ON THE EQUATION $\sigma^*(\sigma^*(n)) = 2n$

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ABSTRACT. As usual $\sigma(n)$ denotes the sum of the divisors of n and $\sigma^*(n)$ the sum of the unitary divisors of n. Suryanarayana defined super perfect numbers as solutions of $\sigma(\sigma(n)) = 2n$ and showed that even super perfect numbers are of the form $n = 2^k$ provided $2^{k+1}-1$ is a prime. He asked for the existence of odd super perfect numbers. J.L. Hunsucker and Carl Pomerance (Indian J. Math. 17 (1975)) showed that there are no such numbers less than $7 \cdot 10^{24}$. In this paper we consider solutions of $\sigma^*(\sigma^*(n)) = 2n$ which may be called unitary super perfect numbers (USP numbers). While $\sigma^*(\sigma^*(n)) = 2n+1$ has no solutions and $\sigma^*(\sigma^*(n)) = 2n-1$ has $n = 1, 3$ as the only solutions, there are both even and odd USP numbers — ten of them up to 24000 (namely 2, 9, 165, 238, 1640, 4320, 10250, 100824, 13500 and 23760) and 22 of them upto 10^8 as listed in the appendix at the end. We do not know of any odd USP numbers other than 9 and 165. Perhaps such numbers are finite in number whereas even USP numbers are likely to be infinite.

§1. Introduction

Let $\sigma(n)$ denote the sum of the divisors of n. It is well-known that a natural number n is called a perfect number if $\sigma(n) = 2n$. In 1969, D. Suryanarayana (cf. [10]) called a natural number n 'super perfect' if $\sigma(\sigma(n)) = 2n$. He proved that an even number n is super perfect if and only if $n = 2^k$ where $2^{k+1}-1$ is a prime number. He asked whether there were odd super perfect numbers. To this question, H.J. Kanold (cf. [5]) proved that an odd super perfect number, if it exists, must be a square. Till today, neither an example of an odd super perfect number has been found nor the non-existence of such a number has been demonstrated. However, J.L. Hunsucker and Carl Pomerance (cf. [4]) proved that there are no odd super perfect numbers less than $7 \cdot 10^{24}$. We believe that this is the best

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result obtained on the upper bound of odd super perfect numbers.

A divisor d of n is called a unitary divisor (cf. [2]) and write $d||n$, if $(d, n/d) = 1$, where (a, b) denotes, as usual, the greatest common divisor of a and b. Let $\sigma^*(n)$ denote the sum of the unitary divisors of n. It is known (cf. [2]) that σ^* is a multiplicative function; that is, $\sigma^*(mn) = \sigma^*(m)\sigma^*(n)$, whenever $(m, n) = 1$. Also, $\sigma^*(1) = 1$ and $\sigma^*(p^\alpha) = p^\alpha + 1$, where p is a prime and α is a positive integer.

Analogous to the notion of perfect numbers, in 1966, M.V. Subbarao and L.J. Warren [9] introduced the notion of a unitary perfect number as follows: a positive integer n is called a unitary perfect number if $\sigma^*(n) = 2n$. They proved that there are no odd unitary perfect numbers. The only five unitary perfect numbers known till now are 6, 60, 90, 87360 and $146361946186458562560000 = 2^{18} \cdot 3 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313$. The first four examples were due to M.V. Subbarao and L.J. Warren [9] and the last being due to C.R. Wall [11]. It is not known whether there exists a unitary perfect number not divisible by 3 (see M.V. Subbarao and L.J. Warren [9]) and also whether there are infinitely many unitary perfect numbers (see M.V. Subbarao [7]). We also refer to the paper of M.V. Subbarao, T.J. Cook, R.S. Newberry and J.M. Weber [8] for certain results on unitary perfect numbers. An open problem is whether there exists a unitary multiperfect number – that is an integer n for which $\sigma^*(n) = kn$ for an integer $k > 2$.

In this paper, analogous to the notion of superperfect numbers, we introduce the notion of unitary super perfect number as follows: a natural number n is called unitary super perfect (USP) if $\sigma^*(\sigma^*(n)) = 2n$. The first ten such numbers are 2, 9, 165, 238, 1640, 4320, 10250, 10824, 13500 and 23760. Thus there are both even and odd USP numbers. To our knowledge such numbers do not seem to have been studied so far. In this paper we obtain some of the simplest properties of these numbers.

Let $\omega(n)$ denote the number of distinct prime factors of n. In Theorems 3.1 and 3.2, we prove that the only USP numbers with $\omega(n) = 1$ are 2 and 9. In Theorem 3.3, we prove that there are no USP numbers with $\omega(n) = 2$. In Theorem 3.4, we prove that if $\omega(n) \geq 3$ and n is an odd USP number such that $3|(n, \sigma^*(n))$, then n must be square-free (that is, a product of distinct primes); if $\omega(n) \geq 4$, then $\omega(n) \geq 46$. If n is an odd USP number not divisible by 3, we show (Theorem 3.5) that $3|\sigma^*(n)$; in such a case we prove that n must be square-free, $\omega(n)$ is even and $\omega(n) \geq 52$. If n is an odd USP number divisible by $3, 3 \nmid \sigma^*(n)$ and $\omega(n) \geq 3$, we show that $\omega(n) \geq 19$ (Theorems 3.6 and 3.8). In Theorem 3.7, we characterize 165 as the only odd USP number with $\omega(n) = 3$. We do not know any examples of odd USP numbers other than 9 and 165. We believe that the set of such numbers is finite.

In §4, we consider even USP numbers. It is easily seen that 238 is a solution of $\omega(n) = 3$ and $\omega(\sigma^*(n)) = 2$. In Theorem 4.1, we show that if n is an even USP number with $\omega(\sigma^*(n)) = 2$, then $3 \mid \sigma^*(n)$, $3 \nmid n$ and n must be square-free. In Theorem 4.2, we characterize 238 as the only even USP number satisfying $\omega(n) = 3$ and $\omega(\sigma^*(n)) = 2$. We are unable to produce any even USP number other than 238 satisfying $\omega(\sigma^*(n)) = 2$. However, we show that if n is an even USP number with $\omega(\sigma^*(n)) = 2$ and $\omega(n) \geq 4$, then $\omega(n)$ is odd (Theorem 4.3) and n must be very large (see Remark 4.3). It is not difficult to see that the numbers 1640 and 10250 are USP numbers with $\omega(n) = \omega(\sigma^*(n)) = 3$; the USP numbers 4320 and 13500 are solutions of the pair of equations $\omega(n) = 3$ and $\omega(\sigma^*(n)) = 4$. We do not know whether these are the only solutions of the corresponding equations. Further, the number 23760 is a solution of $\omega(n) = \omega(\sigma^*(n)) = 4$ and 10824 is a solution of $\omega(n) = 4$ and $\omega(\sigma^*(n)) = 3$; we conjecture that such USP numbers are finite in number.

§2. Preliminaries

Throughout this paper, we use only 'elementary' ideas which, however, may be sometimes tricky, and to save on the length of the paper, we omit proofs in several places which the interested reader should be able to supply with little difficulty.

Lemma 2.1. (a) *If* $2 \leq \alpha < k$ *and* $2^\alpha - 1 | 2^k + 1$, *then* $\alpha = 2$ *and* k *is odd.*
(b) *If* a *is odd,* $a > 1$ *and* $a^\beta - 1 | a^\alpha + 1$ *for some* β, $1 \leq \beta \leq \alpha$ *then* $a = 3$ *and* $\beta = 1$.

Lemma 2.2. (cf. [4], Theorem 13, p. 13). *If* $(a, b) = 1$, *then every odd prime factor of* $a^2 + b^2$ *is of the form* $4n + 1$.

Lemma 2.3. *Let* a *be odd and* $2^a \equiv -1 \ (mod\ p)$, *where* p *is an odd prime. Then* $p \equiv 1$ *or* $3 \ (mod\ 8)$.

Lemma 2.4. *Let* $a > 1$ *and odd. If* $2^x \| a^\alpha + 1$, *where* α *is odd, then* $2^x \| a^d + 1$ *for every divisor* d *of* α.

