} = \Sigma$ or $\mathscr{D} = \mathscr{F}$. The pair $(\mathbb{T}, \mathbb{S})$ is a Quillen equivalence between the categories $\mathscr{D}\mathscr{S}[\mathscr{S}_*]$ and $\mathscr{D}\mathscr{S}[\mathscr{T}]$.*

Since [15] and [35] give the pushout-product and monoid axioms in $\mathscr{D}\mathscr{S}[\mathscr{S}_*]$, $\mathscr{D} = \Sigma$ and $\mathscr{D} = \mathscr{F}$, and we have proven these axioms in $\mathscr{D}[\mathscr{T}]$, we are entitled to the following multiplicative elaborations of Theorem 19.3.

Theorem 19.4. *Let $\mathscr{D} = \Sigma$ or $\mathscr{D} = \mathscr{F}$. The functors $\mathbb{T}$ and $\mathbb{S}$ induce a Quillen equivalence between the categories of $\mathscr{D}$-ring spectra of simplicial sets and $\mathscr{D}$-ring spectra of spaces.*

Theorem 19.5. *Let $\mathscr{D} = \Sigma$ or $\mathscr{D} = \mathscr{F}$. For a $\mathscr{D}$-ring R of simplicial sets, the functors $\mathbb{T}$ and $\mathbb{S}$ induce a Quillen equivalence between the categories of R-module spectra (of simplicial sets) and $\mathbb{T}R$-module spectra (of spaces).*

By Smith's result[2] that the category of commutative symmetric ring spectra of simplicial sets is a Quillen model category with definitions parallel to those in §15, we also have the commutative analogue of Theorem 19.4 in this case.

Theorem 19.6. *The functors $\mathbb{T}$ and $\mathbb{S}$ induce a Quillen equivalence between the categories of commutative symmetric ring spectra of simplicial sets and commutative symmetric ring spectra of spaces.*

Now focus on $\mathscr{F}$-ssets and $\mathscr{F}$-spaces. We must deduce Proposition 18.8 from its simplicial analogue. There is a prolongation functor $\mathbb{P}^{\mathscr{S}_*}$ from $\mathscr{F}$-ssets to the category $\mathscr{S}_*^{\mathscr{S}_*}$ of simplicial functors $\mathscr{S}_* \longrightarrow \mathscr{S}_*$. We can use it to study the topological prolongation functor $\mathbb{P} = \mathbb{P}^{\mathscr{T}}$ from $\mathscr{F}$-spaces to the category $\mathscr{T}^{\mathscr{T}}$ of continuous

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functors $\mathscr{T} \longrightarrow \mathscr{T}$. The advantage of $\mathbb{P}^{\mathscr{S}_*}$ is that, although it is characterized as the left adjoint to the forgetful functor, it has two equivalent explicit descriptions. First, in analogy with $\mathbb{P}^{\mathscr{T}}$ (23.3), for a functor $Y : \mathscr{F} \longrightarrow \mathscr{S}_*$ and a sset K,

$$(\mathbb{P}^{\mathscr{S}_*}Y)(K) = \int^{\mathbf{n}^+ \in \mathscr{F}} K^n \wedge Y_n. \tag{19.7}$$

Since $\mathbb{T}$ commutes with colimits and finite products, this description implies that

$$(\mathbb{P}^{\mathscr{T}}\mathbb{T}Y)(\mathbb{T}K) \cong \mathbb{T}((\mathbb{P}^{\mathscr{S}_*}Y)(K)). \tag{19.8}$$

Of course, this relationship requires us to begin with an $\mathscr{F}$-sset Y. However, there is a simple trick that has the effect of allowing us to use (19.8) to study $\mathscr{F}$-spaces X. Let $\gamma : \mathbb{C}Y \longrightarrow Y$ be a functorial cofibrant approximation in the level model structure on $\mathscr{F}$-ssets and define

$$\xi = \rho \circ \mathbb{T}\gamma : \mathbb{T}\mathbb{C}\mathbb{S}X \longrightarrow X.$$

Since ξ is a level equivalence and $\mathbb{T}$ preserves cofibrant objects, ξ is a functorial cofibrant approximation of $\mathscr{F}$-spaces. If X is cofibrant, then $\mathbb{P}\xi$ is an absolute level equivalence. In effect, this allows us to use $\mathbb{P}^{\mathscr{S}_*}\mathbb{C}\mathbb{S}X$ to study $\mathbb{P}X$.

Remark 19.9. In [35, App B], a map $f : X \longrightarrow Y$ of $\mathscr{F}$-spaces is defined to be a stable equivalence if $\mathbb{S}f$ is a π_*-isomorphism. It is equivalent that $\mathbb{T}\mathbb{C}\mathbb{S}f$ is a π_*-isomorphism. Thus, since $\mathbb{T}\mathbb{C}\mathbb{S}f$ is a cofibrant approximation of f, these stable equivalences are the same as the stable equivalences of Definition 18.1(ii) .

The other description of $\mathbb{P}^{\mathscr{S}_*}$ is given as follows. A based set E can be identified with the colimit of its based finite ordered subsets, and these can be identified with the based injections $\mathbf{n}^+ \longrightarrow E$ for $n \geq 0$. We extend Y to a functor from based sets to simplicial sets by defining $Y(E)$ to be the colimit of the simplicial sets $Y(\mathbf{n}^+)$, where the colimit is taken over the based functions $\mathbf{n}^+ \longrightarrow E$ or, equivalently, over the based injections $\mathbf{n}^+ \longrightarrow E$. We then define $(\mathbb{P}^{\mathscr{S}_*}Y)(K)$ to be the diagonal of the bisimplicial set obtained by applying Y to the set K_q of q-simplices of K for all q. This description is exploited by Bousfield and Friedlander [7] and Lydakis [21] to study the homotopical properties of prolongation. The definitions of special and very special $\mathscr{F}$-ssets are the same as for $\mathscr{F}$-spaces, and an $\mathscr{F}$-space X is (very) special if and only if the $\mathscr{F}$-sset $\mathbb{S}X$ is (very) special.

