Tesler Matrices

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Plan

1. Tesler Matrices
   (with A. Garsia, J. Haglund, B. Rhoades and B. Sagan)

2. Tesler Polytopes
   (with A. Morales and K. Mészáros)

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What is a Tesler Matrix?

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Definition

We say that $A = (a_{i,j}) \in \mathrm{Mat}_n(\mathbb{Z})$ is a Tesler matrix if:

- A is upper triangular.
- For all $1 \leq k \leq n$ we have

$$(a_{k,k} + a_{k,k+1} + \cdots + a_{k,n}) - (a_{1,k} + a_{2,k} + \cdots a_{k-1,k}) = 1.$$

Picture:

$$\begin{array}{cc} & \begin{array}{cccccc} & & k & & & \end{array} \\ k & \left(\begin{array}{cccccc} & & - & & & \\ 0 & & - & & & \\ 0 & 0 & + & + & + & + \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & \end{array}\right) \end{array} = 1 \quad \text{for all } k.$$

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$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} = 1$$

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Recursion

Let Tesler(n) be the set of $n \times n$ Tesler matrices. There is a natural map

$$\text{Tesler}(n) \to \text{Tesler}(n-1)$$

defined by adding $a_{k,n}$ to $a_{k,k}$ then deleting the n-th row and column:

$$\left(\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ \hline 0 & 0 & 0 & 0 & 4 \end{array}\right) \longrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

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Tesler(3)	$\left(\begin{array}{cc\|c} 1 & 0 & 0 \\ & 1 & 0 \\ \hline & & 1 \end{array}\right)$ $\left(\begin{array}{cc\|c} 1 & 0 & 0 \\ & 0 & 1 \\ \hline & & 2 \end{array}\right)$ $\left(\begin{array}{cc\|c} 0 & 0 & 1 \\ & 1 & 0 \\ \hline & & 2 \end{array}\right)$ $\left(\begin{array}{cc\|c} 0 & 0 & 1 \\ & 0 & 1 \\ \hline & & 3 \end{array}\right)$	$\left(\begin{array}{cc\|c} 0 & 1 & 0 \\ & 2 & 0 \\ \hline & & 1 \end{array}\right)$ $\left(\begin{array}{cc\|c} 0 & 1 & 0 \\ & 0 & 2 \\ \hline & & 3 \end{array}\right)$ $\left(\begin{array}{cc\|c} 0 & 1 & 0 \\ & 1 & 1 \\ \hline & & 2 \end{array}\right)$
Tesler(2)	$\begin{pmatrix} 1 & 0 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ & 2 \end{pmatrix}$

What is a Tesler Matrix?

History

- The sequence $\mathrm{Tes}(n) := \#\mathrm{Tesler}(n)$ is A008608 in the OEIS:

$$1, 2, 7, 40, 357, 4820, 96030, 2766572, 113300265, \ldots$$

- Entered by Glenn Tesler (1996), then forgotten for 15 years.
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- Now everybody cares. Why?

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Why Do We Care?

Coinvariants

- The symmetric group $\mathfrak{S}_n$ acts on $\mathrm{S} := \mathbb{C}[x_1, \ldots, x_n]$.
- (Newton) The subalgebra of symmetric polynomials is isomorphic to a polynomial algebra

$$\mathrm{S}^{\mathfrak{S}_n} \approx \mathbb{C}[p_1, \ldots, p_n],$$

  where $p_k = \sum_{i=1}^{n} x_i^k$ are the power sum polynomials.
- (Chevalley) The coinvariant algebra

$$\mathrm{R} := \mathrm{S}/(p_1, \ldots, p_n)$$

  is isomorphic to the regular representation ($R \approx \mathbb{C}\mathfrak{S}_n$) and its Hilbert series is given by the q-factorial:

$$\mathrm{Hilb}(n; q) := \sum_i \dim(\mathrm{R}_i)\, q^i = [n]_q!$$

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- The symmetric group $\mathfrak{S}_n$ acts **diagonally** on the double algebra

$$\mathrm{DS} := \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n].$$

- (Weyl) The subalgebra of diagonal invariants is generated by the polarized power sums

$$p_{k,\ell} = \sum_{i=1}^{n} x_i^k y_i^\ell \quad \text{for } k + \ell > 0,$$

  but these are **not algebraically independent**.

