FREMLIN TENSOR PRODUCTS OF CONCAVIFICATIONS OF BANACH LATTICES

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Abstract. Suppose that E is a uniformly complete vector lattice and $p_1,\dots,p_n$ are positive reals. We prove that the diagonal of the Fremlin projective tensor product of $E_{(p_1)},\dots,E_{(p_n)}$ can be identified with $E_{(p)}$ where $p=p_1+\dots+p_n$ and $E_{(p)}$ stands for the p-concavification of E. We also provide a variant of this result for Banach lattices. This extends the main result of [BBPTT].

1. Introduction and motivation

We start with some motivation. Let E be a vector or a Banach lattice of functions on some set Ω, and consider a tensor product $E\tilde{\otimes}E$ of E with itself. It is often possible to view $E\tilde{\otimes}E$ as a space of functions on the square $\Omega\times\Omega$ with $(f\otimes g)(s,t)=f(s)g(t)$, where $f,g\in E$ and $s,t\in\Omega$. In particular, restricting this function to the diagonal $s=t$ gives just the product fg. Thus, the space of the restrictions of the elements of $E\tilde{\otimes}E$ to the diagonal can be identified with the space $\{fg\ :\ f,g\in E\}$, which is sometimes called the square of E, (see, e.g., [BvR01]). The concept of the square can be extended to uniformly complete vector lattices as the 2-concavification of E. Hence, one can expect that, for a uniformly complete vector lattice, the diagonal of an appropriate tensor product of E with itself can be identified with the 2-concavification of E. This was stated and proved formally in [BBPTT] for Fremlin projective tensor product of Banach lattices. It was shown there that the diagonal of the tensor product is lattice isometric to the 2-concavification of E; the diagonal was defined as the quotient of the product over the ideal generated by all elementary tensors $x\otimes y$ with $x\perp y$.

In the present paper, we extend this result. Let us again provide some motivation. In the case when E is a space of functions on Ω, one can think of its p-concavification as $E_{(p)}=\{f^p\ :\ f\in E\}$ for $p>0$. In this case, for positive real numbers $p_1,\dots,p_n$, the elementary tensors in $E_{(p_1)}\otimes\dots\otimes E_{(p_n)}$ can be thought of as functions on Ω^n of the form

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$f_1^{p_1}\otimes\cdots\otimes f_n^{p_n}(s_1,\dots,s_n)=f_1^{p_1}(s_1)\cdots f_n^{p_n}(s_n)$, where $f_1,\dots,f_n\in E$ and $s_1,\dots,s_n\in\Omega$. The restriction of this function to the diagonal is the product $f_1^{p_1}\cdots f_n^{p_n}$, which is an element of $E_{(p)}$ where $p=p_1+\dots+p_n$. That is, the diagonal of the tensor product of $E_{(p_1)},\dots,E_{(p_n)}$ can be identified with $E_{(p)}$. In this paper, we formally state and prove this fact for the case when E is a uniformly complete vector lattice in Section 2 and when E is a Banach lattice in Section 3. In particular, this extends the result of [BBPTT] to vector lattices and to the product of an arbitrary number of copies of E (a variant of the latter statement was also independently obtained in [BB]).

2. Products of vector lattices

Throughout this section, E will stand for a uniformly complete vector lattice. We need uniform completeness so that we can use positive homogeneous function calculus in E, see, e.g., Theorem 5 in [BvR01]. It is easy to see that every uniformly complete vector lattice is Archimedean.

Following [LT79, p. 53], by t^p, where $t\in\mathbb{R}$ and $p\in\mathbb{R}_+$, we mean $|t|^p\cdot\operatorname{sign}t$; see also the discussion in Section 1.2 of [BBPTT]. In particular, if $p_1,\dots,p_n$ are positive reals and $p=p_1+\dots+p_n$ then $|t|^{p_1}|t|^{p_2}\dots|t|^{p_n}=|t|^p$ while $t^{p_1}|t|^{p_2}\dots|t|^{p_n}=t^p$ for every $t\in R$. It follows that $|x|^{p_1}|x|^{p_2}\dots|x|^{p_n}=|x|^p$ and $x^{p_1}|x|^{p_2}\dots|x|^{p_n}=x^p$ for every $x\in E$. In particular, $(x|x|\cdots|x|)^{\frac{1}{n}}=x$ for every $n\in\mathbb{N}$.

Suppose that p is a positive real number. Using function calculus, we can introduce new vector operations on E via $x\oplus y=(x^p+y^p)^{\frac{1}{p}}$ and $\alpha\odot x=\alpha^{\frac{1}{p}}x$, where $x,y\in E$ and $\alpha\in\mathbb{R}$. Together with these new operations and the original order and lattice structures, E becomes a vector lattice. This new vector lattice is denoted $E_{(p)}$ and called the p-***concavification*** of E. It is easy to see that $E_{(p)}$ is still Archimedean.

