Kreisel and Lawvere on Category Theory and the Foundations of Mathematics

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Impact of categories

- When the impact of categories *on foundations* is discussed with « mainstream » logicians, we often get two responses:

1. Pragmatic scepticism: still waiting for *new significant* results;

2. Philosophically motivated objections.

Claims

1. Kreisel has articulated a view about the foundations of mathematics and category theory that prevails among logicians even today;

2. Although this view had some credibility and force when it was formulated, it ought to be reevaluated;

3. Kreisel’s view is based on certain assumptions which are dubitable and ought to be contrasted with alternatives, in particular with Lawvere’s views.

Kreisel's claims: the sources

1. Appendix to Feferman's paper on the foundations of category theory in 1969;
2. « Observations on popular discussions of foundations » in *Axiomatic Set Theory* 1971;
3. A review of Mac Lane's « Categorical algebra and set-theoretic foundations » in *Axiomatic Set Theory* 1971;
4. Appendix to *Elements of Mathematical Logic* with Krivine 1967.

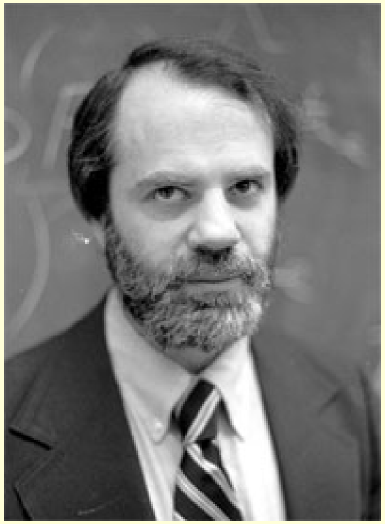
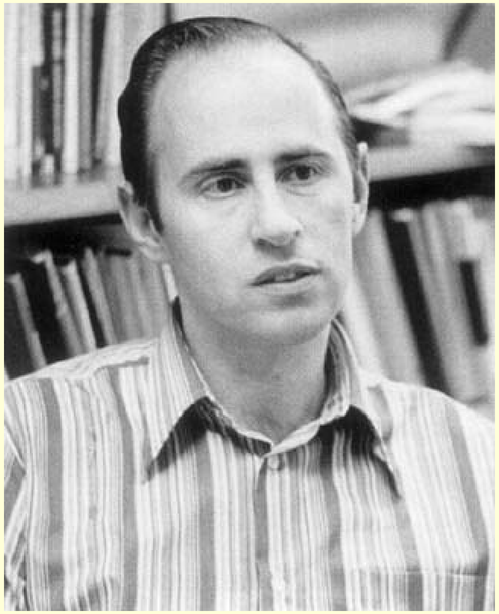
The socio-historical context

1. « Mainstream » developments and research programs in the foundations of mathematics in the 1960’s;