- (Haiman) The algebra of diagonal coinvariants

$$\mathrm{DR} := \mathrm{DS}/(p_{k,\ell} : k + \ell > 0)$$

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The diagonal Hilbert series

$$\mathrm{Hilb}(n; q, t) := \sum_{i,j} \dim(\mathrm{DR}_{i,j})\, q^i t^j$$

satisfies:

- $\mathrm{Hilb}(n; q, t) = \mathrm{Hilb}(n; t, q)$
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Here's Why We Care.

Haglund's Theorem (2011)

Given a Tesler matrix $A \in \mathrm{Tesler}(n)$, let $\mathrm{Pos}(A)$ be the set of positive entries of A and let $\mathrm{pos}(A) := \#\mathrm{Pos}(A)$. We define the weight by

$$\mathrm{weight}(A) := (-M)^{\mathrm{pos}(A)-n} \prod_{a_{i,j} \in \mathrm{Pos}(A)} [a_{i,j}]_{q,t} \in \mathbb{Z}[q,t],$$

where $M = (1-q)(1-t)$ and $[k]_{q,t} = (q^k - t^k)/(q-t)$.

$$\mathrm{weight}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} = (-(1-q)(1-t))^1 [1]_{q,t}^4 [2]_{q,t} [4]_{q,t}$$

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If we define

$$\mathsf{Tes}(n; q, t) := \sum_{A \in \mathsf{Tesler}(n)} \mathsf{weight}(A),$$

then it is true that

$$\mathsf{Tes}(n; q, t) = \mathsf{Hilb}(n; q, t).$$

In particular, we have

$$\mathsf{Tes}(n; q, t) \in \mathbb{N}[q, t],$$

which is **not** obvious.

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Example ($n = 3$)

$$
\begin{array}{c|c|c|c|c|c|c}
\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} &
\begin{pmatrix} 1 & & \\ & & 1 \\ & & 2 \end{pmatrix} &
\begin{pmatrix} & 1 & \\ & 2 & \\ & & 1 \end{pmatrix} &
\color{red}{\begin{pmatrix} & 1 & \\ & 1 & 1 \\ & & 2 \end{pmatrix}} &
\begin{pmatrix} & 1 & \\ & & 2 \\ & & 3 \end{pmatrix} &
\begin{pmatrix} & & 1 \\ & 1 & \\ & & 2 \end{pmatrix} &
\begin{pmatrix} & & 1 \\ & & 1 \\ & & 3 \end{pmatrix} \\
1 & q+t & q+t & \color{red}{-M(q+t)} & (q+t)(q^2+qt+t^2) & q+t & q^2+qt+t^2 \\
\begin{bmatrix} 1 & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} &
\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & \\ & & & \end{bmatrix} &
\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & \\ & & & \end{bmatrix} &
\color{red}{\begin{bmatrix} & -1 & 1 & \\ -1 & 2 & -1 & \\ 1 & -1 & & \\ & & & \end{bmatrix}} &
\begin{bmatrix} & & & 1 \\ & & 2 & \\ & 2 & & \\ 1 & & & \end{bmatrix} &
\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & \\ & & & \end{bmatrix} &
\begin{bmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & \end{bmatrix}
\end{array}
$$

$$
\operatorname{Tes}(3; q, t) = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 3 & 1 & \\ 2 & 1 & & \\ 1 & & & \end{bmatrix}
$$

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A New Hope

- Maybe Haglund's Theorem can be used to prove the famous "Shuffle Conjecture" of HHLRU (2005).
- To do this we must relate Tesler matrices to parking functions.

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Refinement of Haglund's Theorem

Definition

Given a Tesler matrix $A \in \text{Tesler}(n)$ consider the lowest lattice path above the positive entries. Let $\text{Touch}(A) \subseteq \{1, 2, \ldots, n-1\}$ be the set of positions where this path touches the diagonal.