Proof. We can assume that $\alpha > 1$ and $1 \leq d < \alpha$. Let $a^\alpha + 1 = 2^x u$, where $x \geq 1$ and u odd. Since α is odd and $d | \alpha$, $a^d + 1 | a^\alpha + 1$. Hence we can write $a^d + 1 = 2^{x_1} t$ where $x_1 \geq 1$, t odd and $t | u$. Let $r = \alpha / d$ so that $r \geq 3$ and odd.

We have

$$a^{\alpha} = (a^d)^r = (2^{x_1}t - 1)^r = -1 + \sum_{k=1}^{r} \binom{r}{k} (-1)^{r-k} 2^{x_1 k} t^k$$

so that

$$\begin{aligned} a^{\alpha} + 1 &= 2^{x_1} \left\{ rt + \sum_{k=2}^{r} \binom{r}{k} (-1)^{r-k} 2^{x_1(k-1)} t^k \right\} \\ &= 2^{x_1} \cdot m, \quad m \text{ odd}, \end{aligned}$$

since r and t are odd. Hence $x_1 = x$ so that $2^x \| a^d + 1$.

Corollary 2.1. *If a is odd and $a > 1$, then $a^{\alpha} + 1 = 2^x$ implies that $\alpha = 1$.*

Proof. Suppose $\alpha > 1$. If α is odd, then by Lemma 2.4, $a + 1 = 2^x$, which is not possible. If α is even, since $y^2 \equiv 1 \pmod 4$, when y is odd, we have

$$2^x = a^{\alpha} + 1 = (a^{\alpha/2})^2 + 1 \equiv 2 \pmod 4,$$

which is not possible since $x \geq 2$. Hence $\alpha = 1$.

Lemma 2.5. *Let α be an odd prime. If p and q are odd primes such that*

$$p^{\alpha} + 1 = 2^x q^y, \quad x \geq 1, \quad y \geq 1,$$

then $p + 1 = 2^x$.

Proof. Suppose that $p + 1 = 2^x q^z$, $1 \leq z < y$. We have

$$p^{\alpha} = (2^x q^z - 1)^{\alpha} = -1 + \sum_{k=1}^{\alpha} (-1)^{\alpha-k} \binom{\alpha}{k} (2^x q^z)^k,$$

so that

$$2^x q^y = p^\alpha + 1 = \sum_{k=1}^{\alpha} (-1)^{\alpha-k} \binom{\alpha}{k} (2^x q^z)^k$$
$$= 2^x q^z \left\{ \alpha + \sum_{k=2}^{\alpha} (-1)^{\alpha-k} \binom{\alpha}{k} (2^x q^z)^{k-1} \right\}.$$

Hence

$$q^{y-z} = \alpha + \sum_{k=2}^{\alpha} (-1)^{\alpha-k} \binom{\alpha}{k} (2^x q^z)^{k-1}. \tag{2.1}$$

From (2.1), it follows that $q|\alpha$ and hence $q = \alpha$ since α is a prime. Replacing α by q in (2.1) and cancelling q on both sides of (2.1), we obtain,

$$q^{y-z-1} - 1 = \sum_{k=2}^{q} (-1)^{q-k} \binom{q}{k} 2^{x(k-1)} q^{z(k-1)-1} \tag{2.2}$$
$$= \sum_{k=2}^{q-1} (-1)^{q-k} \binom{q}{k} 2^{x(k-1)} q^{z(k-1)-1} + 2^{x(q-1)} q^{z(q-1)-1}.$$

Since $q|\binom{q}{k}$, $1 \le k \le q-1$ (Hardy and Wright [3], Theorem 75, p. 64), it follows that q divides the left hand side of (2.2), which is possible only when $y - z - 1 = 0$ or $y = z + 1$. Thus we must have $p^\alpha + 1 = 2^x q^{z+1}$ and $p + 1 = 2^x q^z$ so that $q = (p^\alpha + 1)/(p + 1)$. Hence

$$p | q - 1. \tag{2.3}$$

Since $p + 1 = 2^x q^z$, $z \ge 1$, we have

$$q | p + 1. \tag{2.4}$$

(2.3) and (2.4) imply that $q = p + 1$, which is impossible since p and q are odd primes. Hence $z = 0$ and $p + 1 = 2^x$.

Lemma 2.6. *Let* α *be odd and composite. If* p *and* q *are odd primes, then* $p^\alpha + 1 = 2^x q^y$, $x \geq 1$ *and* $y \geq 1$, *is impossible.*

Proof. Suppose $p^\alpha + 1 = 2^x q^y$. If s is any prime factor of α, then $p^s + 1 | p^\alpha + 1$. By Corollary 2.1, $p^s + 1 = 2^x$ is not possible. Hence $p^s + 1 = 2^x q^k$ for some $k \geq 1$. By Lemma 2.5, $p + 1 = 2^x$. Let $d|\alpha$ and $1 < d < \alpha$. Let $r = \alpha/d$. We have $p^d + 1 = 2^x q^z$ where $1 \leq z < y$. Also

$$p^\alpha = (p^d)^r = (2^x q^z - 1)^r = -1 + \sum_{k=1}^{r} \binom{r}{k} (-1)^{r-k} 2^{xk} q^{zk},$$

so that

$$2^x q^y = p^\alpha + 1 = 2^x q^z \left\{ r + \sum_{k=2}^{r} \binom{r}{k} (-1)^{r-k} 2^{x(k-1)} q^{z(k-1)} \right\}$$

and hence

$$q^{y-z} = r + \sum_{k=2}^{r} \binom{r}{k} (-1)^{r-k} 2^{x(k-1)} q^{z(k-1)}.$$

It follows that $q | r = \alpha/d$ and so $q|\alpha$. Therefore $p^q + 1 | p^\alpha + 1$ and hence $p^q + 1 = 2^x q^\beta$ for some $\beta \geq 1$ so that

$$p^q \equiv -1 \pmod{q}. \tag{2.5}$$

Since q is a prime, by Fermat's theorem,

$$p^q \equiv p \pmod{q}. \tag{2.6}$$

(2.5) and (2.6) imply that $q | p + 1 = 2^x$, which is impossible since q is an odd prime.

Lemma 2.7. *We have*

(a) $2^x \equiv -1 \pmod 5$ *if and only if* $x \equiv 2 \pmod 4$.
(b) $2^x \equiv -1 \pmod{11}$ *if and only if* $x \equiv 5 \pmod{10}$.
(c) $2^x \equiv -1 \pmod{17}$ *if and only if* $x \equiv 4 \pmod 8$.
(d) $2^x \equiv -1 \pmod{53}$ *if and only if* $x \equiv 26 \pmod{52}$.
(e) $2^x \equiv -1 \pmod{107}$ *if and only if* $x \equiv 53 \pmod{106}$.
(f) $3^x \equiv -1 \pmod 5$ *if and only if* $x \equiv 2 \pmod 4$.

(g) $3^x \equiv -1 \pmod{11}$ *has no solution.*
(h) $3^x \equiv -1 \pmod{17}$ *if and only if* $x \equiv 8 \pmod{16}$.
(i) $3^x \equiv -1 \pmod{53}$ *if and only if* $x \equiv 26 \pmod{52}$.
(j) $3^x \equiv -1 \pmod{107}$ *has no solution.*
(k) *If* a *is odd,* $5 \nmid 2^a + 1$, $17 \nmid 2^a + 1$ *and* $53 \nmid 2^a + 1$.
(l) $5 \nmid 3^x + 1$ *and* x *is even imply that* $4 | x$.
(m) $5 \nmid 3^x + 1$ *and* x *is even imply that* $53 \nmid 3^x + 1$.

Proof. In solving the exponential congruences (a) - (j), we use that 2 is a primitive root mod 5, 11, 53 and 107; 3 is a primitive root modulo 17 and adopt the method as given in Example 3, Apostol [1], Chapter 10, page 215, (k) is a consequence of (a), (c) and (d). (l) follows from (f). (m) follows from (l) and (i).

Lemma 2.8. *The equation* $2^a + 1 = 3^x$ *has no solutions if* $x \geq 3$.

