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The Odd/Even Dichotomy For The Set Of Square-Full Numbers*

Teerapat Srichan†

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Abstract

A positive integer n is called square-full if $p^2|n$ for every prime factor p of n. The asymptotical ratio of odd to even square-full numbers is obtained.

1 Introduction and result

A positive integer n is called square-free if it is a product of different primes. In 2008, Scott [4] conjectured that the ratio of odd to even square-free numbers is asymptotically $2:1$. Two years later, Jameson [3] used some properties of Dirichlet series and convolution to prove that the proportion of square-free numbers is asymptotically $\frac{4}{\pi^2}$ and showed that Scott's conjecture is true. It would be interesting to consider the odd/even dichotomy for the set of other kinds of integers. In this paper we shall consider the asymptotical ratio of odd to even square-full numbers.

A positive integer n is called square-full if $p^2|n$ for every prime factor p of n. Let G be the set of all square-full numbers. Let $G(x)$, $G_o(x)$ and $G_e(x)$ be the set of all square-full numbers, odd square-full numbers and even square-full in the interval $[1, x]$, respectively. We denote by $N(x)$, $N_o(x)$ and $N_e(x)$ the number of members of $G(x)$, $G_o(x)$ and $G_e(x)$, respectively. Erdös and Szekeres [2] were the first to investigate $N(x)$ and showed that

$$N(x) = \frac{\zeta(3/2)}{\zeta(3)}x^{1/2} + O(x^{1/3}), \tag{1}$$

where $\zeta(s)$ denotes the Riemann zeta-function. In 1958, Bateman and Grosswald [1] improved (1) and showed that

$$N(x) = \frac{\zeta(3/2)}{\zeta(3)}x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)}x^{1/3} + O(x^{1/6}). \tag{2}$$

From (1) and (2) one could deduce that

$$N(x) \sim \frac{\zeta(3/2)}{\zeta(3)}x^{1/2}. \tag{3}$$

We obtain the asymptotical ratio of odd to even square-full numbers in the following theorem.

Theorem 1 *As $x \to \infty$, we have*

$$\frac{N_o(x)}{N_e(x)} \sim 2 - \sqrt{2}.$$

Remark 2 *The result in Theorem 1 indicates that the ratio of odd to even square-full numbers is asymptotically $1 : 1 + \frac{\sqrt{2}}{2}$.*

Remark 3 *The result in Theorem 1 can be found as an example in [5]. The author applied Theorem 2.1 and 2.2 in [5] to deduce that in any interval $[1, x]$ of integer, the number of the odd square-full numbers do not exceed the number of the even square-full numbers.*

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†Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok, Thailand

The proof of Theorem 2.1 and 2.2 in [5] is long. Here we give a simple and short proof for Theorem 1.

Notation 4 *$f(x) \sim g(x)$ means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and we say that $f(x)$ is asymptotic to $g(x)$ as $x \to \infty$.*

2 Proof of Theorem 1

Proof. First, we assume that,

$$N_o(x) \sim ax^{1/2} \qquad \text{and} \qquad N_e(x) \sim bx^{1/2}, \qquad \text{for some } a, b \in \mathbb{R}^+. \tag{4}$$

We wish to show that,

$$\frac{a}{b} = 2 - \sqrt{2}. \tag{5}$$

Since there is no square-full number n such that $n \equiv 2\,(\mathrm{mod} 4)$, we have $G_e(x) = \{n \le x,\ n \in G \text{ and } 4|n\}$ and $G_o(x) = \{n \le x,\ n \in G \text{ and } n \equiv 1, 3\,(\mathrm{mod} 4)\}$. Next, we spilt $G_e(x)$ in to the set $G_{e1}(x)$ and the set $G_{e2}(x)$, where $G_{e1}(x) = \{n \le x,\ n \in G_e(x) \text{ and } \frac{n}{4} \in G\}$ and $G_{e2}(x) = \{n \le x,\ n \in G_e(x) \text{ and } \frac{n}{4} \notin G\ \}$. It is obvious that,

$$N_{e1}(x) = N(x/4). \tag{6}$$

Now we will show that,

$$N_{e2}(x) = N_o(x/8). \tag{7}$$

For any positive integer $n \in G_{e2}(x)$, we have $\frac{n}{4} \in \mathbb{Z}^+$. Then, we write $\frac{n}{4} = mr$ with m is square-full, r is square-free and $\gcd(m, r) = 1$. Since $\frac{n}{4} \notin G$, we have $r \ne 1$. Suppose that $r > 2$. We have a contradiction, since $n = 4mr \notin G$. We thus get only $r = 2$ and consequently m is an odd square-full. Then we obtain the one-to-one relation between the sets $G_{e2}(x)$ and $G_o(x/8)$ and (7). In view of (6) and (7), we have

$$N_e(x) = N(x/4) + N_o(x/8). \tag{8}$$

Then

$$N_e(x) = (N_e(x/4) + N_o(x/4)) + N_o(x/8).$$

In view of (4), we have

$$bx^{1/2} \sim \frac{b}{2}x^{1/2} + \frac{a}{2}x^{1/2} + \frac{a}{2\sqrt{2}}x^{1/2}.$$

This shows the asymptotical ratio (5).