We start by extending Theorem 1 and Corollary 2 in [BvR00]. Recall that an n-linear map φ from E^n to a vector lattice F is said to be ***positive*** if $\varphi(x_1,\dots,x_n)\geqslant 0$ whenever $x_1,\dots,x_n\geqslant 0$ and ***orthosymmetric*** if $\varphi(x_1,\dots,x_n)=0$ whenever $|x_1|\wedge\dots\wedge|x_n|=0$; φ is said to be a ***lattice*** n-***morphism*** if $\bigl|\varphi(x_1,\dots,x_n)\bigr|=\varphi\bigl(|x_1|,\dots,|x_n|\bigr)$ for any $x_1,\dots,x_n\in E$.

Theorem 1. *Suppose that $\varphi\colon C(K)^n\to F$, where K is a compact Hausdorff space, F is a vector lattice, $n\in\mathbb{N}$, and φ is an orthosymmetric positive n-linear map. Then $\varphi(x_1,\dots,x_n)=\varphi(x_1\cdots x_n,\mathbb{1},\dots,\mathbb{1})$ for any $x_1,\dots,x_n\in C(K)$.*

Proof. The proof is by induction. The case $n=1$ is trivial. The case $n=2$ follows from Theorem 1 in [BvR00]. Suppose that $n>2$ and the statement is true for

$n-1$. Suppose that $\varphi\colon C(K)^n \to F$ is orthosymmetric positive and n-linear. Fix $0 \leqslant z \in C(K)$ and define $\varphi_z\colon C(K)^{n-1} \to F$ via $\varphi_z(x_1,\dots,x_{n-1}) = \varphi(x_1,\dots,x_n,z)$. Clearly, φ_z is orthosymmetric, positive, and $(n-1)$-linear. By the induction hypothesis, $\varphi_z(x_1,\dots,x_{n-1}) = \varphi_z(x_1\cdots x_{n-1},\mathbb{1},\dots,\mathbb{1})$ for all $x_1,\dots,x_{n-1}$ in $C(K)$. It follows that

$$\varphi(x_1,\dots,x_n) = \varphi(x_1\cdots x_{n-1},\mathbb{1},\dots,\mathbb{1},x_n) \tag{1}$$

for all $x_1,\dots,x_{n-1}$ and all $x_n > 0$. By linearity, (1) remains true for all $x_1,\dots,x_n \in C(K)$ as $x_n = x_n^+ - x_n^-$. Similarly, $\varphi(x_1,\dots,x_n) = \varphi(x_1x_3\cdots x_n,x_2,\mathbb{1},\dots,\mathbb{1})$ for all $x_1,\dots,x_n$ in E. Applying (1) to the latter expression, we get $\varphi(x_1,\dots,x_n) = \varphi(x_1\cdots x_n,\mathbb{1},\dots,\mathbb{1})$. This completes the induction. □

Corollary 2. *Suppose that $\varphi\colon E^n \to F$, where E is a uniformly complete vector lattice, F is a vector lattice, $n \in \mathbb{N}$ and φ is an orthosymmetric positive n-linear map. Then $\varphi(x_1,\dots,x_n)$ is determined by $(x_1\cdots x_n)^{\frac{1}{n}}$. Specifically,*

$$\varphi(x_1,\dots,x_n) = \varphi\big(x,|x|,\dots,|x|\big) \tag{2}$$

where $x = (x_1\cdots x_n)^{\frac{1}{n}}$.

Proof. Suppose that $x_1,\dots,x_n \in E$. Let $e = |x_1|\vee\cdots\vee|x_n|$ and consider the principal ideal I_e. Then $x_1,\dots,x_n \in I_e$. Since I_e is lattice isomorphic to $C(K)$ for some compact Hausdorff space and the restriction of φ to $(I_e)^n$ is still orthosymmetric, positive, and n-linear, by the theorem we get (2). □

Remark 3. The expression $\varphi\big(x,|x|,\dots,|x|\big)$ in (2) may look non-symmetric at the first glance. Lemma 2 may be restated in a more "symmetric" form as follows: $\varphi(x_1,\dots,x_n) = \varphi(x,\dots,x)$ for every *positive* $x_1,\dots,x_n$.

Next, we are going to generalize Corollary 2.

