Row and column length restrictions of some classical Schur function identities and the connection with Howe dual pairs

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Inspired by a conjecture by Joris Van der Jeugt (University of Gent, Belgium) including joint work with Angele Hamel (Wilfred Laurier University, Canada)

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Motivation

- Received query from Joris Van der Jeugt (working with Stijn Lievens and Neli Stoilova)
- Studying representations of the orthosymplectic Lie superalgebra $osp(1|2n)$ built using parabosons
- Identified Fock space modules $\overline{V}(p)$ for any $p \in \mathbb{N}$
- Constructed unitary irreducible infinite-dimensional representations $V(p) = \overline{V}(p)/M(p)$ where $M(p)$ is the maximal submodule of $\overline{V}(p)$, and found that
  - for $p \geq n$ irrep $V(p) = \overline{V}(p)$
  - for $p < n$ irrep $V(p) = \overline{V}(p)/M(p)$
- Also calculated the characters of both $\overline{V}(p)$ and $V(p)$

Van der Jeugt's conjecture

Proposition [Van der Jeugt, Lievens and Stoilova, 2007]

Let $x = (x_1, x_2, \ldots, x_n)$, then

$$\mathrm{ch}\, V(p) = (x_1 x_2 \cdots x_n)^{p/2} \sum_{\lambda:\ell(\lambda)\leq p} s_\lambda(x)$$

Conjecture [Van der Jeugt, Lievens and Stoilova, 2007]

$$\sum_{\lambda:\ell(\lambda)\leq p} s_\lambda(x) = \frac{\sum_\eta (-1)^{c_\eta}\, s_\eta(x)}{\prod_{1\leq i\leq n}(1-x_i)\,\prod_{1\leq j<k\leq n}(1-x_i x_j)}$$

with the sum over all partitions η which in Frobenius notation take the form $\eta = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1+p & a_2+p & \cdots & a_r+p \end{pmatrix}$

Macdonald's Theorem

- Joris Van der Jeugt asked if the result was known
- If so where could it be found, if not could I supply a proof?
- Angele Hamel reminded me of:

  Theorem [Macdonald 79]

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x) = \frac{\left| x_i^{n-j} - x_i^{n+p+j-1} \right|}{\prod_{1\leq i\leq n}(1-x_i)\ \prod_{1\leq j<k\leq n}(x_j-x_k)(1-x_jx_k)}$$

- Need to compare this with an immediate Corollary to Van der Jeugt's Conjecture

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x) = \frac{\sum_\eta (-1)^{c_\eta}\ s_{\eta'}(x)}{\prod_{1\leq i\leq n}(1-x_i)\ \prod_{1\leq j<k\leq n}(1-x_ix_j)}$$

Strategy

- Try to recast the numerator of Macdonald's formula as a signed sum of Schur functions
- Use conjugacy to recover Van der Jeugt's formula
- Try to identify the origin of the row length restriction $\ell(\lambda') \leq p$ in Macdonald's formula
- Try to identify the origin of the column length restriction $\ell(\lambda) \leq p$ in Van der Jeugt's Conjecture
- First some preliminaries on
  - Schur functions and Schur functions series
  - Partitions, Young diagrams, Frobenius notation
  - Determinantal identities and modifications

Schur functions

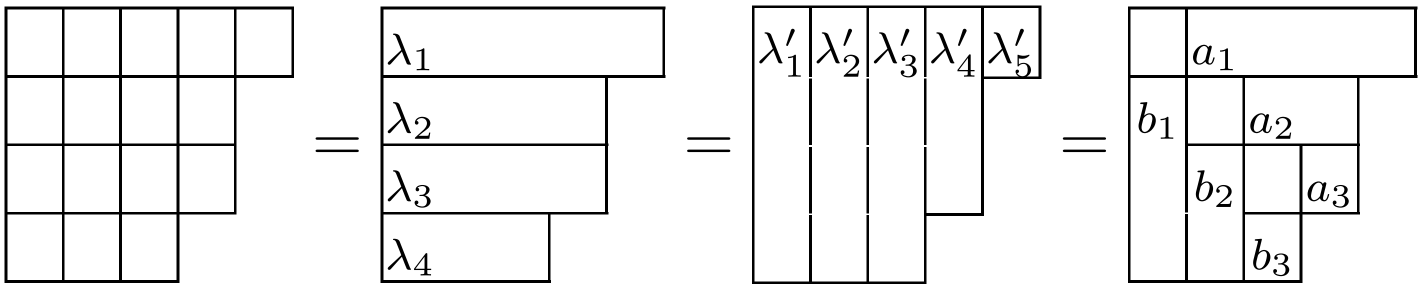
- Let n be a fixed positive integer
- Let $x = (x_1, x_2, \ldots, x_n)$ be a sequence of indeterminates
- Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ be a partition of weight $|\lambda|$ and length $\ell(\lambda) \leq n$
- Then the Schur function $s_\lambda(x)$ is defined by:

$$s_\lambda(x) = \frac{\left| x_i^{\lambda_j+n-j} \right|_{1\leq i,j\leq n}}{\left| x_i^{n-j} \right|_{1\leq i,j\leq n}}$$

- where $\left| x_i^{n-j} \right|_{1\leq i,j\leq n} = \prod_{1\leq i<j\leq n} (x_i - x_j)$
- These Schur functions form a $\mathbb{Z}$-basis of Λ_n, the ring of polynomial symmetric functions of $x_1, \ldots, x_n$.

Partitions and Young diagrams

- Young diagrams F^λ consists of $|\lambda|$ boxes arranged in $\ell(\lambda)$ rows of lengths λ_i for $i = 1, 2, \ldots \ell(\lambda)$
- Conjugate partition λ' is the partition defined by the $\ell(\lambda')$ columns of F^λ of lengths λ'_j for $j = 1, 2 \ldots, \ell(\lambda')$
- Frobenius notation If F^λ has r boxes on the main diagonal, with arm and leg lengths a_k and b_k for $k = 1, 2, \ldots, r$, then $\lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$ has rank $r(\lambda) = r$ with $a_1 > a_2 > \cdots > a_r \geq 0$ and $b_1 > b_2 > \cdots > b_r \geq 0$



Special families of partitions

- Let $\mathcal{P}$ be the set of all partitions, including the zero partition $\lambda = 0 = (0, 0, \ldots, 0)$.
- The zero partition is the unique partition of weight, length and rank zero, ie. $|0| = \ell(0) = r(0) = 0$
- Then for any integer t let

$$\mathcal{P}_t = \left\{ \lambda = \begin{pmatrix} a_1 \, a_2 \cdots a_r \\ b_1 \; b_2 \cdots b_r \end{pmatrix} \in \mathcal{P} \;\middle|\; a_k - b_k = t \quad \begin{array}{l} \text{for} \quad k = 1, 2, \ldots, r \\ \text{and} \quad r = 0, 1, \ldots \end{array} \right\}$$

- Note: The zero partition belongs to $\mathcal{P}_t$ for all integer t

Modification rules

- For $n \in \mathbb{N}$ let $x = (x_1, x_2, \ldots, x_n)$ and $\mathbf{x} = x_1 x_2 \cdots x_n$
- Let $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ with $\kappa_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, n$
- Let $s_\kappa(x) = \dfrac{\left| x_i^{\kappa_j + n - j} \right|_{1 \le i,j \le n}}{\left| x_i^{n-j} \right|_{1 \le i,j \le n}}$
- Either $s_\kappa(x) = 0$ or $s_\kappa(x) = \pm\, \mathbf{x}^k\, s_\lambda(x)$ for some partition λ and some integer k

- Permuting columns leads to various identities, such as
  - $s_\kappa(x) = -s_\mu(x)$ and $s_\kappa(x) = (-1)^{j-1} s_\nu(x)$ with
  - $\mu = (\kappa_1, \ldots, \kappa_{j+1} - 1, \kappa_j + 1, \ldots, \kappa_n)$
  - $\nu = (\kappa_{j+1} - j, \kappa_1 + 1, \ldots, \kappa_j + 1, \kappa_{j+2} \ldots, \kappa_n)$

Example

- If $n = 4$ and $\kappa = (0,4,0,9)$ then $s_\kappa(x) = (-1)^{3+1}\, s_\lambda(x)$ with $\lambda = (6,4,2,1)$ since

$$\frac{\begin{vmatrix} x_i^3 & x_i^6 & x_i & x_i^9 \end{vmatrix}}{\begin{vmatrix} x_i^3 & x_i^2 & x_i & 1 \end{vmatrix}} = \frac{\begin{vmatrix} x_i^9 & x_i^6 & x_i^3 & x_i \end{vmatrix}}{\begin{vmatrix} x_i^3 & x_i^2 & x_i & 1 \end{vmatrix}}$$

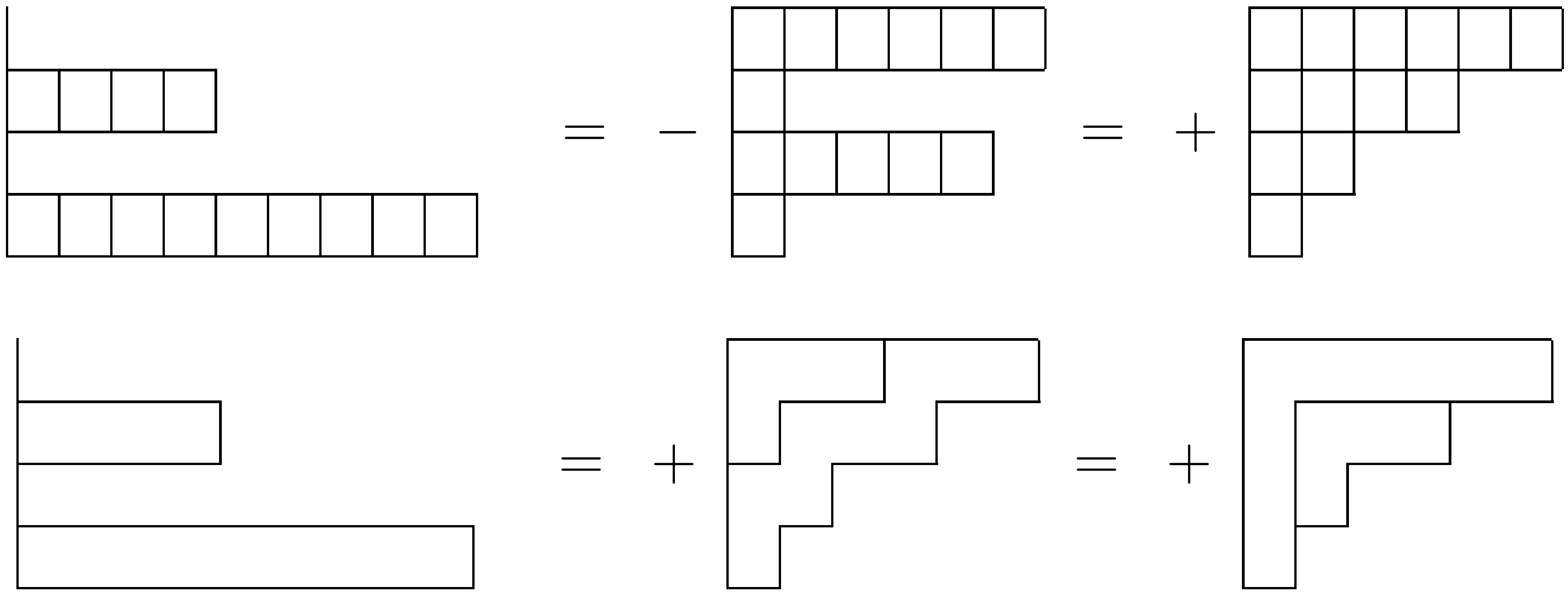
  where just the ith row of each determinant has been displayed

