LUCAS' SQUARE PYRAMID PROBLEM REVISITED

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Abstract. We discuss positive integer solutions to Diophantine equations of the shape

$$x(x+1)(x+2) = ny^2,$$

where n is a fixed positive integer. From the by-no-means-original observation that such solutions guarantee that n is a congruent number, we show that, given a positive integer m and a nonzero integer a, there exist infinitely many congruent numbers in the residue class a modulo m. After sketching relationships to various classical quartic equations, we conclude with some remarks on computational problems.

1. Introduction

> Une pile de boulets à base carée ne contient un nombre de boulets égal au carré d'un nombre entier que lorsqu'elle en contient vingt-quatre sur le côté de la base. *Éduouard Lucas* [26]

This assertion of Lucas, made first in 1875, amounts to the statement that the only solutions in positive integers (s, t) to the Diophantine equation

$$1^2 + 2^2 + \cdots + s^2 = t^2 \tag{1.1}$$

are given by $(s, t) = (1, 1)$ and $(24, 70)$. Putative solutions by Moret-Blanc [32] and Lucas [27] contain fatal flaws (see e.g. [42] for details) and it was not until 1918 that Watson [42] was able to completely solve equation (1.1). His proof depends upon properties of elliptic functions of modulus $1/\sqrt{2}$ and arguably lacks the simplicity one might desire. A second, more algebraic proof was found in 1952 by Ljunggren [24], though it also is somewhat on the complicated side. Attempts to repair this perceived defect have, in recent years, resulted in a number of elementary proofs, by Ma [28] and [29], Cao and Yu [7] , Cucurezeanu [11] and Anglin [2]. Various generalizations, distinct from that considered here, have been addressed in [12] and [36].

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We rewrite equation (1.1) as

$$\frac{s(s+1)(2s+1)}{6} = t^2$$

and, multiplying by 24 and setting $x = 2s, y = 2t$, find that

$$x(x+1)(x+2) = 6y^2. \tag{1.2}$$

In this paper, we will consider the generalization of this equation obtained by replacing the constant 6 in (1.2) by an arbitrary square free integer n; viz

$$x(x+1)(x+2) = ny^2. \tag{1.3}$$

This corresponds to finding integral "points" on quadratic twists of the elliptic curve $y^2 = u^3 - u$. We begin by proving a general upper bound on the number of integral solutions to (1.3) which implies Lucas' problem as a special case.

2. Solutions to equation (1.3)

If b and d are positive integers, let us denote by $N(b,d)$ the number of solutions in positive integers (x,y) to the Diophantine equation

$$b^2x^4 - dy^2 = 1. \tag{2.1}$$

Our first result is the following :

Theorem 2.1. *If n is a squarefree positive integer, then equation (1.3) has precisely*

$$\sum N(b,d) \leq 2^{\omega(n)} - 1$$

solutions in positive integers x and y. Here, the summation runs over positive integers b and d with $bd = n$ and $\omega(n)$ denotes the number of distinct prime factors of n.

Proof. From (1.3), we may write

$$x = 2^\delta a u^2, \quad x+1 = bv^2, \quad x+2 = 2^\delta c w^2$$

where a, b, c, u, v and w are positive integers, $\delta \in \{0,1\}$ and

$$(a,b) = (a,c) = (b,c) = 1.$$

Setting $d = ac$, it follows that

$$b^2v^4 - d\left(2^\delta uw\right)^2 = 1$$

where $bd = n$. Conversely, if X and Y are positive integers for which $b^2X^4 - dY^2 = 1$, where b and d are positive integers with $bd = n$, writing $x = bX^2 - 1$ and $y = XY$, we find that

$$x(x+1)(x+2) = bdy^2 = ny^2.$$

To prove the inequality in Theorem 2.1, we note, since we assume n to be squarefree, that there are precisely $2^{\omega(n)}$ pairs of positive integers (b, d) with $bd = n$. Since $N(b, 1) = 0$, the stated bound is essentially a consequence of theorems of Cohn [10] and the author and Gary Walsh [4]. To state this result, we require some notation. Let $d > 1$ be a squarefree integer and let $T + U\sqrt{d}$ be the fundamental solution to $X^2 - dY^2 = 1$; i.e. T and U are the smallest positive integers with $T^2 - dU^2 = 1$. Define T_k and U_k via the equation

$$T_k + U_k\sqrt{d} = \left(T + U\sqrt{d}\right)^k$$

and let the *rank of apparition* $\alpha(b)$ be the smallest positive integer k such that b divides T_k (where we set $\alpha(b) = \infty$ if no such integer exists).