Theorem 4. *Suppose that $\varphi\colon E_{(p_1)}\times\cdots\times E_{(p_n)} \to F$, where E is a uniformly complete vector lattice, F is a vector lattice, $n \in \mathbb{N}$, $p_1,\dots,p_n$ are positive reals, and φ is an orthosymmetric positive n-linear map. Then the following are true.*

(i) *For all $x_1,\dots,x_n \in E$, we have $\varphi(x_1,\dots,x_n) = \varphi\big(x,|x|,\dots,|x|\big)$ where $x = x_1^{p_1/p}\cdots x_n^{p_n/p}$ with $p = p_1+\cdots+p_n$.*

(ii) *The map $\hat{\varphi}\colon E_{(p)} \to F$ defined by $\hat{\varphi}(x) = \varphi\big(x,|x|,\dots,|x|\big)$ is a positive linear map. If φ is a lattice n-morphism then $\hat{\varphi}$ is a lattice homomorphism.*

Proof. (i) First, we prove the statement for the case $E = C(K)$ for some Hausdorff compact space K. Define $\psi\colon E^n \to F$ via $\psi(x_1,\dots,x_n) = \varphi\big(x_1^{1/p_1},\dots,x_n^{1/p_n}\big)$. It is easy to see that ψ is an orthosymmetric positive n-linear map. Hence, applying Theorem 1 to ψ, we get

$$\begin{aligned}\varphi(x_1,\dots,x_n) &= \psi\big(x_1^{p_1},\dots,x_n^{p_n}\big) = \psi\big(x_1^{p_1}\cdots x_n^{p_n},\mathbb{1},\dots,\mathbb{1}\big)\\ &= \psi\big(x^p,\mathbb{1},\dots,\mathbb{1}\big) = \psi\big(x^{p_1},\lvert x\rvert^{p_2},\dots,\lvert x\rvert^{p_n}\big) = \varphi\big(x,\lvert x\rvert,\dots,\lvert x\rvert\big).\end{aligned}$$

Now suppose that E is a uniformly complete vector lattice. Choose $e \in E_+$ such that $x_1,\dots,x_n \in I_e$. Recall that I_e is lattice isomorphic to $C(K)$ for some Hausdorff compact space K. It is easy to see that $(I_e)_{(p_i)}$ is an ideal in $E_{(p_i)}$. The restriction of φ to $(I_e)_{(p_1)} \times \dots \times (I_e)_{(p_n)}$ is again an orthosymmetric positive n-linear map, so the conclusion follows from the first part of the proof.

(ii) The proof that $\hat{\varphi}(\alpha \odot x) = \alpha\hat{\varphi}(x)$ is straightforward. We proceed to check additivity. Again, suppose first that $E = C(K)$ for some compact Hausdorff space K; let ψ be as before. Take any $x,y \in E$ and put $z = x \oplus y$ in $E_{(p)}$, i.e., $z = (x^p+y^p)^{1/p}$. Then, again applying Theorem 1 to ψ, we have

$$\begin{aligned}\varphi\big(z,\lvert z\rvert,\dots,\lvert z\rvert\big) &= \psi\big(z^{p_1},\lvert z\rvert^{p_2},\dots,\lvert z\rvert^{p_n}\big) = \psi(z^p,\mathbb{1},\dots,\mathbb{1})\\ &= \psi(x^p,\mathbb{1},\dots,\mathbb{1}) + \psi(y^p,\mathbb{1},\dots,\mathbb{1}) = \varphi\big(x,\lvert x\rvert,\dots,\lvert x\rvert\big) + \varphi\big(y,\lvert y\rvert,\dots,\lvert y\rvert\big).\end{aligned}$$

Hence,

$$\varphi\big((x^p+y^p)^{\frac{1}{p}},\lvert x^p+y^p\rvert^{\frac{1}{p}},\dots\lvert x^p+y^p\rvert^{\frac{1}{p}}\big) = \varphi\big(x,\lvert x\rvert,\dots,\lvert x\rvert\big) + \varphi\big(y,\lvert y\rvert,\dots,\lvert y\rvert\big) \tag{3}$$

Now suppose that E is an arbitrary uniformly complete vector lattice and $x,y \in E$. Taking $e = \lvert x\rvert \vee \lvert y\rvert$ and proceeding as in (i), one can see that (3) still holds, which yields $\hat{\varphi}(x \oplus y) = \hat{\varphi}(x) + \hat{\varphi}(y)$. □

Corollary 5. *Let E be a uniformly complete vector lattice and $p_1,\dots,p_n$ positive reals; put $p = p_1+\dots+p_n$. For $x_1,\dots,x_n \in E$, define $\mu(x_1,\dots,x_n) = x_1^{p_1/p}\cdots x_n^{p_n/p}$. Then*

(i) *$\mu\colon E_{(p_1)} \times \dots \times E_{(p_n)} \to E_{(p)}$ is an orthosymmetric lattice n-morphism;*

(ii) *For every vector lattice F there is a one to one correspondence between orthosymmetric positive n-linear maps $\varphi\colon E_{(p_1)} \times \dots \times E_{(p_n)} \to F$ and positive linear maps $T\colon E_{(p)} \to F$ such that $\varphi = T\mu$ and $Tx = \varphi\big(x,\lvert x\rvert,\dots,\lvert x\rvert\big)$. Moreover, φ is a lattice n-morphism iff T is a lattice homomorphism.*

Proof. (i) is straightforward. Note that $\mu\big(x,\lvert x\rvert,\dots,\lvert x\rvert\big) = x$ for every $x \in E$.