Proof. Let $2^a + 1 = 3^x$ for some positive integers a and $x \geq 3$. Then we have

$$2^a \equiv -1 \pmod{3^x} \Longleftrightarrow a \equiv \frac{\varphi(3^x)}{2} \pmod{\varphi(3^x)}$$
$$\Longleftrightarrow a \equiv 3^{x-1} \pmod{2 \cdot 3^{x-1}}$$

where φ is Euler-totient function. In particular $3^{x-1} | a$ so that $a \geq 3^{x-1}$. Since $2^n > 3n$ for $n \geq 4$, we obtain

$$3^x = 2^a + 1 \geq 2^{3^{x-1}} + 1 > 2^{3^{x-1}} > 3 \cdot 3^{x-1} = 3^x,$$

a contradiction.

Lemma 2.9. *Let* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ *be odd and* USP. *Then we can find positive integers* $a \geq r$, b *and an odd prime* q *such that*

$$(p_1^{\alpha_1} + 1)(p_2^{\alpha_2} + 1) \cdots (p_r^{\alpha_r} + 1) = 2^a q^b,$$

and

$$(2^a + 1)(q^b + 1) = 2 p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}.$$

Proof. Follows from the definition of USP number.

Lemma 2.10. *Let n be an odd* USP *number and* $3|n$. *Let* $\sigma^*(n) = 2^a q^b$, *where q is an odd prime* $\neq 3$. *Let* $\omega(n) \geq 3$. *We have*

(a) $\omega(n)$ *is odd* $\Longleftrightarrow 3|(2^a + 1, q^b + 1)$.

If $\omega(n)$ is odd, we have

(b) $3^{\alpha_1} \| n \Longrightarrow \alpha_1 \geq 2$.
(c) a *and* b *are odd and* $b \geq 3$.
(d) $q \equiv 5 \pmod{12}$.
(e) $5|n \Longleftrightarrow q \equiv 4 \pmod 5$.
(f) $7|n \Longleftrightarrow b \equiv 3 \pmod 6$ *and either* $q \equiv 3 \pmod 7$ *or* $q \equiv 5 \pmod 7$.
(g) *If* $q = 5$ *and* $7|n$ *then* $\alpha_1 \geq 6$, *where* $3^{\alpha_1} \| n$.

Lemma 2.11. *Let* $3|n$, n *odd and n be* USP. *Let* $\sigma^*(n) = 2q^b$, *where* $q \equiv 1 \pmod 4$ *and b odd. If* $\omega(n) \geq 4$, *then* $b \geq 3$.

§3. Main Results

We first prove the following:

Theorem 3.1. 2^k *is not a* USP *number if* $k \geq 2$.

Proof. Suppose 2^k is USP. We must have

$$2^{k+1} = \sigma^*(\sigma^*(2^k)) = \sigma^*(2^k + 1). \tag{3.1}$$

If $2^k + 1 = p^\alpha$, p an odd prime, then we must have

$$2^{k+1} = \sigma^*(p^\alpha) = p^\alpha + 1 = 2^k + 2,$$

so that $2^k = 2^{k-1} + 1$ and hence $k = 1$. However $k \geq 2$. Hence we can assume that

$$2^k + 1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}, \tag{3.2}$$

where p_i's are distinct odd primes and $r \geq 2$. From (3.2) and (3.1) we have

$$2^{k+1} = \sigma^*(p_1^{\alpha_1} \cdots p_r^{\alpha_r}) = \prod_{i=1}^{r} (p_i^{\alpha_i} + 1), \tag{3.3}$$

so that

$$p_i^{\alpha_i}+1=2^{a_i}, \quad i=1,2,\ldots,r, \tag{3.4}$$

where $a_i \geq 2$ for each i and

$$\sum_{i=1}^{r} a_i = k+1. \tag{3.5}$$

Further $a_i \neq a_j$ for $i \neq j$. From (3.4) and (3.2),

$$2^k+1=\prod_{i=1}^{r}(2^{a_i}-1). \tag{3.6}$$

From (3.6) and Lemma 2.1(a), it follows that $a_i = 2$, for $i = 1,2,\ldots,r$. This is a contradiction since $r \geq 2$ and $a_i \neq a_j$ for $i \neq j$. Hence 2^k can not be USP if $k \geq 2$.

Theorem 3.2. *If p^α is a USP number, where p is an odd prime, then $p=3$ and $\alpha=2$.*

Proof. If p^α is a USP number, we can find positive integers a, b and an odd prime q such that

$$p^\alpha+1=2^a q^b \tag{3.7}$$

and

$$(2^a+1)(q^b+1)=2p^\alpha. \tag{3.8}$$

From (3.7) and (3.8), we obtain that $2^a+3=q^b(2^a-1)$, so that $a=1$. Hence $q=5$ and $b=1$. Using these results in (3.8), we obtain that $p=3$ and $\alpha=2$.

Corollary 3.1. *The only USP numbers n with $\omega(n)=1$ are $n=2$ and $n=9$.*

Theorem 3.3. *If $n=p_1^{\alpha_1}p_2^{\alpha_2}$, p_1 and p_2 being distinct primes, then n can not be a USP number.*

Proof. First we assume that n is even. Without loss of generality we may suppose that $p_1=2$ and p_2 is an odd prime. We must have

$$2^{\alpha_1+1}p_2^{\alpha_2}=\sigma^*\big(\sigma^*(2^{\alpha_1}p_2^{\alpha_2})\big)=\sigma^*\big((2^{\alpha_1}+1)(p_2^{\alpha_2}+1)\big). \tag{3.9}$$

Let

$$(2^{\alpha_1}+1)(p_2^{\alpha_2}+1) = 2^a q_1^{b_1}\cdots q_r^{b_r}, \tag{3.10}$$

where q_i's are distinct odd primes. From (3.9) and (3.10), we obtain

$$(2^a+1)(q_1^{b_1}+1)\cdots(q_r^{b_r}+1) = 2^{\alpha_1+1}p_2^{\alpha_2}. \tag{3.11}$$

From (3.11), $2^a+1 = p_2^{\beta_2}$, $1 \le \beta_2 \le \alpha_2$ and from (3.10), $2^a | p_2^{\alpha_2}+1$, so that $p_2^{\beta_2}-1 | p_2^{\alpha_2}+1$. By Lemma 2.1(b), we must have $p_2 = 3$, $\beta_2 = 1$ so that $a = 1$. Using these values in (3.10) and (3.11), we obtain

$$(2^{\alpha_1}+1)(3^{\alpha_2}+1) = 2\cdot q_1^{b_1}q_2^{b_2}\cdots q_r^{b_r}, \tag{3.12}$$

and

$$(q_1^{b_1}+1)\cdots(q_r^{b_r}+1) = 2^{\alpha_1+1}3^{\alpha_2-1}. \tag{3.13}$$

From (3.12), $2 \| 3^{\alpha_2}+1$ so that α_2 is even. We distinguish the following cases:

CASE 1. Let α_1 be odd so that $3 | 2^{\alpha_1}+1$. From (3.12), $q_i = 3$ for some i. Let $q_1 = 3$. From (3.13), it follows that $r \ge 2$ and $3^{b_1}+1 = 2^x$, $2 \le x \le \alpha_1+1$, so that $b_1 = 1$, by Corollary 2.1 so that $x = 2$. We obtain from (3.12) and (3.13) that

$$(2^{\alpha_1}+1)(3^{\alpha_2}+1) = 6q_2^{b_2}\cdots q_r^{b_r}, \tag{3.14}$$

and

$$(q_2^{b_2}+1)\cdots(q_r^{b_r}+1) = 2^{\alpha_1-1}3^{\alpha_2-1}. \tag{3.15}$$

From (3.15) we have for $2 \le i \le r$,

$$q_i^{b_i}+1 = 2^{x_i}3^{y_i},$$

wich $x_i > 0$, $y_i \ge 0$, $\sum_{i=2}^{r} x_i = \alpha_1 - 1$ and $\sum_{i=2}^{r} y_i = \alpha_2 - 1$. From (3.14), we then obtain

$$
\begin{aligned}
(2^{\alpha_1}+1)(3^{\alpha_2}+1) &= 6\prod_{i=2}^{r}(2^{x_i}3^{y_i}-1) \\
&< 6\prod_{i=2}^{r}2^{x_i}3^{y_i} \\
&= 2^{\alpha_1}3^{\alpha_2},
\end{aligned}
$$

a contradiction.