2. Category theory in the 1960’s.

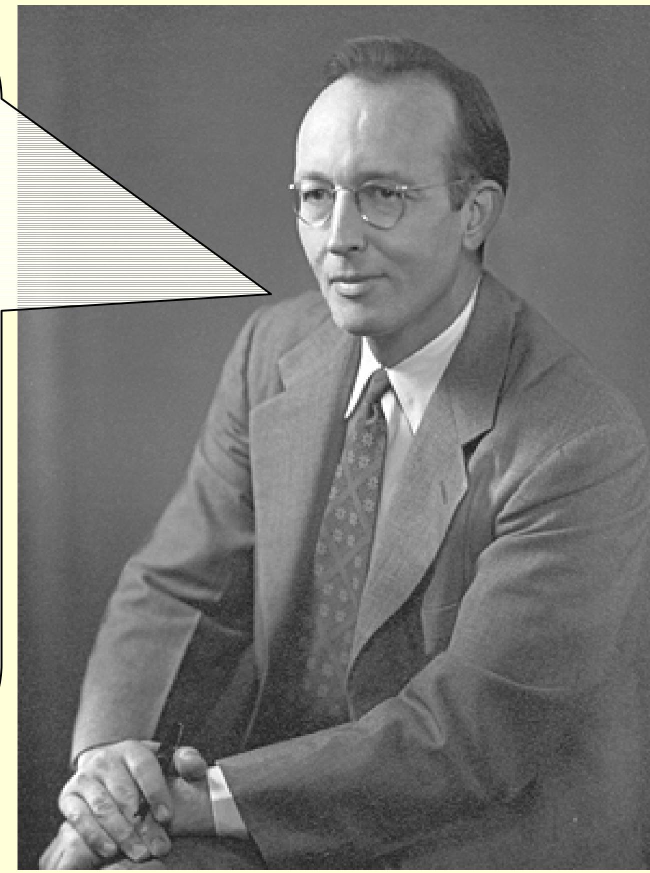
1963: Berkeley

- Meeting on model theory:
  - It is the who's who of logic and the foundations of mathematics.

1963

- We can be category theorists:
  Lecture on categorical algebra at the AMS;
  Grothendieck’s SGA4;
  Freyd’s presentation of his AFT;
  Lawvere’s thesis;
  Ehresmann’s « catégories structurées »;
  First coherence theorem;
  Adjoint functors and limits



1963: Lawvere's thesis

- Algebraic categories and algebraic functors;
- The category of categories as a foundation for mathematics;
- Sets within categories;
- Central role to adjoint functors.

1964

- ETCS: we should develop set theory within category theory;
- It can be done: here is how.

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1964

- As I told you, you can't do that Bill! Membership is primitive in set theory; But I will let you publish it anyway…

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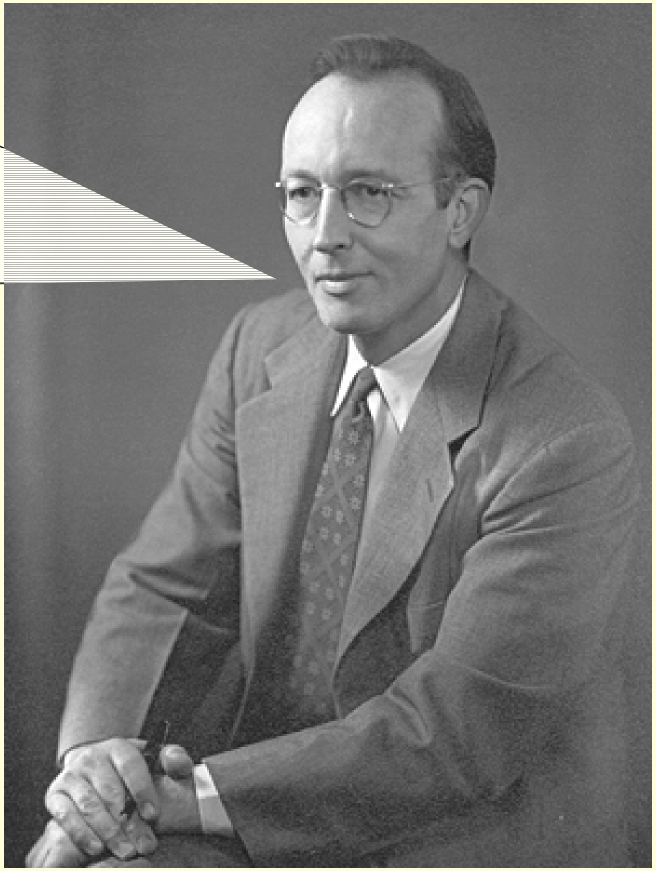

1966

- Now, I will tell everyone, the category of categories should be the foundations of mathematics;
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Lawvere’s views

1. « In the mathematical development of recent decades one sees clearly the rise of the conviction that the relevant properties of mathematical objects are those which can be stated in terms of their abstract structure rather than in terms of the elements which the objects were thought to be made of. The question thus naturally arises whether one can give a foundation for mathematics which expresses wholeheartedly this conviction concerning what mathematics is about, and in particular in which classes and membership in classes do not play any role. *Here by “foundation” we mean a single system of first-order axioms in which all usual mathematical objects can be defined and all their usual properties proved.* » (Lawvere, 1966, 1)

The socio-historical context

- Summing up:

1. There are exciting developments and much to do in « mainstream » foundational research; it is an active « research program »;

2. There *are* set-theoretical ways to handle category theory.

The socio-historical context

- Summing up:

1. Category *theory* is arising as an autonomous mathematical discipline;
2. It is not *global*;
3. Its foundational role is still problematic and presented in a classical tarskian manner (first-order theory).