FIGURE 1

$$\begin{tikzcd} E_{(p_1)}\times\dots\times E_{(p_n)} \arrow[r, "\varphi"] \arrow[d, "\mu"'] & F \\ E_{(p)} \arrow[ur, "T"'] & \end{tikzcd}$$

(ii) If $T\colon E_{(p)}\to F$ is a positive linear map then setting $\varphi := T\mu$ defines an orthosymmetric positive n-linear map on $E_{(p_1)}\times\dots\times E_{(p_n)}$ and

$$\varphi\big(x,|x|,\dots,|x|\big) = T\mu\big(x,|x|,\dots,|x|\big) = Tx.$$

Conversely, suppose that $\varphi\colon E_{(p_1)}\times\dots\times E_{(p_n)}\to F$ is an orthosymmetric positive n-linear map; define $T\colon E_{(p)}\to F$ via $Tx := \varphi\big(x,|x|,\dots,|x|\big)$. Then T is a positive linear operator by Theorem 4(ii). Given $x_1,\dots,x_n\in E$, put $x=\mu(x_1,\dots,x_n)$. It follows from Theorem 4(i) that

$$T\mu(x_1,\dots,x_n) = Tx = \varphi\big(x,|x|,\dots,|x|\big) = \varphi(x_1,\dots,x_n),$$

so that $T\mu=\varphi$. $\square$

We will use the fact, due to Luxemburg and Moore, that if J is an ideal in a vector lattice F then the quotient vector lattice F/J is Archimedean iff J is uniformly closed, see, e.g., Theorem 2.23 in [AB06] and the discussion preceding it. Recall that given a set A in a vector lattice F, A is uniformly closed if the limit of every uniformly convergent net in A is contained in A (it is easy to see that it suffices to consider sequences). The uniform closure of a set A in F is the set of the uniform limits of sequences in A; it can be easily verified that this set is uniformly closed. Clearly, the uniform closure of an ideal is an ideal. Hence, for every set A, the uniform closure of the ideal generated by A is the smallest uniformly closed ideal containing A.

For Archimedean vector lattices $E_1,\dots,E_n$, we write $E_1\bar{\otimes}\dots\bar{\otimes}E_n$ for their Fremlin vector lattice tensor product; see [Frem72, Frem74].

Theorem 6. *Let E be a uniformly complete vector lattice, $p_1,\dots,p_n$ positive reals, and I_{o} the uniformly closed ideal in $E_{(p_1)}\bar{\otimes}\dots\bar{\otimes}E_{(p_n)}$ generated by the elementary tensors of form $x_1\otimes\dots\otimes x_n$ with $\bigwedge_{i=1}^n|x_i|=0$. Then the quotient $\big(E_{(p_1)}\bar{\otimes}\dots\bar{\otimes}E_{(p_n)}\big)/I_{\mathrm{o}}$ is lattice isomorphic to $E_{(p)}$.*

Proof. Consider the diagram

$$E_{(p_1)}\times\dots\times E_{(p_n)} \xrightarrow{\otimes} E_{(p_1)}\bar{\otimes}\dots\bar{\otimes}E_{(p_n)} \xrightarrow{q} \big(E_{(p_1)}\bar{\otimes}\dots\bar{\otimes}E_{(p_n)}\big)/I_{\mathrm{o}} \tag{4}$$

where q is the quotient map; $q(u)=u+I_{\mathrm{o}} =: \tilde{u}$ for $u\in E_{(p_1)}\bar{\otimes}\dots\bar{\otimes}E_{(p_n)}$.

Let μ be as in Corollary 5. By the universal property of the tensor product (see, e.g., [Frem72, Theorem 4.2(i)]), there exists a lattice homomorphism $M\colon E_{(p_1)}\bar{\otimes}\dots\bar{\otimes}E_{(p_n)}\to E_{(p)}$ such that $M(x_1\otimes\dots\otimes x_n)=\mu(x_1,\dots,x_n)$ for all $x_1,\dots,x_n$. Since μ is orthosymmetric, $M(x_1\otimes\dots\otimes x_n)=0$ whenever $\bigwedge_{i=1}^n\lvert x_i\rvert=0$. Since M is a lattice homomorphism, it follows that M vanishes on I_{o}. Therefore, the quotient operator $\widetilde{M}$ is well defined: for $u\in E_{(p_1)}\bar{\otimes}\dots\bar{\otimes}E_{(p_n)}$ we have $\widetilde{M}\tilde{u}=Mu$. Furthermore, since q is a lattice homomorphism (see, e.g., [AB06, Theorem 2.22]), it is easy to see that $\widetilde{M}$ is a lattice homomorphism as well.

Note that M is onto because for every $x\in E_{(p)}$ we have $x=M\bigl(x\otimes\lvert x\rvert\otimes\dots\otimes\lvert x\rvert\bigr)$. It follows that $\widetilde{M}$ is onto. It is left to show that $\widetilde{M}$ is one-to-one.