CASE 2. *Let* α_1 *be even.* Since α_2 is also even, it follows from Lemma 2.2 and (3.12), that $q_i \equiv 1 \pmod 4$ for $1 \le i \le r$. Hence $2 \| q_i^{b_i}+1$, for $i = 1, 2, \ldots, r$. Hence (3.13) implies that $r = \alpha_1 + 1$ so that r is odd. We show that $3 | q_i^{b_i}+1$ for $1 \le i \le r$. If $3 \nmid q_i^{b_i}+1$, for some i, then (3.13) implies that $q_i^{b_i}+1 = 2^x$, $x \ge 2$, so that $q_i^{b_i} \equiv -1 \pmod 4$. However $q_i^{b_i} \equiv 1 \pmod 4$, since $q_i \equiv 1 \pmod 4$. Thus $3 | q_i^{b_i}+1$, for $i = 1, 2, \ldots, r$. Hence b_i is odd for $i = 1, 2, \ldots, r$ so that $q_i^{b_i}+1 = 2 \cdot 3^{y_i}$, $y_i \ge 1$, $1 \le i \le r$. Lemma 2.6 implies that b_i is a prime or $b_i = 1$ for $1 \le i \le r$. If b_i is a prime for some i, Lemma 2.5 implies that $q_i + 1 = 2$, which is impossible. Hence $b_i = 1$ for $1 \le i \le r$. Thus we must have

$$(2^{\alpha_1}+1)(3^{\alpha_2}+1) = 2q_1 \ldots q_r \tag{3.16}$$

$$(q_1+1)\cdots(q_r+1) = 2^{\alpha_1+1}3^{\alpha_2-1} \tag{3.17}$$

and

$$q_i + 1 = 2 \cdot 3^{y_i}, \quad 1 \le i \le r. \tag{3.18}$$

If $y_i \ge 2$ for $i = 1, 2, \ldots, r$ then (3.18) implies that $q_i \equiv -1 \pmod 9$. Since $\alpha_2 \ge 2$, $3^{\alpha_2}+1 \equiv 1 \pmod 9$. Using these results in (3.16) and the fact that r is odd, we get that

$$2^{\alpha_1}+1 \equiv 2 \cdot (-1)^r \equiv -2 \pmod 9$$

or

$$2^{\alpha_1} \equiv -3 \pmod 9,$$

which is impossible. Hence $y_i = 1$ for exactly one i say "$i = 1$" and $y_i \ge 2$ for $i = 2, 3, \ldots, r$. In such a case we have $q_1 = 5$ and $q_i \equiv -1$

(mod 9) for $i = 2, 3, \ldots, r$. Using these results in (3.16), we obtain,

$$\begin{aligned} 2^{\alpha_1} + 1 &\equiv 2 \cdot 5 \cdot (-1)^{r-1} \pmod 9 \\ &\equiv 1 \pmod 9, \end{aligned}$$

which is again not possible.

Thus in all cases we could obtain a contradiction. Hence $n = 2^{\alpha_1} p_2^{\alpha_2}$ can not be USP. The case when n is odd can be easily dealt with.

Theorem 3.4. *Let n be an odd* USP *number. If $3 | (n, \sigma^*(n))$, then we have*

(a) n *is square-free if* $\omega(n) \geq 3$.
(b) $\omega(n) = 3$ *implies that* $n = 165$.
(c) *If* $\omega(n) \geq 4$, *then* $17 \nmid n$, $53 \nmid n$ *and* $107 \nmid n$.
(d) *If* $\omega(n) \geq 4$ *and* $\omega(n)$ *is odd, we have*
(d_1) $\omega(n) \geq 93$, *if* $5 \nmid n$ *and* $11 \nmid n$.
(d_2) $\omega(n) \geq 63$, *if* $5 \nmid n$ *and* $11 \mid n$.
(d_3) $\omega(n) \geq 227$, *if* $5 \mid n$ *and* $11 \mid n$.
(d_4) $\omega(n) \geq 51$, *if* $5 \mid n$ *and* $11 \nmid n$.
(e) *If* $\omega(n) \geq 4$ *and* $\omega(n)$ *is even, we have*
(e_1) $\omega(n) \geq 50$, *if* $5 \nmid n$ *and* $11 \nmid n$.
(e_2) $\omega(n) \geq 178$, *if* $5 \nmid n$ *and* $11 \mid n$.
(e_3) $\omega(n) \geq 238$, *if* $5 \mid n$ *and* $11 \mid n$.
(e_4) $\omega(n) \geq 46$, *if* $5 \mid n$ *and* $11 \nmid n$.

Proof. Let $n = 3^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be an odd unitary super-perfect number, with $3 < p_2 < p_3 < \cdots < p_r$. By Lemma 2.9 $(q = 3)$ we can find positive integers $a \geq r$ and b such that

$$(3^{\alpha_1} + 1)(p_2^{\alpha_2} + 1) \cdots (p_r^{\alpha_r} + 1) = 2^a 3^b, \tag{3.19}$$

and

$$(2^a + 1)(3^b + 1) = 2 \cdot 3^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}. \tag{3.20}$$

From (3.25), $3^{\alpha_1} + 1 = 2^x$ for some $x \geq 1$. Hence by Corollary 2.1, $\alpha_1 = 1$. Also, from (3.26), $3 | 2^a + 1$, so that a is odd. Again from (3.20), $2 \| 3^b + 1$, so that b is even. Thus (3.19) and (3.20) can be written as

$$(p_2^{\alpha_2} + 1)(p_3^{\alpha_3} + 1) \cdots (p_r^{\alpha_r} + 1) = 2^{a-2} 3^b, \tag{3.21}$$

and

$$(2^a+1)(3^b+1) = 6p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_r^{\alpha_r} \tag{3.22}$$

with a *odd* and b *even.*

Proof of (a). If α_i is even for some i, $2 \le i \le r$, then by Corollary 2.1, $p_i^{\alpha_i}+1 = 2^x$ is impossible. Hence for even α_i, from (3.21) we must have $p_i^{\alpha_i}+1 = 2^x3^y$ for some positive integers x and y. However by Lemma 2.2, $3 \nmid p_i^{\alpha_i}+1$, if α_i is even. Hence α_i must be odd for $2 \le i \le r$. By Lemmas 2.2, 2.3 and (3.28), $p_i \equiv 1 \pmod 4$ or $p_i \equiv 3 \pmod 8$ or $p_i \equiv 1 \pmod 8$ for $2 \le i \le r$. If $3 \nmid p_i^{\alpha_i}+1$ for some i, $2 \le i \le r$, then (3.27) implies that $p_i^{\alpha_i}+1 = 2^x$. Corollary 2.1 implies that $\alpha_i = 1$ so that $p_i + 1 = 2^x$. If $p_i \equiv 1 \pmod 4$, then $x = 1$ which is not possible. If $p_i \equiv 3 \pmod 8$, then $x = 2$. This is also not possible since $p_i \ne 3$. Thus $3 | p_i^{\alpha_i}+1$ for $2 \le i \le r$, so that $p_i^{\alpha_i}+1 = 2^x3^y$, where $1 \le x \le 2$ and $y \ge 1$. Since α_i is odd, it follows from Lemmas 2.5 and 2.6 that $\alpha_i = 1$ or α_i is a prime. If α_i is a prime, then by Lemma 2.5, $p_i + 1 = 2^x$, $1 \le x \le 2$, which is not possible. Hence $\alpha_i = 1$ for $2 \le i \le r$. Thus (a) follows.

From (a), (3.21) and (3.22), we have the following;

$$(p_2+1)(p_3+1)\cdots(p_r+1) = 2^{a-2}3^b \tag{3.23}$$

and

$$(2^a+1)(3^b+1) = 6p_2p_3\cdots p_r, \tag{3.24}$$

with a odd, $a \ge r$ and b even.