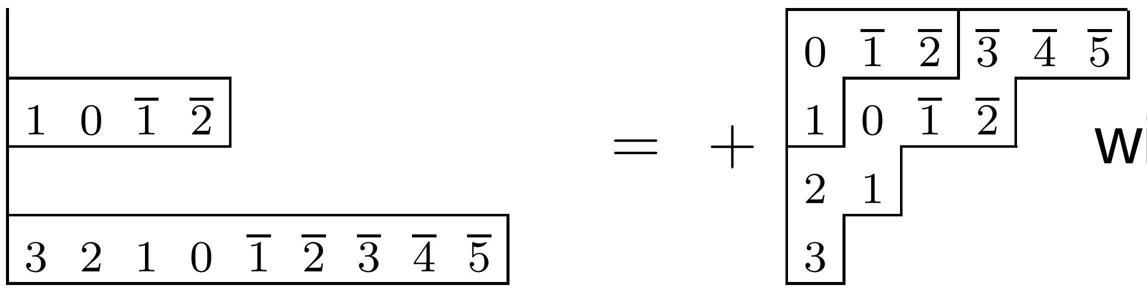
- Alternatively, one can proceed iteratively using the previous identities

$$s_{0409}(x) \;=\; -\; s_{6151}(x) \;=\; +\; s_{6421}(x)$$

Diagrammatically

Ex: $s_\kappa(x) = s_{0409}(x) = -s_{6151}(x) = +s_{6421}(x) = +s_\lambda(x)$





with $\lambda = (6421) = \begin{pmatrix} 5\,2 \\ 3\,1 \end{pmatrix}$

Frobenius notation and modifications

- Let $\kappa_j = 0$ unless $j \in \{b_1+1, b_2+1, \ldots, b_r+1\}$
- Let $b_1 > b_2 > \cdots > b_r \geq 0$ without loss of generality
- Let $\kappa(j) = a_k + b_k + 1$ if $j = b_k+1$ so that

$$\kappa = (0^{b_r}, a_r+b_r+1, 0^{b_{r-1}-b_r-1}, \ldots, a_2+b_2+1, 0^{b_1-b_2-1}, a_1+b_1+1)$$

- Then, if $a_1 > a_2 > \cdots > a_r \geq 0$,

$$s_\kappa(x) \;=\; (-1)^{b_1+b_2+\cdots+b_r}\; s_\lambda(x)$$

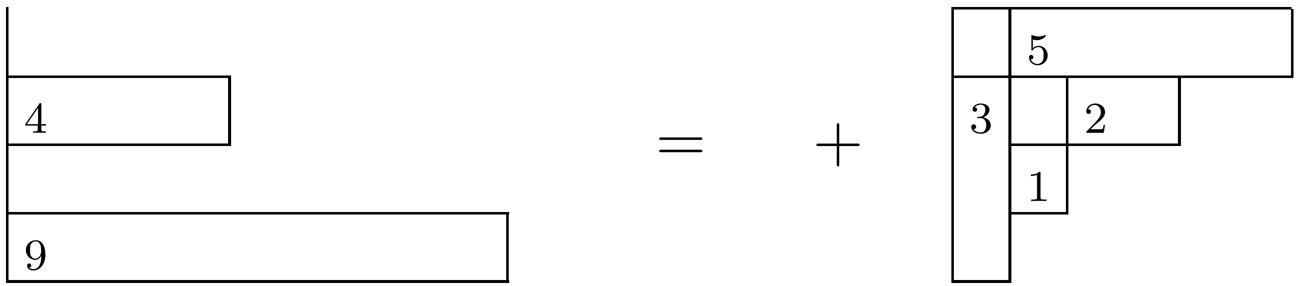
with

$$\lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

and $r = r(\lambda)$

Example

- For $\kappa = (0, 4, 0, 9)$ we have $\kappa = 0$ unless $j \in \{2, 4\}$
- Hence $r = 2$ and $b_1 = 3 > b_2 = 1 \geq 0$
- Since $\kappa_4 = a_1 + b_1 + 1 = 9$ and $\kappa_2 = a_2 + b_2 + 1 = 4$ we have $a_1 = 5 > a_2 = 2 \geq 0$
- Hence if we set $\lambda = \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix} = (6, 4, 2, 1)$
- we have $s_{0409}(x) = (-1)^{3+1} \, s_{6421}(x)$ as before



Schur function series

- Littlewood [1940] For all $n \geq 1$ and $x = (x_1, x_2, \ldots, x_n)$:

$$\sum_{\lambda} s_\lambda(x) = \prod_{1\leq i\leq n} (1 - x_i)^{-1} \prod_{1\leq j<k\leq n} (1 - x_j x_k)^{-1}$$

$$\sum_{\lambda\,\text{even}} s_\lambda(x) = \prod_{1\leq j\leq k\leq n} (1 - x_j x_k)^{-1}$$

$$\sum_{\lambda'\,\text{even}} s_\lambda(x) = \prod_{1\leq j<k\leq n} (1 - x_j x_k)^{-1}$$

- A partition is even if all its non-zero parts are even
- The infinite sums over λ involve no restriction on either $\ell(\lambda)$ or $\ell(\lambda')$, but $s_\lambda(x) = 0$ if $\ell(\lambda) > n$.

Inverse Schur function series

- Littlewood [1940] For all $n \geq 1$ and $x = (x_1, x_2, \ldots, x_n)$

$$\sum_{\lambda \in \mathcal{P}_0} (-1)^{(|\lambda|+r(\lambda))/2}\, s_\lambda(x) = \prod_{1 \leq i \leq n} (1 - x_i) \prod_{1 \leq j < k \leq n} (1 - x_j x_k)$$

$$\sum_{\lambda \in \mathcal{P}_1} (-1)^{|\lambda|/2}\, s_\lambda(x) = \prod_{1 \leq j \leq k \leq n} (1 - x_j x_k)$$

$$\sum_{\lambda \in \mathcal{P}_{-1}} (-1)^{|\lambda|/2}\, s_\lambda(x) = \prod_{1 \leq j < k \leq n} (1 - x_j x_k)$$

- These series are finite for all finite n
- For finite n both $\ell(\lambda)$ and $\ell(\lambda')$ are restricted, since for $\lambda \in \mathcal{P}_t$ these differ by t

Determinantal identities

- Littlewood [1940] For all $n \geq 1$ and $x = (x_1, x_2, \ldots, x_n)$

$$\frac{\left|x_i^{n-j} - x_i^{n+j-1}\right|}{\left|x_i^{n-j}\right|} = \sum_{\lambda \in \mathcal{P}_0} (-1)^{[|\lambda| + r(\lambda)]/2} \, s_\lambda(x)$$

$$\frac{\left|x_i^{n-j} - x_i^{n+j}\right|}{\left|x_i^{n-j}\right|} = \sum_{\lambda \in \mathcal{P}_1} (-1)^{|\lambda|/2} \, s_\lambda(x)$$

$$\frac{\left|x_i^{n-j} + \chi_{j>1} \, x_i^{n+j-2}\right|}{\left|x_i^{n-j}\right|} = \sum_{\lambda \in \mathcal{P}_{-1}} (-1)^{|\lambda|/2} \, s_\lambda(x)$$

- the determinants are all $n \times n$ with $i, j = 1, 2, \ldots, n$
- and, for any proposition P, $\chi_P = \begin{cases} 1 & \text{if } P \text{ is true} \\ 0 & \text{if } P \text{ is false} \end{cases}$

General determinantal identity

- Lemma K [2008] For all $n \geq 1$ and $x = (x_1, x_2, \ldots, x_n)$

$$\frac{\left| x_i^{n-j} + q\,\chi_{j>-t}\; x_i^{n+t+j-1} \right|}{\left| x_i^{n-j} \right|} = \sum_{\lambda \in \mathcal{P}_t} (-1)^{[|\lambda| - r(\lambda)(t+1)]/2}\, q^{r(\lambda)}\, s_\lambda(x)$$

  - where t is any integer, and q is arbitrary
  - and the determinants are all $n \times n$
  - so that $i, j = 1, 2, \ldots, n$

- The special cases:

  $$q = -1, t = 0; \qquad q = -1, t = 1; \qquad q = 1, t = -1,$$

  correspond to Littlewood's previous formulae

Algebraic proof

$$\frac{\left| x_i^{n-j} + q\,\chi_{j>-t}\, x_i^{n+t+j-1} \right|}{\left| x_i^{n-j} \right|} = \frac{\left| x_i^{n-j} + q\,\chi_{j>-t}\, x_i^{2j-1+t+n-j} \right|}{\left| x_i^{n-j} \right|}$$

$$= \sum_{r=0}^{n} \sum_{\kappa} q^r\, s_\kappa(x) = \sum_{\lambda \in \mathcal{P}_t} (-1)^{(j_r-1)+\cdots+(j_2-1)+(j_1-1)}\, q^r\, s_\lambda(x)$$

- $\kappa_j = 2j - 1 + t$ for $j \in \{j_1, j_2, \ldots, j_r\}$ and $\kappa_j = 0$ otherwise
- with $n \geq j_1 > j_2 > \cdots > j_r \geq 1 - \chi_{t<0} t$
- $\lambda = \begin{pmatrix} j_1 - 1 + t & j_2 - 1 + t & \cdots & j_r - 1 + t \\ j_1 - 1 & j_2 - 1 & \cdots & j_r - 1 \end{pmatrix} \in \mathcal{P}_t$
- $r = r(\lambda)$
- $|\lambda| = 2((j_1 - 1) + (j_2 - 1) + \cdots + (j_r - 1)) + r(t + 1)$

Example with $n = 4$ and $t = 2$

$$\frac{\left| x_i^{4-j} + q\,\chi_{j>-2}\; x_i^{5+j} \right|}{\left| x_i^{4-j} \right|}$$

$$= \frac{\begin{vmatrix} x_i^3 - q\,x_i^6 & x_i^2 - q\,x_i^7 & x_i - q\,x_i^8 & 1 - q\,x_i^9 \end{vmatrix}}{\begin{vmatrix} x_i^3 & x_i^2 & x_i & 1 \end{vmatrix}}$$

$$= s_{0000} - q\,(s_{3000} + s_{0500} + s_{0070} + s_{0009})$$

$$+q^2\,(s_{3500} + s_{3070} + s_{0570} + s_{3009} + s_{0509} + s_{0079})$$

$$-q^3\,(s_{3570} + s_{3509} + s_{3079} + s_{0579}) + q^4\,s_{3579}$$

$$= 1 + q\,(-s_3 + s_{41} - s_{511} + s_{6111})$$

$$+q^2\,(-s_{44} + s_{541} - s_{552} - s_{6411} + s_{6521} - s_{6622})$$

$$+q^3\,(s_{555} - s_{6551} + s_{6652} - s_{6663}) + q^4\,s_{6666}$$

Example with $n = 4$ and $t = 2$ contd.

