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An Explicit Formula for Computing the Partition Numbers $p(n)$

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The main result of the paper is an explicit formula for computing the number $p(n)$ of partitions of an integer n, i.e., the cardinal number $p(n) = |P(n)|$ of the set $P(n) = \{(k_1, k_2, \ldots, k_n)|\ k_i \in \mathbb{Z},\ k_i \geq 0,\ \sum_{i=1}^{n} ik_i = n\}$. Besides that, several iterative formulas for computing the number $p(n)$ and an iterative procedure for construction of the set $P(n)$ are given.

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1. Preliminaries

A partition number $p(n)$ of a positive integer n is said to be the number of different representations of n as sum of positive integers. The partition numbers play important role in combinatorics and they appear also in other mathematical theories, like group theory, Lie algebras, graph theory, analysis, etc. The partition number $p(n)$ is in fact the cardinal number of the set

$$P(n) = \{(k_1, k_2, \ldots, k_n)|\ k_i \in \mathbb{Z},\ k_i \geq 0,\ \sum_{i=1}^{n} ik_i = n\},$$

i.e., $p(n) = |P(n)|$.

The interest for the integer function $p(n)$ started intensively after Euler's discovery of a generating function for the numbers $p(n)$. Nowadays several other generating functions and several iterative formulas for computing $p(n)$ are known. The paper of Ahlgren and Ono [1] is a nice reference of this matter, as well as the web site "On-line Encyclopedia of Integer Sequences" [5]. For example, in [1] and/or [5] one can find the following iterative formulas:

$p(n) = \frac{1}{n}\sum_{k=0}^{n-1}\sigma(n-k)p(k)$, where $\sigma(k)$ is the sum of divisors of k,

$p(n) = p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+p(n-15)- \cdots \pm p(n-m(3m-1)/2) \pm p(n-m(3m+1)/2) \pm \cdots$

Hardy and Ramanujan discovered the asymptotic formula

$$p(n) \approx \frac{1}{(4n\sqrt{3})}e^{\pi\sqrt{2n/3}}$$

and Rademacher improved it to a rather precise extent.

According to Knuth [3], an explicit formula for $p(n)$ is not known. In this paper we give such a formula, as stated in Theorem 3.1 of Section 3. For that aim we present in Section 2 a new iterative way for computing the numbers $p(n)$ and, by using the obtained results, in Section 3 we extract an explicit formula.

2. Iterative computing of the numbers $p(n)$

Define iteratively the sets $C(n)$ for $n = 1, 2, 3, \ldots$ by $C(1) = \{(1)\}$ and, if $C(n)$ is defined, then define $C(n+1)$ as follows.

Given $(k_1, \ldots, k_n) \in C(n)$, let

$$\begin{aligned} S(k_1, k_2, \ldots, k_n) = \\ = \{&(k_1+1, k_2, \ldots, k_n, 0), \\ &(k_1-1, k_2+1, k_3, \ldots, k_n, 0), \\ &(k_1, k_2-1, k_3+1, k_4, \ldots, k_n, 0), \ldots, \\ &(k_1, k_2, \ldots, k_{n-2}, k_{n-1}-1, k_n+1, 0), \\ &(k_1, \ldots, k_{n-1}, k_n-1, 1)\}, \end{aligned} \tag{1}$$

where the subtraction $k_i - 1$ is applied only if $k_i \geq 1$ (i.e., the element $(k_1, \ldots, k_{i-1}, k_i-1, k_{i+1}+1, \ldots, k_n, 0)$ does not appear in the set $S(k_1, k_2, \ldots, k_n)$ when $k_i = 0$).

Then we define

$$C(n+1) = \bigcup_{(k_1,\ldots,k_n)\in C(n)} S(k_1, \ldots, k_n). \tag{2}$$

Proposition 2.1. *$P(n) = C(n)$ for each $n \geq 1$.*

P r o o f. We use indiction on n.