The composition map $q\otimes$ in (4) is an orthosymmetric lattice n-morphism. By Corollary 5, there is a lattice homomorphism $T\colon E_{(p)}\to\bigl(E_{(p_1)}\bar{\otimes}\dots\bar{\otimes}E_{(p_n)}\bigr)/I_{\mathrm{o}}$ such that $q\otimes=T\mu$ and

$$Tx=(q\otimes)\bigl(x,\lvert x\rvert,\dots,\lvert x\rvert\bigr)=x\otimes\lvert x\rvert\otimes\dots\otimes\lvert x\rvert+I_{\mathrm{o}}$$

for every $x\in E_{(p)}$. It follows that for every $x_1,\dots,x_n$ we have

$$\begin{aligned}T\widetilde{M}(x_1\otimes\dots\otimes x_n+I_{\mathrm{o}})&=TM(x_1\otimes\dots\otimes x_n)\\&=T\mu(x_1,\dots,x_n)=(q\otimes)(x_1,\dots,x_n)=x_1\otimes\dots\otimes x_n+I_{\mathrm{o}},\end{aligned}$$

so that $T\widetilde{M}$ is the identity on the quotient of algebraic tensor product $\bigl(E_{(p_1)}\otimes\dots\otimes E_{(p_n)}\bigr)/I_{\mathrm{o}}$. We claim that it is still the identity map on $\bigl(E_{(p_1)}\bar{\otimes}\dots\bar{\otimes}E_{(p_n)}\bigr)/I_{\mathrm{o}}$; this would imply that $\widetilde{M}$ is one-to-one and complete the proof.

Suppose that $u\in E_{(p_1)}\bar{\otimes}\dots\bar{\otimes}E_{(p_n)}$. By [Frem72, Theorem 4.2(i)], there exist $w:=z_1\otimes\dots\otimes z_n$ in $E_{(p_1)}\otimes\dots\otimes E_{(p_n)}$ with $z_1,\dots,z_n\geqslant 0$ such that for every positive real δ there exists $v\in E_{(p_1)}\otimes\dots\otimes E_{(p_n)}$ with $\lvert u-v\rvert\leqslant\delta w$. Since the quotient map q is a lattice homomorphism, we get $\lvert\tilde{u}-\tilde{v}\rvert\leqslant\delta\tilde{w}$. Since T and $\widetilde{M}$ are lattice homomorphisms and, by the preceding paragraph, $T\widetilde{M}$ preserves $\tilde{v}$ and $\tilde{z}$, we get $\lvert T\widetilde{M}\tilde{u}-\tilde{v}\rvert\leqslant\delta\tilde{w}$. It follows that $\lvert T\widetilde{M}\tilde{u}-\tilde{u}\rvert\leqslant\lvert T\widetilde{M}\tilde{u}-\tilde{v}\rvert+\lvert\tilde{u}-\tilde{v}\rvert\leqslant 2\delta\tilde{w}$. Since δ is arbitrary, it follows by the Archimedean property that $T\widetilde{M}\tilde{u}=\tilde{u}$. □

Remark 7. Note that the lattice isomorphism constructed in the proof of the theorem sends $x_1\otimes\dots\otimes x_n+I_{\mathrm{o}}$ into $x_1^{p_1/p}\cdots x_n^{p_n/p}$, while its inverse sends x to $x\otimes\lvert x\rvert\otimes\dots\otimes\lvert x\rvert+I_{\mathrm{o}}$ for every x.

Remark 8. The ideal I_{o} was introduced for $n=2$ in [BvR01]; a variant of I_{o} for a general n was introduced in [BB06]; its Banach lattice counterpart (see I_{oc} in the next section) was introduced in [BB12].

3. Products of Banach lattices

Now suppose that E is a Banach lattice and p is a positive real number. Following [LT79, p.54] (see also Remark 4 in [BBPTT]), for each $x \in E_{(p)}$ we define

$$\|x\|_{(p)} = \inf\Bigl\{\sum_{i=1}^{k}\|v_i\|^p \ :\ |x| \leqslant v_1 \oplus \dots \oplus v_k,\ v_1,\dots,v_k \geqslant 0\Bigr\}.$$

It is easy to see that this is a lattice seminorm on $E_{(p)}$. We will write $x \sim y$ if the difference $x \ominus y$ is in the kernel of this seminorm. For $x \in E$ we will write $[x]$ for the equivalence class of x. Let $E_{[p]}$ be the completion of $E_{(p)}/\ker\|\cdot\|_{(p)}$. Then $E_{[p]}$ is a Banach lattice.