Theorem 2.2. *Let b and d be squarefree positive integers. Then $N(b, d) \leq 1$ unless $(b, d) = (1, 1785)$ in which case there are two positive solutions to (2.1), given by $(x, y) = (13, 4)$ and $(239, 1352)$. If $N(b, d) = 1$, so that (2.1) has a solution in positive integers (x, y), then, if $b = 1$, we may conclude that $x^2 \in \{T_1, T_2\}$. If, on the other hand, $b > 1$, then $bx^2 = T_{\alpha(b)}$.*

For $n = 1785 = 3 \cdot 5 \cdot 7 \cdot 17$, it remains to show that (1.3) has at most 15 positive integral solutions (x, y). This is immediate from Theorem 2.2 upon noting that (2.1) is insoluble modulo 3 if $(b, d) = (255, 7)$.

□

Since (1.2) has the solutions

$$(x, y) = (1, 1), (2, 2), \quad \text{and} \quad (48, 140),$$

we conclude from Theorem 2.1 that it has no others with x and y positive. These lead to precisely the solutions $(s, t) = (1, 1)$ and $(24, 70)$ in Lucas' original problem.

Theorem 2.1 implies that equation (1.3) has at most a single solution in positive integers, if n is prime. In fact, work of Ljunggren [24] on $N(1, p)$ immediately enables one to strengthen this :

Corollary 2.3. *If n is prime, then equation (1.3) has no solutions in positive integers x and y, unless $n \in \{5, 29\}$. In each of these cases, there is precisely one such solution, given by $(x, y) = (8, 12)$ and $(9800, 180180)$, respectively.*

It is reasonable to suppose that the dependence in Theorem 2.1 on $\omega(n)$ is an artificial one. Indeed, a conjecture of Lang (see e.g. Abramovich [1] and Pacelli [34]) implies that the number of integral solutions to (1.3) should be absolutely bounded. We present some computations in support of this in our final section.

3. Congruent Numbers

A positive integer n is called a *congruent number* if there exists a right triangle with sides of rational length and area n. It is a classical result (and elementary to prove; see e.g. Chahal [9], Theorems 1.34 and 7.24) that n is congruent precisely when the elliptic curve

$$E_n : \; Y^2 = X^3 - n^2 X$$

has positive Mordell rank; i.e. $E_n(\mathbb{Q})$ is infinite. This leads to

Proposition 3.1. *If n is a positive integer for which equation (1.3) has a solution in positive $x, y \in \mathbb{Q}$, then n is a congruent number or, equivalently, $E_n(\mathbb{Q})$ has positive rank.*

Proof. As is well known (see e.g Corollary 7.23 of [9]), the torsion subgroup of $E_n(\mathbb{Q})$ consists of the point at infinity, together with $(0,0), (n,0)$ and $(-n,0)$ (i.e. the obvious points of order 2). Writing $X = n(x+1)$ and $Y = n^2 y$, it follows that a positive rational solution (x,y) to (1.3) corresponds to a point with positive rational coordinates (X,Y) on E_n, which is necessarily of infinite order. By our above remarks, this implies that n is a congruent number. □

In [8], Chahal applied an identity of Desboves to show that there are infinitely many congruent numbers in each residue class modulo 8 (and, in particular, infinitely many squarefree congruent numbers, congruent to $1, 2, 3, 5, 6$ and 7 modulo 8). We can generalize this as follows :

Theorem 3.2. *If m is a positive integer and a is any integer, then there exist infinitely many (not necessarily squarefree) congruent numbers n with $n \equiv a \pmod{m}$. If, further, $\gcd(a,m)$ is squarefree, then there exist infinitely many (squarefree) congruent numbers n with $n \equiv a \pmod{m}$.*

