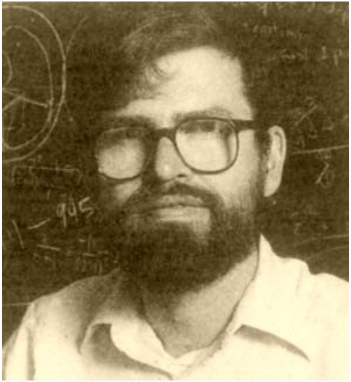
Foliations : What's next after Thurston ?



The mathematical legacy of Bill Thurston,

Étienne Ghys, CNRS ENS Lyon

A dozen publications between 1972 and 1976

MR0425985 Reviewed Thurston, W. P. Existence of codimension-one foliations. *Ann. of Math. (2)* **104** (1976), no. 2, 249–268. (Reviewer: D. B. Fuks) 57D30
PDF | Clipboard | Journal | Article

MR0488087 Reviewed Thurston, William On the construction and classification of foliations. *Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974), Vol* 547–549. *Canad. Math. Congress, Montreal, Que.*, 1975. 57D30
PDF | Clipboard | Journal | Article

MR0380828 Reviewed Thurston, William P. A local construction of foliations for three-manifolds. *Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Stanford Univ., Stanford* *1973), Part 1*, pp. 315–319. *Amer. Math. Soc., Providence, R.I.*, 1975. (Reviewer: John W. Wood) 57D30
PDF | Clipboard | Journal | Article

MR0375366 Reviewed Thurston, W. P.; Winkelnkemper, H. E. On the existence of contact forms. *Proc. Amer. Math. Soc.* **52** (1975), 345–347. (Reviewer: H. B. Griffiths) 58A10 (57D3
PDF | Clipboard | Journal | Article

MR0375345 Reviewed Thurston, William P. The theory of foliations of codimension greater than one. *Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Stanford Univ., Sta* *Calif., 1973), Part 1*, pp. 321. *Amer. Math. Soc., Providence, R.I.*, 1975. 57D30
PDF | Clipboard | Journal | Article

MR0370615 Reviewed Hirsch, Morris W.; Thurston, William P. Foliated bundles, invariant measures and flat manifolds. *Ann. Math. (2)* **101** (1975), 369–390. (Reviewer: D. B. Fuks) 57
PDF | Clipboard | Journal | Article

MR0370619 Reviewed Thurston, William The theory of foliations of codimension greater than one. *Comment. Math. Helv.* **49** (1974), 214–231. (Reviewer: John W. Wood) 57D30
PDF | Clipboard | Journal | Article

MR0339267 Reviewed Thurston, William Foliations and groups of diffeomorphisms. *Bull. Amer. Math. Soc.* **80** (1974), 304–307. (Reviewer: M. Craioveanu) 58D05 (57D30)
PDF | Clipboard | Journal | Article

MR0339211 Reviewed Rosenberg, H.; Thurston, W. Some remarks on foliations. *Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)*, pp. 463–478. *Academic Press, New* 1973. (Reviewer: Bruce L. Reinhart) 57D30 (58F99)
PDF | Clipboard | Journal | Article

MR2940155 Thesis Thurston, William Paul FOLIATIONS OF THREE-MANIFOLDS WHICH ARE CIRCLE BUNDLES. Thesis (Ph.D.)–University of California, Berkeley. 1972. (no paging), *LLC*
PDF | Clipboard | Series | Thesis

MR0298692 Reviewed Thurston, William Noncobordant foliations of S^3. *Bull. Amer. Math. Soc.* **78** (1972), 511–514. (Reviewer: F. J. Echarte Reula) 57D30
PDF | Clipboard | Journal | Article

On proofs and progress in mathematics, Thurston 1994

"First I will discuss briefly the theory of foliations, which was my first subject, starting when I was a graduate student. [...]
I fairly rapidly proved some dramatic theorems. I proved a classification theorem for foliations, giving a necessary and sufficient condition for a manifold to admit a foliation. I proved a number of other significant theorems. I wrote respectable papers and published at least the most important theorems. It was hard to find the time to write to keep up with what I could prove, and I built up a backlog."

Foliations ?

"An interesting phenomenon occurred. Within a couple of years, a dramatic evacuation of the field started to take place. I heard from a number of mathematicians that they were giving or receiving advice not to go into foliations—they were saying that Thurston was cleaning it out. People told me (not as a complaint, but as a compliment) that I was killing the field. Graduate students stopped studying foliations, and fairly soon, I turned to other interests as well."