$C(n+1) \subseteq P(n+1)$ since, if $n = 1k_1 + 2k_2 + \cdots + nk_n$, then

$$n+1 = 1(k_1+1) + 2k_2 + \cdots + nk_n + (n+1)0,$$
$$n+1 = 1(k_1-1) + 2(k_2+1) + 3k_3 + \cdots + nk_n + (n+1)0,$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$n+1 = 1k_1 + 2k_2 + \cdots + (n-2)k_{n-2} + (n-1)(k_{n-1}-1) + \\ +n(k_n+1) + (n+1)0,$$
$$n+1 = 1k_1 + \cdots + (n-1)k_{n-1} + n(k_n-1) + (n+1)1.$$

Conversely, let $n+1 = 1k_1 + 2k_2 + \cdots + (n+1)k_{n+1}$, where $k_i \geq 0$. Then $k_{n+1} = 1$ implies $k_1 = \cdots = k_n = 0$ and $(\underbrace{0,\dots,0}_{n},1) \in P(n+1)$ appears as an element in $S(\underbrace{0,\dots,0}_{n-1},1)$. If $k_{n+1} = 0$ then there is largest index i, $1 \leq i \leq n$, such that $k_i > 0$. Then, if $i > 1$ we have that the set $S(k_1,\dots,k_{i-2},k_{i-1}+1,k_i-1,k_{i+1},\dots,k_n)$ contains the element $(k_1,\dots,k_n,0)$ since, by the inductive hypothesis, $(k_1,\dots,k_{i-2},k_{i-1}+1,k_i-1,k_{i+1},\dots,k_n) \in C(n)$. (Namely, $\sum_{s=1}^{i-2} sk_s + (i-1)(k_{i-1}+1) + i(k_i-1) + \sum_{i+1}^{n} sk_s = -1 + \sum_{s=1}^{n+1} sk_s = n$.) The last case is $i = 1$, hence $k_1 = n+1$, and then $(n+1,\underbrace{0,\dots,0}_{n}) \in S(n,\underbrace{0,\dots,0}_{n-1})$. ∎

By (2) and Proposition 2.1, we have

Corollary 2.1. $P(1) = \{(1)\}$, $P(n+1) = \bigcup_{(k_1,\dots,k_n)\in P(n)} S(k_1,\dots,k_n)$.

Proposition 2.1 gives an iterative way for construction of the sets $P(n)$, and here we propose a much simpler one. We will use it for iterative computing of the numbers $p(n)$ too.

We order the elements of the set $P(n)$ by using an opposite lexicographic ordering:

$$(k_1,\dots,k_n) < (s_1,\dots,s_n) \Leftrightarrow (\exists\ i)\ k_i < s_i\ \&\ k_{i+1} = s_{i+1}\ \&\ \dots\ \&\ k_n = s_n.$$

Then the set $P(n)$ is linearly ordered. In the sequel, when we speak on the set $P(n)$ or subsets of $P(n)$, we take that they are ordered sets and that their elements are presented in order.

We say that the sets $S(k_1,\dots,k_{n-1})$ are (ordered) classes of the set $P(n)$.

We concatenate all classes $S(k_1,\dots,k_{n-1})$ of $P(n)$ as

(3) $$S(k_1,\dots,k_{n-1})||S(m_1,\dots,m_{n-1})||\dots||S(p_1,\dots,p_{n-1})$$

by using the ordering of $P(n-1)$, i.e. $(k_1,\dots,k_{n-1}) < (m_1,\dots,m_{n-1}) < \dots < (p_1,\dots,p_{n-1})$. Then we reduce the concatenated classes (3) by using the following erasure procedure: Take an element $(t_1,\dots,t_n)\in P(n)$ and erase all of its appearances in (3) except the last right. Apply the erasure procedure until no erasing is possible. Then, from (3), we will obtain the string of sets

$$R(k_1,\dots,k_{n-1})||R(m_1,\dots,m_{n-1})||\dots||R(p_1,\dots,p_{n-1}), \tag{4}$$

where $R(q_1,\dots,q_{n-1})$ is obtained from $S(q_1,\dots,q_{n-1})$. The ordered set $R(q_1,\dots$ $\dots,q_{n-1})$ is said to be the reduced class of $P(n)$ and, as a consequence of Corollary 2.1, we have:

Corollary 2.2. $P(1)=\{(1)\}$, $P(n+1)=\bigcup\limits_{(k_1,\dots,k_n)\in P(n)} R(k_1,\dots,k_n)$.