Let's compare this definition with the concepts of the p-convexification and the p-concavification of a Banach lattice, e.g., in [LT79]. If $p > 1$ and E is p-convex then $\|\cdot\|_{(p)}$ is a complete norm on $E_{(p)}$, hence $E_{[p]} = E_{(p)}$, and this is exactly the p-concavification of E in the sense of [LT79]. In particular, if E is p-convex with constant 1 then $\|\cdot\|^p$ is already a norm, so that, by the triangle inequality, we have $\|\cdot\|_{(p)} = \|\cdot\|^p$. On the other hand, let $0 < p < 1$. Put $q = \frac{1}{p} > 1$. As in the construction of the q-convexification $E^{(q)}$ of E in [LT79], we see that $\|\cdot\|^p$ is already a norm on $E_{(p)}$, so that $\|\cdot\|_{(p)} = \|\cdot\|^p$. In this case, $E_{[p]} = E_{(p)} = E^{(q)}$. Thus, the $E_{[p]}$ notation allows us to unify convexifications and concavifications, and it does not make any assumptions on E besides being a Banach (or even a normed) lattice.

If $E_1,\dots,E_n$ are Banach lattices, we write $E_1 \otimes_{|\pi|} \dots \otimes_{|\pi|} E_n$ for the Fremlin projective tensor of $E_1,\dots,E_n$ as in [Frem74]; we denote the norm on this product by $\|\cdot\|_{|\pi|}$. We will make use of the following universal property of this tensor product, which is essentially Theorem 1E(iii,iv) in [Frem74] (see also Part (d) of Section 2 in [Sch84]).

Lemma 9. *Suppose $E_1,\dots,E_n$ and F are Banach lattices. There is an one-to-one norm preserving correspondence between continuous positive n-linear maps $\varphi : E_1 \times \dots \times E_n \to F$ and positive operators $\varphi^{\otimes} \colon E_1 \otimes_{|\pi|} \dots \otimes_{|\pi|} E_n \to F$ such that $\varphi(x_1,\dots,x_n) = \varphi^{\otimes}(x_1 \otimes \dots \otimes x_n)$. Moreover, $\varphi^{\otimes}$ is a lattice homomorphism if and only if φ is a lattice n-morphism.*

Lemma 10. *Let E be a Banach lattice and μ be as in Corollary 5. Then $\|\mu\| \leqslant 1$.*

Proof. By Proposition 1.d.2(i) of [LT79], we have

$$\bigl\|\mu(x_1,\dots,x_n)\bigr\| \leqslant \|x_1\|^{\frac{p_1}{p}} \cdots \|x_n\|^{\frac{p_n}{p}} \tag{5}$$

for every $x_1,\dots,x_n$. Fix $x_1,\dots,x_n\in E$. As in the definition of $\|\cdot\|_{(p)}$, suppose that

$$|x_1|\leqslant v_1^{(1)}\oplus\dots\oplus v_{k_1}^{(1)},\quad \dots,\quad |x_n|\leqslant v_1^{(n)}\oplus\dots\oplus v_{k_n}^{(n)} \tag{6}$$

for some positive $v_i^{(m)}$'s. Since μ is a lattice n-morphism, we have

$$\bigl|\mu(x_1,\dots,x_n)\bigr|=\mu\bigl(|x_1|,\dots,|x_n|\bigr)\leqslant\mu\Bigl(\bigoplus_{i_1=1}^{k_1}v_{i_1}^{(1)},\dots,\bigoplus_{i_n=1}^{k_n}v_{i_n}^{(n)}\Bigr)=\bigoplus_{i_1,\dots,i_n}\mu\bigl(v_{i_1}^{(1)},\dots,v_{i_n}^{(n)}\bigr)$$

where each i_m runs from 1 to k_m. The definition of $\|\cdot\|_{(p)}$ yields

$$\bigl\|\mu(x_1,\dots,x_n)\bigr\|_{(p)}\leqslant\sum_{i_1,\dots,i_n}\bigl\|\mu(v_{i_1}^{(1)},\dots,v_{i_n}^{(n)})\bigr\|^p.$$

It follows from (5) that

$$\bigl\|\mu(x_1,\dots,x_n)\bigr\|_{(p)}\leqslant\sum_{i_1,\dots,i_n}\|v_{i_1}^{(1)}\|^{p_1}\cdots\|v_{i_n}^{(n)}\|^{p_n}=\Bigl(\sum_{i_1=1}^{k_1}\|v_{i_1}^{(1)}\|^{p_1}\Bigr)\cdots\Bigl(\sum_{i_n=1}^{k_n}\|v_{i_n}^{(n)}\|^{p_n}\Bigr).$$

Taking infimum over all positive $v_i^{(m)}$'s in (6), we get

$$\bigl\|\mu(x_1,\dots,x_n)\bigr\|_{(p)}\leqslant\|x_1\|_{(p_1)}\cdots\|x_n\|_{(p_n)}.$$

□

Theorem 11. *Let E be a Banach lattice and $p_1,\dots,p_n$ positive reals. Put $F=E_{[p_1]}\otimes_{|\pi|}\dots\otimes_{|\pi|}E_{[p_n]}$. Let I_{oc} be the norm closed ideal in F generated by elementary tensors $[x_1]\otimes\dots\otimes[x_n]$ with $\bigwedge_{i=1}^n|x_i|=0$. Then F/I_{oc} is lattice isometric to $E_{[p]}$ where $p=p_1+\dots+p_n$.*