Foliations

Codimension q *foliation* on a manifold X :

- An open covering U_i of X.
- Submersions $f_i : U_i \to \mathbf{R}^q$.
- A cocycle $\theta_{i,j}$ of C^∞ diffeomorphisms between open sets of $\mathbf{R}^q$ such that $\theta_{j,k} \circ \theta_{i,j} = \theta_{i,k}$ where it is defined and $f_j = \theta_{i,j} \circ f_i$.

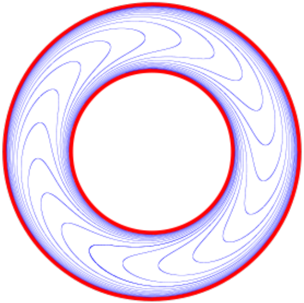
Première Leçon.

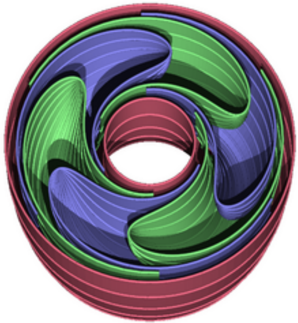
Considérations générales sur les singularités des Equations différentielles.

L'objet principal de ces leçons est l'étude des équations différentielles dont l'intégrale générale est une fonction analytique uniforme ou à un nombre fini n de déterminations.

Pour comprendre l'importance de cette étude, il suffit de remarquer qu'une équation dont l'intégrale est uniforme doit être regardée comme intégrée au sens moderne de ce mot. Dans ces dernières années, en effet, grâce surtout aux travaux de M. Weierstrass et de M. Mittag-Leffler, la représentation des fonctions [illegible] fait de tels pro-

The Reeb component (1948)





- 1895 : Leçons de Stockholm (Painlevé).
- 1944-1948 : Foliation on the 3-sphere (Reeb).
- 1955-1958 : Inexistence of codimension 1 analytic foliations on spheres (Haefliger).
- 1964 : Every codimension 1 foliation on the 3-sphere has a compact leaf (Novikov).
- 1968 : Topological obstruction to integrability : certain plane fields are not homotopic to a foliation (Bott).
- 1970 : Classifying space $B\Gamma$ (Haefliger).

Thurston's helical wobble (1971)

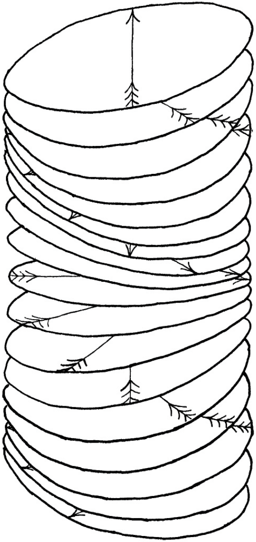
NONCOBORDANT FOLIATIONS OF S^3

BY WILLIAM THURSTON[1]

Communicated by Emery Thomas, December 15, 1971

In this note, we will sketch the construction of uncountably many noncobordant foliations of S^3, and a surjective homomorphism $\pi_3(B\Gamma_1^r) \twoheadrightarrow R$ $[2 \leqq r \leqq \infty]$, where $B\Gamma_1^r$ is the classifying space for singular C^r codimension one foliations constructed by Haefliger ([3], [4]).

Helical wobble



On proofs and progress in mathematics, Thurston 1994

"I threw out prize cryptic tidbits of insight, such as "the Godbillon-Vey invariant measures the helical wobble of a foliation", that remained mysterious to most mathematicans who read them. This created a high entry barrier : I think many graduate students and mathematicians were discouraged that it was hard to learn and understand the proofs of key theorems."

Helical wobble

Alejandra Ruddoff "Diacronia" 2005

Godbillon-Vey invariant (1971)

- A (transversaly orientable) codimension 1 foliation $\mathcal{F}$ on M is defined by a 1-form ω.
- Integrability of $\mathcal{F}$ implies $\omega \wedge d\omega = 0$.
- There exists α such that $d\omega = \omega \wedge \alpha$.
- The 3-form $\alpha \wedge d\alpha$ is closed.
- Its cohomology class in $H^3(M, \mathbf{R})$ is independent of all choices : this is the Godbillon-Vey invariant of $\mathcal{F}$.
- If $dim(M) = 3$ and if M is oriented, this is a number : $gv(\mathcal{F}) \in \mathbf{R}$.
- Two cobordant foliations have the same Godbillon-Vey number.