If can be shown that for $2 \le i \le r$,

$$p_i + 1 = \begin{cases} 2\cdot 3^{y_i},\ y_i & \text{odd, if} \quad p_i \equiv -3 \pmod 8 \\ 2\cdot 3^{e_i},\ e_i & \text{even, if} \quad p_i \equiv 1 \pmod 8 \\ 4\cdot 3^{y_i'},\ y_i' & \text{odd, if} \quad p_i \equiv 3 \pmod 8. \end{cases} \tag{3.25}$$

Proof of (b). Let $r = 3$ and

$$\begin{aligned} k_1 &= \#\{p_i : 2 \le i \le 3,\ p_i \equiv 3 \pmod 8) \\ k_2 &= \#\{p_i : 2 \le i \le 3,\ p_i \equiv 1 \pmod 8) \\ k_3 &= \#\{p_i : 2 \le i \le 3,\ p_i \equiv -3 \pmod 8), \end{aligned} \tag{3.26}$$

so that

$$k_1 + k_2 + k_3 = 2. \tag{3.27}$$

From (3.25), (3.26) and (3.23), we have

$$a - 2 = 2k_1 + k_2 + k_3. \tag{3.28}$$

From (3.25) and (3.24), we have

$$b = \sum_{p_i \in S_1} y_i' + \sum_{p_i \in S_3} y_i + \sum_{p_i \in S_2} e_i,$$

where S_1, S_2 and S_3 are the sets defining k_1, k_2 and k_3 respectively. Since y_i' and y_i are odd, it follows that k_1 and k_3 are of the same parity. Since a is odd, it follows from (3.28) that $k_2 + k_3$ is odd so that k_2 and k_3 are of opposite parity. Hence if k_1 is even then k_3 is also even and k_2 must be odd. This does not occur in virtue of (3.27). It follows from (3.27) that $k_1 = k_3 = 1$ and $k_2 = 0$. From (3.28) we must have $a = 5$. From (3.24) $(r = 3)$ it follows that either p_2 or p_3 is 11, say $p_3 = 11$. Using this, in place of (3.23) and (3.24), we obtain the equations $12(p_2+1) = 2^3 \cdot 3^b$ and $3^b + 1 = 2p_2$, which imply that $b = 2$ and $p_2 = 5$. Hence $n = 3 \cdot 5 \cdot 11 = 165$ and the proof of (b) is complete.

Proof of (c). Suppose that $17|n$. From (3.30) it follows that $17|2^a + 1$ or $17|3^b + 1$. Since a is odd, Lemma 2.7(c) implies that $17 \nmid 2^a + 1$. Hence $17|3^b + 1$. Again by Lemma 2.7(h), $b \equiv 8 \pmod{16}$ so that $b = 8u$ where u is odd. We note that $3^8 + 1|3^b + 1$ and $3^8 + 1 = 2 \cdot 17 \cdot 193$. Thus $193|3^b + 1$ and 193 is a prime. From (3.30) and (3.29), we must have that $193 \equiv -1 \pmod 3$ which is false. Hence $17 \nmid n$.

We assume that $53|n$. From (3.27), $53|2^a + 1$ or $53|3^b + 1$. Since a is odd, Lemma 2.7(d) implies that $53 \nmid 2^a + 1$. Hence $53|3^b + 1$ so that $b \equiv 26 \pmod{52}$, by Lemma 2.7(i). Let $b = 26u$ where u is odd. It follows that $3^{26} + 1|3^b + 1$. But $3^{26} + 1 = 2541865828330 = 2 \cdot 5 \cdot 53 \cdot r$, where $r = 4795973261$ is a prime, and $r + 1$ is not of the form $2^x 3^y$. Hence $53 \nmid n$.

Suppose that $107|n$. From (3.24) and Lemma 2.7(e) and (j), it follows that $a \equiv 53 \pmod{106}$, so that $2^{53} + 1|2^a + 1$. We have

$$2^{53} + 1 = 3 \cdot 107 \cdot 28059810762433.$$

Hence $107 \nmid n$.

The proof of (c) is complete.

We assume that $r \geq 4$ so that $b \geq 4$. Hence we have

$$3^b + 1 \equiv 1 \pmod{81}. \tag{3.29}$$

(d) Let r be odd.

Proof of (d_1). Let $5 \nmid n$ and $11 \nmid n$. It follows from (3.29) and (c) that $y_i \geq 5$, $e_i \geq 4$ and $y_i' \geq 5$ so that for $2 \leq i \leq r$, $p_i \equiv -1 \pmod{81}$. Using this in (3.24) we obtain from (3.35), $2^a + 1 \equiv 6 \cdot (-1)^{r-1} = 6 \pmod{81}$, so that $2^a \equiv 5 \pmod{81}$. Using that 2 is a primitive root (mod 81), we find that this congruence is equivalent to the congruence $a \equiv \text{ind}_2 5 \pmod{54}$ and $\text{ind}_2 5 = 23$. Hence $a \equiv 23 \pmod{54}$. We have $2^{23} + 1 = 3p$, where $p = 2796203$ is a prime. Also, $p + 1 = 12.233017 = 12 \cdot 43 \cdot 5419$ and $3 \nmid 233017$. From (3.24) and (3.23) it follows that $a \neq 23$. Hence $a \geq 23 + 54 = 77$. Since $43 | 129 = 2^7 + 1 | 2^{77} + 1$ and $43 \not\equiv -1 \pmod{3}$, it follows that $a \neq 77$. Hence $a \geq 77 + 54 = 131$. Thus, here $4744297 | 2^{131} + 1$ and $3 \nmid 4744298$, so $a \geq 131 + 54 = 185$. From (3.24) and (3.23) we see that $a - 2 \leq 2(r-1)$ or $a \leq 2r$. Since a is odd, we must have $a \leq 2r - 1$ so that $2r - 1 \geq a \geq 185$ and hence $r \geq 93$.

Proof of (d_2). Let $5 \nmid n$ and $11 | n$. Let $p_2 = 11$ so that $p_i \equiv -1 \pmod{81}$ for $3 \leq i \leq r$. Hence from (3.24) and (3.29), we obtain $2^a + 1 \equiv 6 \cdot 11(-1)^{r-2} \equiv -66 \pmod{81}$, so that $2^a \equiv -67 \equiv 14 \pmod{81}$, which is equivalent to $a \equiv 17 \pmod{54}$. Also since $11 | n$, Lemma 2.7 ((b) and (g)) implies that $a \equiv 5 \pmod{10}$. This together with the previous congruence relation implies that $a \equiv 125 \pmod{270}$. Hence $r \geq 63$.

Proof of (d_3). Let $5 | n$ and $11 | n$. Let $p_2 = 5$ and $p_3 = 11$ so that by (3.24) and (c), $p_i \equiv -1 \pmod{81}$, for $4 \leq i \leq r$. We have by (3.28) and (3.23), $2^a + 1 \equiv 6 \cdot 5 \cdot 11 \cdot (-1)^{r-3} \equiv 6 \pmod{81}$, so that $2^a \equiv 5 \pmod{81}$. Hence $a \equiv 23 \pmod{54}$. Since $11 | n$ from parts (b) and (g) of Lemma 2.7 and (3.23), it follows that $a \equiv 5 \pmod{10}$; this together with the previous congruence implies that $a \equiv 185 \pmod{270}$. Since $1777 | 2^{37} + 1 | 2^{185} + 1$ and 1777 is a prime not congruent to $-1 \pmod{3}$, it follows that $a \neq 185$. Hence $a \geq 185 + 270 = 455$, so that $r \geq 227$.

Proof of (d_4). Let $5 | n$ and $11 \nmid n$. Let $p_2 = 5$ so that $p_i \equiv -1 \pmod{81}$ for $3 \leq i \leq r$. Hence by(3.29) and (3.24),

$$2^a + 1 \equiv 6.5(-1)^{r-2} \equiv -30 \pmod{81},$$

so that $2^a \equiv 50 \pmod{81}$. Hence $a \equiv 47 \pmod{54}$. Since 283 is a prime factor of $2^{47}+1$ and $283 \not\equiv -1 \pmod 3$, it follows that $a \not\equiv 47$. Hence $a \geq 47+54 = 101$ so that $r \geq 51$.

(e) Let r be even and $r \geq 4$.