1968-1969

- The foundations of category theory raises difficulties for set theory;
- Here are some ways of handling them (one universe, common foundations).

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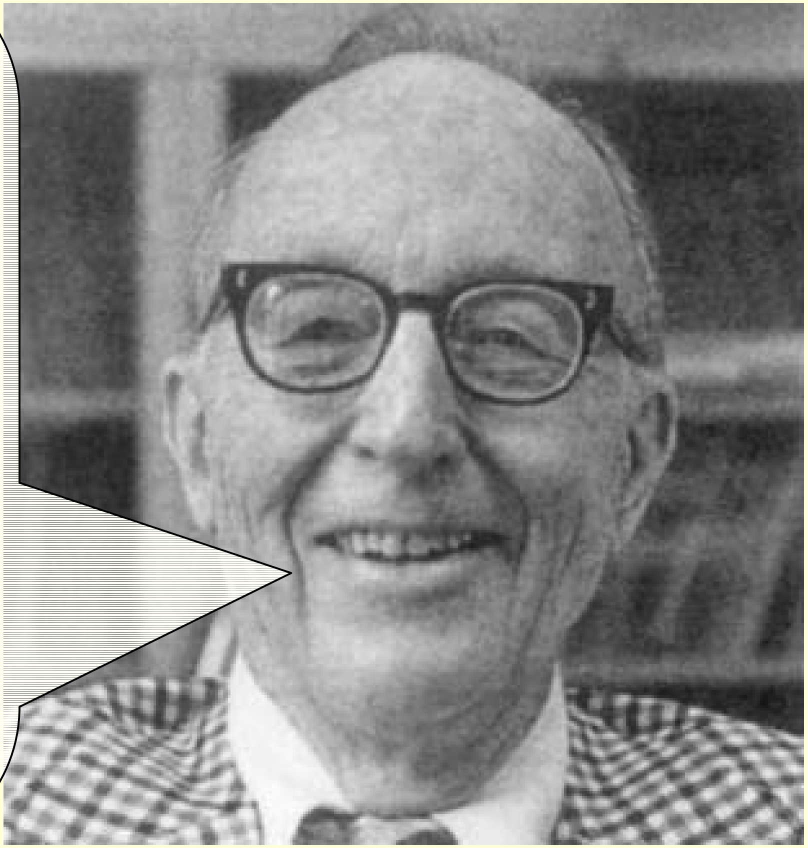


Kreisel jumps in

This can't be right! The whole issue is based on a confusion and a failure to understand the true purpose of the foundational enterprise.

Kreisel's claims

1. Kreisel distinguishes *foundations* and *organization* of mathematics.
   a) The distinction is useful « for analyzing the nature of the problems presented by (existing) category theory, and, more generally, for analyzing the role of foundations for working mathematicians. » (Kreisel, 1971, 189)
   b) Foundations have to do with *validity, justification*;
   c) Organization has to do with *efficiency*.

Kreisel's claims

1. « For foundations it is important to know what we are talking about; we make the subject as *specific* as possible. In this way we have a chance to make *strong* assertions. For practice, to make a proof intelligible, we want to eliminate all properties which are not relevant to the result proved, in other words, we make the subject matter less specific. »

Kreisel’s claims

1. « Foundations provide an *analysis* of practice. To deserve this name, foundations must be expected to introduce notions which do *not* occur in practice. Thus in foundations of set theory, *types* of sets are treated explicitly while in practice they are generally absent; and in foundations of constructive mathematics, the analysis of the logical operations involves (intuitive) *proofs* while in practice there is no explicit mention of the latter. » (Kreisel, 1971, 192)