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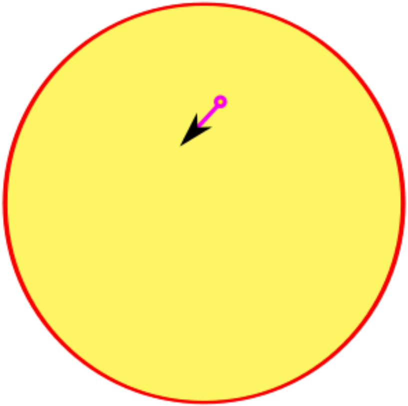
- Two cobordant foliations have the same Godbillon-Vey number.

Theorem (Thurston 1971) :

There exists of family $\mathcal{F}_\lambda$ of foliations on $\mathbf{S}^3$ such that $gv(\mathcal{F}_\lambda)$ varies continuously.

Helical wobble

Unit tangent bundle of the Poincaré disc $T^1(\mathbf{D})$.



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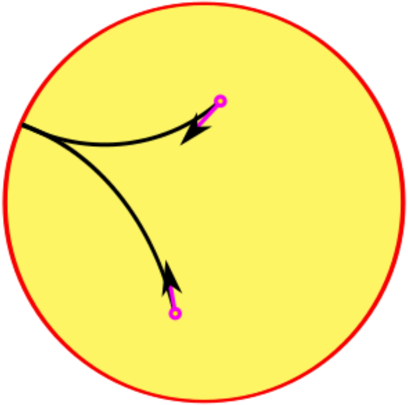

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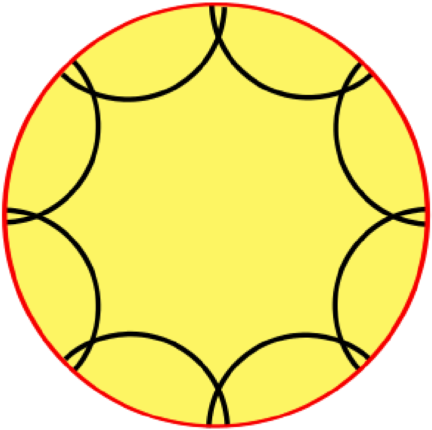

Main open problem

Suppose that the Godbillon-Vey invariant of a codimension 1 foliation on a 3-manifold is 0. Does that imply that the foliation is cobordant to zero ?

$$gv : \mathrm{Cobordism}(\mathrm{Foliations\,on\,3\,manifolds}) \to \mathbf{R}$$

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Theorem (Thurston) 1972
If M is a circle bundle over a compact surface, every codimension 1 foliation on M with no compact leaf can be isotoped to a foliation transversal to the fibers, therefore associated to a group of diffeomorphisms of the circle.

Haefliger Γ-structures

Codimension q Haefliger Γ-structure on a manifold X :

- An open covering U_i of X.
- Continuous maps $f_i : U_i \to \mathbf{R}^q$,
- A cocycle $\theta_{i,j}$ of C^∞ diffeomorphisms of open sets of $\mathbf{R}^q$ such that $\theta_{j,k} \circ \theta_{i,j} = \theta_{i,k}$ where it is defined and $f_j = \theta_{i,j} \circ f_i$.

André Haefliger (1970) There exists a classifying space $B\Gamma_q^\infty$. Every codimension q Γ-structure is the pull-back of a universal structure by some map $f : X \to B\Gamma_q^\infty$.

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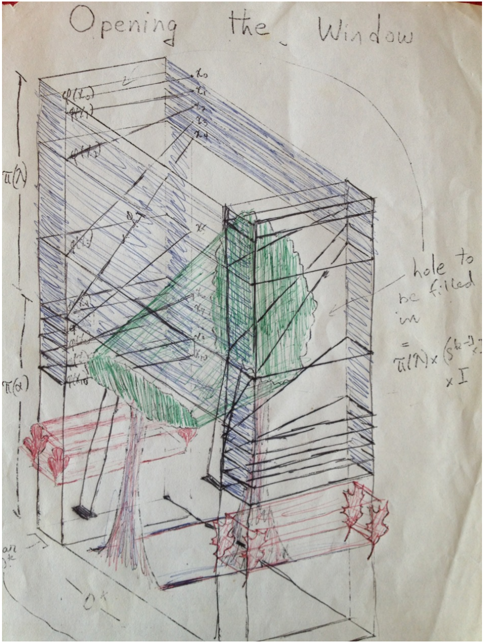
Existence of codimension q foliations

Theorem (Thurston) 1973 : A codimension $q \geq 2$ Γ-structure on a compact manifold M is homotopic to a foliation if and only if its (abstract) normal bundle embeds in the tangent bundle of M.