In Frobenius notation $s_\lambda(x) = \begin{pmatrix} a_1\, a_2 & \cdots & a_r \\ b_1\, b_2 & \cdots & b_r \end{pmatrix}$, we have

$$1 + q\left[-\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 3 \end{pmatrix}\right]$$

$$+q^2\left[-\begin{pmatrix} 32 \\ 10 \end{pmatrix} + \begin{pmatrix} 42 \\ 20 \end{pmatrix} - \begin{pmatrix} 43 \\ 21 \end{pmatrix} - \begin{pmatrix} 52 \\ 30 \end{pmatrix} + \begin{pmatrix} 53 \\ 31 \end{pmatrix} - \begin{pmatrix} 54 \\ 32 \end{pmatrix}\right]$$

$$+q^3\left[\begin{pmatrix} 432 \\ 210 \end{pmatrix} - \begin{pmatrix} 532 \\ 310 \end{pmatrix} + \begin{pmatrix} 542 \\ 320 \end{pmatrix} - \begin{pmatrix} 543 \\ 321 \end{pmatrix}\right]$$

$$+q^4\begin{pmatrix} 5432 \\ 3210 \end{pmatrix}$$

Example with $n = 4$ and $t = -2$

$$\frac{\left| x_i^{4-j} + q\,\chi_{j>2}\, x_i^{1+j} \right|}{\left| x_i^{4-j} \right|}$$

$$= \frac{\begin{vmatrix} x_i^3 & x_i^2 & x_i - q\,x_i^4 & 1 - q\,x_i^5 \end{vmatrix}}{\begin{vmatrix} x_i^3 & x_i^2 & x_i & 1 \end{vmatrix}}$$

$$= s_{0000} - q\,(s_{0030} + s_{0005}) + q^2\, s_{0035}$$

$$= 1 + q\,(-s_{111} + s_{2111}) - q^2\, s_{2222}$$

$$= 1 - q \begin{pmatrix} 0 \\ 2 \end{pmatrix} + q \begin{pmatrix} 1 \\ 3 \end{pmatrix} - q^2 \begin{pmatrix} 1\;0 \\ 3\;2 \end{pmatrix}$$

Row length restricted Schur function series

Theorem [Macdonald 79; Désarménien 87, Stembridge 90; Bressoud 98, Okada 98]

For all $n \geq 1$, $x = (x_1, x_2, \ldots, x_n)$ and $p \geq 0$:

$$\sum_{\lambda : \ell(\lambda') \leq p} s_\lambda(x) = \frac{\left| x_i^{n-j} - x_i^{n+p+j-1} \right|}{\left| x_i^{n-j} \right| \prod_{1 \leq i \leq n} (1 - x_i) \prod_{1 \leq j < k \leq n} (1 - x_j x_k)}$$

$$\sum_{\lambda \,\text{even}\, : \ell(\lambda') \leq 2p} s_\lambda(x) = \frac{\left| x_i^{n-j} - x_i^{n+2p+j} \right|}{\left| x_i^{n-j} \right| \prod_{1 \leq j \leq k \leq n} (1 - x_j x_k)}$$

$$\sum_{\lambda' \,\text{even}\, : \ell(\lambda') \leq p} s_\lambda(x) = \frac{\frac{1}{2} \left| x_i^{n-j} - x_i^{n+p+j-2} \right| + \frac{1}{2} \left| x_i^{n-j} + x_i^{n+p+j-2} \right|}{\left| x_i^{n-j} \right| \prod_{1 \leq j < k \leq n} (1 - x_j x_k)}$$

Row length restricted Schur function series

Using the Lemma for given q and t as indicated, we find

Corollary For all $x=(x_1,x_2,\ldots)$

$q=-1, t=p$

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_p} (-1)^{[|\mu|-r(\mu)(p-1)]/2}\, s_\mu(x)}{\prod_{1\leq i\leq n}(1-x_i)\prod_{1\leq j<k\leq n}(1-x_jx_k)}$$

$q=-1, t=2p+1$

$$\sum_{\lambda\,\text{even}\,:\ell(\lambda')\leq 2p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_{2p+1}} (-1)^{[|\mu|-r(\mu)(2p)]/2}\, s_\mu(x)}{\prod_{1\leq j\leq k\leq n}(1-x_jx_k)}$$

$q=\pm 1, t=p-1$

$$\sum_{\lambda'\,\text{even}\,:\ell(\lambda')\leq p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_{p-1}:r(\mu)\,\text{even}} (-1)^{[|\mu|-r(\mu)p]/2}\, s_\mu(x)}{\prod_{1\leq j<k\leq n}(1-x_jx_k)}$$

Row length restricted Schur function series

Littlewood's inverse Schur function series formulae then give:

Corollary For all $x = (x_1, x_2, \ldots)$

$q = -1, t = p$

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_p} (-1)^{[|\mu|-r(\mu)(p-1)]/2}\, s_\mu(x)}{\sum_{\nu\in\mathcal{P}_0} (-1)^{[|\nu|+r(\nu)]/2}\, s_\nu(x)}$$

$q = -1, t = 2p+1$

$$\sum_{\lambda\,\text{even}\,:\ell(\lambda')\leq 2p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_{2p+1}} (-1)^{[|\mu|-r(\mu)(2p)]/2}\, s_\mu(x)}{\sum_{\nu\in\mathcal{P}_1} (-1)^{|\nu|/2}\, s_\nu(x)}$$

$q = \pm 1, t = p-1$

$$\sum_{\lambda'\,\text{even}\,:\ell(\lambda')\leq p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_{p-1}:r(\mu)\,\text{even}} (-1)^{[|\mu|-r(\mu)p]/2}\, s_\mu(x)}{\sum_{\nu\in\mathcal{P}_{-1}} (-1)^{|\nu|/2}\, s_\nu(x)}$$

Column length restricted Schur function series

- Using the involution $\omega : s_\lambda(x) \mapsto s_{\lambda'}(x)$ for all λ
- and noting that $\lambda \in \mathcal{P}_t \implies \lambda' \in \mathcal{P}_{-t}$ for all t, we have

Corollary For all $x = (x_1, x_2, \ldots)$

$$\sum_{\lambda:\ell(\lambda)\le p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_{-p}} (-1)^{[|\mu|-r(\mu)(p-1)]/2}\, s_\mu(x)}{\sum_{\nu\in\mathcal{P}_0} (-1)^{[|\nu|+r(\nu)]/2}\, s_\nu(x)}$$

$$\sum_{\lambda' \text{ even}:\ell(\lambda)\le 2p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_{-2p-1}} (-1)^{[|\mu|-r(\mu)(2p)]/2}\, s_\mu(x)}{\sum_{\nu\in\mathcal{P}_{-1}} (-1)^{|\nu|/2}\, s_\nu(x)}$$

$$\sum_{\lambda \text{ even}:\ell(\lambda)\le p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_{-p+1}:r(\mu)\text{ even}} (-1)^{[|\mu|-r(\mu)p]/2}\, s_\mu(x)}{\sum_{\nu\in\mathcal{P}_1} (-1)^{|\nu|/2}\, s_\nu(x)}$$

Column length restricted Schur function series

Littlewood's inverse Schur function series formulae then give:

Corollary For all $x = (x_1, x_2, \ldots)$

$$\sum_{\lambda:\ell(\lambda)\leq p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_{-p}} (-1)^{[|\mu|-r(\mu)(p-1)]/2}\, s_\mu(x)}{\prod_{1\leq i\leq n}(1-x_i)\prod_{1\leq j<k\leq n}(1-x_jx_k)}$$

$$\sum_{\lambda' \text{ even }:\ell(\lambda)\leq 2p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_{-2p-1}} (-1)^{[|\mu|-r(\mu)(2p)]/2}\, s_\mu(x)}{\prod_{1\leq j<k\leq n}(1-x_jx_k)}$$

$$\sum_{\lambda \text{ even }:\ell(\lambda)\leq p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_{-p+1}:r(\mu)\text{ even}} (-1)^{[|\mu|-r(\mu)p]/2}\, s_\mu(x)}{\prod_{1\leq j\leq k\leq n}(1-x_jx_k)}$$

- Note: The first of these was Van der Jeugt's Conjecture

Column length restricted Schur function series

Using $(q,t)=(-1,-p)$, $(\pm 1,-p+1)$ and $(-1,-2p-1)$ in our Lemma, we find

Theorem For all $n\geq 1$, $x=(x_1,x_2,\ldots,x_n)$ and $p\geq 0$:

$$\sum_{\lambda:\ell(\lambda)\leq p} s_\lambda(x) = \frac{\left|x_i^{n-j}-(-1)^p\chi_{j>p}\, x_i^{n-p+j-1}\right|}{\left|x_i^{n-j}\right|\prod_{1\leq i\leq n}(1-x_i)\prod_{1\leq j<k\leq n}(1-x_jx_k)}$$

$$\sum_{\lambda\,\text{even}\,:\ell(\lambda)\leq p} s_\lambda(x) = \frac{\frac{1}{2}\left|x_i^{n-j}-\chi_{j\geq p}\, x_i^{n-p+j}\right|+\frac{1}{2}\left|x_i^{n-j}+\chi_{j\geq p}\, x_i^{n-p+j}\right|}{\left|x_i^{n-j}\right|\prod_{1\leq j\leq k\leq n}(1-x_jx_k)}$$

$$\sum_{\lambda'\,\text{even}\,:\ell(\lambda)\leq 2p} s_\lambda(x) = \frac{\left|x_i^{n-j}+\chi_{j>2p+1}\, x_i^{n-2p+j-2}\right|}{\left|x_i^{n-j}\right|\prod_{1\leq j<k\leq n}(1-x_jx_k)}$$

Row length restricted Schur function series

Alternative universal expressions giving each restricted series as a product of an unrestricted series and a correction factor for all $x=(x_1,x_2,\ldots)$ take the form

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x) \;=\; \sum_{\lambda} s_\lambda(x) \cdot \sum_{\mu\in\mathcal{P}_p} (-1)^{[|\mu|-r(\mu)(p-1)]/2}\, s_\mu(x)$$

$$\sum_{\lambda\,\text{even}\,:\ell(\lambda')\leq 2p} s_\lambda(x) \;=\; \sum_{\lambda\,\text{even}} s_\lambda(x) \cdot \sum_{\mu\in\mathcal{P}_{2p+1}} (-1)^{[|\mu|-r(\mu)(2p)]/2}\, s_\mu(x)$$

$$\sum_{\lambda'\,\text{even}\,:\ell(\lambda')\leq p} s_\lambda(x) \;=\; \sum_{\lambda'\,\text{even}} s_\lambda(x) \cdot \sum_{\mu\in\mathcal{P}_{p-1}:r(\mu)\,\text{even}} (-1)^{[|\mu|-r(\mu)p]/2}\, s_\mu(x)$$