Proof of (e_1) . Suppose $5 \nmid n$ and $11 \nmid n$. From (3.29) and (3.24), we obtain that $2^a \equiv -7 \pmod{81}$ so that $a \equiv 43 \pmod{54}$. We have

$$2^{43}+1 = 3.2932031007403.$$

If $s = (2^{43}+1)/3$, then s is a prime and $s+1$ is not of the form $2^x 3^y$. Hence $a \neq 43$ and so $a \geq 43+54 = 97$ so that $r \geq 50$.

Proof of (e_2) . Let $5 \nmid n$ and $11|n$. We obtain that $2^a \equiv -16 \pmod{81}$ so that $a \equiv 31 \pmod{54}$. Since $11|n$, we also have $a \equiv 5 \pmod{10}$; combining this with the previous congruence relation we obtain that $a \equiv 85 \pmod{270}$. We have $43691|2^{85}+1$ and $p+1 = 4\cdot 3\cdot 11\cdot 331$. It follows that $a \neq 85$ so that $a \geq 355$. Hence $r \geq 178$.

Proof of (e_3) . Suppose $5|n$ and $11|n$. We obtain that $2^a \equiv -7 \pmod{81}$ and hence $a \equiv 43 \pmod{54}$. Also, $11|n$ implies that $a \equiv 5 \pmod{10}$ so that $a \equiv 205 \pmod{270}$. We have $83|2^{205}+1$. Since $83+1 = 84 = 4\cdot 3\cdot 7 \neq 2^x 3^y$, it follows that $a \neq 205$ so that $a \geq 475$. Hence $r \geq 238$.

Proof of (e_4) . Let $5|n$ and $11 \nmid n$. We obtain that $2^a \equiv 29 \pmod{81}$ so that $a \equiv 37 \pmod{54}$. We note that $a \neq 37$ since 1777 is a prime factor of $2^{37}+1$ and $1777 \not\equiv -1 \pmod 3$. Hence $a \geq 91$, and $r \geq 46$.

The proof of Theorem 3.4 is complete.

Theorem 3.5. *Let* n *be an odd* USP *and* $3 \nmid n$. *Then we have*

(a) $3|\sigma^*(n)$.
(b) *Each prime factor* p *of* n *is congruent to* $1 \pmod 4$ *and*

$$p = \begin{cases} 2\cdot 3^y - 1, & y \text{ odd, if } p \equiv -3 \pmod 8 \\ 2\cdot 3^e - 1, & e \text{ even, if } p \equiv 1 \pmod 8). \end{cases}$$

(c) $\omega(n)$ *is even.*
(d) n *is square-free.*
(e) *If* $5 \nmid n$ *then* $53 \nmid n$.
(f) *If* $5 \nmid n$ *and* $17 \mid n$, *then* $\omega(n) \equiv 12 \pmod{216}$ *and* $\omega(n) \geq 228$.

(g) *If* $5 \nmid n$ *and* $17 \nmid n$, *then* $\omega(n) \equiv 0 \pmod{216}$ *and* $\omega(n) \geq 432$.
(h) *If* $5|n$, $17|n$ *and* $53|n$ *then* $\omega(n) \equiv 52 \pmod{216}$.
(i) *If* $5|n$, $17|n$ *and* $53 \nmid n$, *then* $\omega(n) \equiv 124 \pmod{216}$.
(j) *If* $5|n$, $17 \nmid n$ *and* $53|n$, *then either* "$\omega(n) \equiv 22 \pmod{108}$ *and* $\omega(n) \geq 130$" *or* "$\omega(n) \equiv 76 \pmod{108}$".
(k) *If* $5|n$, $17 \nmid n$ *and* $53 \nmid n$, *then either* "$\omega(n) = 94 \pmod{108}$" *or* "$\omega(n) \equiv 40 \pmod{108}$ *and* $\omega(n) \geq 148$".

Theorem 3.6. *Let* n *be odd,* $3|n$ *and* n *be* USP. *Let* $q|\sigma^*(n)$ *and* $q \neq 3$. *If* $\omega(n) \geq 5$ *and odd, then* $\omega(n) \geq 19$.

Proof. Let $n = 3^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, so that we have

$$(3^{\alpha_i}+1)(p_2^{\alpha_2}+1)\cdots(p_r^{\alpha_r}+1) = 2^a q^b, \tag{3.30}$$

and

$$(2^a+1)(q^b+1) = 2 \cdot 3^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}, \tag{3.31}$$

for some positive integers $a \geq r$ and b.

From (3.30) and (3.31), we obtain

$$2 = \left(1+\frac{1}{3^{\alpha_1}}\right) \prod_{i=2}^{r} \left(1+\frac{1}{p_i^{\alpha_i}}\right) \left(1+\frac{1}{2^a}\right)\left(1+\frac{1}{q^b}\right) = N, \tag{3.32}$$

say. By Lemma 2.10, we have $\alpha_1 \geq 2$ and $b \geq 3$. Using these in (3.32) for $r = 5$, we obtain

$$2 \leq \left(1+\frac{1}{9}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\left(1+\frac{1}{2^5}\right)\left(1+\frac{1}{5^3}\right) < 2,$$

a contradiction.

Let $7 \leq r \leq 17$. We distinguish the following cases:

Case 1. Let $5|n$ and $p_2 = 5$. By Lemma 2.10, we have $q \equiv 5 \pmod{12}$ and $q \equiv 4 \pmod 5$ so that $q \equiv 29 \pmod{60}$; hence $q \geq 29$. Also, we note that $q \equiv 29 \pmod{60}$ and $p_2 = 5$ imply that $\alpha_2 \geq 6$ and $\alpha_1 \geq 4$. Using these inequalities, $a \geq 7$ and $b \geq 3$, we obtain from (3.38),

$$2 = N \leq \left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{5^6}\right) \prod_{i=4}^{18} \left(1+\frac{1}{q_i}\right)\left(1+\frac{1}{2^7}\right)\left(1+\frac{1}{29^3}\right) < 2,$$

a contradiction, where q_i is the i-th prime with $q_1 = 2$, $q_2 = 3$, $q_3 = 5$ and so on.

CASE 2. Suppose $5 \nmid n$ and $7|n$.

(a) Let $q = 5$. By (g) of Lemma 2.10, we have $\alpha_1 \geq 6$.
We have

$$2 = N \leq \left(1 + \frac{1}{2^7}\right)\left(1 + \frac{1}{3^6}\right)\left(1 + \frac{1}{5^3}\right)\prod_{i=4}^{19}\left(1 + \frac{1}{q_i}\right) < 1 \cdot 994,$$

a contradiction.

(b) Let $q \neq 5$. It follows from Lemma 2.10(d) that $q \geq 17$. Also, $\alpha_1 \geq 7$.
We obtain

$$2 = N \leq \left(1 + \frac{1}{2^7}\right)\left(1 + \frac{1}{3^7}\right)\left(1 + \frac{1}{17^3}\right)\prod_{i=4}^{19}\left(1 + \frac{1}{q_i}\right) < 1 \cdot 98,$$

a contradiction.

CASE 3. Let $5 \nmid n$ and $7 \nmid n$. Using $\alpha_1 \geq 2$, $a \geq 7$, $b \geq 3$ and $q \geq 5$, we obtain from (3.38)

$$2 = N \leq \left(1 + \frac{1}{3^2}\right)\prod_{i=5}^{20}\left(1 + \frac{1}{q_i}\right)\left(1 + \frac{1}{2^7}\right)\left(1 + \frac{1}{5^3}\right) < 2,$$

a contradiction.

Hence the proof of Theorem 3.6 is complete.

Theorem 3.7. *Let n be an odd,* USP *number with $\omega(n) = 3$. Then $n = 165$.*

Proof. By Theorem 3.5, $3|n$. If $3|\sigma^*(n)$, it follows from Theorem 3.4 that $n = 165$. Let $q|\sigma^*(n)$ and $q \neq 3$. Let $n = 3^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}$, a and b be as in Theorem 3.6. Taking $r = 3$, using $a \geq 3$, $b \geq 3$, $\alpha_1 \geq 2$ and $q \geq 5$, we obtain from (3.32) $(r = 3)$,

$$2 \leq \left(1 + \frac{1}{3^2}\right)\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{7}\right)\left(1 + \frac{1}{2^3}\right)\left(1 + \frac{1}{5^3}\right) < 2,$$

a contradiction.