Proof of Proposition 18.8. Since any finite CW complex is homotopy equivalent to $\mathbb{T}K$ for some finite simplicial complex K, we may restrict attention to spaces of the form $\mathbb{T}K$ in $\mathscr{W}$. Let X be a cofibrant $\mathscr{F}$-space and let $Y = \mathbb{C}\mathbb{S}X$. By the absolute level equivalence $\xi : \mathbb{T}Y \longrightarrow X$, it suffices to prove the result for $\mathbb{T}Y$, and

$$(\mathbb{P}\mathbb{T}Y)(\mathbb{T}K) \cong \mathbb{T}(\mathbb{P}^{\mathscr{S}_*}Y(K)). \tag{19.10}$$

By [7, 4.10], $(\mathbb{P}^{\mathscr{S}_*}Y)(K)$ is n-connected if K is n-connected. Since a simplicial set L is n-connected if and only $\mathbb{T}L$ is n-connected, this shows that $(\mathbb{P}\mathbb{T}Y)(\mathbb{T}K)$ is n-connected if $\mathbb{T}K$ is n-connected, so that $\mathbb{T}Y$ is strictly connective. Now assume that X and therefore Y is very special. By Lemma 17.9, it suffices to prove that $\tilde{\sigma} : (\mathbb{P}\mathbb{T}Y)(\mathbb{T}K) \longrightarrow \Omega(\mathbb{P}\mathbb{T}Y)(\Sigma\mathbb{T}K)$ is a weak equivalence for all finite simplicial complexes K, and we may replace the target of $\tilde{\sigma}$ by the homotopy fiber of the evident map $(\mathbb{P}\mathbb{T}Y)(C\mathbb{T}K) \longrightarrow (\mathbb{P}\mathbb{T}Y)(\Sigma\mathbb{T}K)$. By [7, 4.3], $(\mathbb{P}^{\mathscr{S}_*}Y)(K)$ maps by a weak equivalence to the homotopy fiber of the map $(\mathbb{P}^{\mathscr{S}_*}Y)(CK) \longrightarrow (\mathbb{P}^{\mathscr{S}_*}Y)(\Sigma K)$.

Since $\mathbb{T}$ commutes with cones, suspensions, and homotopy fibers, the conclusion follows upon applying $\mathbb{T}$ and using (19.10). $\square$

Finally, as promised in the introduction, we compare the category $\mathscr{W}\mathscr{T}$ with Lydakis' category $\mathscr{S}\mathscr{F}$ of "simplicial functors", namely simplicial functors from the category of based finite ssets to the category of all based ssets; see [22].

Theorem 19.11. *There is a Quillen equivalence* $(\mathbb{P}\mathbb{T}, \mathbb{S}\mathbb{U})$ *from the category* $\mathscr{S}\mathscr{F}$ *to* $\mathscr{W}\mathscr{T}$ *with its absolute stable model structure. The functor* $\mathbb{P}\mathbb{T}$ *is strong symmetric monoidal and the functor* $\mathbb{S}\mathbb{U}$ *is lax symmetric monoidal.*

Proof. For based ssets K and L, let $F(K,L)$ denote the usual sset of based maps $K \longrightarrow L$. Define a topological category $\mathscr{V}$ with objects the based finite ssets and whose space of maps $K \longrightarrow L$ is $\mathbb{T}F(K,L)$. There is a natural inclusion of ssets

$$F(K,L) \longrightarrow \mathbb{S}F(\mathbb{T}K, \mathbb{T}L).$$

Its adjoint is a natural continuous map

$$t : \mathbb{T}F(K,L) \longrightarrow F(\mathbb{T}K, \mathbb{T}L).$$

There results a continuous functor $t : \mathscr{V} \longrightarrow \mathscr{W}$ that sends K to $\mathbb{T}K$, hence we have an adjoint pair $(\mathbb{P}, \mathbb{U})$ relating $\mathscr{V}$-spaces to $\mathscr{W}$-spaces. For a simplicial functor Y, we obtain a continuous functor $\mathbb{T}Y : \mathscr{V} \longrightarrow \mathscr{T}$ such that $(\mathbb{T}Y)(K) = \mathbb{T}Y(K)$; on morphism spaces, $\mathbb{T}Y$ is given by the composites

$$\mathbb{T}F(K,L) \xrightarrow{\mathbb{T}Y} \mathbb{T}F(Y(K), Y(L)) \xrightarrow{t} F(\mathbb{T}Y(K), \mathbb{T}Y(L)).$$

For based spaces A and B, the adjoint of the evident map

$$\mathbb{S}F(A,B) \wedge \mathbb{S}A \cong \mathbb{S}(F(A,B) \wedge A) \longrightarrow \mathbb{S}B$$

is a natural map

$$s : \mathbb{S}F(A,B) \longrightarrow F(\mathbb{S}A, \mathbb{S}B).$$

For a $\mathscr{V}$-space X, we obtain a simplicial functor $\mathbb{S}X$ such that $(\mathbb{S}X)(K) = \mathbb{S}X(K)$; on morphism ssets, $\mathbb{S}X$ is given by the composites

$$F(K,L) \xrightarrow{\nu} \mathbb{S}\mathbb{T}F(K,L) \xrightarrow{\mathbb{S}X} \mathbb{S}F(X(K), X(L)) \xrightarrow{s} F(\mathbb{S}X(K), \mathbb{S}X(L)).$$