Opening the Window
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$\pi(N)$
$\pi(\alpha)$
O^*

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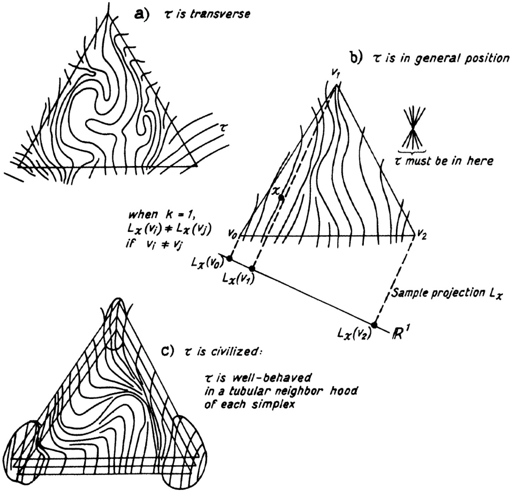

Fig. 1

Existence of codimension 1 foliations

Theorem (Thurston) 1973 : Every C^∞ hyperplane field is homotopic to a foliation.

Homotopy type of the classifying space : Mather et Thurston

Theorem 1973 : There exists a "natural" continuous map

$$B\,\mathit{Diff}_c^r(\mathbf{R}^q) \to \Omega^q(B\Gamma_q^r)$$

inducing an isomorphism in integral homology.

Corollaries :

- Every plane field, in any dimension, is homotopic to a C^0 foliation.
- $\mathrm{Cobordism(Foliations\,on\,3\,manifolds)} \simeq H_3(B\Gamma_1^\infty, \mathbf{Z}) \simeq H_2(\mathrm{Diff}_c^\infty(\mathbf{R}), \mathbf{Z})$.

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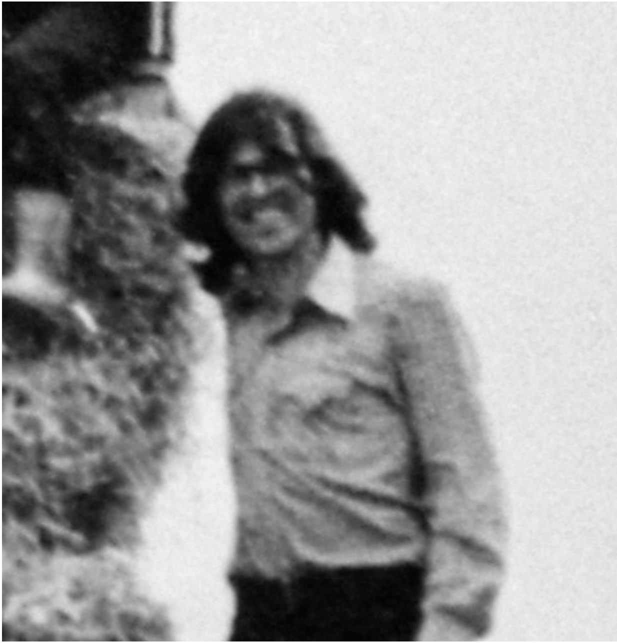
Warwick, Summer 76



On proofs and progress in mathematics, Thurston 1994

"I believe that two ecological effects were much more important in putting a damper on the subject than any exhaustion of intellectual resources that occurred. First, the results I proved [...] were documented in a conventional, formidable mathematician's style. They depended heavily on readers who shared certain background and certain insights. [...] The papers I wrote did not (and could not) spend much time explaining the background culture. They documented top-level reasoning and conclusions that I often had achieved after much reflection and effort."

On proofs and progress in mathematics, Thurston 1994

"Second is the issue of what is in it for other people in the subfield. When I started working on foliations, I had the conception that what people wanted was to know the answers. I thought that what they sought was a collection of powerful proven theorems that might be applied to answer further mathematical questions. But that's only one part of the story. More than the knowledge, people want personal understanding. And in our credit-driven system, they also want and need theorem-credits."

Back to Godbillon-Vey invariant.

What is the "qualitative meaning" of a non zero Godbillon-Vey number ?