Theorem 3.8. *Let* $3|n$, n *odd and* n *be* USP. *Let* $q|\sigma^*(n)$ *and* $q \neq 3$. *Let* $\omega(n)$ *be even and* $\omega(n) \geq 4$. *Then* $\omega(n) \geq 19$.

§4. Even Unitary Super Perfect Numbers

Throughout this section, we assume that n is an even unitary super perfect number. First we prove the following:

Theorem 4.1. *If* $\omega(\sigma^*(n)) = 2$, *then* $3|\sigma^*(n)$, $3 \nmid n$ *and* n *is square-free.*

Proof. Let $n = 2^\alpha p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where p_i's are distinct odd primes with $p_1 < p_2 < \cdots < p_r$. We can find positive integers $a \geq r$, b and an odd prime q such that

$$(2^\alpha + 1)(p_1^{\alpha_1} + 1) \cdots (p_r^{\alpha_r} + 1) = 2^a q^b, \tag{4.1}$$

and

$$(2^a + 1)(q^b + 1) = 2^{\alpha+1} p_1^{\alpha_1} \cdots p_r^{\alpha_r}. \tag{4.2}$$

From (4.1), $2^\alpha + 1 = q^c$ for some c, $1 \leq c \leq b$ and from (4.2), $2^{\alpha+1}|q^b + 1$. Hence $q^c - 1|q^b + 1$. By Lemma 2.1, we must have $c = 1$ and $q = 3$ so that $\alpha = 1$. Thus $3|\sigma^*(n)$. Taking $\alpha = 1$ and $q = 3$ in (4.1) and (4.2), we obtain

$$(p_1^{\alpha_1} + 1) \cdots (p_r^{\alpha_r} + 1) = 2^a 3^{b-1}, \tag{4.3}$$

and

$$(2^a + 1)(3^b + 1) = 4 p_1^{\alpha_1} \cdots p_r^{\alpha_r}. \tag{4.4}$$

Suppose $3|n$ so that $p_1 = 3$. From (4.3) $(p_1 = 3)$, it follows that $3^{\alpha_1} + 1 = 2^x$ and Corollary 2.1 implies that $\alpha_1 = 1$. Taking $\alpha_1 = 1$ and $p_1 = 3$ in (4.3) and (4.4), we obtain

$$1 = \left(1 + \frac{1}{p_2^{\alpha_2}}\right)\left(1 + \frac{1}{p_3^{\alpha_3}}\right) \cdots \left(1 + \frac{1}{p_r^{\alpha_r}}\right)\left(1 + \frac{1}{2^a}\right)\left(1 + \frac{1}{3^b}\right) > 1,$$

a contradiction. Hence $3 \nmid n$.

We now prove that n is square-free. Since $4|3^b + 1$ (from (4.4)), b must be odd. Also, $b \geq 3$. If $b = 1$, then $p_2^{\alpha_2} + 1 = 2^x$ for some $x \geq 3$ and $2^x - 1|2^a + 1$, which is not possible by Lemma 2.1. Since $3 \nmid n$, from (4.4) it follows that $3 \nmid 2^a + 1$ and hence a is even. By Lemma 2.2, every prime factor of $2^a + 1$ is congruent to 1 (mod 4). Since b is odd, every odd prime factor of $3^b + 1$ is congruent to 1 or -5 (mod 12). Thus for each i, $1 \le i \le r$, $p_i \equiv 1 \pmod 4$ or $p_i \equiv 1 \pmod{12}$ or $p_i \equiv -5 \pmod{12}$. We fix i, $1 \le i \le r$. From (4.3), $p_i^{\alpha_i} + 1 = 2^{x_i}$ or $p_i^{\alpha_i} + 1 = 2^{x_i}3^{y_i}$ for some $x_i \ge 1$ and $y_i \ge 1$. If $p_i^{\alpha_i} + 1 = 2^{x_i}$, then $\alpha_i = 1$ by Corollary 2.1. If $p_i^{\alpha_i} + 1 = 2^{x_i}3^{y_i}$, then α_i must be odd, by Lemma 2.2. If $\alpha_i > 1$, then by Lemma 2.6 and 2.5, α_i is a prime and $p_i + 1 = 2^{x_i}$. Hence $x_i \ge 2$ so that $p_i \equiv 3 \pmod 4$. So, $p_i \equiv -5$ (mod 12). Since α_i is odd,

$$-5^{\alpha_i} + 1 = (-5)^{\alpha_i} + 1 \equiv p_i^{\alpha_i} + 1 = 2^{x_i}3^{y_i} \equiv 0 \pmod{12},$$

so that $5^{\alpha_i} \equiv 1 \pmod{12}$. But $5^{\alpha_i} \equiv 2^{\alpha_i} \equiv -1 \pmod 3$, since α_i is odd. This is a contradiction. Hence $\alpha_i = 1$. This being true for $i = 1, 2, \ldots, r$, it follows that n is square-free.

Theorem 4.2. *If* $\omega(n) = 3$ *and* $\omega(\sigma^*(n)) = 2$, *then* $n = 238$.

Proof. Taking $r = 2$ and $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 1$ in (4.3) and (4.4), we obtain

$$(p_1 + 1)(p_2 + 1) = 2^a 3^{b-1} \tag{4.5}$$

and

$$(2^a + 1)(3^b + 1) = 4p_1p_2. \tag{4.6}$$

From (4.6) we obtain

$$2^a + 1 = p_2 \tag{4.7}$$

and

$$3^b + 1 = 4p_1.$$

From (4.5), $p_2 + 1 = 2 \cdot 3^y$ for some $y \ge 1$, so that by (4.7), $2 \cdot 3^y = p_2 + 1 = 2^a + 2 = 2(2^{a-1} + 1)$ and hence $2^{a-1} + 1 = 3^y$. By Lemma 2.8, it follows that either "$a - 1 = 1$, $y = 1$" or "$a - 1 = 3$, $y = 2$". If $a = 2$, then from (4.7), $p_2 = 5$. From (4.5) we obtain that $p_1 + 1 = 2 \cdot 3^{b-2}$. From (4.6) we get that

$$4p_1 = 3^b + 1 = \frac{9(p_1 + 1)}{2} + 1 = \frac{9p_1 + 11}{2},$$

so that $8p_1 = 9p_1 + 11$, a contradiction. Hence $a - 1 = 3$, that is, $a = 4$ and $p_2 = 17$. Using these values in (4.5) and (4.6), we get that $p_1 = 7$. Hence $n = 2 \cdot 7 \cdot 17 = 238$ and Theorem 4.2 follows.

Remark 4.1. From the proof of Theorem 4.1, it follows that in case $\omega(\sigma^*(n)) = 2$, then every prime factor p of n is either congruent to $1 \pmod 4$ or congruent to 1 or $-5 \pmod{12}$. If $p \equiv 1 \pmod 4$ and $p|n$, then $p = 2 \cdot 3^y - 1$ for some $y \geq 1$. We also note that if $p \equiv -5 \pmod{12}$ and $p|n$, then p must be a Mersenne prime.

Remark 4.2. It is not difficult to show that $\omega(n) = 3$ and $3^3 \| \sigma^*(n)$ imply that $n = 238$.

Theorem 4.3. *If* $\omega(\sigma^*(n)) = 2$, *then* $\omega(n)$ *is odd.*

Proof. Let $n = 2p_1p_2 \cdots p_r$ and $t = \#\{1 \leq i \leq r : \ p_i \equiv -5 \pmod{12}\}$. Let S denote the set defining t. If $i \in S$, by Remark 4.1, we must have $p_i = 2^{x_i} - 1$, where x_i is an odd prime. Also, $p_i \notin S$ implies that $p_i = 2 \cdot 3^{y_i} - 1$ for some $y_i \geq 1$. From (4.3) $(\alpha_1 = \alpha_2 = \cdots = \alpha_r = 1)$, we obtain

$$a = \sum_{i \in S} x_i + r - t. \tag{4.8}$$

Since a is even, from (4.8) it follows that r is even. Since $\omega(n) = r + 1$, $\omega(n)$ must be odd.

