for any even n the maximal length of a centrally symmetric inscribed n-gon in $\mathcal{E}$ having a given point P of $\mathcal{E}$ as a vertex is independent of P. Theorem 2 can be proved and generalized along the same lines. The generalization reads as follows: if $\mathcal{E}_1, \dots, \mathcal{E}_k$ are confocal ellipses, then the maximal length of a centrally symmetric $2k$-gon $(P_1, \dots, P_k, -P_1, \dots, -P_k)$ with P_i in $\mathcal{E}_i$, as $P_2, \dots, P_k$ vary with P_1 fixed, is independent of P_1.

REFERENCES

1. M. Berger, *Géométrie*, vol. 2, Nathan, Paris 1990.
2. J.-M. Richard, Safe domain and elementary geometry, *Eur. J. Phys.* **25** (2004) 835–844.
3. S. Tabachnikov, *Billiards*, Panorames et Synthèses **1**, Société Mathématique de France, Paris, 1995.
4. ———, *Geometry and Billiards*, American Mathematical Society, Providence, RI, 2005.

Collège de France, 3, rue d'Ulm, 75005 Paris, France
Alain.Connes@college-de-france.fr

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
zagier@mpim-bonn.mpg.de

An Elementary Proof of the Wallis Product Formula for pi

Johan Wästlund

1. THE WALLIS PRODUCT FORMULA. In 1655, John Wallis wrote down the celebrated formula

$$\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots = \frac{\pi}{2}. \tag{1}$$

Most textbook proofs of (1) rely on evaluation of some definite integral like

$$\int_0^{\pi/2} (\sin x)^n \, dx$$

by repeated partial integration. The topic is usually reserved for more advanced calculus courses. The purpose of this note is to show that (1) can be derived using only the mathematics taught in elementary school, that is, basic algebra, the Pythagorean theorem, and the formula $\pi \cdot r^2$ for the area of a circle of radius r.

Viggo Brun gives an account of Wallis's method in [**1**] (in Norwegian). Yaglom and Yaglom [**2**] give a beautiful proof of (1) which avoids integration but uses some quite sophisticated trigonometric identities.

2. A NUMBER SEQUENCE. We define a sequence of numbers by $s_1 = 1$, and for $n \geq 2$,

$$s_n = \frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n-1}{2n-2}.$$

The partial products of (1) with an odd number of factors can be written as

$$o_n = \frac{2^2 \cdot 4^2 \cdots (2n-2)^2 \cdot (2n)}{1 \cdot 3^2 \cdots (2n-1)^2} = \frac{2n}{s_n^2}, \tag{2}$$

while those with an even number of factors are of the form

$$e_n = \frac{2^2 \cdot 4^2 \cdots (2n-2)^2}{1 \cdot 3^2 \cdots (2n-3)^2 \cdot (2n-1)} = \frac{2n-1}{s_n^2}. \tag{3}$$

Here $e_1 = 1$ should be interpreted as an empty product. Clearly $e_n < e_{n+1}$ and $o_n > o_{n+1}$, and by comparing (2) and (3) we see that $e_n < o_n$. Therefore we must have

$$e_1 < e_2 < e_3 < \cdots < o_3 < o_2 < o_1.$$

Thus if $1 \leq i \leq n$,

$$\frac{2i}{s_i^2} = o_i \geq o_n$$

and

$$\frac{2i-1}{s_i^2} = e_i \leq e_n,$$

from which it follows that

$$\frac{2i-1}{e_n} \leq s_i^2 \leq \frac{2i}{o_n}. \tag{4}$$

It will be convenient to define $s_0 = 0$. Notice that with this definition, (4) holds also for $i = 0$. We denote the difference $s_{n+1} - s_n$ by a_n. Observe that $a_0 = 1$, and for $n \geq 1$,

$$a_n = s_{n+1} - s_n = s_n \left(\frac{2n+1}{2n} - 1 \right) = \frac{s_n}{2n} = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n}.$$

We first derive the identity

$$a_i a_j = \frac{j+1}{i+j+1} a_i a_{j+1} + \frac{i+1}{i+j+1} a_{i+1} a_j. \tag{5}$$

Proof. After the substitutions

$$a_{i+1} = \frac{2i+1}{2(i+1)} a_i$$

and

$$a_{j+1} = \frac{2j+1}{2(j+1)} a_j,$$

the right hand side of (5) becomes

$$a_i a_j \left(\frac{2j+1}{2(j+1)} \cdot \frac{j+1}{i+j+1} + \frac{2i+1}{2(i+1)} \cdot \frac{i+1}{i+j+1} \right) = a_i a_j. \qquad \blacksquare$$

If we start from a_0^2 and repeatedly apply (5), we obtain the identities

$$\begin{aligned} 1 = a_0^2 &= a_0a_1 + a_1a_0 = a_0a_2 + a_1^2 + a_2a_0 = \cdots \\ &= a_0a_n + a_1a_{n-1} + \cdots + a_na_0. \end{aligned} \tag{6}$$