To complete the proof of Theorem 1, we have to show the existence of a and b in (4). In view of (8), we have

$$\begin{cases} N_e(x) = N(x/4) + N_o(x/8), \\ N(x) - N_o(x) = N(x/4) + N_o(x/8), \\ N(x) - N(x/4) = N_o(x) + N_o(x/8). \end{cases} \tag{9}$$

We write $f(x) = N(x) - N(x/4)$. In view of (3), we know that,

$$f(x) \sim cx^{1/2}, \tag{10}$$

for a certain $c > 0$. In view of (9), we have

$$f(x) - f(x/8) = N_o(x) + N_o(x/8) - (N_o(x/8) + N_o(x/8^2)) = N_o(x) - N_o(x/8^2). \tag{11}$$

Replace x in (11) by $x/8^2$, we have

$$f(x/8^2) - f(x/8^3) = N_o(x/8^2) + N_o(x/8^3) - (N_o(x/8^3) + N_o(x/8^4)) = N_o(x/8^2) - N_o(x/8^4). \tag{12}$$

In view of (11) and (12), we have

$$N_o(x) - N_o(x/8^4) = f(x) - f(x/8) + f(x/8^2) - f(x/8^3).$$

Repeating this, we see that

$$N_o(x) - N_o(x/8^{2k}) = \sum_{0\le i\le k-1} f(x/8^{2i}) - \sum_{0\le j\le k-1} f(x/8^{2j+1}). \tag{13}$$

Since the asymptotic value (10), for $\epsilon > 0$, we take x_0 such that $(c-\epsilon)x^{1/2} \le f(x) \le (c+\epsilon)x^{1/2}$, for $x > x_0$. Then we take k such that $x/8^{2k} < x_0 \le x/8^{2k-1}$. We note that $N_0(x/8^{2k}) \le N_0(x_0) < x_0$. From this and (13), we have

$$\begin{aligned} N_o(x) &\le \sum_{0\le i\le k-1} f(x/8^{2i}) - \sum_{0\le j\le k-1} f(x/8^{2j+1}) + x_0 \\ &\le \sum_{i=0}^{\infty} f(x/8^{2i}) - \sum_{j=0}^{\infty} f(x/8^{2j+1}) + x_0 \\ &= \sum_{i=0}^{\infty}\Big((c+\epsilon)\frac{x^{1/2}}{8^i}\Big) - \sum_{j=0}^{\infty}\Big((c-\epsilon)\frac{x^{1/2}}{8^{j+1/2}}\Big) + x_0 \\ &= \frac{c\sqrt{8}}{\sqrt{8}+1}x^{1/2} + \frac{\epsilon\sqrt{8}}{\sqrt{8}-1}x^{1/2} + x_0 \\ &\le \frac{c\sqrt{8}}{\sqrt{8}+1}x^{1/2} + 2\epsilon x^{1/2} + x_0. \end{aligned}$$

Thus, for $x > (\frac{x_0}{\epsilon})^2$, we have

$$N_o(x) \le \Big(\frac{c\sqrt{8}}{\sqrt{8}+1} + 3\epsilon\Big)x^{1/2}. \tag{14}$$

Next, we estimate the lower bound for $N_o(x)$. In view of (13), we can write

$$N_o(x) = \sum_{i=0}^{\infty} f(x/8^{2i}) - \sum_{j=0}^{\infty} f(x/8^{2j+1}).$$

Thus, for $x > x_0$, and we get

$$\begin{aligned} N_o(x) &\ge \sum_{i=0}^{\infty}\Big((c-\epsilon)\frac{x^{1/2}}{8^i}\Big) - \sum_{j=0}^{\infty}\Big((c+\epsilon)\frac{x^{1/2}}{8^{j+1/2}}\Big) \\ &= \frac{c\sqrt{8}}{\sqrt{8}+1}x^{1/2} - \frac{\epsilon\sqrt{8}}{\sqrt{8}-1}x^{1/2} \\ &\ge \frac{c\sqrt{8}}{\sqrt{8}+1}x^{1/2} - 2\epsilon x^{1/2}. \end{aligned}$$

Thus, for $x > x_0$, we have

$$N_o(x) \ge \Big(\frac{c\sqrt{8}}{\sqrt{8}+1} - 2\epsilon\Big)x^{1/2}. \tag{15}$$

In view of (14) and (15), the value a exists. Similarly for the existence of b. ■

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