Example 2.1 We have $P(1)=\{(1)\}$, $S(1)=\{(2,0),(0,1)\}=P(2)$ and then:

$S(2,0)=\{(3,0,0),(1,1,0)\}$, $S(0,1)=\{(1,1,0),(0,0,1)\}$,
$P(3)=S(2,0)\cup S(0,1)=\{(3,0,0),(1,1,0),(0,0,1)\}$.

$S(3,0,0)=\{(4,0,0,0),(2,1,0,0)\}$,
$S(1,1,0)=\{(2,1,0,0),(0,2,0,0),(1,0,1,0)\}$,
$S(0,0,1)=\{(1,0,1,0),(0,0,0,1)\}$,
$P(4)=S(3,0,0)\cup S(1,1,0)\cup S(0,0,1)=$
$\{(4,0,0,0),(2,1,0,0),(0,2,0,0),(1,0,1,0),(0,0,0,1)\}$.

The reduced classes are: $R(1)=S(1)$,

$R(2,0)=\{(3,0,0)\}$, $R(0,1)=\{(1,1,0),(0,0,1)\}$,

$R(3,0,0)=\{(4,0,0,0)\}$, $R(1,1,0)=\{(2,1,0,0),(0,2,0,0)\}$,
$R(0,0,1)=\{(1,0,1,0),(0,0,0,1)\}$.

Note that the obtained reduced classes contain only one or two elements and that the ordering $(3,0,0)<(1,1,0)<(0,0,1)$ for $n=3$ induces the ordering $(4,0,0,0)<(2,1,0,0)<(0,2,0,0)<(1,0,1,0)<(0,0,0,1)$ for $n=4$.

Lemma 2.1. *Each reduced class $R(k_1,\dots,k_{n-1})$ of $P(n)$ contains one or two elements: the first element $(k_1+1,k_2\dots,k_{n-1},0)$ and possibly the second element of $S(k_1,\dots,k_{n-1})$.*

Proof. First we show that the smallest element $(k_1+1,k_2\dots,k_{n-1},0)$ in the class $S(k_1,\dots,k_{n-1})$ will be not erased by the erasure procedure. Namely, the erasing can happen only if this element is appearing in a class $S(p_1,\dots,p_{n-1})$ for some $(p_1,\dots,p_{n-1})>(k_1,\dots,k_{n-1})$. Then $(p_1+1,\dots,p_{n-1},0)\neq(k_1+1,\dots,k_{n-1},0)$ and hence, for some $i:\ 2\le i\le n-2$, we have:

$(p_1,\dots,p_{i-1},p_i-1,p_{i+1}+1,p_{i+2},\dots,p_{n-1},0)=(k_1+1,\dots,k_{n-1},0)\Longrightarrow$
$p_{i+1}+1=k_{i+1},\ p_{i+2}=k_{i+2},\dots\ \Longrightarrow (p_1,\dots,p_{n-1})<(k_1,\dots,k_{n-1})$.
We conclude that the reduced classes contain at least one element.

We show next that each element in the class $S(k_1,\dots,k_{n-1})$, after the second one, will be erased by the erasure procedure. Consider first the case when $S(k_1,\dots,k_{n-1})$ contains at least three elements that can be ordered as $(k_1+1,k_2,\dots,k_{n-1},0)<\cdots<(k_1,\dots,k_{j-2},k_{j-1}-1,k_j+1,k_{j+1},\dots,k_{n-1},0)<\cdots<(k_1,\dots,k_{i-2},k_{i-1}-1,k_i+1,k_{i+1},\dots,k_{n-1},0)$ for some $j<i,\ 3\le i\le n-1,\ 2\le j$. Then $k_{j-1}>0$. Now we put

$$(p_1,\dots,p_{n-1})=$$
$$=\begin{cases}(k_1,\dots,k_{j-2}+1,k_{j-1}-1,\dots,k_{i-1}-1,k_i+1,\dots,k_{n-1}), & j>2\\ (k_1-1,k_2,\dots,k_{i-1}-1,k_i+1,\dots,k_{n-1}), & j=2\end{cases}$$