Column length restricted Schur function series

Alternative universal expressions giving each restricted series as a product of an unrestricted series and a correction factor for all $x = (x_1, x_2, \ldots)$ take the form

$$\sum_{\lambda:\ell(\lambda)\leq p} s_\lambda(x) = \sum_{\lambda} s_\lambda(x) \cdot \sum_{\mu\in\mathcal{P}_{-p}} (-1)^{[|\mu|-r(\mu)(p-1)]/2}\, s_\mu(x)$$

$$\sum_{\lambda' \text{ even}\,:\ell(\lambda)\leq 2p} s_\lambda(x) = \sum_{\lambda' \text{ even}} s_\lambda(x) \cdot \sum_{\mu\in\mathcal{P}_{-2p-1}} (-1)^{[|\mu|-r(\mu)(2p)]/2}\, s_\mu(x)$$

$$\sum_{\lambda \text{ even}\,:\ell(\lambda)\leq p} s_\lambda(x) = \sum_{\lambda \text{ even}} s_\lambda(x) \cdot \sum_{\mu\in\mathcal{P}_{-p+1}:r(\mu) \text{ even}} (-1)^{[|\mu|-r(\mu)p]/2}\, s_\mu(x)$$

Special case - row length ≤ 1

Setting $p = 1$ we recover a very well known series:

$$\sum_{\lambda:\ell(\lambda')\leq 1} s_\lambda(x) = \sum_{\lambda} s_\lambda(x) \cdot \sum_{\mu\in\mathcal{P}_1} (-1)^{|\mu|/2} s_\mu(x)$$

$$= \prod_{1\leq i\leq n} (1-x_i)^{-1} \prod_{1\leq j<k\leq n} (1-x_j x_k)^{-1} \cdot \prod_{1\leq j\leq k\leq n} (1-x_j x_k)$$

$$= \prod_{1\leq i\leq n} (1+x_i) = \sum_{m=0}^{n} s_{1^m}(x) = \sum_{m=0}^{n} e_m(x)$$

- where the sums over m are finite, but could be extended to ∞ since $s_{1^m}(x) = e_m(x) = 0$ for all $m > n$.

Special case - column length ≤ 1

Again setting $p = 1$ we recover another very well known series:

$$\sum_{\lambda:\ell(\lambda)\leq 1} s_\lambda(x) = \sum_{\lambda} s_\lambda(x) \cdot \sum_{\mu\in\mathcal{P}_{-1}} (-1)^{|\mu|/2}\, s_\mu(x)$$

$$= \prod_{1\leq i\leq n} (1-x_i)^{-1} \prod_{1\leq j<k\leq n} (1-x_jx_k)^{-1} \cdot \prod_{1\leq j<k\leq n} (1-x_jx_k)$$

$$= \prod_{1\leq i\leq n} (1-x_i)^{-1} = \sum_{m=0}^{\infty} s_m(x) = \sum_{m=0}^{\infty} h_m(x)$$

- where the sums over m are infinite since for all $n \geq 1$ we have $s_m(x) = h_m(x) \neq 0$ for any $m \geq 0$.

So far

- We have recast the numerator of Macdonald’s formula as a signed sum of Schur functions
- We have then used conjugacy to prove Van der Jeugt’s conjecture
- We have obtained three determinantal formulae on column length restricted partitions analogous to those for row length restricted partitions
- We have not explained why the various determinants lead to row or column length restrictions
- To do this we need to exploit the fact that they define characters of particular representations of classical groups, as emphasized by Okada

Classical groups and their characters

Let $x = (x_1, x_2, \ldots, x_n)$ and $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$
with $x_i = e^{\epsilon_i}$ and $\overline{x}_i = x_i^{-1} = e^{-\epsilon_i}$ for $i = 1, 2, \ldots, n$

$$\mathrm{ch}\, V^{\lambda}_{GL(n)} = \frac{\left| x_i^{\lambda_j+n-j} \right|}{\left| x_i^{n-j} \right|}$$

$$\mathrm{ch}\, V^{\lambda}_{SO(2n+1)} = \frac{\left| x_i^{\lambda_j+n-j+\frac{1}{2}} - \overline{x}_i^{\lambda_j+n-j+\frac{1}{2}} \right|}{\left| x_i^{n-j+\frac{1}{2}} - \overline{x}_i^{n-j+\frac{1}{2}} \right|}$$

$$\mathrm{ch}\, V^{\lambda}_{Sp(2n)} = \frac{\left| x_i^{\lambda_j+n-j+1} - \overline{x}_i^{\lambda_j+n-j+1} \right|}{\left| x_i^{n-j+1} - \overline{x}_i^{n-j+1} \right|}$$

$$\mathrm{ch}\, V^{\lambda}_{SO(2n)} = \frac{\left| x_i^{\lambda_j+n-j} + \overline{x}_i^{\lambda_j+n-j} \right| + \left| x_i^{\lambda_j+n-j} - \overline{x}_i^{\lambda_j+n-j} \right|}{\left| x_i^{n-j} + \overline{x}_i^{n-j} \right|}$$

Row length restricted series and characters

Theorem [Macdonald, Stembridge, Okada]

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x) = \frac{\left|x_i^{n-j} - x_i^{n+p+j-1}\right|}{\left|x_i^{n-j} - x_i^{n+j-1}\right|} = \mathbf{x}^{p/2}\ \mathrm{ch}\, V_{SO(2n+1)}^{(p/2)^n}(x,\overline{x},1)$$

$$\sum_{\lambda\ \mathrm{even}\ :\ell(\lambda')\leq 2p} s_\lambda(x) = \frac{\left|x_i^{n-j} - x_i^{n+2p+j}\right|}{\left|x_i^{n-j} - x_i^{n+j}\right|} = \mathbf{x}^{p}\ \mathrm{ch}\, V_{Sp(2n)}^{p^n}(x,\overline{x})$$

$$\sum_{\lambda'\ \mathrm{even}\ :\ell(\lambda')\leq p} s_\lambda(x) = \frac{\left|x_i^{n-j} - x_i^{n+p+j-2}\right| + \left|x_i^{n-j} + x_i^{n+p+j-2}\right|}{\left|x_i^{n-j} + x_i^{n+j-2}\right|}$$

$$= \mathbf{x}^{p/2}\ \mathrm{ch}\, V_{SO(2n)}^{(p/2)^n_{(-)^n}}(x,\overline{x})$$

where $\mathbf{x} = x_1 x_2 \cdots x_n = \mathrm{ch}\, V_{GL(n}^{1^n}(x)$

Proof of formulae in terms of characters

- Start from the original determinantal formulae
- In each determinant permute columns under

$$j \to n-j+1$$

- Extract factors $(-1)^n$ by changing signs of all terms of the form $x_i^a - x_i^b$
- Extract factors
  - $x_i^{n-\frac{1}{2}+\frac{p}{2}}$ and $x_i^{n-\frac{1}{2}}$
  - x_i^{n+p} and x_i^n
  - $x_i^{n-1+\frac{p}{2}}$ and x_i^{n-1}

  from each row of numerator and denominator determinants

Howe dual pairs of groups

Definition [Howe 85]

- Let groups G and H act on a linear vector space V
- Let their actions mutually commute
- As a representation of $G \times H$, let

$$V = \oplus_{k \in K} V_G^{\lambda(k)} \otimes V_H^{\mu(k)}$$

  - k varies over some index set K
  - $V_G^{\lambda(k)}$ and $V_H^{\mu(k)}$ are irreps of G and H
  - $V_G^{\lambda(k)}$ and $V_H^{\mu(k)}$ vary without repetition
- In such a case we say that G and H form a (Howe) dual pair with respect to V.

Howe dual pairs of classical groups

- In some cases V is an irrep of a group $F \supseteq G \times H$
- On restriction to the subgroup $G \times H$

$$\mathrm{ch}\, V^F_{G\times H} = \sum_{k\in K} \mathrm{ch}\, V_G^{\lambda(k)}\ \mathrm{ch}\, V_H^{\mu(k)}$$

Ex: [Howe 89, Hasegawa 89] For V the spin irrep of an othogonal group with character $\mathrm{ch}\, V^\Delta$, dual pairs are defined through each of the following restrictions:

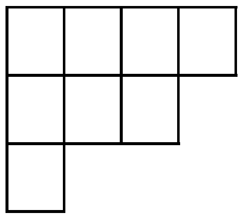
$$\begin{aligned}
O(4np) &\supseteq SO(2n)\times O(2p)\\
O(4np+2p) &\supseteq SO(2n+1)\times O(2p)\\
O(4np+2n) &\supseteq SO(2n)\times O(2p+1)\\
O(4np+2n+2p+1) &\supseteq SO(2n+1)\times O(2p+1)\\
O(4np) &\supseteq Sp(2n)\times Sp(2p)
\end{aligned}$$

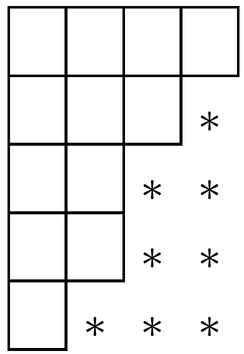
Notation for p^n-complements

- For any partition $\lambda \subseteq n^p$ we have $\lambda' \subseteq p^n$
- In such a case, let $\lambda^\dagger = (p - \lambda'_n, \ldots, p - \lambda'_2, p - \lambda'_1)$
- Then $\lambda^\dagger$ is also a partition

  Ex: If $p = 4$, $n = 5$ and $\lambda = (4,3,1)$

  then $\lambda' = (3,2,2,1)$ and $\lambda^\dagger = (4,3,2,2,1)$

$F^\lambda =$ 

$F^{\lambda^\dagger} =$ 

- Note: $0^\dagger = p^n = (p, p, \ldots, p)$

The spin module and Howe dual pairs

Theorem [Morris 58,60; Hasegawa 89; Terada 93; Bump and Gamburd 05] On restriction to the appropriate subgroup:

$$\mathrm{ch}\, V^{\Delta}_{O(4np)} = \sum_{\lambda \subseteq n^p} \mathrm{ch}\, V^{\lambda^\dagger}_{SO(2n)}\ \mathrm{ch}\, V^{\lambda}_{O(2p)}$$

$$\mathrm{ch}\, V^{\Delta}_{O(4np+2p)} = \sum_{\lambda \subseteq n^p} \mathrm{ch}\, V^{\lambda^\dagger}_{SO(2n+1)}\ \mathrm{ch}\, V^{\Delta;\lambda}_{O(2p)}$$

$$\mathrm{ch}\, V^{\Delta}_{O(4np+2n)} = \sum_{\lambda \subseteq n^p} \mathrm{ch}\, V^{\Delta;\lambda^\dagger}_{SO(2n)}\ \mathrm{ch}\, V^{\lambda}_{O(2p+1)}$$

$$\mathrm{ch}\, V^{\Delta}_{O(4np+2n+2p+1)} = \sum_{\lambda \subseteq n^p} \mathrm{ch}\, V^{\Delta;\lambda^\dagger}_{SO(2n+1)}\ \mathrm{ch}\, V^{\Delta;\lambda}_{O(2p+1)}$$

$$\mathrm{ch}\, V^{\Delta}_{O(4np)} = \sum_{\lambda \subseteq n^p} \mathrm{ch}\, V^{\lambda^\dagger}_{Sp(2n)}\ \mathrm{ch}\, V^{\lambda}_{Sp(2p)}$$