The pair $(\mathbb{T}, \mathbb{S})$ relating $\mathscr{S}\mathscr{F}$ and $\mathscr{V}\mathscr{T}$ is adjoint, and we have the following diagram of pairs of adjoint functors:

$$\begin{array}{ccccc}
\mathscr{S}\mathscr{F} & \underset{\mathbb{S}}{\overset{\mathbb{T}}{\rightleftarrows}} & \mathscr{V}\mathscr{T} & \underset{\mathbb{U}}{\overset{\mathbb{P}}{\rightleftarrows}} & \mathscr{W}\mathscr{T} \\
\mathbb{U}\downarrow\uparrow\mathbb{P} & & & & \mathbb{U}\downarrow\uparrow\mathbb{P} \\
\Sigma\mathscr{S}[\mathscr{S}_*] & & \underset{\mathbb{S}}{\overset{\mathbb{T}}{\rightleftarrows}} & & \Sigma\mathscr{S}[\mathscr{T}].
\end{array}$$

The diagram of right adjoints commutes by inspection, hence the diagram of left adjoints commutes up to isomorphism. By comparing our characterizations of absolute q-fibrations and absolute acyclic q-fibrations in Propositions 17.10 and 17.14 with the analogous characterizations [22, 9.4, 9.8] given by Lydakis, we see that $\mathbb{S}\mathbb{U}$ preserves q-fibrations and acyclic q-fibrations, so that $(\mathbb{P}\mathbb{T}, \mathbb{S}\mathbb{U})$ is a Quillen adjoint pair. The right pair $(\mathbb{P}, \mathbb{U})$ is a Quillen equivalence by Theorem 0.1, the left pair $(\mathbb{P}, \mathbb{U})$ is a Quillen equivalence by the simplicial analogue of that result, and the bottom pair $(\mathbb{T}, \mathbb{S})$ is a Quillen equivalence by Theorem 19.4. Therefore $(\mathbb{P}\mathbb{T}, \mathbb{S}\mathbb{U})$

is a Quillen equivalence. The monoidal properties of these functors follow from Proposition 3.3 and the properties of $\mathbb{T}$ and $\mathbb{S}$. □

Part III. Symmetric monoidal categories and FSP's

We fix language about symmetric monoidal categories in §20, and we discuss symmetric monoidal diagram categories in §21. Briefly, there is an elementary external smash product that takes a pair of $\mathscr{D}$-spaces to a $(\mathscr{D} \times \mathscr{D})$-space. Left Kan extension internalizes this product to give a smash product that takes a pair of $\mathscr{D}$-spaces to a $\mathscr{D}$-space. We show how the functors $\mathbb{U}$ and $\mathbb{P}$ behave with respect to internal smash products in §23.

Such internal products on functor categories were studied by Day [8], in a general categorical setting. The construction made a first brief appearance in stable homotopy theory in work of Anderson [3], but its real importance only became apparent with Jeff Smith's introduction of symmetric spectra.

In §22, we show how functors with smash product, FSP's, fit into the picture. For a commutative monoid R in $\mathscr{D}\mathscr{T}$, we define $\mathscr{D}$-FSP's over R in terms of the external smash product, and we show that the category of $\mathscr{D}$-FSP's over R is isomorphic to the category of R-algebras, as defined with respect to the internal smash product. We are mainly interested in the case $R = S_{\mathscr{D}}$, where $\mathscr{D}$ is one of our standard examples. Here the conclusion is that $\mathscr{D}$-FSP's are equivalent to $\mathscr{D}$-ring spectra.

The notion of an FSP was introduced by Bökstedt [6], who used it to define topological Hochschild homology. His FSP's were essentially the same as our $\mathscr{T}$-FSP's (although his definition was simplicial and he imposed convergence and connectivity conditions). Under the name "strictly associative ring spectrum", Σ-FSP's first appeared in work of Gunnarson [12]. The name "FSP defined on spheres" has also been used. Jeff Smith first recognized the relationship between these externally defined FSP's and his symmetric ring spectra. Similarly, an $\mathscr{F}$-FSP is equivalent to a Gamma-ring, as defined by Lydakis and Schwede [21, 35]. Under the unprepossessing name "$\mathscr{I}_*$-prefunctor", commutative $\mathscr{I}$-FSP's already appeared in work of May, Quinn, and Ray [28], where they were shown to give rise to E_∞ ring spectra.

20. Symmetric monoidal categories

We fix some language to avoid confusion. A *monoidal category* is a category $\mathscr{D}$ together with a product $\square = \square_{\mathscr{D}} : \mathscr{D} \times \mathscr{D} \longrightarrow \mathscr{D}$ and a unit object $u = u_{\mathscr{D}}$ such that $\square$ is associative and unital up to coherent natural isomorphism; $\mathscr{D}$ is *symmetric monoidal* if $\square$ is also commutative up to coherent natural isomorphism. See [16, 17, 23] for the precise meaning of coherence here. A symmetric monoidal category $\mathscr{D}$ is *closed* if it has internal hom objects $F(d, e)$ with adjunction isomorphisms

$$\mathscr{D}(d \square e, f) \cong \mathscr{D}(d, F(e, f)).$$

There are evident notions of monoids in monoidal categories and commutative monoids in symmetric monoidal categories. The (strict) ring spectra in any of the modern approaches to stable homotopy theory are the monoids and commutative monoids in the relevant symmetric monoidal ground category. To compare such objects in different ground categories, we need language to describe when functors and natural transformations preserve monoids and commutative monoids.

Definition 20.1. A functor $T : \mathscr{A} \longrightarrow \mathscr{B}$ between monoidal categories is *lax monoidal* if there is a map $\lambda : u_{\mathscr{B}} \longrightarrow T(u_{\mathscr{A}})$ and there are maps

$$\phi : T(A)\Box_{\mathscr{B}}T(A') \longrightarrow T(A\Box_{\mathscr{A}}A')$$

that specify a natural transformation $\phi : \Box_{\mathscr{B}} \circ (T \times T) \longrightarrow T \circ \Box_{\mathscr{A}}$; it is required that all coherence diagrams relating the associativity and unit isomorphisms of $\mathscr{A}$ and $\mathscr{B}$ to the maps λ and ϕ commute. If $\mathscr{A}$ and $\mathscr{B}$ are symmetric monoidal, then T is *lax symmetric monoidal* if all coherence diagrams relating the associativity, unit, and commutativity isomorphisms of $\mathscr{A}$ and $\mathscr{B}$ commute. The functor T is *strong monoidal* or *strong symmetric monoidal* if λ and ϕ are isomorphisms.