Examples:

Refinement of Haglund's Theorem

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Examples:

$$\text{Touch}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 0 & 1 & 1 \\ & & & 0 & 2 \\ & & & & 4 \end{pmatrix} = \{\ \}$$

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Examples:

$$\text{Touch}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ & 2 & 0 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 0 & 1 \\ & & & & 2 \end{pmatrix} = \{2, 3\}$$

Refinement of Haglund's Theorem

Definition

We say $A \in \text{Tesler}(n)$ is connected if $\text{Touch}(A) = \{\ \}$. Let

$$\text{CTesler}(n) := \{A \in \text{Tesler}(n) : A \text{ is connected}\}$$

and define

$$\text{CTes}(n; q, t) := \sum_{A \in \text{CTesler}(n)} \text{weight}(A).$$

Refinement of Haglund's Theorem

Example

Here are the first few values of $\mathrm{CTes}(n; q, t)$:

$n = 1$	$n = 2$	$n = 3$	$n = 4$
$\begin{bmatrix} 1 \end{bmatrix}$	$\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$	$\begin{bmatrix} & & 2 & 1 \\ & 3 & 1 & \\ 2 & 1 & & \\ 1 & & & \end{bmatrix}$	$\begin{bmatrix} & & & 4 & 5 & 3 & 1 \\ & & 9 & 9 & 4 & 1 & \\ & 9 & 9 & 4 & 1 & & \\ 4 & 9 & 4 & 1 & & & \\ 5 & 4 & 1 & & & & \\ 3 & 1 & & & & & \\ 1 & & & & & & \end{bmatrix}$

Refinement of Haglund's Theorem

Theorem

It is sufficient to study connected Tesler matrices:

$$\mathrm{Tes}(n; q, t) = \sum_{\{s_1 < \cdots < s_k\} \subseteq [n-1]} \prod_{i=1}^{k-1} \mathrm{CTes}(s_{i+1} - s_i; q, t).$$

Refinement of Haglund's Theorem

Example

$$\mathrm{Tes}(3; q, t) = \mathrm{CTes}(1; q, t)^3 + 2\,\mathrm{CTes}(1; q, t)\mathrm{Tes}(2; q, t) + \mathrm{CTes}(3; q, t)$$

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 3 & 1 & \\ 2 & 1 & & \\ 1 & & & \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}^3 + 2\begin{bmatrix} 1 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} + \begin{bmatrix} & & 2 & 1 \\ & 3 & 1 & \\ 2 & 1 & & \\ 1 & & & \end{bmatrix}$$

Refinement of Haglund's Theorem

Conjectures and Problems

- $\mathrm{CTes}(n; q, t) \in \mathbb{N}[q, t]$ for all $n \in \mathbb{N}$.
- $\mathrm{CTes}(n; 1, 1) = \#$ "connected parking functions" $\mathrm{CPF}(n)$ in the sense of Novelli and Thibon (2007). This is A122708 in OEIS:

$$1, 2, 11, 92, 1014, 13795, 223061, 4180785, 89191196, \ldots$$

- Find statistics qstat, tstat : $\mathrm{CPF}(n) \to \mathbb{N}$ such that

$$\mathrm{CTes}(n; q, t) = \sum_{cpf \in \mathrm{CPF}(n)} q^{\mathrm{qstat}(cpf)} t^{\mathrm{tstat}(cpf)}.$$

- Prove the Shuffle Conjecture.

Refinement of Haglund's Theorem

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- Prove the Shuffle Conjecture.

Pause

Generalized Tesler Matrices

Definition

Given any positive integer vector $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{N}^n$ we say that $A = (a_{i,j}) \in \mathrm{Mat}_n(\mathbb{Z})$ is an **r**-Tesler matrix if:

- A is upper triangular.
- For all $1 \leq k \leq n$ we have

$$(a_{k,k} + a_{k,k+1} + \cdots + a_{k,n}) - (a_{1,k} + a_{2,k} + \cdots a_{k-1,k}) = r_k.$$

Picture: The k-th "hook sum" equals r_k

$$\begin{array}{cc} & \begin{array}{cccccc} & & k & & & \end{array} \\ \begin{array}{c} \\ \\ k \\ \\ \\ \\ \end{array} & \left(\begin{array}{cccccc} & & - & & & \\ 0 & & - & & & \\ 0 & 0 & + & + & + & + \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & \end{array}\right) \end{array} = r_k \qquad \text{for all } k.$$