Kreisel’s claims

2. « Foundations and organization are similar in that both provide some sort of more systematic exposition. But a step in this direction may be crucial for organization, yet foundationally trivial, for instance a new *choice of language* when (i) old theorems are simpler to state but (ii) the primitive notions of the new language are defined in terms of the old, that is if they are logically dependent on the latter. Quite often, (i) will be achieved by using new notions with more ‘structure’, that is less analyzed notions, which is a step in the *opposite* direction to a foundational analysis. In short, foundational and organizational aims are liable to be actually contradictory. » (Kreisel, 1971, 192)

Kreisel’s claims

3. « Organization and foundations are incomparable. Organization involves a proper choice of language; we have already seen that this is not necessarily provided by set-theoretical foundations. On the other hand, we may have a very successful organization which leaves open the verification of adequacy conditions, at least for a given foundational scheme.

   Generally speaking, the aims of foundations and organization will be in conflict. Being an *analysis* of practice foundations must be expected to involve concepts that do not occur in practice (just as fundamental theories in physics deal with objects that do not occur in ordinary life). Organization is directly concerned with practice;… » (Kreisel, in Feferman 1969, 244-245.)

Kreisel's claims

4. « Before going further into the relation between mathematical practice and foundations, it is worth noting the obvious distinction between (i) foundational analysis (which is specifically concerned with validity) and (ii) general conceptual analysis (which, in the traditional sense of the word, is certainly a philosophical activity). As mentioned above, the working mathematician is rarely concerned with (i), but he does engage in (ii), for instance when establishing *definitions* of such concepts as length or area or, for that matter, natural transformation. For this activity to be called an *analysis* the principal issue must be whether the definitions are *correct*, not merely, for instance, whether they are useful technically for deriving results not involving the concepts (when their correctness is irrelevant). In short, it's not (only) what you do it's the way that you do it. » (Kreisel, 1971, 192)

Kreisel's claims

- What does it mean for a definition (theory) to be *correct*?
- « In the next section the distinction beween organization and foundation is elaborated. In this section a further distinction is made - conceptual analysis versus foundational analysis [which the reviewer found more confusing than clarifying]. » (Halpern, 1973, 1)

Kreisel on Mac Lane

- Mac Lane claims that category theory may suggest or require revisions in axiomatic set theory as a foundation.
- According to Kreisel, Mac Lane does not provide sufficient grounds for this claim. The category of all groups is not more nor less problematic than the « set » of all ordered sets.

Kreisel’s claims

5. « … in foundations we try to find (a theoretical framework permitting the formulation of) good reasons *for* the basic principles accepted in mathematical practice, while the latter is only concerned with derivations *from* these principles. The methods used in a deeper analysis of mathematical practice often lead to an extension of our theoretical understanding. A particularly important example is the search for new axioms, which is nothing more than a continuation of the process which led to the discovery of the currently accepted principles. » (Kreisel, 1967, 161)

2. Kreisel’s claims

- Examples of valuable foundational programs according to Kreisel:
  1. Set theoretic semantic foundations;
  2. Combinatorial foundations (constructivists).
- Both have worthy elements but both also have defects:
  - « a conceptual framework is defective if it does not allow (theoretical) explanations of facts for which an alternative theory has an explanation, one purpose of theory being the extension of the range of theoretical understanding.» (Kreisel, 1971, 228)

Kreisel's claims

- According to Kreisel, the set theoretical foundations provide a realistic analysis of mathematical practice:
  1. It presents mathematics as being about certain abstract objects;
  2. It does so by reducing each mathematical structure *U* to a set;
  3. An adequate axiomatization of the reduction of a structure *U* to set theory is a set of axioms A_U satisfying the following conditions:

Kreisel’s claims

1. A_U is purely logical (in the language of predicate calculus)
2. *U* satisfies A_U and hence, there exists a structure that satisfies A_U;
3. All structures that satisfy A_U are isomorphic;
4. All intuitive properties of *U* can be expressed or defined in terms of those explicitly mentioned in A_U;
5. All assertions about *U* that can be proved intuitively follow logically from A_U.