The relevant coherence diagrams are specified in [16, 17]. The direction of the arrows λ and ϕ leads to the following conclusion.

Lemma 20.2. *If $T : \mathscr{A} \longrightarrow \mathscr{B}$ is lax monoidal and M is a monoid in $\mathscr{A}$ with unit $\eta : u_{\mathscr{A}} \longrightarrow M$ and product $\mu : M\Box_{\mathscr{A}}M \longrightarrow M$, then $T(M)$ is a monoid in $\mathscr{B}$ with unit $T(\eta) \circ \lambda : u_{\mathscr{B}} \longrightarrow T(u_{\mathscr{A}}) \longrightarrow T(M)$ and product*

$$T(\mu) \circ \phi : T(M)\Box_{\mathscr{B}}T(M) \longrightarrow T(M\Box_{\mathscr{A}}M) \longrightarrow T(M).$$

If $T : \mathscr{A} \longrightarrow \mathscr{B}$ is lax symmetric monoidal and M is a commutative monoid in $\mathscr{A}$, then $T(M)$ is a commutative monoid in $\mathscr{B}$.

We also need the concomitant notion of a monoidal natural transformation. Here we needn't use an adjective "lax" or "strong" since the definition is the same for either lax or strong monoidal functors.

Definition 20.3. Let S and T be lax monoidal or lax symmetric monoidal functors $\mathscr{A} \longrightarrow \mathscr{B}$. A natural transformation $\alpha : S \longrightarrow T$ is *monoidal* if the following diagrams commute:

$$\begin{array}{ccc} & u_{\mathscr{B}} & \\ {}^{\lambda_S}\swarrow & & \searrow^{\lambda_T} \\ S(u_{\mathscr{A}}) & \xrightarrow[\alpha]{} & T(u_{\mathscr{A}}) \end{array} \quad \text{and} \quad \begin{array}{ccc} S(A)\Box_{\mathscr{B}}S(A') & \xrightarrow{\alpha\Box\alpha} & T(A)\Box_{\mathscr{B}}T(A') \\ \downarrow{\scriptstyle \phi_S} & & \downarrow{\scriptstyle \phi_T} \\ S(A\Box_{\mathscr{A}}A') & \xrightarrow[\alpha]{} & T(A\Box_{\mathscr{A}}A'). \end{array}$$

The following assertion is obvious from the definition and the previous lemma.

Lemma 20.4. *If α is monoidal and A is a monoid in $\mathscr{A}$, then $\alpha : S(A) \longrightarrow T(A)$ is a map of monoids in $\mathscr{B}$. If α is symmetric monoidal and A is a commutative monoid in $\mathscr{A}$, then $\alpha : S(A) \longrightarrow T(A)$ is a map of commutative monoids in $\mathscr{B}$.*

21. Symmetric monoidal categories of $\mathscr{D}$-spaces

Let $\mathscr{D}$ be a symmetric monoidal (based) topological category with unit object u and continuous product $\Box$. We describe the symmetric monoidal structure on the category $\mathscr{D}\mathscr{T}$ of $\mathscr{D}$-spaces in this section. After the following definition and lemma, we assume that $\mathscr{D}$ has a small skeleton $sk\mathscr{D}$; $sk\mathscr{D}$ inherits a symmetric monoidal structure such that the inclusion $sk\mathscr{D} \subset \mathscr{D}$ is strong symmetric monoidal.

Definition 21.1. For $\mathscr{D}$-spaces X and Y, define the *"external" smash product* $X \mathbin{\bar{\wedge}} Y$ by

$$X \mathbin{\bar{\wedge}} Y = \wedge \circ (X \times Y) : \mathscr{D} \times \mathscr{D} \longrightarrow \mathscr{T};$$

thus, for objects d and e of $\mathscr{D}$, $(X \barwedge Y)(d,e) = X(d) \wedge Y(e)$. For a $\mathscr{D}$-space Y and a $(\mathscr{D} \times \mathscr{D})$-space Z, define the *external function $\mathscr{D}$-space* $\bar{F}(Y,Z)$ by

$$\bar{F}(Y,Z)(d) = \mathscr{D}\mathscr{T}(Y, Z\langle d\rangle),$$

where $Z\langle d\rangle(e) = Z(d,e)$. Then, for $\mathscr{D}$-spaces X and Y and a $(\mathscr{D} \times \mathscr{D})$-space Z,

$$(\mathscr{D} \times \mathscr{D})\mathscr{T}(X \barwedge Y, Z) \cong \mathscr{D}\mathscr{T}(X, \bar{F}(Y,Z)). \tag{21.2}$$

Recall the functors F_d from Definition 1.3.

Lemma 21.3. *There is a natural isomorphism*

$$F_d A \barwedge F_e B \longrightarrow F_{(d,e)}(A \wedge B).$$

Proof. Using (21.2), (1.4), and the definitions, we see that

$$(\mathscr{D} \times \mathscr{D})\mathscr{T}(F_d A \barwedge F_e B, Z) \cong \mathscr{T}(A \wedge B, Z(d,e)) \cong (\mathscr{D} \times \mathscr{D})\mathscr{T}(F_{(d,e)}(A \wedge B), Z)$$

for a $(\mathscr{D} \times \mathscr{D})$-space Z. □

We internalize the external smash product $X \barwedge Y$ by taking its topological left Kan extension along $\square$ [23, Ch.X]. This gives $\mathscr{D}\mathscr{T}$ a smash product $\wedge$ under which it is a closed symmetric monoidal topological category. For an object d of $\mathscr{D}$, let $\square/d$ denote the category of objects $\square$-over d; its objects are the maps $\alpha : e\square f \to d$ and its morphisms are the pairs of maps $(\phi, \psi) : (e,f) \longrightarrow (e', f')$ such that $\alpha'(\phi\square\psi) = \alpha$. This category inherits a topology from $\mathscr{D}$, and a map $d \to d'$ induces a continuous functor $\square/d \longrightarrow \square/d'$.