Proof. Let μ be as in Corollary 5. Fix $x_1,\dots,x_n$ in E. Take any $x_1',\dots,x_n'$ in E such that $\|x_i'-x_i\|_{(p_i)}=0$ as $i=1,\dots,n$. Then it follows from Lemma 10 that

$$\bigl\|\mu(x_1,x_2,\dots,x_n)\ominus\mu(x_1',x_2,\dots,x_n)\bigr\|_{(p)}=\bigl\|\mu(x_1\ominus x_1',x_2,\dots,x_n)\bigr\|_{(p)}=0,$$

so that $\mu(x_1,x_2,\dots,x_n)\sim\mu(x_1',x_2,\dots,x_n)$ in $E_{(p)}$. Iterating this process, we see that $\mu(x_1',\dots,x_n')\sim\mu(x_1,\dots,x_n)$. It follows that μ induces a map

$$\tilde\mu\colon\bigl(E_{(p_1)}/\ker\|\cdot\|_{(p_1)}\bigr)\times\dots\times\bigl(E_{(p_n)}/\ker\|\cdot\|_{(p_n)}\bigr)\to E_{(p)}/\ker\|\cdot\|_{(p)}$$

via $\tilde\mu\bigl([x_1],\dots,[x_n]\bigr)=\bigl[\mu(x_1,\dots,x_n)\bigr]$. Lemma 10 implies that $\|\tilde\mu\|\leqslant1$, so that it extends by continuity to a map $\varphi\colon E_{[p_1]}\times\dots\times E_{[p_n]}\to E_{[p]}$. It is easy to see that φ is still an orthosymmetric lattice n-morphism and $\|\varphi\|\leqslant1$. As in Lemma 9, φ gives rise to a lattice homomorphism $\varphi^{\otimes}\colon F\to E_{[p]}$ such that $\|\varphi^{\otimes}\|\leqslant1$ and

$$\varphi^{\otimes}\bigl([x_1]\otimes\dots\otimes[x_n]\bigr)=\varphi\bigl([x_1],\dots,[x_n]\bigr)=\bigl[\mu(x_1,\dots,x_n)\bigr]. \tag{7}$$

The latter implies that $\varphi^{\otimes}$ vanishes on I_{oc}. This, in turn, implies that $\varphi^{\otimes}$ induces a map $\widetilde{\varphi^{\otimes}}\colon F/I_{\mathrm{oc}} \to E_{[p]}$, which is again a lattice homomorphism and $\|\widetilde{\varphi^{\otimes}}\| \leqslant 1$.

Consider the map $\psi\colon E_{(p_1)} \times \cdots \times E_{(p_n)} \to F/I_{\mathrm{oc}}$ defined by $\psi(x_1,\dots,x_n) = [x_1] \otimes \cdots \otimes [x_n] + I_{\mathrm{oc}}$. It can be easily verified that ψ is an orthosymmetric lattice n-morphism. It follows from Corollary 5 that there exists a lattice homomorphism $T\colon E_{(p)} \to F/I_{\mathrm{oc}}$ such that $\psi = T\mu$ and

$$Tx = \psi\bigl(x,|x|,\dots,|x|\bigr) = [x] \otimes \bigl[|x|\bigr] \otimes \cdots \otimes \bigl[|x|\bigr] + I_{\mathrm{oc}}$$

for every $x \in E_{(p)}$.

We claim that $\|Tx\| \leqslant \|x\|_{(p)}$. Note first that as $\|\cdot\|_{|\pi|}$ is a cross-norm, we have

$$\|Tx\| \leqslant \bigl\|[x]\bigr\|_{E_{[p_1]}} \cdots \bigl\|[x]\bigr\|_{E_{[p_n]}} \leqslant \|x\|_{(p_1)} \cdots \|x\|_{(p_n)} \leqslant \|x\|^{p_1} \cdots \|x\|^{p_n} = \|x\|^{p}. \tag{8}$$

Suppose that $|x| \leqslant v_1 \oplus \cdots \oplus v_m$ for some positive $v_1,\dots,v_m$, as in the definition of $\|\cdot\|_{(p)}$. Then $|Tx| = T|x| \leqslant \sum_{i=1}^{m} Tv_i$, so that $\|Tx\| \leqslant \sum_{i=1}^{m}\|Tv_i\| \leqslant \sum_{i=1}^{m}\|v_i\|^{p}$ by (8). It follows that $\|Tx\| \leqslant \|x\|_{(p)}$.