Exploitation of Howe duality

- Let (G,H) be a Howe dual pair with $F \supseteq G \times H$ such that $\mathrm{ch}\, V^F_{G\times H} = \sum_{k\in K} \mathrm{ch}\, V_G^{\lambda(k)}\ \mathrm{ch}\, V_H^{\mu(k)}$
- The character $\mathrm{ch}\, V_G^{\lambda(k)}$ is just the coefficient of $\mathrm{ch}\, V_H^{\mu(k)}$ in any formula we can devise for $\mathrm{ch}\, V^F_{G\times H}$
- In the case of the spin character identities all that is needed are:
  - dual Cauchy formula
  - expressions for classical group characters in terms of Schur functions [Littlewood 1940]
  - some modification rules [Newell 1951]

Spin characters and their decomposition

In terms of appropriate parameters

$$\mathrm{ch}\, V^{\Delta}_{O(2n)}(x,\overline{x}) = \prod_{i=1}^{n} (x_i^{\frac{1}{2}} + x_i^{-\frac{1}{2}}) = \mathbf{x}^{-1} \prod_{i=1}^{n} (1 + x_i)$$

$$\mathrm{ch}\, V^{\Delta}_{O(4np)}(xy, x\overline{y}, \overline{x}y, \overline{xy})$$

$$= \prod_{i=1}^{n}\prod_{j=1}^{p} (x_i^{\frac{1}{2}} y_j^{\frac{1}{2}} + x_i^{-\frac{1}{2}} y_j^{-\frac{1}{2}})(x_i^{\frac{1}{2}} y_j^{-\frac{1}{2}} + x_i^{-\frac{1}{2}} y_j^{\frac{1}{2}})$$

$$= \prod_{i=1}^{n}\prod_{j=1}^{p} (x_i + \overline{x}_i + y_j + \overline{y}_j)$$

$$= \mathbf{x}^{-p} \prod_{i=1}^{n}\prod_{j=1}^{p} (1 + x_i y_j)(1 + x_i \overline{y}_j) = \mathbf{x}^{-p} \sum_{\zeta \subseteq n^{2p}} s_{\zeta'}(x)\, s_{\zeta}(y,\overline{y})$$

Application to Howe dual pair contd.

$$= \mathbf{x}^{-p} \sum_{\zeta\subseteq n^{2p}} s_{\zeta'}(x)\, s_\zeta(y,\overline{y}) = \mathbf{x}^{-p} \sum_{\zeta\subseteq n^{2p}} s_{\zeta'}(x)\ \mathrm{ch}\, V^{\zeta}_{GL(n)}(y,\overline{y})$$

$$= \mathbf{x}^{-p} \sum_{\zeta\subseteq n^{2p}} s_{\zeta'}(x) \sum_{\beta:\beta'\, even} \mathrm{ch}\, V^{\zeta/\beta}_{Sp(2p)}(y,\overline{y})$$

$$= \mathbf{x}^{-p} \sum_{\eta\subseteq n^{2p}} \mathcal{W}_{2p}\Bigg(\sum_{\beta:\beta'\, even} s_{\eta'}(x)\, s_{\beta'}(x)\Bigg)\ \mathrm{ch}\, V^{\eta}_{Sp(2p)}(y,\overline{y})$$

$$= \mathbf{x}^{-p} \sum_{\eta\subseteq n^{2p}} \mathcal{W}_{2p}\Bigg(\sum_{\delta\, even} s_{\eta'}(x)\, s_{\delta}(x)\Bigg)\ \mathrm{ch}\, V^{\eta}_{Sp(2p)}(y,\overline{y})$$

$$= \sum_{\lambda\subseteq n^{p}} \mathrm{ch}\, V^{\lambda^\dagger}_{Sp(2n)}(x,\overline{x})\ \ \mathrm{ch}\, V^{\lambda}_{Sp(2p)}(y,\overline{y})$$ dual pair Theorem

where $\mathcal{W}_{2p}$ restricts any sum of Schur functions $s_\nu(x)$ to those having $\nu_1 = \ell(\nu') \leq 2p$

Character formula

It follows that

$$\mathrm{ch}\, V^{\lambda^\dagger}_{Sp(2n)}(x,\overline{x}) = \; \mathbf{x}^{-p} \sum_{\eta\subseteq n^{2p}} \varepsilon_{\eta,\lambda}\, \mathcal{W}_{2p}\Bigg(\sum_{\delta\; even} s_{\eta'}(x)\, s_\delta(x)\Bigg)$$

where the modification rules for $Sp(2p)$ characters are such that

$$\varepsilon_{\eta,\lambda} = \begin{cases} \pm 1 & \text{if } \mathrm{ch}\, V^{\eta}_{Sp(2p)}(y,\overline{y}) = \pm\, \mathrm{ch}\, V^{\lambda}_{Sp(2p)}(y,\overline{y}) \\ 0 & \text{otherwise} \end{cases}$$

In the special case $\lambda = 0$, so that $\lambda^\dagger = p^n$ this gives

$$\mathrm{ch}\, V^{p^n}_{Sp(2n)}(x,\overline{x}) = \; \mathbf{x}^{-p}\, \mathcal{W}_{2p}\Bigg(\sum_{\delta\; even} s_\delta(x)\Bigg) = \; \mathbf{x}^{-p} \sum_{\delta\; even:\ell(\delta')\le 2p} s_\delta(x)$$

The spin module and Howe dual pairs

- Thus we have recovered the Stembridge formula for the symplectic group characters as a sum of row length restricted Schur functions specified by even partitions
- The formulae of Macdonald and Okada may be recovered in the same way from the Howe dual pairs listed earlier
- In each case the row length restriction owes its origin to the bijective correspondence between irreps of the dual groups specified by $\lambda^\dagger$ and λ
- We would like to identify other Howe dual pairs that might lead to characters expressible as our sums of column length restricted Schur functions
- Such characters are necessarily infinite dimensional

The metaplectic module and Howe dual pairs

- We need an infinite-dimensional analogue of the spin representation of the orthogonal group
- This is provided by the metaplectic representation of the symplectic group

Ex: [Howe 89] For V the metaplectic irrep of a symplectic group with character $\mathrm{ch}\, V^{\tilde{\Delta}}$, dual pairs are defined through each of the following restrictions:

$$
\begin{aligned}
Sp(4np) &\supseteq Sp(2n) \times O(2p) \\
Sp(4np+2p) &\supseteq Sp(2n) \times O(2p+1) \\
Sp(4np) &\supseteq SO(2n) \times Sp(2p)
\end{aligned}
$$

Metaplectic dual pair character formula

Theorem [Moshinsky and Quesne 71, Kashiwara and Vergne 78, Howe 85, K and Wybourne 85]

On restriction to the appropriate subgroup:

$$\mathrm{ch}\, V^{\tilde{\Delta}}_{Sp(4np)} = \sum_{\lambda:\lambda'_1+\lambda'_2\leq 2p,\, \lambda'_1\leq n} \mathrm{ch}\, V^{p(\lambda)}_{Sp(2n)}\ \mathrm{ch}\, V^{\lambda}_{O(2p)}$$

$$\mathrm{ch}\, V^{\tilde{\Delta}}_{Sp(4np+2n)} = \sum_{\lambda:\lambda'_1+\lambda'_2\leq 2p+1,\, \lambda'_1\leq n} \mathrm{ch}\, V^{p+\frac{1}{2}(\lambda)}_{Sp(2n)}\ \mathrm{ch}\, V^{\lambda}_{O(2p+1)}$$

$$\mathrm{ch}\, V^{\tilde{\Delta}}_{Sp(4np)} = \sum_{\lambda:\lambda'_1\leq \min(p,n)} \mathrm{ch}\, V^{p(\lambda)}_{SO(2n)}\ \mathrm{ch}\, V^{\lambda}_{Sp(2p)}$$

Metaplectic characters and their decomposition

In terms of appropriate parameters

$$\mathrm{ch}\, V^{\tilde{\Delta}}_{Sp(2n)}(x,\overline{x}) = \prod_{i=1}^{n} (x_i^{-\frac{1}{2}} - x_i^{\frac{1}{2}})^{-1} = \mathbf{x} \prod_{i=1}^{n} (1-x_i)^{-1}$$

$$\mathrm{ch}\, V^{\tilde{\Delta}}_{Sp(4np)}(xy, x\overline{y}, \overline{x}y, \overline{xy})$$

$$= \prod_{i=1}^{n}\prod_{j=1}^{p} (x_i^{-\frac{1}{2}} y_j^{-\frac{1}{2}} - x_i^{\frac{1}{2}} y_j^{\frac{1}{2}})^{-1} (x_i^{-\frac{1}{2}} y_j^{\frac{1}{2}} - x_i^{\frac{1}{2}} y_j^{-\frac{1}{2}})^{-1}$$

$$= \mathbf{x}^p \prod_{i=1}^{n}\prod_{j=1}^{p} (1 - x_i y_j)^{-1} (1 - x_i \overline{y}_j)^{-1}$$

$$= \mathbf{x}^p \sum_{\zeta:\ell(\zeta)\leq \min(n,2p)} s_\zeta(x)\; s_\zeta(y,\overline{y})$$

Application to Howe dual pair contd.

$$= \mathbf{x}^p \sum_{\zeta:\ell(\zeta)\le\min(n,2p)} s_\zeta(x)\ s_\zeta(y,\overline{y})$$

$$= \mathbf{x}^p \sum_{\zeta:\ell(\zeta)\le\min(n,2p)} s_\zeta(x)\ \mathrm{ch}\, V^{\zeta}_{GL(2p)}(y,\overline{y})$$

$$= \mathbf{x}^p \sum_{\zeta:\ell(\zeta)\le\min(n,2p)} s_\zeta(x)\ \sum_{\delta\ even} \mathrm{ch}\, V^{\zeta/\delta}_{O(2p)}(y,\overline{y})$$

$$= \mathbf{x}^p \sum_{\eta:\ell(\eta)\le\min(n,2p)} \mathcal{L}_{2p}\left(\sum_{\delta\ even} s_\eta(x)\ s_\delta(x)\right)\ \mathrm{ch}\, V^{\eta}_{O(2p)}(y,\overline{y})$$

$$= \sum_{\lambda:\lambda'_1+\lambda'_2\le 2p,\ \lambda'_1\le n} \mathrm{ch}\, V^{p(\lambda)}_{Sp(2n)}(x,\overline{x})\ \ \mathrm{ch}\, V^{\lambda}_{O(2p)}(y,\overline{y})$$ dual pair

where $\mathcal{L}_{2p}$ restricts any sum of Schur functions $s_\nu(x)$ to those having $\nu'_1 = \ell(\nu) \le 2p$

Character formula

It follows that

$$\mathrm{ch}\, V^{p(\lambda)}_{Sp(2n)}(x,\overline{x}) = \mathbf{x}^p \sum_{\eta:\ell(\zeta)\leq \min(n,2p)} \varepsilon_{\eta,\lambda}\, \mathcal{L}_{2p}\left(\sum_{\delta\, even} s_\eta(x)\, s_\delta(x) \right)$$

where the modification rules for $O(2p)$ characters are such that

$$\varepsilon_{\eta,\lambda} = \begin{cases} \pm 1 & \text{if } \mathrm{ch}\, V^{\eta}_{O(2p)}(y,\overline{y}) = \pm\, \mathrm{ch}\, V^{\lambda}_{O(2p)}(y,\overline{y}) \\ 0 & \text{otherwise} \end{cases}$$