Definition 21.4. Let X and Y be $\mathscr{D}$-spaces. Define the *internal smash product* $X \wedge Y$ to be the topological left Kan extension of $X \barwedge Y$ along $\square$. It is characterized by the universal property

$$\mathscr{D}\mathscr{T}(X \wedge Y, Z) \cong (\mathscr{D} \times \mathscr{D})\mathscr{T}(X \barwedge Y, Z \circ \square). \tag{21.5}$$

On an object d, it is specified explicitly as the colimit

$$(X \wedge Y)(d) = \mathrm{colim}_{e\square f \to d}\, X(e) \wedge Y(f)$$

indexed on $\square/d$; this makes sense since $\square/d$ has a small cofinal subcategory. When $\mathscr{D}$ itself is small, $(X \wedge Y)(d)$ can also be described as the coend

$$(X \wedge Y)(d) = \int^{(e,f)\in\mathscr{D}\times\mathscr{D}} \mathscr{D}(e\square f, d) \wedge (X(e) \wedge Y(f))$$

with its topology as a quotient of $\vee_{(e,f)}\mathscr{D}(e\square f, d) \wedge (X(e) \wedge Y(f))$. By the functoriality of colimits, maps $d \to d'$ in $\mathscr{D}$ induce maps $(X \wedge Y)(d) \longrightarrow (X \wedge Y)(d')$ that make $X \wedge Y$ into a $\mathscr{D}$-space.

Definition 21.6. Let X, Y, and Z be $\mathscr{D}$-spaces. Define the *internal function $\mathscr{D}$-space* $F(Y,Z)$ by

$$F(Y,Z) = \bar{F}(Y, Z \circ \square).$$

Then (21.1) and (21.5) immediately imply the adjunction

$$\mathscr{D}(X \wedge Y, Z) \cong \mathscr{D}(X, F(Y,Z)). \tag{21.7}$$

With these definitions, the proof of Theorem 1.7 is formal; see Day [8]. For Lemma 1.8, we see by use of (21.5), (21.7), (1.4), and the definitions that

$$\mathscr{D}\mathscr{T}(F_d A \wedge F_e B, X) \cong \mathscr{T}(A \wedge B, X(d\square e)) \cong \mathscr{D}\mathscr{T}(F_{d\square e}(A \wedge B), X)$$

for a $\mathscr{D}$-space X.

22. Diagram spectra and functors with smash product

Fix a skeletally small symmetric monoidal category $\mathscr{D}$. We have the symmetric monoidal category $\mathscr{D}\mathscr{T}$ of $\mathscr{D}$-spaces, and we consider its monoids and commutative monoids and their modules and algebras. These are defined in terms of the internal smash product in $\mathscr{D}\mathscr{T}$, and we shall explain their reinterpretations in terms of the more elementary external smash product $\bar{\wedge}$. The proofs of the comparisons are direct applications of the defining universal properties of $\wedge$ (21.5) and F_d (1.4).

Recall the definitions in §20. We have the category of lax monoidal functors $\mathscr{D} \longrightarrow \mathscr{T}$ and monoidal transformations and its full subcategory of lax symmetric monoidal functors. These are the structures defined in terms of the external smash product that correspond to monoids and commutative monoids in $\mathscr{D}\mathscr{T}$.

Proposition 22.1. *The category of monoids in $\mathscr{D}\mathscr{T}$ is isomorphic to the category of lax monoidal functors $\mathscr{D} \longrightarrow \mathscr{T}$. The category of commutative monoids in $\mathscr{D}\mathscr{T}$ is isomorphic to the category of lax symmetric monoidal functors $\mathscr{D} \longrightarrow \mathscr{T}$.*

Proof. Let $R : \mathscr{D} \longrightarrow \mathscr{T}$ be lax monoidal. We have a unit map $\lambda : S^0 \longrightarrow R(u)$ and product maps $\phi : R(d) \wedge R(e) \longrightarrow R(d\Box e)$ that make all coherence diagrams commute. We may view ϕ as a natural transformation $R \,\bar{\wedge}\, R \longrightarrow R \circ \Box$. By the defining properties of F_u and $\wedge$, λ and ϕ determine and are determined by maps $\tilde{\lambda} : u^* \longrightarrow R$ and $\tilde{\phi} : R \wedge R \longrightarrow R$ that give R a structure of monoid in $\mathscr{D}\mathscr{T}$. □

Now assume given a lax monoidal functor $R : \mathscr{D} \longrightarrow \mathscr{T}$. Definition 1.9 gives the notion of a $\mathscr{D}$-spectrum X over R, and we see that X is defined by means of a continuous natural transformation $\sigma : X \,\bar{\wedge}\, R \longrightarrow X$. Regarding R as a monoid in $\mathscr{D}\mathscr{T}$, we have the notion of a (right) R-module X defined in terms of a map $X \wedge R \longrightarrow X$. Proposition 1.10 states that R-modules and $\mathscr{D}$-spectra over R are the internal and external versions of the same notion, and the proof of that result is immediate from (21.5). We mimic the definitions of tensor product and Hom functors in algebra to define functors $\wedge_R$ and F_R. For a right R-module X and a left R-module Y, $X \wedge_R Y$ is the coequalizer of $\mathscr{D}$-spaces displayed in the diagram

$$X \wedge R \wedge Y \underset{\mathrm{id}\wedge\mu'}{\overset{\mu\wedge\mathrm{id}}{\rightrightarrows}} X \wedge Y \longrightarrow X \wedge_R Y, \tag{22.2}$$

where μ and μ' are the actions of R on X and Y. For right R-modules Y and Z, $F_R(Y,Z)$ is the equalizer of $\mathscr{D}$-spaces displayed in diagram

$$F_R(Y,Z) \longrightarrow F(Y,Z) \underset{\omega}{\overset{\mu^*}{\rightrightarrows}} F(Y \wedge R, Z). \tag{22.3}$$

Here $\mu^* = F(\mu, \mathrm{id})$ and ω is the adjoint of the composite

$$F(Y,Z) \wedge Y \wedge R \xrightarrow{\varepsilon\wedge\mathrm{id}} Z \wedge R \xrightarrow{\nu} Z,$$

where μ and ν are the actions of R on Y and Z.