Suppose two codimension one foliations of class C^∞ on a 3 manifold are topologically equivalent. Do they have the same Godbillon-Vey number ?

Godbillon-Vey : some kind of self linking number of a foliation ?

Dennis Sullivan :

Let $\mathcal{F}$ be a codimension 1 foliation on M^3.
Choose a flow ϕ^t transverse to the foliation.
Think of $\mathcal{F}$ as a 2-current : approximate by a large number of large balls in leaves.

Compute $link(\mathcal{F}, \phi^t(\mathcal{F}) = \int_M d\omega \wedge (\phi^t)^\star\omega$

$$gv(\mathcal{F}) = \frac{d^2}{dt^2} link(\mathcal{F}, (\phi^t)^\star(\mathcal{F}))_{|t=0}$$

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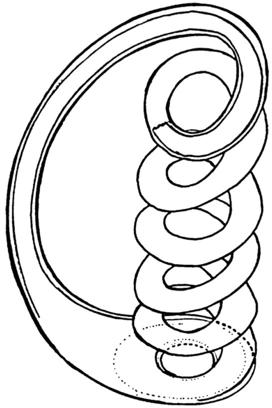
Theorem (Duminy) 1982 : If $gv(\mathcal{F}) \neq 0$, there is a resilient leaf.



Godbillon-Vey and projective group

Let $R : \pi_1(\Sigma) \to \mathrm{Diff}^\infty_+(\mathbf{S}^1)$.

- Obstruction to be projective in

$$schwarz(R) \in H^1(\mathrm{Diff}^\infty_+(\mathbf{S}^1), \{u(x)dx^2\})$$

- If R_t depends on a parameter,

$$\frac{dR_t}{dt} \in H^1(\mathrm{Diff}^\infty_+(\mathbf{S}^1), \{v(x)\frac{\partial}{\partial x}\})$$

- Pairing

$$H^1(\pi_1(\Sigma)), \{u(x)dx^2\}) \otimes H^1(\pi_1(\Sigma), \{v(x)\frac{\partial}{\partial x}\})$$

$$\to H^2(\pi_1(\Sigma), \{w(x)dx\}) \to \mathbf{R}$$

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Godbillon-Vey and projective group

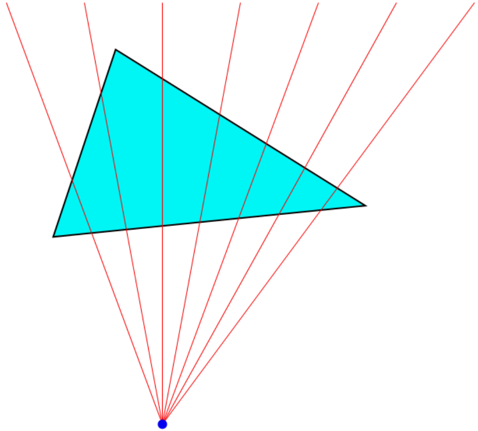
Theorem (Maszczyk) 1999

$$\frac{d\,gv(R_t)}{dt} = schwarz(R_t).\frac{dR_t}{dt}$$

Gelfand and Fuchs simple model : "piecewise projective foliations"

Start with a simplicial complex.
Foliate each simplex by a pencil of hyperplanes containing a codimension 2 subspace, disjoint from the simplex.
All these foliations should be coherent on boundaries of simplices.

Gelfand and Fuchs simple model : "piecewise projective foliations"



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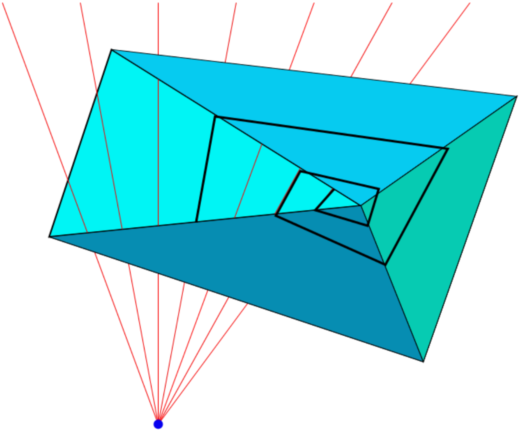

Gelfand and Fuchs simple model : "piecewise projective foliations"

Theorem (Gelfand and Fuchs)

- There is a classifying space B_{PL}.
- There is a non trivial "Godbillon-Vey invariant" $H^3(B_{PL}, \mathbf{R})$.