In the special case $\lambda = 0$ this gives

$$\mathrm{ch}\, V^{p(0)}_{Sp(2n)}(x,\overline{x}) = \mathbf{x}^p\, \mathcal{L}_{2p}\left(\sum_{\delta\, even} s_\delta(x) \right) = \mathbf{x}^p \sum_{\delta\, even:\ell(\delta)\leq 2p} s_\delta(x)$$

The metaplectic module and Howe dual pairs

- Thus we have obtained a formula for a particular symplectic group character as a sum of column length restricted Schur functions specified by even partitions
- Our other column length restricted Schur function formula may be also be identifed with characters in the same way
- In each case the column length restriction owes its origin to the bijective correspondence between irreps of the dual groups specified by $p(\lambda)$ and λ

Generalisation to Lie supergroups

- Howe's original work on dual pairs encompassed supergroups, such as $GL(m/n)$ and $OSp(m/n)$
- All our identities can be extended to orthosymplectic versions [Cheng and Zhang 04]
- In particular, one can recover our starting point:

Proposition [Van der Jeugt, Lievens and Stoilova, 2007]
Let $x = (x_1, x_2, \ldots, x_n)$, then for each positive integer p the irrep $V(p)$ of $OSp(1|2n)$ has character

$$\mathrm{ch}\, V(p) = (x_1 x_2 \cdots x_n)^{p/2} \sum_{\lambda : \ell(\lambda) \leq p} s_\lambda(x)$$

Cauchy formula and its inverse

- Let m, n be positive integers
- Then for all $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$

$$\sum_{\lambda} s_\lambda(x)\, s_\lambda(y) \;=\; \prod_{i=1}^{m} \prod_{j=1}^{n} (1 - x_i y_j)^{-1}$$

$$\sum_{\lambda} (-1)^{|\lambda|}\, s_\lambda(x)\, s_{\lambda'}(y) \;=\; \prod_{i=1}^{m} \prod_{j=1}^{n} (1 - x_i y_j)$$

- The first sum over λ is infinite with non-zero terms arising for all $\ell(\lambda) \leq \min\{m, n\}$, and no restriction on $\ell(\lambda')$
- The second sum over λ is finite, with $\lambda \subseteq n^m$, since $s_\lambda(x) = 0$ if $\ell(\lambda) > m$ and $s_{\lambda'}(y) = 0$ if $\ell(\lambda') > n$

Determinantal identity

- For all $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$

$$\frac{1}{\left|x_i^{m-j}\right| \;\left|y_a^{n-b}\right|} \cdot \begin{vmatrix} y_{n+1-i}^{j-1} \\ \cdots \\ x_{i-n}^{m+n-j} \end{vmatrix} = \prod_{i=1}^{m} \prod_{a=1}^{n} (1 - x_i y_a)$$

  where the $m+n \times m+n$ determinant in the numerator is partitioned after the nth row, and the determinants in the denominator are $m \times m$ and $n \times n$

Proof: Use the fact that the three determinants are Vandermonde determinants in the m variables x_i, the n variables y_a and the $m+n$ variables x_i and $\overline{y}_a$, with $\overline{y}_a = y_a^{-1}$

Nature of key determinant

- Ex: $m = 3,\ n = 4,$ setting $\overline{y}_a = y_a^{-1}$ for $a = 1, 2, 3, 4$

$$\begin{vmatrix} 1 & y_4 & y_4^2 & y_4^3 & y_4^4 & y_4^5 & y_4^6 \\ 1 & y_3 & y_3^2 & y_3^3 & y_3^4 & y_3^5 & y_3^6 \\ 1 & y_2 & y_2^2 & y_2^3 & y_2^4 & y_2^5 & y_2^6 \\ 1 & y_1 & y_1^2 & y_1^3 & y_1^4 & y_1^5 & y_1^6 \\ x_1^6 & x_1^5 & x_1^4 & x_1^3 & x_1^2 & x_1 & 1 \\ x_2^6 & x_2^5 & x_2^4 & x_2^3 & x_2^2 & x_2 & 1 \\ x_3^6 & x_3^5 & x_3^4 & x_3^3 & x_3^2 & x_3 & 1 \end{vmatrix} \sim \begin{vmatrix} \overline{y}_4^6 & \overline{y}_4^5 & \overline{y}_4^4 & \overline{y}_4^3 & \overline{y}_4^2 & \overline{y}_4 & 1 \\ \overline{y}_3^6 & \overline{y}_3^5 & \overline{y}_3^4 & \overline{y}_3^3 & \overline{y}_3^2 & \overline{y}_3 & 1 \\ \overline{y}_2^6 & \overline{y}_2^5 & \overline{y}_2^4 & \overline{y}_2^3 & \overline{y}_2^2 & \overline{y}_2 & 1 \\ \overline{y}_1^6 & \overline{y}_1^5 & \overline{y}_1^4 & \overline{y}_1^3 & \overline{y}_1^2 & \overline{y}_1 & 1 \\ x_1^6 & x_1^5 & x_1^4 & x_1^3 & x_1^2 & x_1 & 1 \\ x_2^6 & x_2^5 & x_2^4 & x_2^3 & x_2^2 & x_2 & 1 \\ x_3^6 & x_3^5 & x_3^4 & x_3^3 & x_3^2 & x_3 & 1 \end{vmatrix}$$

- Factors $(x_i - x_j),\ (y_a - y_b),\ (1 - x_i y_a) \sim (\overline{y}_a - x_i)$
- Leading term $y_4^0\, y_3^1\, y_2^2\, y_1^3\, x_1^2\, x_2^1\, x_3^0$

Corollary

- For all $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$

$$\sum_{\lambda \subseteq n^m} (-1)^{|\lambda|} \, s_\lambda(x) \, s_{\lambda'}(y) \;=\; \frac{1}{\left|x_i^{m-j}\right| \; \left|y_i^{n-j}\right|} \cdot \begin{vmatrix} y_{n+1-i}^{j-1} \\ \cdots \\ x_{i-n}^{m+n-j} \end{vmatrix}$$

- where the $m{+}n \times m{+}n$ determinant in the numerator is partitioned after the nth row, and the determinants in the denominator are $m \times m$ and $n \times n$

Note: This may also be proved directly through using the Laplace expansion of the determinant

Row length restricted Cauchy formula

Theorem Let $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$ with $m, n \geq 1$. Then for all $p \geq 0$ we have

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x)\, s_\lambda(y)$$

$$= \frac{1}{\left|x_i^{m-j}\right| \; \left|y_a^{n-b}\right| \; \prod_{i=1}^m \prod_{a=1}^n (1 - x_i y_a)} \cdot \left| \begin{array}{c} y_{n+1-i}^{\,j-1+\chi_{j>n}p} \\ \cdots \\ x_{i-n}^{\,m+n-j+\chi_{j\leq n}p} \end{array} \right|$$

- Since $s_\lambda(x) = 0$ if $\ell(\lambda) > m$ and $s_\lambda(y) = 0$ if $\ell(\lambda) > n$ the sum is restricted to $\lambda \subseteq p^q$ with $q = \min\{m, n\}$
- If $p = 0$ then both sides are just 1, as confirmed by the inverse Cauchy determinantal identity

Proof of row length restricted Cauchy formula

$$\left| \begin{array}{c} y_{n+1-i}^{j-1+\chi_{j>n} p} \\ \cdots \\ x_{i-n}^{m+n-j+\chi_{j\le n} p} \end{array} \right| = \prod_{a=1}^{n} y_a^{m+n-1+p} \left| \begin{array}{ccc} \overline{y}_{n+1-i}^{m+n-j+p} & \vdots & \overline{y}_{n+1-i}^{m+n-j} \\ \cdots & \cdots & \cdots \\ x_{i-n}^{m+n-j+p} & \vdots & x_{i-n}^{m+n-j} \end{array} \right|$$

$$= \prod_{a=1}^{n} y_a^{m+n-1+p} \quad s_{p^n}(\overline{y}, x) \left| \begin{array}{c} \overline{y}_{n+1-i}^{m+n-j} \\ \cdots \\ x_{i-n}^{m+n-j} \end{array} \right|$$

$$= \prod_{a=1}^{n} y_a^{p} \quad s_{p^n}(\overline{y}, x) \left| \begin{array}{c} y_{n+1-i}^{j-1} \\ \cdots \\ x_{i-n}^{m+n-j} \end{array} \right|$$

Proof contd.

where

$$s_{p^n}(\overline{y},x) = \sum_{\lambda\subseteq p^n} s_\lambda(x)\, s_{p^n/\lambda}(\overline{y}) = \prod_{a=1}^{n} \overline{y}_a^p \sum_{\lambda\subseteq p^n} s_\lambda(x)\, s_\lambda(y)$$

so that

$$\left|\begin{array}{c} y_{n+1-i}^{j-1+\chi_{j>n}p} \\ \cdots \\ x_{i-n}^{m+n-j+\chi_{j\le n}p} \end{array}\right| \Bigg/ \left|\begin{array}{c} y_{n+1-i}^{j-1} \\ \cdots \\ x_{i-n}^{m+n-j} \end{array}\right| = \sum_{\lambda:\ell(\lambda')\le p} s_\lambda(x)\, s_\lambda(y)$$

with

$$\left|\begin{array}{c} y_{n+1-i}^{j-1} \\ \cdots \\ x_{i-n}^{m+n-j} \end{array}\right| = \left|x_i^{m-j}\right| \left|y_a^{n-b}\right| \prod_{i=1}^{m}\prod_{a=1}^{n}(1-x_iy_a)$$

Extended dual Cauchy formula

Lemma [K, 2008]

- Let $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$
- Then for each pair of integers p and q we have

$$\frac{1}{\left|x_i^{m-j}\right| \; \left|y_i^{n-j}\right|} \cdot \left| \begin{array}{c:c} y_{n+1-i}^{j-1} & \chi_{j>n-q} \, y_{n+1-i}^{j-1+q} \\ \cdots & \cdots \\ \chi_{j \le n+p} \, x_{i-n}^{m+n-j+p} & x_{i-n}^{m+n-j} \end{array} \right|$$

$$= \sum_{\zeta \subseteq n^m} (-1)^{|\zeta|} \, s_{\sigma}(x) \, s_{\tau}(y)$$

- where $\sigma = (\zeta + p^r)$ and $\tau = (\zeta' + q^r)$ with $r = r(\zeta)$

Extended dual Cauchy formula

- where the determinant is $(m+n)\times(m+n)$, and is partitioned after the nth row and nth column
- and if $\zeta = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix} \in (n^m)$
- with $a_1 < n$, $b_1 < m$ and $r = r(\zeta)$, then
  - $\sigma = \begin{pmatrix} a_1+p & a_2+p & \cdots & a_r+p \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$
  - $\tau = \begin{pmatrix} b_1+q & b_2+q & \cdots & b_r+q \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}$
  - with $a_r \geq \max\{0, -p\}$ and $b_r \geq \max\{0, -q\}$