In the rest of this section, we assume that R is a commutative monoid in $\mathscr{D}\mathscr{T}$; that is, R is a lax symmetric monoidal functor $\mathscr{D} \longrightarrow \mathscr{T}$. Here the categories of left and right R-modules are isomorphic. Moreover, $X \wedge_R Y$ and $F_R(X,Y)$ inherit R-module structures from X or, equivalently, Y. For R-modules X, Y, and Z,

$$\mathscr{D}\mathscr{S}_R(X \wedge_R Y, Z) \cong \mathscr{D}\mathscr{S}_R(X, F_R(Y,Z)). \tag{22.4}$$

It is formal to prove Theorem 1.7 from the definitions of $\wedge_R$ and F_R.

The external version of an R-algebra is called a $\mathscr{D}$-FSP (functor with smash product) over R. We write τ consistently for symmetry isomorphisms.

Definition 22.5. A $\mathscr{D}$-FSP over R is a $\mathscr{D}$-space X together with a unit map $\eta : R \longrightarrow X$ of $\mathscr{D}$-spaces and a continuous natural product map $\mu : X \bar{\wedge} X \longrightarrow X \circ \Box$ of functors $\mathscr{D} \times \mathscr{D} \longrightarrow \mathscr{T}$ such that the composite

$$X(d) \cong X(d) \wedge S^0 \xrightarrow{\mathrm{id} \wedge \lambda} X(d) \wedge R(u) \xrightarrow{\mathrm{id} \wedge \eta} X(d) \wedge X(u) \xrightarrow{\mu} X(d \Box u) \cong X(d)$$

is the identity and the following unity, associativity, and centrality of unit diagrams commute:

$$\begin{array}{ccc}
R(d) \wedge R(e) & \xrightarrow{\eta \wedge \eta} & X(d) \wedge X(e) \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \mu} \\
R(d \Box e) & \xrightarrow[\eta]{} & X(d \Box e),
\end{array}$$

$$\begin{array}{ccc}
X(d) \wedge X(e) \wedge X(f) & \xrightarrow{\mu \wedge \mathrm{id}} & X(d \Box e) \wedge X(f) \\
{\scriptstyle \mathrm{id} \wedge \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\
X(d) \wedge X(e \Box f) & \xrightarrow[\mu]{} & X(d \Box e \Box f),
\end{array}$$

and

$$\begin{array}{ccccc}
R(d) \wedge X(e) & \xrightarrow{\eta \wedge \mathrm{id}} & X(d) \wedge X(e) & \xrightarrow{\mu} & X(d \Box e) \\
{\scriptstyle \tau}\downarrow & & & & \downarrow{\scriptstyle X(\tau)} \\
X(e) \wedge R(d) & \xrightarrow[\mathrm{id} \wedge \eta]{} & X(e) \wedge X(d) & \xrightarrow[\mu]{} & X(e \Box d)
\end{array}$$

A $\mathscr{D}$-FSP is commutative if the following diagram commutes, in which case the centrality of unit diagram just given commutes automatically:

$$\begin{array}{ccc}
X(d) \wedge X(e) & \xrightarrow{\mu} & X(d \Box e) \\
{\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle X(\tau)} \\
X(e) \wedge X(d) & \xrightarrow[\mu]{} & X(e \Box d).
\end{array}$$

Observe that X has an underlying $\mathscr{D}$-spectrum over R with structure map

$$\sigma = \mu \circ (\mathrm{id} \bar{\wedge} \eta) : X \bar{\wedge} R \longrightarrow X \circ \Box.$$

Proposition 22.6. *The category of R-algebras is isomorphic to the category of $\mathscr{D}$-FSP's over R. The category of commutative R-algebras is isomorphic to the category of commutative $\mathscr{D}$-FSP's over R.*

23. Categorical results on diagram spaces and diagram spectra

We prove the categorical results stated in §§2, 3. First, we use (21.5) to prove Theorem 2.2, which states that the categories of $\mathscr{D}_R$-spaces and $\mathscr{D}$-spectra over R are isomorphic.

Proof of Theorem 2.2. We return to the notations of Construction 2.1. We have

$$\begin{aligned} \mathscr{D}_R(d,e) &= \mathscr{D}\mathscr{S}_R(e^* \wedge R, d^* \wedge R) \\ &\cong \mathscr{D}\mathscr{T}(e^*, d^* \wedge R) \\ &\cong (d^* \wedge R)(e) \\ &\cong \operatorname{colim}_{\alpha: f\square g \longrightarrow e} \mathscr{D}(d,f) \wedge R(g). \end{aligned}$$

Taking α to be the identity map of $d\square e$ and using the identity map $d \longrightarrow d$, we obtain an inclusion $\nu : R(e) \longrightarrow \mathscr{D}_R(d, d\square e)$. Let X be a $\mathscr{D}_R$-space. Pullback along δ gives X a structure of $\mathscr{D}$-space. Pullback along ν of the evaluation map $\mathscr{D}_R(d, d\square e) \wedge X(d) \longrightarrow X(d\square e)$ gives the components $X(d) \wedge R(e) \longrightarrow X(d\square e)$ of a map $X \bar{\wedge} R \longrightarrow X \circ \square$. Via (21.5), this gives an action of R on X. These two actions determine the original action of $\mathscr{D}_R$. Indeed, working conversely, if X is an R-module and $\alpha : f\square g \longrightarrow e$ is a morphism of $\mathscr{D}$, then the composites displayed in the following diagram pass to colimits to define the evaluation maps $(d^* \wedge R)(e) \wedge X(d) \longrightarrow X(e)$ of a functor $X : \mathscr{D}_R \longrightarrow \mathscr{T}$.