A combinatorical cocyle for "piecewise projective foliations"

Rogers L function for $0 < x < 1$.

$$L(x) = -\frac{1}{2}\int_0^x \left(\frac{\ln(1-t)}{t} + \frac{t}{1-t}\right) dt - \frac{\pi^2}{6}$$

Theorem

- The Gelfand-Fuchs-Godbillon-Vey invariant on piecewise projective foliations is represented by L evaluated on the cross ratio of four hyperplanes.
- $H_3(B_{PL}, \mathbf{Z}) \to \mathbf{R}$ is injective !

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Thurston cocycle on the group of diffeomorphisms of the circle

$$Thurston(f, g, h) = \int_{\mathbf{S}^1} \begin{vmatrix} 1 & \ln Df & d \ln Df \\ 1 & \ln Dg & d \ln Dg \\ 1 & \ln Dh & d \ln Dh \end{vmatrix} dt$$

is a homogeneous 2-cocycle on Diff which represents the Godbillon-Vey class.

What is the "natural" domain of definition of Godbillon-Vey ?

$\int_{\mathbf{S}^1} x(t)\, dy(t)$ is the area of a curve $(x(t), y(t))$ in the plane.

- C^2 foliations.
- f of class C^1 such that $\ln Df$ has bounded variation (Duminy and Sergiescu).
- f is of class $C^{1+\alpha}$ with $\alpha > 1/2$ (Katok-Hurder)
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Malliavin-Shavgulidze "Haar measure".

How to choose a diffeomorphism of the circle "at random"? Choose it so that the log of its derivative is a random function on the circle.

- Choose a random path $t \in [0,1] \mapsto u(t) \in \mathbf{R}$ with $u(0) = 0$.
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$$f(t) = f(0) + \frac{\exp(\int_0^t b(t)\, dt)}{\exp(\int_0^1 b(t)\, dt)}$$

where $f(0)$ is random with respect to the Lebesgue measure.

This defines the Malliavin-Shavgulidze probability measure on $\mathrm{Diff}_1^+(\mathbf{S}^1)$.
Almost surely, the derivative of a circle diffeomomorphism is Holder 1/2.

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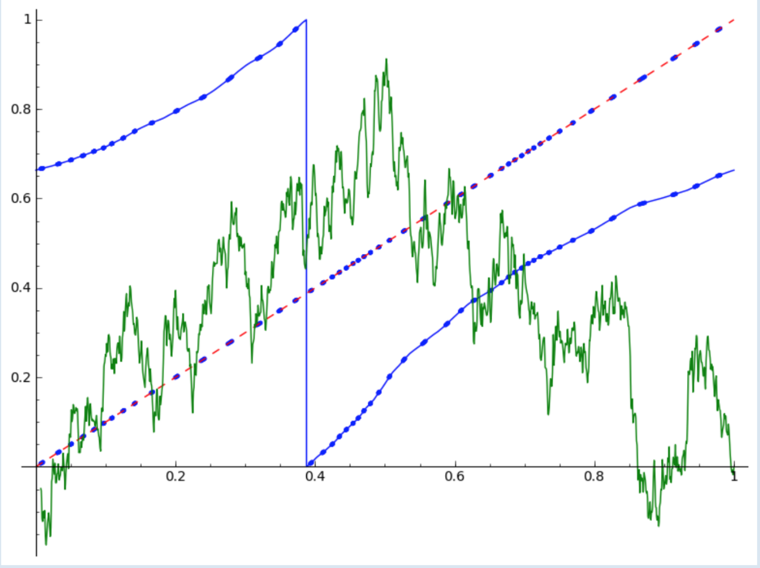
1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1

Theorem (Malliavin / Shavgulidze) :

This probability measure is quasi-invariant under left translations by C^3-difffeomorphism.

$$\frac{d(L_\phi)_\star\mu}{d\mu}(f) = \exp\left(\int_{\mathbf{S}^1} S_\phi(f(t))(f'(t))^2 dt\right)$$

Michele Triestino

Using stochastic integration, the Thurston cocycle can be defined almost everywhere in $\mathrm{Diff}^1_+(\mathbf{S}^1)$ and defines a measurable Godbillon-Vey cohomology class.

$$\int B_1(t)\, dB_2(t)$$

Problem : Compute the "measurable Gelfand-Fuchs cohomology" $\mathrm{Diff}^1_+(\mathbf{S}^1)$ with respect to Malliavin-Shavgulidze measure.

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"I forgot !"