If $j>2$ then we have

$$\sum_{s=1}^{n-1}sp_s = \sum_{s=1}^{j-3}sk_s+(j-2)(k_{j-2}+1)+(j-1)(k_{j-1}-1)+$$
$$\sum_{s=j}^{i-2}sk_s+(i-1)(k_{i-1}-1)+i(k_i+1)+\sum_{s=i+1}^{n-1}sk_s=n-1$$

In a similar way we have $\sum sp_s=n-1$ for $j=2$. Thus, the element $(k_1,\dots,k_{i-2},k_{i-1}-1,k_i+1,k_{i+1},\dots,k_{n-1},0)$ will be erased by the erasure procedure, since $(p_1,\dots,p_{n-1})>(k_1,\dots,k_{n-1})$ and $(k_1,\dots,k_{i-2},k_{i-1}-1,k_i+1,k_{i+1},\dots,k_{n-1},0)\in S(p_1,\dots,p_{n-1})$.

Finally, we note that $(0,\dots,0,1)\in S(k_1,\dots,k_{n-1})$ iff $(k_1,\dots,k_{n-1})=(0,\dots,0,1)$, and then $S(0,\dots,0,1)=\{(1,0,\dots,0,1,0),(0,\dots,0,1)\}$. ■

It follows from Lemma 2.1 that $(k_1+1,k_2,\dots,k_{n-1},0)\in R(k_1,\dots,k_{n-1})$ for each $(k_1,\dots,k_{n-1})\in P(n-1)$ and that it is possible $R(k_1,\dots,k_{n-1})$ to contain the second element from $S(k_1,\dots,k_{n-1})$. The next property gives the cases when the reduced classes contain two elements.

Lemma 2.2. *The second element* $(p_1,p_2,p_3,\dots,p_n)$ *of the class* $S(k_1,\dots\ \dots,k_{n-1})$ *belongs to the reduced class* $R(k_1,\dots,k_{n-1})$ *if and only if the first nonzero component of* $(k_1,\dots,k_{n-1})$ *is equal to* 1.

Proof. Let $(k_1,\dots,k_{n-1})=(0,\dots,0,1,k_{j+1},\dots,k_{n-1})$ for some $j\ge 1$. Then the second element in $S(k_1,\dots,k_{n-1})$ is $(0,\dots,0,0,k_{j+1}+1,\dots,k_{n-1},0)$. Assume that there is an $(m_1,\dots,m_{n-1})>(k_1,\dots,k_{n-1})$ such that $(0,\dots,0,0,k_{j+1}+1,\dots,k_{n-1},0)\in S(m_1,\dots,m_{n-1})$. Then $(0,\dots,0,0,k_{j+1}+1,\dots,$

$k_{n-1}, 0) = (m_1, \dots, m_{i-1}, m_i - 1, m_{i+1} + 1, \dots, m_{n-1}, 0)$ for some $i \geq j$. It follows that $m_{i+1} + 1 = k_{i+1}, m_{i+2} = k_{i+2}, \dots$ when $i > j$ and $(m_1, \dots, m_{n-1}) = (k_1, \dots, k_{n-1})$ when $i = j$, a contradiction to $(m_1, \dots, m_{n-1}) > (k_1, \dots, k_{n-1})$. We have that the second element in the class $S(k_1, \dots, k_{n-1})$ cannot be erased.

If $(k_1, \dots, k_{n-1}) = (0, \dots, 0, k_j, \dots, k_{n-1})$, $j \geq 1$, and $k_j > 1$, then the second element in $S(k_1, \dots, k_{n-1})$ is $(0, \dots, 0, k_j - 1, k_{j+1} + 1, k_{j+2}, \dots, k_{n-1}, 0)$ and it belongs also to $S(m_1, \dots, m_{n-1})$, where

$$(m_1, \dots, m_{n-1}) = \begin{cases} (k_1 - 2, k_2 + 1, k_3 \dots, k_{n-1}), & j = 1 \\ (0, \dots, 0, 1, k_j - 2, k_{j+1} + 1, k_{j+2}, \dots, k_{n-1}), & j > 1 \end{cases}$$