$$\begin{array}{ccccc}
\mathscr{D}(d,f) \wedge R(g) \wedge X(d) & & & & \\
\Big\downarrow{\scriptstyle \mathrm{id}\wedge\tau} & & & & \\
\mathscr{D}(d,f) \wedge X(d) \wedge R(g) & \xrightarrow{\varepsilon\wedge\mathrm{id}} & X(f) \wedge R(g) & & \\
\Big\downarrow{\scriptstyle ((-)\square\ \mathrm{id})\wedge\mu} & & \Big\downarrow{\scriptstyle \mu} & & \\
\mathscr{D}(d\square g, f\square g) \wedge X(d\square g) & \xrightarrow[\varepsilon]{} & X(f\square g) & \xrightarrow{X(\alpha)} & X(e).
\end{array}$$

Here ε is the evaluation map of X and μ is the action of R on X. This gives the desired isomorphism of categories between $\mathscr{D}_R\mathscr{T}$ and $\mathscr{D}\mathscr{S}_R$.

Now let R be commutative. To show that the smash products agree under the isomorphism between $\mathscr{D}\mathscr{S}_R$ and $\mathscr{D}_R\mathscr{T}$, we can either compare the definitions of the respective smash products directly or compare the defining universal properties. The unit $(u_{\mathscr{D}_R})^*$ of the smash product of $\mathscr{D}_R$-spaces is isomorphic to R since

$$(u_{\mathscr{D}_R})^*(d) = \mathscr{D}_R(u_{\mathscr{D}_R}, d) = ((u_{\mathscr{D}})^* \wedge R)(e) \cong R(e). \quad \square$$

Returning to the context of §3, let $\iota : \mathscr{C} \longrightarrow \mathscr{D}$ be a continuous functor, where $\mathscr{C}$ is skeletally small. The following definition includes the proof of Proposition 3.2.

Definition 23.1. Define $\mathbb{P} : \mathscr{C}\mathscr{T} \longrightarrow \mathscr{D}\mathscr{T}$ on $\mathscr{C}$-spaces X by letting $\mathbb{P}X$ be the topological left Kan extension of X along ι. It is characterized by the adjunction

$$\mathscr{D}\mathscr{T}(\mathbb{P}X, Y) \cong \mathscr{C}\mathscr{T}(X, \mathbb{U}Y). \tag{23.2}$$

Let ι/d be the topological category of objects ι-over d; its objects are the maps $\alpha : \iota c \longrightarrow d$ in $\mathscr{D}$ and its morphisms are the maps $\psi : c \longrightarrow c'$ in $\mathscr{C}$ such that $\alpha'(\iota\psi) = \alpha$. On an object d, $\mathbb{P}X$ is specified explicitly as the colimit

$$\mathbb{P}X(d) = \operatorname{colim}_{\iota c \to d} X(c)$$

indexed on ι/d. If $\mathscr{C}$ is small, $\mathbb{P}X(d)$ can also be described as the coend

$$\mathbb{P}X(d) = \int^{c\in\mathscr{C}} \mathscr{D}(\iota c, d) \wedge X(c). \tag{23.3}$$

If $\iota : \mathscr{C} \longrightarrow \mathscr{D}$ is fully faithful and $c \in \mathscr{C}$, then the identity map of ιc is a terminal object in $\iota/\iota c$ and therefore $\eta : X \longrightarrow \mathbb{U}\mathbb{P}X$ is an isomorphism.

Now assume that $\mathscr{C}$ and $\mathscr{D}$ are skeletally small symmetric monoidal categories and that ι is a strong symmetric monoidal functor.

Proof of Proposition 3.3. Observe that left Kan extension also gives a functor

$$\mathbb{P} : (\mathscr{C} \times \mathscr{C})\mathscr{T} \longrightarrow (\mathscr{D} \times \mathscr{D})\mathscr{T}.$$

A direct comparison of colimits shows that

$$\mathbb{P}(X \barwedge X') \cong \mathbb{P}X \barwedge \mathbb{P}X', \tag{23.4}$$

and it is trivial to check the analogous isomorphism

$$\mathbb{U}(Y \barwedge Y') \cong \mathbb{U}Y \barwedge \mathbb{U}Y'. \tag{23.5}$$

We have a unit isomorphism $\lambda : u_{\mathscr{D}} \longrightarrow \iota u_{\mathscr{C}}$ and a product isomorphism ϕ : $\Box_{\mathscr{D}} \circ (\iota \times \iota) \longrightarrow \iota \circ \Box_{\mathscr{C}}$. For $(\mathscr{D} \times \mathscr{D})$-spaces Z, ϕ induces a natural isomorphism

$$\mathbb{U}(Z \circ \Box_{\mathscr{D}}) \cong (\mathbb{U}Z) \circ \Box_{\mathscr{C}}. \tag{23.6}$$