Proof. By applying (5) to every term, the sum $a_0a_{n-1} + \cdots + a_{n-1}a_0$ becomes

$$\left(a_0a_n + \frac{1}{n}a_1a_{n-1}\right) + \left(\frac{n-1}{n}a_1a_{n-1} + \frac{2}{n}a_2a_{n-2}\right) + \cdots + \left(\frac{1}{n}a_{n-1}a_1 + a_na_0\right). \tag{7}$$

After collecting terms, this simplifies to $a_0a_n + \cdots + a_na_0$. ∎

3. A GEOMETRIC CONSTRUCTION. We divide the positive quadrant of the xy-plane into rectangles by drawing the straight lines $x = s_n$ and $y = s_n$ for all n. Let $R_{i,j}$ be the rectangle with lower left corner (s_i, s_j) and upper right corner (s_{i+1}, s_{j+1}). The area of $R_{i,j}$ is a_ia_j. Therefore the identity (6) states that the total area of the rectangles $R_{i,j}$ for which $i + j = n$ is 1. We let P_n be the polygonal region consisting of all rectangles $R_{i,j}$ for which $i + j < n$. Hence the area of P_n is n (see Figure 1).

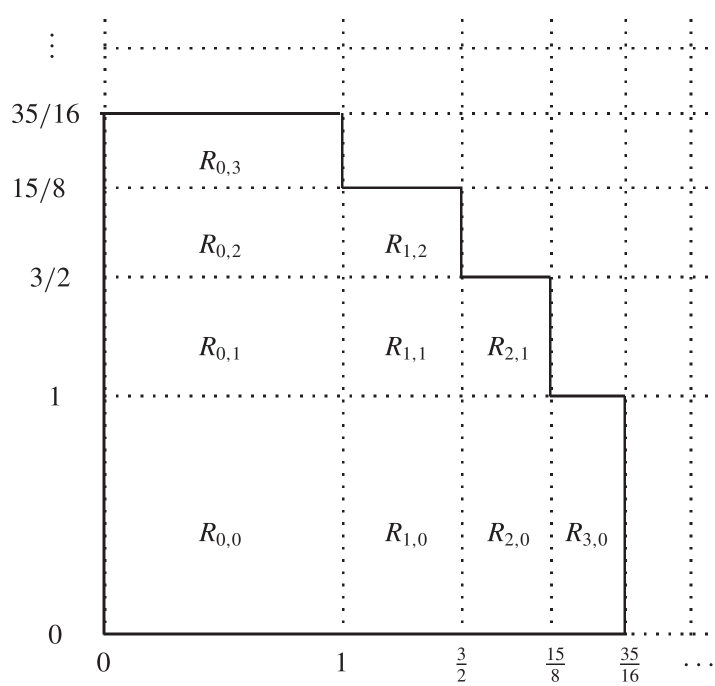

Figure 1. The region P_4 of area 4.

The outer corners of P_n are the points (s_i, s_j) for which $i + j = n + 1$ and $1 \le i, j \le n$. By the Pythagorean theorem, the distance of such a point to the origin is

$$\sqrt{s_i^2 + s_j^2}.$$

By (4), this is bounded from above by

$$\sqrt{\frac{2(i+j)}{o_n}} = \sqrt{\frac{2(n+1)}{o_n}}.$$

Similarly, the inner corners of P_n are the points (s_i, s_j) for which $i + j = n$ and $0 \le i, j \le n$. The distance of such a point to the origin is bounded from below by

$$\sqrt{\frac{2(i+j-1)}{e_n}} = \sqrt{\frac{2(n-1)}{e_n}}.$$

Therefore P_n contains a quarter circle of radius $\sqrt{2(n-1)/e_n}$, and is contained in a quarter circle of radius $\sqrt{2(n+1)/o_n}$. Since the area of a quarter circle of radius r is equal to $\pi r^2/4$ while the area of P_n is n, this leads to the bounds

$$\frac{(n-1)\pi}{2e_n} < n < \frac{(n+1)\pi}{2o_n},$$

from which it follows that

$$\frac{(n-1)\pi}{2n} < e_n < o_n < \frac{(n+1)\pi}{2n}.$$

It is now clear that as $n \to \infty$, e_n and o_n both approach $\pi/2$.

REFERENCES

1. V. Brun, Wallis's og Brounckers formler for π (in Norwegian), *Norsk matematisk tidskrift* **33** (1951) 73–81.
2. A. M. Yaglom and I. M. Yaglom, An elementary derivation of the formulas of Wallis, Leibnitz and Euler for the number π (in Russian), *Uspechi matematiceskich nauk.* (N. S.) **57** (1953) 181–187.

Department of Mathematics, Chalmers University of Technology, S-412 96 Göteborg, Sweden
wastlund@gmail.com

Automorphisms of Finite Abelian Groups

Christopher J. Hillar and Darren L. Rhea

1. INTRODUCTION. In introductory abstract algebra classes, one typically encounters the classification of finite Abelian groups [**1**]:

Theorem 1.1. *Let G be a finite Abelian group. Then G is isomorphic to a product of groups of the form*

$$H_p = \mathbb{Z}/p^{e_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{e_n}\mathbb{Z},$$

in which p is a prime number and $1 \le e_1 \le \cdots \le e_n$ are positive integers.