Now, since $(m_1, \dots, m_{n-1}) > (k_1, \dots, k_{n-1})$, the second element in the class $S(k_1, \dots, k_{n-1})$ will be erased during the erasure procedure. ■

Lemma 2.3. *If $(m_1, \dots, m_{n-1}) > (k_1, \dots, k_{n-1})$ and if the second element in the reduced class $R(k_1, \dots, k_{n-1})$ exists, then it is smaller than all elements in the reduced class $R(m_1, \dots, m_{n-1})$.*

Proof. Let $(m_1, \dots, m_{n-1}) > (k_1, \dots, k_{n-1}) = (0, \dots, 0, 1, k_{j+1}, \dots$ $\dots, k_{n-1})$, i.e., $m_{n-1} = k_{n-1}, \dots, m_{i+1} = k_{i+1},\ m_i > k_i$ for some $i \geq 1$. (Note that $m_1 \geq k_1 = 0, \dots, m_{j-1} \geq k_{j-1} = 0$). It is enough to show that the second element $(0, \dots, 0, 0, k_{j+1}{+}1, k_{j+2}, \dots, k_{n-1}, 0)$ in $R(k_1, \dots, k_{n-1})$ is smaller than the first element $(m_1 + 1, m_2, \dots, m_{n-1}, 0)$ of $R(m_1, \dots, m_{n-1})$. That is clear for $i > j + 1$ or $i = j + 1,\ m_{j+1} > k_{j+1} + 1$, and we show that $m_i > k_i$ is not possible for the other cases. If $i < j$ and $m_i > k_i = 0$ or $i = j$ and $m_i > k_i = 1$, then we would have $\sum_{s=1}^{n-1} sm_s \geq\ im_i + \sum_{s=i+1}^{n-1} sm_s > ik_i + \sum_{s=i+1}^{n-1} sk_s = n-1$, a contradiction to $\sum_{s=1}^{n-1} sm_s = n - 1$. The remaining case $i = j + 1,\ m_{j+1} = k_{j+1} + 1$ is not possible either, since then $\sum_{s=1}^{n-1} sm_s \geq\ (j + 1)(k_{j+1} + 1) + \sum_{s=j+2}^{n-1} sk_s = 1 + j \cdot 1 + (j + 1)k_{j+1} + \sum_{s=j+2}^{n-1} sk_s = 1 + \sum_{s=1}^{n-1} sk_s = n$. ■

We have from Lemma 2.3 that the elements in the sequence of reduced sets (4) appear ordered. Now, the Lemmas 2.1, 2.2 and 2.3 give proof of the following theorem.

Theorem 2.1. *Each reduced class $R(k_1, \dots, k_{n-1})$ of $P(n)$ contains only one or two elements, the first element of the ordered class $S(k_1, \dots, k_{n-1})$ and possibly the second element of $S(k_1, \dots, k_{n-1})$; the second element belongs to $R(k_1, \dots, k_{n-1})$ if and only if its first nonzero component is equal to* 1. *The ordering of the elements of $P(n)$ is given by their appearance in the sequence of reduced sets* (4).

Now, by Theorem 2.1, we have the following iterative procedure for construction of the sets $P(n)$, $n > 0$.

Theorem 2.2 *$P(1) = \{(1)\}$. Let the first nonzero component of $(k_1, \dots, k_n) \in P(n)$ be k_i. Then $P(n+1)$ is obtained from $P(n)$ by applying the following replacement of each element $(k_1, \dots, k_n) \in P(n)$:*

if $k_i > 1$ replace $(k_1, \dots, k_n)$ by $(k_1 + 1, k_2, \dots, k_n, 0)$;

if $k_i = 1,\ i = n = 1$, replace (1) by $(2, 0)$ and $(0, 1)$;

if $k_i = 1,\ i = 1,\ n > 1$, replace $(1, k_2, \dots, k_n)$ by two elements $(2, k_2, \dots \dots, k_n, 0)$ and $(0, k_2 + 1, k_3, \dots, k_n, 0)$;

if $k_i = 1,\ 1 < i < n$, replace $(0, \dots, 0, 1, k_{i+1}, \dots, k_n)$ by two elements $(1, 0, \dots \dots, 0, 1, k_{i+1}, \dots, k_n, 0)$ and $(0, \dots, 0, k_{i+1} + 1, k_{i+2}, \dots, k_n, 0)$;

if $k_i = 1,\ i = n > 1$, replace $(0, \dots, 0, 1)$ by two elements $(1, 0, \dots, 0, 1, 0)$ and $(0, 0, \dots, 0, 0, 1)$.