The unit isomorphism $\mathbb{P}u^*_{\mathscr{C}} \cong u^*_{\mathscr{D}}$ is given by the last statement of Proposition 3.2, and its adjoint gives the unit isomorphism $u^*_{\mathscr{C}} \cong \mathbb{U}u^*_{\mathscr{D}}$. The defining universal properties of $\wedge$ and $\mathbb{P}$, together with (23.4) and (23.6), give a natural isomorphism

$$\mathscr{D}\mathscr{T}(\mathbb{P}X \wedge \mathbb{P}X', Y) \xrightarrow{\cong} \mathscr{D}\mathscr{T}(\mathbb{P}(X \wedge X'), Y),$$

and this implies the product isomorphism $\mathbb{P}X \wedge \mathbb{P}X' \cong \mathbb{P}(X \wedge X')$. Note the direction of the displayed arrow: $\mathbb{P}$ would not even be lax monoidal if ι were only lax, rather than strong, monoidal. Similarly, the defining universal properties of $\wedge$ and $\mathbb{P}$, together with (23.5) and (23.6), give a composite natural map

$$\begin{array}{rcl}
\mathscr{D}\mathscr{T}(Y \wedge Y', Y \wedge Y') & \cong & (\mathscr{D} \times \mathscr{D})\mathscr{T}(Y \barwedge Y', (Y \wedge Y') \circ \Box_{\mathscr{D}}) \\
& \xrightarrow{\varepsilon^*} & (\mathscr{D} \times \mathscr{D})\mathscr{T}(\mathbb{P}\mathbb{U}(Y \barwedge Y'), (Y \wedge Y') \circ \Box_{\mathscr{D}}) \\
& \cong & (\mathscr{C} \times \mathscr{C})\mathscr{T}(\mathbb{U}(Y \barwedge Y'), \mathbb{U}((Y \wedge Y') \circ \Box_{\mathscr{D}})) \\
& \cong & (\mathscr{C} \times \mathscr{C})\mathscr{T}(\mathbb{U}Y \barwedge \mathbb{U}Y', \mathbb{U}(Y \wedge Y') \circ \Box_{\mathscr{C}}) \\
& \cong & \mathscr{C}\mathscr{T}(\mathbb{U}Y \wedge \mathbb{U}Y', \mathbb{U}(Y \wedge Y')).
\end{array}$$

The product map $\mathbb{U}Y \wedge \mathbb{U}Y' \longrightarrow \mathbb{U}(Y \wedge Y')$ is the image of the identity map of $Y \wedge Y'$ along this composite, and one cannot expect this map to be an isomorphism. $\Box$

Proof of Proposition 3.4. We are given a monoid R in $\mathscr{D}\mathscr{T}$. For objects a and b of $\mathscr{C}$, we have

$$\mathscr{C}_{\mathbb{U}R}(a, b) \cong \operatorname{colim}_{\alpha : c \Box c' \longrightarrow b} \mathscr{C}(a, c) \wedge R\iota(c').$$

Smash products of maps $\iota : \mathscr{C}(a, c) \longrightarrow \mathscr{D}(\iota a, \iota c)$ and identity maps of the spaces $R(\iota c')$ pass to colimits to give maps

$$\mathscr{C}_{\mathbb{U}R}(a, b) \longrightarrow \mathscr{D}_R(\iota(a), \iota(b)).$$

These specify the required extension $\kappa : \mathscr{C}_{\mathbb{U}R} \longrightarrow \mathscr{D}_R$ of $\iota : \mathscr{C} \longrightarrow \mathscr{D}$ on morphism spaces. By inspection, κ is symmetric monoidal when R is commutative. $\Box$

Appendix A. Recollections about equivalences of model categories

We have made heavy use of basic facts about adjoint functors and adjoint equivalences between model categories. We recall these facts for the reader's convenience.

Definition A.1. Let $\mathbb{P} : \mathscr{A} \longrightarrow \mathscr{B}$ and $\mathbb{U} : \mathscr{B} \longrightarrow \mathscr{A}$ be left and right adjoints between model categories $\mathscr{A}$ and $\mathscr{B}$. The functors $\mathbb{P}$ and $\mathbb{U}$ are a *Quillen adjoint pair* if $\mathbb{U}$ preserves q-fibrations and acyclic q-fibrations or, equivalently, if $\mathbb{P}$ preserves q-cofibrations and acyclic q-cofibrations. A Quillen adjoint pair is a *Quillen equivalence* if, for all cofibrant $A \in \mathscr{A}$ and all fibrant $B \in \mathscr{B}$, a map $\mathbb{P}A \longrightarrow B$ is a weak equivalence if and only if its adjoint $A \longrightarrow \mathbb{U}B$ is a weak equivalence.

These notions are discussed thoroughly in [14, §1.3], and the following result is immediate from [14, I.3.13, I.3.16].

Lemma A.2. *Let $\mathbb{P} : \mathscr{A} \longrightarrow \mathscr{B}$ and $\mathbb{U} : \mathscr{B} \longrightarrow \mathscr{A}$ be a Quillen adjoint pair.*

(i) *The total derived functors*

$$\mathbb{LP} : Ho(\mathscr{A}) \longrightarrow Ho(\mathscr{B}) \quad \textit{and} \quad \mathbb{RU} : Ho(\mathscr{B}) \longrightarrow Ho(\mathscr{A})$$

exist and are adjoint.

(ii) $(\mathbb{P}, \mathbb{U})$ *is a Quillen equivalence if and only if $\mathbb{RU}$ or, equivalently, $\mathbb{LP}$ is an equivalence of categories.*

(iii) *If $\mathbb{U}$ creates the weak equivalences of $\mathscr{B}$ and $\eta : A \longrightarrow \mathbb{UP}A$ is a weak equivalence for all cofibrant objects A, then $(\mathbb{P}, \mathbb{U})$ is a Quillen equivalence.*

The following observation [14, 4.3.3] is relevant to Theorems 0.3 and 0.10.

Lemma A.3. *Let $\mathbb{P} : \mathscr{A} \longrightarrow \mathscr{B}$ and $\mathbb{U} : \mathscr{B} \longrightarrow \mathscr{A}$ be a Quillen equivalence, where $\mathbb{P}$ is a strong monoidal functor between monoidal categories (under products $\wedge$). The natural isomorphism $\mathbb{P}X \wedge \mathbb{P}Y \longrightarrow \mathbb{P}(X \wedge Y)$ induces a natural isomorphism*

$$\mathbb{LP}X \wedge^{\mathbb{L}} \mathbb{LP}Y \longrightarrow \mathbb{LP}(X \wedge^{\mathbb{L}} Y).$$

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