Kreisel’s claims

- Note:

1. The language does not have to be finite nor does it have to be first-order;
2. Although all 19th century informal mathematics can be reduced adequately according to Kreisel, set theory itself cannot be so reduced;
3. For the latter, a generalized notion of realization (model) is required: predicate symbols are added for which variables range over all sets;
4. The adequacy conditions become: « axioms are set theoretically justified if one has a (precise) concept which satisfies the axioms in the wider sense of realization ».

Kreisel’s claims

- Fundamental or foundational notions must be *logically simple.*
- Contrast:
  - « Mathematics is a study which, when we start from its most familiar portions, may be pursued in either of two opposite directions. The more familiar direction is constructive, towards gradually increasing complexity: from integers to fractions, real numbers, complex numbers; from addition and multiplication to differentiation and integration, and on to higher mathematics. The other direction, which is less familiar, proceeds, by analyzing, to greater and greater abstractness and logical simplicity. » (Russell, 1903)

Kreisel's claims

- Kreisel's basic analogy:
  - There are basically three levels:
    1. The level of « ordinary » mathematics, e.g. number theory, analysis, geometry, topology, algebra, etc.
    2. Underlying this level, we have 'foundations', e.g. logic, set theory, proof theory, etc.
    3. Above ordinary mathematics, we have organizational tools, e.g. category theory.

Summing up

1. Category theory is an organizational tool;
2. Category theory does not raise any *new* set-theoretical foundational problems;
3. One has to distinguish organization and foundations; they are opposite or contradictory;
4. Category theory, as it is in the sixties, cannot provide a foundational framework:
   i. It does not satisfy the adequacy conditions;
   ii. Its concepts are not logically simple.
   iii. Its concepts are logically dependent on other concepts.

Lawvere’s views

« A foundation of the sort we have in mind would seemingly be much more natural and readily-useable than the classical one when developing such subjects as algebraic topology, functional analysis, model theory of general algebraic systems, etc. Clearly any such foundation would have to reckon with the Eilenberg-MacLane (sic) theory of categories and functors.» (Lawvere, 1966, 1)

Lawvere's views

« Foundations will mean here the study of what is universal in mathematics. Thus Foundations in this sense cannot be identified with any « starting-point » or « justification » for mathematics, though partial results in these directions may be among its fruits. But among the other fruits of Foundations so defined would presumably be guide-lines for passing from one branch of mathematics to another and for gauging to some extent which directions of research are likely to be relevant. » (Lawvere, 1969, 281)

Lawvere’s views

More recently, the search for universals has also taken a conceptual turn in the form of Category Theory, which began with viewing as a new mathematical object the totality of all morphisms of the mathematical objects of a given species A, and then recognizing that these new mathematical objects all belong to a common non-trivial species C *which is independent* of A. (Lawvere, 1969b, 281)[our emphasis]

Lawvere’s views

« A foundation makes explicit the essential general features, ingredients, and operations of a science as well as its origins and general laws of development. The purpose of making these explicit is to provide a guide to the learning, use, and further development of the science. A “pure” foundation that forgets this purpose and pursues a speculative “foundations” for its own sake is clearly a nonfoundation. » (Lawvere & Rosebrugh, 2003, 235)

Lawvere’s views

- Some facts known by Lawvere in the sixties:
  - Adjoint functors pervade mathematics : they unify an unexpectedly large quantity of mathematical concepts; they are seen as exhibiting the *correctness* of these concepts;
  - In particular, quantifiers can be conceived as adjoints; *all* logical operations arise as adjoints to *elementary (simple?)* functors;
  - Category theory allows an invariant construction of theories (in the logical sense), e.g. there is *one* theory of groups in this sense.
  - Cartesian closed categories can be used to encode type theories and shed a new light on fundamental paradoxes (1969).
  - Variables sets are just as relevant as abstract sets.