Examples of key determinant

- $m = 3$, $n = 4$, $p = 2$, $q = 1$

$$\begin{vmatrix} 1 & y_4 & y_4^2 & y_4^3 & \vdots & y_4^5 & y_4^6 & y_4^7 \\ 1 & y_3 & y_3^2 & y_3^3 & \vdots & y_3^5 & y_3^6 & y_3^7 \\ 1 & y_2 & y_2^2 & y_2^3 & \vdots & y_2^5 & y_2^6 & y_2^7 \\ 1 & y_1 & y_1^2 & y_1^3 & \vdots & y_1^5 & y_1^6 & y_1^7 \\ \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ x_1^8 & x_1^7 & x_1^6 & x_1^5 & \vdots & x_1^2 & x_1 & 1 \\ x_2^8 & x_2^7 & x_2^6 & x_2^5 & \vdots & x_2^2 & x_2 & 1 \\ x_3^8 & x_3^7 & x_3^6 & x_3^5 & \vdots & x_3^2 & x_3 & 1 \end{vmatrix}$$

Examples of key determinant

- $m = 3,\ n = 4,\ p = -2,\ q = 1$

$$\begin{vmatrix} 1 & y_4 & y_4^2 & y_4^3 & \vdots & y_4^5 & y_4^6 & y_4^7 \\ 1 & y_3 & y_3^2 & y_3^3 & \vdots & y_3^5 & y_3^6 & y_3^7 \\ 1 & y_2 & y_2^2 & y_2^3 & \vdots & y_2^5 & y_2^6 & y_2^7 \\ 1 & y_1 & y_1^2 & y_1^3 & \vdots & y_1^5 & y_1^6 & y_1^7 \\ \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ x_1^4 & x_1^3 & - & - & \vdots & x_1^2 & x_1 & 1 \\ x_2^4 & x_2^3 & - & - & \vdots & x_2^2 & x_2 & 1 \\ x_3^4 & x_3^3 & - & - & \vdots & x_3^2 & x_3 & 1 \end{vmatrix}$$

Examples of key determinant

- $m = 3,\; n = 4,\; p = 2,\; q = -1$

$$\begin{vmatrix}
1 & y_4 & y_4^2 & y_4^3 & \vdots & - & y_4^4 & y_4^5 \\
1 & y_3 & y_3^2 & y_3^3 & \vdots & - & y_3^4 & y_3^5 \\
1 & y_2 & y_2^2 & y_2^3 & \vdots & - & y_2^4 & y_2^5 \\
1 & y_1 & y_1^2 & y_1^3 & \vdots & - & y_1^4 & y_1^5 \\
\cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\
x_1^8 & x_1^7 & x_1^6 & x_1^5 & \vdots & x_1^2 & x_1 & 1 \\
x_2^8 & x_2^7 & x_2^6 & x_2^5 & \vdots & x_2^2 & x_2 & 1 \\
x_3^8 & x_3^7 & x_3^6 & x_3^5 & \vdots & x_3^2 & x_3 & 1
\end{vmatrix}$$

Examples of key determinant

- $m = 3, \; n = 4, \; p = -2, \; q = -1$

$$\begin{vmatrix}
1 & y_4 & y_4^2 & y_4^3 & \vdots & - & y_4^4 & y_4^5 \\
1 & y_3 & y_3^2 & y_3^3 & \vdots & - & y_3^4 & y_3^5 \\
1 & y_2 & y_2^2 & y_2^3 & \vdots & - & y_2^4 & y_2^5 \\
1 & y_1 & y_1^2 & y_1^3 & \vdots & - & y_1^4 & y_1^5 \\
\cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\
x_1^4 & x_1^3 & - & - & \vdots & x_1^2 & x_1 & 1 \\
x_2^4 & x_2^3 & - & - & \vdots & x_2^2 & x_2 & 1 \\
x_3^4 & x_3^3 & - & - & \vdots & x_3^2 & x_3 & 1
\end{vmatrix}$$

Ex 1: $m=3$, $n=4$, $p=2$, $q=1$

Ex. $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 5 & 7 & 1 & 4 & 6 \end{pmatrix}$

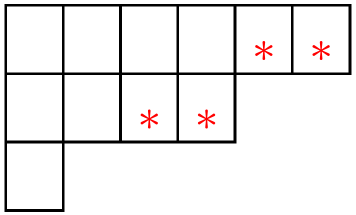
$$\begin{vmatrix} 1 & y_4 & y_4^2 & y_4^3 & \vdots & y_4^5 & y_4^6 & y_4^7 \\ 1 & y_3 & y_3^2 & y_3^3 & \vdots & y_3^5 & y_3^6 & y_3^7 \\ 1 & y_2 & y_2^2 & y_2^3 & \vdots & y_2^5 & y_2^6 & y_2^7 \\ 1 & y_1 & y_1^2 & y_1^3 & \vdots & y_1^5 & y_1^6 & y_1^7 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ x_1^8 & x_1^7 & x_1^6 & x_1^5 & \vdots & x_1^2 & x_1 & 1 \\ x_2^8 & x_2^7 & x_2^6 & x_2^5 & \vdots & x_2^2 & x_2 & 1 \\ x_3^8 & x_3^7 & x_3^6 & x_3^5 & \vdots & x_3^2 & x_3 & 1 \end{vmatrix} \sim - \begin{vmatrix} y_4 & y_4^2 & y_4^5 & y_4^7 \\ y_3 & y_3^2 & y_3^5 & y_3^7 \\ y_2 & y_2^2 & y_2^5 & y_2^7 \\ y_1 & y_1^2 & y_1^5 & y_1^7 \end{vmatrix} \cdot \begin{vmatrix} x_1^8 & x_1^5 & x_1 \\ x_2^8 & x_2^5 & x_2 \\ x_3^8 & x_3^5 & x_3 \end{vmatrix}$$

$$= - \; s_{4311}(y) \; \left| y_a^{4-b} \right| \cdot s_{641}(x) \; \left| x_i^{3-j} \right|$$

Ex 1: $m = 3$, $n = 4$, $p = 2$, $q = 1$

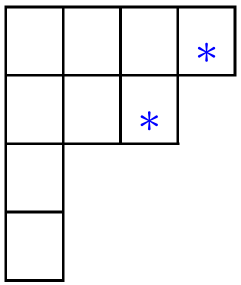
Ex. $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 5 & 7 & 1 & 4 & 6 \end{pmatrix}$ $\quad (-1)^{\pi} = (-1)^{|\zeta|} = -1$

$$\zeta = (4, 2, 1) = \begin{pmatrix} 3 & 0 \\ 2 & 0 \end{pmatrix}$$



$$\sigma = \begin{pmatrix} 3+2 & 0+2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 0 \end{pmatrix} = (6, 4, 1)$$

$$\zeta' = (3, 2, 1, 1) = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}$$



$$\tau = \begin{pmatrix} 2+1 & 0+1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & 0 \end{pmatrix} = (4, 3, 1, 1)$$

Ex 2: $m = 3,\ n = 4,\ p = -2,\ q = -1$

Ex. $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 6 & 7 & 1 & 2 & 5 \end{pmatrix}$

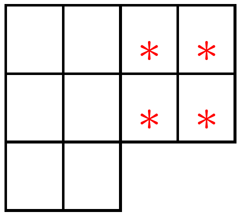
$$\begin{vmatrix} 1 & y_4 & y_4^2 & y_4^3 & \vdots & - & y_4^4 & y_4^5 \\ 1 & y_3 & y_3^2 & y_3^3 & \vdots & - & y_3^4 & y_3^5 \\ 1 & y_2 & y_2^2 & y_2^3 & \vdots & - & y_2^4 & y_2^5 \\ 1 & y_1 & y_1^2 & y_1^3 & \vdots & - & y_1^4 & y_1^5 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ x_1^4 & x_1^3 & - & - & \vdots & x_1^2 & x_1 & 1 \\ x_2^4 & x_2^3 & - & - & \vdots & x_2^2 & x_2 & 1 \\ x_3^4 & x_3^3 & - & - & \vdots & x_3^2 & x_3 & 1 \end{vmatrix} \sim - \begin{vmatrix} {y_4}^2 & y_4^3 & y_4^4 & y_4^5 \\ y_3^2 & y_3^3 & y_3^4 & y_3^5 \\ y_2^2 & y_2^3 & y_2^4 & y_2^5 \\ y_1^2 & y_1^3 & y_1^4 & y_1^5 \end{vmatrix} \cdot \begin{vmatrix} x_1^4 & x_1^3 & x_1^2 \\ x_2^4 & x_2^3 & x_2^2 \\ x_3^4 & x_3^3 & x_3^2 \end{vmatrix}$$

$$= - \quad s_{2222}(y)\ \left| y_a^{4-b} \right| \cdot s_{222}(x)\ \left| x_i^{3-j} \right|$$

Ex 2: $m = 3$, $n = 4$, $p = -2$, $q = -1$

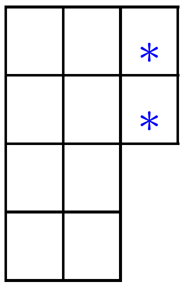
Ex. $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 6 & 7 & 1 & 2 & 5 \end{pmatrix}$ $\quad (-1)^{\pi} = (-1)^{|\zeta|} = +1$

$$\zeta = (4,4,2) = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$



$$\sigma = \begin{pmatrix} 3-2 & 2-2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = (2,2,2)$$

$$\zeta' = (3,3,2,2) = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$



$$\tau = \begin{pmatrix} 2-1 & 1-1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} = (2,2,2,2)$$

Row length restricted Cauchy formula

- We can combine our Theorem for $p \geq 0$ with our Lemma for $p = q \geq 0$

- $$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x)\, s_\lambda(y)$$
$$= \frac{1}{\left|x_i^{m-j}\right| \; \left|y_a^{n-b}\right| \; \prod_{i=1}^m \prod_{a=1}^n (1 - x_i y_a)} \cdot \begin{vmatrix} y_{n+1-i}^{j-1+\chi_{j>n}p} \\ \cdots \\ x_{i-n}^{m+n-j+\chi_{j\leq n}p} \end{vmatrix}$$

- $$\sum_\zeta (-1)^{|\zeta|}\, s_{\zeta+p^r}(x)\, s_{\zeta'+q^r}(y)$$
$$= \frac{1}{\left|x_i^{m-j}\right| \; \left|y_a^{n-b}\right|} \cdot \begin{vmatrix} y_{n+1-i}^{j-1} & \vdots & \chi_{j>n-q}\, y_{n+1-i}^{j-1+q} \\ \cdots & & \cdots \\ \chi_{j\leq n+p}\, x_{i-n}^{m+n-j+p} & \vdots & x_{i-n}^{m+n-j} \end{vmatrix}$$

Row length restricted Cauchy formula

- For all $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots)$ and $p \geq 0$

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x)\, s_\lambda(y)$$

$$= \frac{1}{\prod_{i=1}^{m}\prod_{a=1}^{n}(1 - x_i y_a)} \cdot \sum_{\zeta} (-1)^{|\zeta|}\, s_{\zeta+p^r}(x)\, s_{\zeta'+p^r}(y)$$

$$= \sum_{\lambda} s_\lambda(x)\, s_\lambda(y) \cdot \sum_{\zeta} (-1)^{|\zeta|}\, s_{\zeta+p^r}(x)\, s_{\zeta'+p^r}(y)$$