Therefore, T induces an operator from $E_{(p)}/\ker\|\cdot\|_{(p)}$ to I_{oc} and, furthermore, an operator from $E_{[p]}$ to F/I_{oc}, which we will denote $\widetilde{T}$, such that $\widetilde{T}[x] = Tx$ for every $x \in E_{(p)}$. Clearly, $\widetilde{T}$ is still a lattice homomorphism and $\|\widetilde{T}\| \leqslant 1$. We will show that $\widetilde{T}$ is the inverse of $\widetilde{\varphi^{\otimes}}$. This will complete the proof because this would imply that $\widetilde{\varphi^{\otimes}}$ is a surjective lattice isomorphism; it would follow from $\|\widetilde{\varphi^{\otimes}}\| \leqslant 1$ and $\|\widetilde{T}\| \leqslant 1$ that $\widetilde{\varphi^{\otimes}}$ is an isometry.

Take any $x \in E$ and consider the corresponding class $[x]$ in $E_{[p]}$. Using (7), we get

$$\widetilde{\varphi^{\otimes}}\widetilde{T}[x] = \widetilde{\varphi^{\otimes}}Tx = \varphi^{\otimes}\bigl([x] \otimes \bigl[|x|\bigr] \otimes \cdots \otimes \bigl[|x|\bigr]\bigr) = \bigl[\mu\bigl(x,|x|,\dots,|x|\bigr)\bigr] = [x].$$

Therefore, $\widetilde{\varphi^{\otimes}}\widetilde{T}$ is the identity on $E_{[p]}$. Conversely, for any $x_1,\dots,x_n$ in E it follows by (7) that

$$\begin{aligned}\widetilde{T}\widetilde{\varphi^{\otimes}}\bigl([x_1] \otimes \cdots \otimes [x_n] + I_{\mathrm{oc}}\bigr) &= \widetilde{T}\bigl[\mu(x_1,\dots,x_n)\bigr] = T\mu(x_1,\dots,x_n) \\ &= \psi(x_1,\dots,x_n) = [x_1] \otimes \cdots \otimes [x_n] + I_{\mathrm{oc}}.\end{aligned}$$

Therefore, $\widetilde{T}\widetilde{\varphi^{\otimes}}$ is the identity on the linear subspace of F/I_{oc} that corresponds to the algebraic tensor product, i.e., on $q\bigl(E_{[p_1]} \otimes \cdots \otimes E_{[p_n]}\bigr)$, where q is the canonical quotient map from F to F/I_{oc}. Since $E_{[p_1]} \otimes \cdots \otimes E_{[p_n]}$ is dense in F, it follows that q maps it into a dense subspace of F/I_{oc}. Therefore, $\widetilde{T}\widetilde{\varphi^{\otimes}}$ is the identity on a dense subspace of F/I_{oc}, hence on all of F/I_{oc}. $\square$

Remark 12. Note that the isometry from F/I_{oc} onto $E_{[p]}$ constructed in the proof of Theorem 11 sends $[x_1]\otimes\cdots\otimes[x_n]+I_{\mathrm{oc}}$ into $\bigl[x_1^{p_1/p}\cdots x_n^{p_n/p}\bigr]$, while its inverse sends $[x]$ to $[x]\otimes\bigl[|x|\bigr]\otimes\cdots\otimes\bigl[|x|\bigr]+I_{\mathrm{oc}}$ for every x.

Applying the theorem with $p_1=\cdots=p_n=1$, we obtain the following corollary, which extends the main result of [BBPTT]; see also [BB].

Corollary 13. *Suppose that E is a Banach lattice. Let I_{oc} be the closed ideal in $E\otimes_{|\pi|}\ldots\otimes_{|\pi|}E$ generated by the elementary tensors $x_1\otimes\cdots\otimes x_n$ where $\bigwedge_{i=1}^n|x_i|=0$. Then $\bigl(E\otimes_{|\pi|}\ldots\otimes_{|\pi|}E\bigr)/I_{\mathrm{oc}}$ is lattice isometric to $E_{[n]}$.*

Recall that if $p<1$ then $E_{[p]}=E^{(q)}$, the q-convexification of E where $q=\frac{1}{p}$. Hence, putting $q_i=\frac{1}{p_i}$ in the theorem, we obtain the following.

Corollary 14. *Suppose that E is a Banach lattice $q_1,\ldots,q_n$ are positive reals such that their geometric mean $q:=\bigl(\frac{1}{q_1}+\cdots+\frac{1}{q_n}\bigr)^{-1}$ satisfies $q\geqslant 1$. Let I_{oc} be the closed ideal in $E^{(q_1)}\otimes_{|\pi|}\ldots\otimes_{|\pi|}E^{(q_n)}$ generated by the elementary tensors $x_1\otimes\cdots\otimes x_n$ where $\bigwedge_{i=1}^n|x_i|=0$. Then $\bigl(E^{(q_1)}\otimes_{|\pi|}\ldots\otimes_{|\pi|}E^{(q_n)}\bigr)/I_{\mathrm{oc}}$ is lattice isometric to $E^{(q)}$.*

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