Lawvere's views

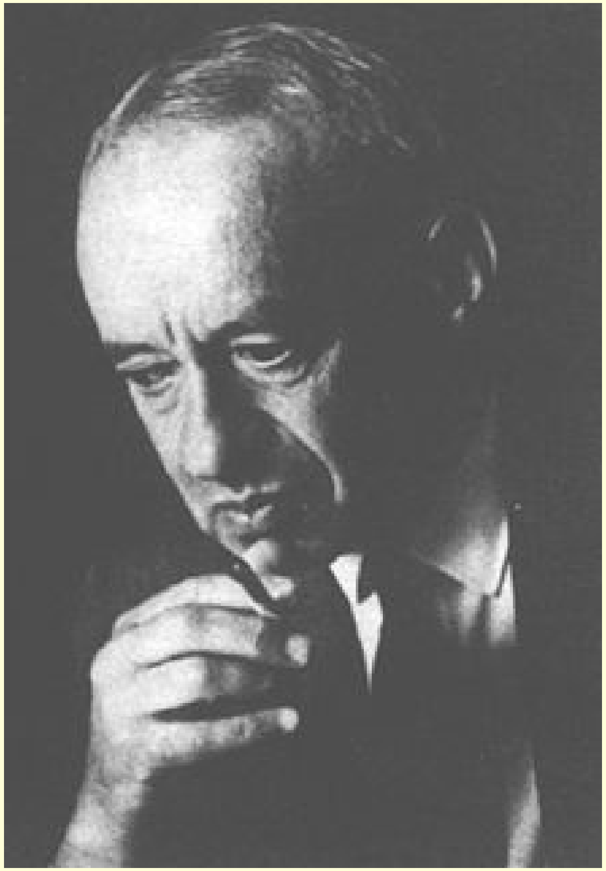
$$T^{\mathrm{op}} \underset{\text{structure}}{\overset{\text{semantics}}{\rightleftarrows}} \mathrm{Mod}(T, \mathit{Set})$$

$$\text{Formal} \rightleftarrows \text{Theories}$$

$$\text{Formal}^{\mathrm{op}} \rightleftarrows \text{Conceptual}$$

Tarski's reaction

- This man confuses mathematics and metamathematics, syntax and semantics!



Another evaluation

"Lawvere himself proposed in an article of 1969 to connect the concept of duality, and other categorical concepts, with the epistemological issues related to the philosophy of mathematics. In order to do that, he identified two "dual aspects" of mathematical knowledge — the conceptual and the formal aspects — which appear in many domains of mathematics. …. Now Lawvere proposed to dedicate efforts to develop the second aspect, the conceptual one, embodied in category theory. This proposal, however, remained at the programmatic level and no one seems to have developed it further." (Corry, 1996, 388)

Lawvere’s views

- Foundations is *part* of mathematics.
- These developments are still at the programmatic stage.
- *Categorical logic*, launched mainly by Lawvere, was still to be elaborated, developed and presented.
- The advent of Elementary topos theory opened the way to *categorical doctrines*.

What now?

- We can have both worlds!
  - Categorical doctrines are logical systems:
    - Regular categories = regular logic
    - Coherent categories = coherent logic
    - Heyting categories = intuitionistic logic
    - Boolean categories = classical logic
    - Monoidal categories = linear logic
  - This provides an organization of logic itself!

What now?

- There is no distinction between mathematics and metamathematics: a logical analysis *is* a categorical analysis.
- New concepts appear in these analyses, e.g. adjunctions.
- These analyses have to do with validity.

What now?

- We can have both worlds!
  - Toposes *are* higher order type theories (internal language);
  - Toposes provide tools of analysis of concepts of mathematical practice, even for 19th century mathematics, e.g. the reals;
  - Furthermore, it is now possible to relate constructive perspectives with set theoretical semantics!

What now?

- We can have both worlds!
  - Work done on higher-dimensional categories raise foundational issues:
    - The nature of sets themselves
    - The nature of identity of mathematical objects
    - The overall structure of a mathematical universe
    - The adequacy criterion for a logical framework

What now?

- But what is the basic picture?
- Neurath’s ship has become a spaceship.