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Generalized Tesler Matrices

Definition

Given $\mathbf{r} \in \mathbb{N}^n$, let Tesler($\mathbf{r}$) be the set of $\mathbf{r}$-Tesler matrices and define

$$\begin{aligned}\mathrm{Tes}(\mathbf{r}; q, t) &:= \sum_{A \in \mathrm{Tesler}(\mathbf{r})} \mathrm{weight}(A) \\ &= \sum_{A \in \mathrm{Tesler}(\mathbf{r})} (-M)^{\mathrm{pos}(A)-n} \prod_{a_{i,j} \in \mathrm{Pos}(A)} [a_{i,j}]_{q,t},\end{aligned}$$

as before.

Conjecture (Haglund)

If $\mathbf{r} \in \mathbb{N}^n$ is increasing $(1 \leq r_1 \leq r_2 \leq \cdots \leq r_n)$ then we have

$$\mathrm{Tes}(\mathbf{r}; q, t) \in \mathbb{N}[q, t].$$

Question

For which other $\mathbf{r} \in \mathbb{N}^n$ is $\mathrm{Tes}(\mathbf{r}; q, t) \in \mathbb{N}[q, t]$?

Generalized Tesler Matrices

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Generalized Tesler Matrices

Theorem (Armstrong → Sagan → Haglund)

Recall from Haglund's and Haiman's Theorems that we have

$$\text{Tes}(n; 1, 1) = \text{Hilb}(n; 1, 1) = (n + 1)^{n-1}.$$

For general $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{N}^n$ we have

$$\text{Tes}(\mathbf{r}; 1, 1) = r_1(r_1 + nr_2)(r_1 + r_2 + (n-1)r_3) \cdots (r_1 + \cdots + r_{n-1} + 2r_n).$$

Conjectured by me; proved by Bruce.

Generalized Tesler Matrices

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Generalized Tesler Matrices

Theorem (Armstrong → Sagan → Haglund)

Then Jim defined **r**-parking functions $\mathrm{PF}(\mathbf{r})$ such that

$$\mathrm{Tes}(\mathbf{r}; 1, 1) = \#\mathrm{PF}(\mathbf{r}).$$

Special Case: m-parking functions

$$\mathrm{Tes}((1, m, m, \ldots, m); 1, 1) = (mn + 1)^{n-1}.$$

Generalized Parking Functions

Conjecture

For all $\mathbf{r} \in \mathbb{N}^n$ we have

$$q^{(\sum_i i \cdot r_{n-i+1})-n} \cdot \mathrm{Tes}(\mathbf{r}; q, q^{-1}) =$$
$$[r_1]_{q^{n+1}}[r_1 + nr_2]_q[r_1 + r_2 + (n-1)r_3]_q \cdots [r_1 + \cdots + r_{n-1} + 2r_n]_q.$$

Problem

Find an algebraic interpretation of $\mathrm{Tes}(\mathbf{r}; q, t)$.

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Pause

Tesler Polytopes (New!)

Tesler Polytopes

Definition

We say that $A = (a_{i,j}) \in \mathrm{Mat}_n(\mathbb{R})$ is a "null-hook matrix" if

- A is upper triangular.
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$$(a_{k,k} + a_{k,k+1} + \cdots + a_{k,n}) - (a_{1,k} + a_{2,k} + \cdots a_{k-1,k}) = 0.$$

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Tesler Polytopes

Observations

Let Hook$_0 \subseteq \mathrm{Mat}_n(\mathbb{R})$ be the set of null-hook matrices. These form an $\mathbb{R}$-subspace of $\mathrm{Mat}_n(\mathbb{R})$ of dimension $\binom{n}{2}$:

$$\mathsf{Hook}_0 \approx \mathbb{R}^{\binom{n}{2}}$$

with basis given by the following "null-transposition" matrices:

$$T(i,j) := \begin{array}{cc} & \begin{array}{ccccc} & & i & & j \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} 0 & & & & \\ 0 & 0 & -1 & & 1 \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \text{for all } 1 \le i < j \le n.$$