Theorem 4.4. *Let* $\omega(\sigma^*(n)) = 2$ *and* $\omega(n) \geq 4$. *Then* n *must be divisible by a Mersenne prime.*

Proof. Let $n = 2p_1p_2 \cdots p_r$ so that we have

$$(p_1 + 1) \cdots (p_r + 1) = 2^a 3^{b-1}, \tag{4.9}$$

and

$$(2^a + 1)(3^b + 1) = 4p_1 \cdots p_r, \tag{4.10}$$

with a even and b odd. Cleary $b = 1$ can not occur. If $b = 3$, then $7|n$ and hence the conclusion of the theorem holds. We may assume that $b \geq 4$ so that $3^b + 1 \equiv \pmod{81}$.

Suppose that n is not divisible by any Mersenne prime. By Remark 4.1, it follows that every prime factor of n is of the form $2 \cdot 3^y - 1$ for some $y \geq 1$. Hence if $p_i \notin \{5, 17, 53\}$, then $p_i \equiv -1 \pmod{81}$.

Therefore, if no $p_i \in \{5, 17, 53\}$, from (4.10) and since r is even, we obtain

$$2^a + 1 \equiv 4 \cdot (-1)^r \equiv 4 \pmod{81},$$

a contradiction.

Suppose $p_i \in \{5, 17, 53\}$ for some i, $1 \le i \le r$. From Lemma 2.7, it follows that $5|n \Longrightarrow 17 \nmid n$, $5 \nmid n \Longrightarrow 53 \nmid n$ and $17|n \Longrightarrow 53 \nmid n$.

Let $53|n$. Then $5|n$ and $17 \nmid n$. From (4.10), we obtain

$$2^a + 1 \equiv 7 \pmod{81},$$

a contradiction. Hence $53 \nmid n$.

Let $5|n$. Then $17 \nmid n$. From (4.10), we obtain

$$2^a + 1 \equiv 4 \cdot 5 \cdot (-1)^{r-1} \equiv -20 \pmod{81},$$

a contradiction. Hence $5 \nmid n$.

Let $17|n$. From (4.10), we obtain,

$$2^a + 1 \equiv 4 \cdot 17 \cdot (-1)^{r-1} \equiv -68 \pmod{81},$$

again a contradiction. The proof of Theorem 4.4 is complete.

Remark 4.3. Let $\omega(\sigma^*(n)) = 2$ and $\omega(n) \ge 4$ (so that $\omega(n) \ge 5$ since $\omega(n)$ is odd). If n is divisible by a Mersenne prime $\ge 2^{89} - 1$, then obviously we must have $n \ge 2 \cdot 5 \cdot 7 \cdot 11 \cdot (2^{89} - 1) \ge 385 \cdot 2^{89}$. Let the largest Mersenne prime factor of n be less than $2^{89} - 1$. Let $M_p = 2^p - 1$, so that the only Mersenne primes dividing n are among $M_3, M_5, M_7, M_{13}, M_{17}, M_{19}, M_{31}$ and M_{61}. Let S denote the set of these eight Mersenne primes. We observe that $5|n \Longrightarrow 17 \nmid n$, $53|n \Longrightarrow 5|n$ and $17|n \Longrightarrow 53 \nmid n$. Hence we can distinguish four cases viz., (i) $53|n$ (ii) $5|n$ and $53 \nmid n$ (iii) $5 \nmid n$ and $17|n$ (iv) $5 \nmid n$ and $17 \nmid n$. In each case we obtain 255 possibilities by assuming that n is divisible by exactly k primes from S where $1 \le k \le 8$. Suppose $5|n$, $53|n, M_3|n$ and n is not divisible by any of the other seven Mersenne primes in S. In this case from (4.10) we obtain that $2^a + 1 \equiv 32 \pmod{81}$ so that $2^a \equiv 31 \pmod{81}$, which is equivalent to $a \equiv 20 \pmod{54}$. Since $53|n$ if and only if $a \equiv 26 \pmod{52}$, it follows that $a \equiv 182 \pmod{1404}$. In this case ((4.8), $k = 1, \sum x_i = 3$) $a = r - 1 + 3 = r + 2 = \omega(n) - 1 + 2 = \omega(n) + 1$, so that $\omega(n) \equiv 181 \pmod{1404}$. In this way we have examined all the $4 \cdot 255 = 1020$ possibilities. In case (i) it has been found that out of the 255 possibilities only 62 are feasible (in the remaining $255 - 62$ possibilities either we obtain that a is odd or that the exponential congruence $2^a \equiv y \pmod{81}$ has no solution) and the least value of $\omega(n)$ is 45. In case (ii) we found that 62 possibilities are feasible and $\omega(n) \geq 19$. In case (iv), 58 possibilities are feasible and $\omega(n) \geq 21$. In case (iii), there are 62 possibilities which are feasible and $\omega(n) \equiv 11 \pmod{108}$ or $\omega(n) \equiv 17 \pmod{108}$ or $\omega(n) \equiv 9 \pmod{108}$ or $\omega(n) \geq 21$. If $\omega(n) \equiv 11 \pmod{108}$, then the only Mersenne primes dividing n are M_3, M_{13}, M_{17} and M_{61}; or $M_3, M_5, M_{13}, M_{19}, M_{31}$ and M_{61}; or $M_3, M_{13}, M_{17}, M_{19}, M_{31}$ and M_{61}. If $\omega(n) \equiv 17 \pmod{108}$, the only Mersenne primes dividing n are M_{17}, M_{19} and M_{61}. If $\omega(n) \equiv 9 \pmod{108}$, the only Mersenne primes dividing n are M_3, M_5, M_{13}, M_{17} and M_{19}. Thus we find that n is very large.

§5. Concluding Remarks

It is not difficult to show that the equation $\sigma^*(\sigma^*(n)) = 2n + 1$ has no solution and the only solutions of $\sigma^*(\sigma^*(n)) = 2n - 1$ are 1 and 3.

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Appendix

All numbers $n \leq 10^8$ such that $\sigma^*\sigma^*(n) = kn$

k	n
2	$2 = 2$
2	$9 = 3^2$
3	$10 = 2 \cdot 5$
4	$18 = 2 \cdot 3^2$
3	$30 = 2 \cdot 3 \cdot 5$
2	$165 = 3 \cdot 5 \cdot 11$
2	$238 = 2 \cdot 7 \cdot 17$
3	$288 = 2^5 \cdot 3^2$
3	$660 = 2^2 \cdot 3 \cdot 5 \cdot 11$
3	$720 = 2^4 \cdot 3^2 \cdot 5$
2	$1640 = 2^3 \cdot 5 \cdot 41$
2	$4320 = 2^5 \cdot 3^3 \cdot 5$
2	$10250 = 2 \cdot 5^3 \cdot 41$
2	$10824 = 2^3 \cdot 3 \cdot 11 \cdot 41$
2	$13500 = 2^2 \cdot 3^3 \cdot 5^3$
2	$23760 = 2^4 \cdot 3^3 \cdot 5 \cdot 11$
2	$58500 = 2^2 \cdot 3^2 \cdot 5^3 \cdot 13$
2	$66912 = 2^5 \cdot 3 \cdot 17 \cdot 41$
2	$425880 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13^2$
2	$520128 = 2^6 \cdot 3^3 \cdot 7 \cdot 43$
2	$873180 = 2^2 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 11$
2	$931392 = 2^6 \cdot 3^3 \cdot 7^2 \cdot 11$
2	$1899744 = 2^6 \cdot 3^3 \cdot 7 \cdot 11 \cdot 257$
2	$2129400 = 2^3 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13^2$
3	$2146560 = 2^8 \cdot 3 \cdot 5 \cdot 13 \cdot 43$
2	$2253888 = 2^6 \cdot 3^2 \cdot 7 \cdot 13 \cdot 43$
2	$3276000 = 2^5 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 13$
2	$4580064 = 2^5 \cdot 3^5 \cdot 19 \cdot 31$
2	$4668300 = 2^2 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13 \cdot 19$

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