- This expresses the row length restricted series as a product of the unrestricted series times a correction factor

Column length restricted Cauchy formula

- Using the involutions $\omega_x : s_\lambda(x) \mapsto s_{\lambda'}(x)$ and $\omega_y : s_\lambda(y) \mapsto s_{\lambda'}(y)$ for all λ, either separately or together, we obtain three more restricted formula.
- Using $\omega_x\omega_y$ we find that for all x, y and for all $p \geq 0$

$$\sum_{\lambda:\ell(\lambda)\leq p} s_\lambda(x)\, s_\lambda(y)$$

$$= \sum_\lambda s_\lambda(x)\, s_\lambda(y) \cdot \sum_\zeta (-1)^{|\zeta|}\, s_{(\zeta+p^r)'}(x)\, s_{(\zeta'+p^r)'}(y)$$

$$= \sum_\lambda s_\lambda(x)\, s_\lambda(y) \cdot \sum_\eta (-1)^{|\eta|}\, s_{\eta-p^r}(x)\, s_{\eta'-p^r}(y)$$

where the sum over η is restricted to those η such that both $\eta - p^r$ and $\eta' - p^r$ are partitions

Column length restricted Cauchy formula

- Using our Lemma with both p and q set equal to $-p$, with $p \geq 0$ gives

Theorem Let $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$ with $m, n \geq 1$. Then for all $p \geq 0$ we have

$$\sum_{\lambda:\ell(\lambda)\leq p} s_\lambda(x)\, s_\lambda(y) = \frac{1}{\left|x_i^{m-j}\right| \; |y_a^{n-b}| \; \prod_{i=1}^{m} \prod_{a=1}^{n} (1 - x_i y_a)}$$

$$\cdot \left| \begin{array}{c:c} y_{n+1-i}^{j-1} & \chi_{\,j>n+p}\; y_{n+1-i}^{j-1-p} \\ \cdots & \cdots \\ \chi_{\,j\leq n-p}\; x_{i-n}^{m+n-j-p} & x_{i-n}^{m+n-j} \end{array} \right|$$

Qutients by infinite series

- So far we have discussed infinite series of Schur functions, including their expression in determinantal form
- These have been used to provide generalisations of formulae of both Littlewood and Cauchy
- Characters of the orthogonal and symplectic groups may be expressed as quotients of given Schur functions by infinite series of Schur functions in $\mathcal{P}_t$ for $t = 0, \pm 1$
- Each of these characters has a simple determinantal form, given by Weyl's character formula
- It is natural to ask if the same is true if t is extended to other positive and negative integer values.

Bressoud-Wei identities

- Bressoud and Wei [1992] For all integers $t \geq -1$:

$$2^{(t-|t|)/2} \left| h_{\lambda_i - i + j}(x) + (-1)^{(t+|t|)/2} h_{\lambda_i - i - j + 1 - t}(x) \right|$$

$$= \sum_{\sigma \in \mathcal{P}_t} (-1)^{[|\sigma| + r(\sigma)(|t|-1)]/2} s_{\lambda/\sigma}(x)$$

- Hamel and K [2008] For all integers t and all q:

$$\left| h_{\lambda_i - i + j}(x) + q \, \chi_{j > -t} \, h_{\lambda_i - i - j + 1 - t}(x) \right|$$

$$= \sum_{\sigma \in \mathcal{P}_t} (-1)^{[|\sigma| - r(\sigma)(t+1)]/2} q^{r(\sigma)} s_{\lambda/\sigma}(x)$$

Algebraic proof

$$\begin{aligned}
&|\, h_{\lambda_i-i+j}(x) + q\ \chi_{j>-t}\ h_{\lambda_i-i-j+1-t}(x)\,| \\
&\qquad = \sum_{r=0}^{n} \sum_{\kappa} q^r\ \left|\, h_{\lambda_i-i+j-\kappa_j}(x)\,\right| \\
&\qquad = \sum_{\sigma\in\mathcal{P}_t} (-1)^{(j_r-1)+\cdots+(j_2-1)+(j_1-1)}\ q^r\ \left|\, h_{\lambda_i-i+j-\sigma_j}(x)\,\right|
\end{aligned}$$

- $\kappa_j = 2j-1+t$ for $j \in \{j_1, j_2, \ldots, j_r\}$ and $\kappa_j = 0$ otherwise
- with $n \geq j_1 > j_2 > \cdots > j_r \geq 1 - \chi_{t<0}t$
- $\sigma = \begin{pmatrix} j_1-1+t & j_2-1+t & \cdots & j_r-1+t \\ j_1-1 & j_2-1 & \cdots & j_r-1 \end{pmatrix} \in \mathcal{P}_t$
- $r = r(\sigma)$

Cauchy-type extension

Theorem [Hamel and K, 2008]

- Let $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$
- Let λ and μ have lengths $\ell(\lambda) \leq m$ and $\ell(\mu) \leq n$
- Then for each pair of integers p and q we have

$$\left| \begin{array}{c|c} h_{\mu_{n+1-i}+i-j}(y) & \chi_{j>n-q}\, h_{\mu_{n+1-i}+i-j-q}(y) \\ \cdots & \cdots \\ \chi_{j\leq n+p}\, h_{\lambda_{i-n}-i+j-p}(x) & h_{\lambda_{i-n}-i+j}(x) \end{array} \right|$$

$$= \sum_{\zeta \subseteq n^m} (-1)^{|\zeta|}\, s_{\lambda/\sigma(\zeta)}(x)\, s_{\mu/\tau(\zeta)}(y)$$

Cauchy-type extension

- where the determinant is $(m+n)\times(m+n)$, and is partitioned after the nth row and nth column
- and if $\zeta = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix} \in (n^m)$
- with $a_1 < n$, $b_1 < m$ and $r = r(\zeta)$, then
  - $\sigma(\zeta) = \zeta + p^r = \begin{pmatrix} a_1{+}p & a_2{+}p & \cdots & a_r{+}p \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$
  - $\tau(\zeta) = \zeta' + q^r = \begin{pmatrix} b_1{+}q & b_2{+}q & \cdots & b_r{+}q \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}$
  - with $a_r \geq \max\{0, -p\}$ and $b_r \geq \max\{0, -q\}$

Example

- $m = 3$, $n = 4$, $p = -2$, $q = -1$ $\lambda = (5, 3, 2)$, $\mu = (4, 3, 2, 2)$
- Let $\{k\} = h_k(x)$ and $\{k\} = h_k(y)$ for all integers k

$$\left| \begin{array}{cccccccc} \{2\} & \{1\} & \{0\} & - & \vdots & - & - & - \\ \{3\} & \{2\} & \{1\} & \{0\} & \vdots & - & - & - \\ \{5\} & \{4\} & \{3\} & \{2\} & \vdots & - & \{1\} & \{0\} \\ \{7\} & \{6\} & \{5\} & \{4\} & \vdots & - & \{3\} & \{2\} \\ \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ \{3\} & \{4\} & - & - & \vdots & \{5\} & \{6\} & \{7\} \\ \{0\} & \{1\} & - & - & \vdots & \{2\} & \{3\} & \{4\} \\ - & - & - & - & \vdots & \{0\} & \{1\} & \{2\} \end{array} \right|$$

Typical term in Laplace expansion

- $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 6 & 2 & 5 & 7 \end{pmatrix} \quad (-1)^{\pi} = (-1)^{0+1+1+2} = +1$

- $\begin{vmatrix} \{2\} & \{0\} & - & - \\ \{3\} & \{1\} & \{0\} & - \\ \{5\} & \{3\} & \{2\} & \{1\} \\ \{7\} & \{5\} & \{4\} & \{3\} \end{vmatrix} \times \begin{vmatrix} \{4\} & \{5\} & \{7\} \\ \{1\} & \{2\} & \{4\} \\ - & \{0\} & \{2\} \end{vmatrix}$

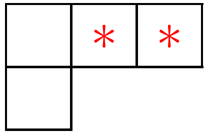
- $(\zeta'_n, \ldots, \zeta'_1 \,|\, \zeta_1, \ldots, \zeta_m) =$
  $(1-1, 3-2, 4-3, 6-4 \,|\, 5-2, 6-5, 7-7) = (0, 1, 1, 2 \,|\, 3, 1, 0)$

- $\zeta = (3, 1, 0) = \begin{pmatrix} 5-2-1 \\ 6-4-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \quad (-1)^{|\zeta|} = +1$

- $r(\zeta) = 1$

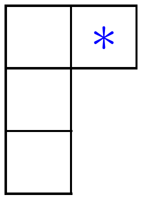
Determination of $\sigma(\zeta)$ and $\tau(\zeta)$

- $\zeta = (310) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



$$\Longrightarrow \sigma(\zeta) = \zeta + p^r = (310) - (200) = (110) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

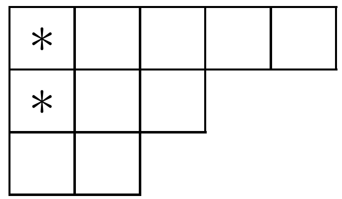
- $\zeta' = (2110) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



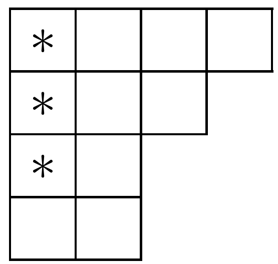
$$\Longrightarrow \tau(\zeta) = \zeta' + q^r = (2110) - (1000) = (1110) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Identification of constituent determinants

- Recall that $\lambda = (532)$ and $\mu = (4322)$
- while $\sigma(\zeta) = (110)$ and $\tau(\zeta) = (1110)$
- $$\begin{vmatrix} \{4\} & \{5\} & \{7\} \\ \{1\} & \{2\} & \{4\} \\ - & \{0\} & \{2\} \end{vmatrix} = s_{532/110}(x)$$



- $$\begin{vmatrix} \{2\} & \{0\} & - & - \\ \{3\} & \{1\} & \{0\} & - \\ \{5\} & \{3\} & \{2\} & \{1\} \\ \{7\} & \{5\} & \{4\} & \{3\} \end{vmatrix} = s_{4322/1110}(y)$$



Conclusions

- Both the classical Schur function series of Littlewood and the Cauchy identity may be restricted with respect to row lengths or column lengths through determinantal formulae
- In each case the correction factors to the original multiplicative formulae may be expressed as a signed sum of Schur functions or pairs of Schur functions specified by partitions having a particularly simple form in Frobenius notation
- Each row or column restricted Schur function series is nothing other than the character of some (rather simple) finite or infinite-dimensional irrep of a classical group

Conclusions

- To evaluate these characters (and thereby derive the restricted Schur function series) use may be made of Howe dual pairs with respect to spin and metaplectic representations of (the covering groups) of the orthogonal and symplectic groups
- All the Schur function identities may be extended to the case of supersymmetric Schur functions
- At present a dual pair approach has not been given for the row or column restricted Cauchy identities
- The dual pair approach enables many other characters to be evaluated, although in doing so it is usually necessary to invoke classical group modification rules

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