Tesler Polytopes

Observations

Let $\mathbb{Z}\mathsf{Hook}_0$ denote the lattice of integer points in Hook_0:

$$\mathbb{Z}\mathsf{Hook}_0 := \mathbb{Z}\{T(i,j) : 1 \leq i < j \leq n\} \leq \mathsf{Hook}_0.$$

And given any $\mathbf{r} \in \mathbb{N}^n$, let $\mathbb{Z}\mathsf{Hook}_{\mathbf{r}}$ be the lattice of integer points in the affine space of "$\mathbf{r}$-hook matrices":

$$\mathbb{Z}\mathsf{Hook}_{\mathbf{r}} \leq \mathsf{Hook}_{\mathbf{r}} := \mathsf{Hook}_0 + \begin{pmatrix} r_1 & & & \\ & r_2 & & \\ & & \ddots & \\ & & & r_n \end{pmatrix}.$$

Given any two $\mathbf{r}$-Tesler matrices $A, B \in \mathsf{Tesler}(\mathbf{r})$ observe that we have $A - B \in \mathsf{Hook}_0$, and hence

$$\mathsf{Tesler}(\mathbf{r}) \subseteq \mathbb{Z}\mathsf{Hook}_{\mathbf{r}}.$$

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Tesler Polytopes

Observations

In fact, the **r**-Tesler matrices are precisely the integer points in the polytope of "positive **r**-hook matrices":

$$\mathsf{Tesler}(\mathbf{r}) \subseteq \mathsf{Hook}_r \cap (\mathbb{R}_{\geq 0})^{\binom{n}{2}}$$

Tesler Polytopes

Suggestion

Study this polytope. Call it the "**r**-Tesler polytope":

$$\Delta_{\mathsf{Tes}}(\mathbf{r}) := \mathsf{Hook}_{\mathbf{r}} \cap (\mathbb{R}_{\geq 0})^{\binom{n}{2}}.$$

One example has been studied before. If $\mathbf{r} = (1, 2, 3, \ldots, n)$ then

$$\mathsf{CRY_n} := \Delta_{\mathsf{Tes}}(\mathbf{r})$$

is called the Chan-Robbins-Yuen polytope. Its volume was conjectured by Chan-Robbins-Yuen and proved by Zeilberger (1998) to be a product of Catalan numbers:

$$\mathsf{Vol}(\mathsf{CRY}_n) = \prod_{i=1}^{n-2} \frac{1}{i+1}\binom{2i}{i}.$$

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Tesler Polytopes

More generally, the **r**-Tesler polytope is an example of a flow polytope. These have been studied by Postnikov-Stanley, Baldoni-Vergne, Mèszàros and Morales. They have beautiful properties related to the Kostant partition function for type A root systems.

Conjecture (Morales)

If $\mathbf{r} = (1, 1, \ldots, 1) \in \mathbb{N}^n$ we say that

$$\Delta_{\mathrm{Tes}}(n) := \Delta_{\mathrm{Tes}}(\mathbf{r})$$

is the standard Tesler polytope. Its volume seems to be

$$\mathrm{Vol}(\Delta_{\mathrm{Tes}}(n)) = 2^{\binom{n}{2}} \frac{\binom{n}{2}!}{1!2!\cdots n!}.$$

Final Suggestions

- Can one learn more about diagonal Hilbert series by studing Tesler polytopes?
- Is there a way to incorporate q and t into the Ehrhart series?
- It seems that the vertices of $\Delta_{\mathrm{Tes}}(\mathbf{r})$ are the monomial $\mathbf{r}$-Tesler matrices:

$$\mathrm{MTesler}(\mathbf{r}) := \{A \in \mathrm{Tesler}(\mathbf{r}) : \mathrm{pos}(A) - n = 0\}.$$

  Paul Levande gave a natural bijection $\mathrm{MTesler}(\mathbf{r}) \leftrightarrow \mathfrak{S}_n$.
- More genrally, given $A \in \mathrm{Tesler}(\mathbf{r})$ it seems that $\mathrm{pos}(A) - n$ is the dimension of the face of $\Delta_{\mathrm{Tes}}(\mathbf{r})$ containing A. That should be important.

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Happy Birthday Bruce!