Proof. Suppose that l is a positive integer and set

$$n = m^4 l^3 - l = (m^2 l - 1)(m^2 l + 1) l.$$

It follows that $(x,y) = (m^2 l - 1, m)$ is a positive solution to (1.3). Since $n \equiv -l \pmod{m}$, every $l \equiv -a \pmod{m}$ yields a value of n with $n \equiv a \pmod{m}$ and, by Proposition 3.1, n congruent. If, further, $\gcd(a,m)$ is squarefree, we may apply work of Mirsky [30] to conclude that n is squarefree for infinitely many $l \equiv -a \pmod{m}$. Indeed, writing $l = mk - a$ for $k \in \mathbb{N}$, Theorems 1 and 2 of [30] show, if we denote by $N(X)$ the cardinality of the set of positive integers $k \leq X$ for which n is squarefree, that

$$N(X) = AX + O\left(X^{2/3+\epsilon}\right) \text{ as } X \to \infty,$$

for any $\epsilon > 0$. Here $A = A(a,m) > 0$ is a computable constant.

□

It is worth remarking that a much more refined version of the above result should follow from the work of Gouvea and Mazur [13].

4. Quartic Equations

There is a vast literature on equations of the form $Ax^4 - By^2 = \pm 1$ (the reader is directed to the survey paper of Walsh [41] for more details). In particular, there are many papers giving explicit characterizations of $N(b,d)$ when $\omega(bd)$ is suitably small (see e.g. [5], [6], [14], [15], [16], [17], [18], [19], [20]). The preceding observations (specifically Theorem 2.1 and Proposition 3.1) imply that $N(b,d) = 0$ whenever bd is noncongruent. Together with criteria for noncongruent numbers (see e.g. Table 3.8 of [37]), this enables one to recover many classical vanishing results for $N(b,d)$. It also leads to various new statements, the simplest of which is the following:

Corollary 4.1. *If b and d are positive integers with $bd = 2pq$, where p and q are distinct primes with*

$$p \equiv q \equiv 5 \pmod 8,$$

then equation (2.1) has no solution in positive integers x and y.

For the state of the art on the problem of determining congruent numbers, the reader is directed to, for example, [31], [33] and [39]. A good overview of this subject can be found in [21].

5. Computations

Given $n \in \mathbb{N}$, as noted previously, the set of positive integer solutions to (1.3) corresponds to a subset of the integer "points" on E_n. We could thus apply standard computational techniques based either on the solution of Thue equations (see e.g. [40]) or on lower bounds for linear forms in elliptic logarithms (see e.g. [38]) to find all integer solutions (X,Y) to $Y^2 = X^3 - n^2X$ and check to see which, if any, yield solutions to (1.3). To find positive integral solutions to (1.3), for all squarefree n up to some bound, say $n \leq N$, it is computationally much more efficient however, to rely upon Theorem 2.2. With this approach, we begin by computing fundamental units in $\mathbb{Q}(\sqrt{d})$ for each squarefree $d \leq N$ (see e.g. [23]). For each squarefree n, we then retrieve the data for the $2^{\omega(n)} - 1$ quadratic fields corresponding to nontrivial divisors n_1 of n, and determine $N(n_1, n/n_1)$ by combining Theorem 2.2 with the following lemma due to Lehmer [22]:

Lemma 5.1. *Let $\epsilon = T + U\sqrt{d}$ be the fundamental solution to $X^2 - dY^2 = 1$, and $T_k + U_k\sqrt{d} = \epsilon^k$ for $k \geq 1$. Let p be prime and $\alpha(p)$ denote, as before,* *the rank of apparition of p in the sequence $\{T_k\}$.*

(i) *If $p = 2$ then $\alpha(p) = 1$ or ∞*
(ii) *If $p > 2$ divides d then $\alpha(p) = \infty$*
(iii) *If $p > 2$ fails to divide d then either $\alpha(p) | \frac{p - (\frac{d}{p})}{2}$ or $\alpha(p) = \infty$.*

Here $(\frac{d}{p})$ denotes the usual Legendre symbol.

We carry out this program with $n \leq N = 10^5$ and note that, in each instance, equation (1.3) has at most three solutions in positive integers x and y. In fact, of the 60794 squarefree n, $1 \leq n \leq 10^5$, only 280 corresponding equations of the shape (1.3) possess positive solutions. Moreover, only for

$$n = 6, 210, 546, 915, 1785, 7230, 13395, 16206, 17490, 20930, 76245$$

do we find more than a single such solution (with the first two values having three positive solutions and the remaining ones having two apiece).

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