Denote by $a(n)$ the number of elements in $P(n)$ whose first nonzero component is equal to 1. Then we have the following formula for the partition numbers $p(n)$:

Corollary 2.3. *$p(1) = 1,\ p(n+1) = p(n) + a(n)$.*

Next we give an iterative procedure for computing the numbers $a(n)$. Note that the only element of $P(n)$ having exactly one nonzero component equal to 1 is $(0, \dots, 0, 1)$, and it generates two elements in $P(n+1)$: $(1, 0, \dots, 0, 1, 0)$ and $(0, \dots, 0, 1)$. The other elements of $P(n)$ having the first nonzero component equal to 1 are of the form

$$(0, \dots, 0, 1, 0, \dots, 0, k_i, \dots, k_{n-1}, 0)$$

for some $k_i > 0,\ 2 \le i \le n-1$.

Define the following subsets of $P(n)$:

(a.1) $A^{n-1,0}(n) = \{(\underbrace{0, \dots, 0}_{n-1}, 1)\}$,

(a.2) $A^{i,j}(n) = \{(\underbrace{0, \dots, 0}_{i}, 1, \underbrace{0, \dots, 0}_{j}, k_{i+j+2}, \dots, k_{n-1}, 0) |\ k_{i+j+2} > 0\}$

for $i, j \ge 0,\ i + j \le n - 3$,

(a.3) $A^{i,j}(n) = \emptyset$ otherwise,

(b.1) $O^j(n) = \{(\underbrace{0,\dots,0}_{j}, k_{j+1},\dots,k_{n-1},0) |\ k_{j+1} > 1\}$

for $0 < j \le \lfloor \frac{n}{2} \rfloor$,

(b.2) $O^j(n) = \emptyset$ otherwise.

Denote $a^{i,j}(n) = |A^{i,j}(n)|$, $o^j(n) = |O^j(n)|$.

Lemma 2.4 *The following iterative formulas are true for* $n > 1$:

$$
\begin{array}{llll}
(i) & a^{i,j}(n) & = & a^{i-1,j+1}(n-1), \ \ i > 0, \ j \ge 0, \\
(ii) & a^{0,j}(n) & = & o^{j+1}(n-1) + \sum_{r=0}^{n-5-j} a^{j+1,r}(n-1), \\
(iii) & o^j(n) & = & a^{j-1,0}(n-1), \quad j > 0.
\end{array}
$$

P r o o f. (i) Let $\mathbf{x} = (\underbrace{0,\dots,0}_{i}, 1, \underbrace{0,\dots,0}_{j}, k_{i+j+2},\dots,k_{n-1},0) \in A^{i,j}(n)$, where $i > 0$, $j \ge 0$. By Theorem 2.1, since all first elements in the reduced classes have nonzero first component, $\mathbf{x}$ can be obtained only as the second element from the set $R(\mathbf{y})$, where $\mathbf{y} = (\underbrace{0,\dots,0}_{i-1}, 1, \underbrace{0,\dots,0}_{j+1}, k_{i+j+2},\dots,k_{n-1})$. Now $j+1 \ge 1$ implies $k_{n-1} = 0$, and then $\mathbf{x} \mapsto \mathbf{y}$ defines a bijection between the sets $A^{i,j}(n)$ and $A^{i-1,j+1}(n-1)$.

(ii) Let $(1, \underbrace{0,\dots,0}_{j}, k_{j+2},\dots,k_{n-1},0) \in A^{0,j}(n)$. We can obtain that element only as first element in the set $R(\underbrace{0,\dots,0}_{j+1}, k_{j+2},\dots,k_{n-1})$. If $k_{j+2} > 1$, then $(\underbrace{0,\dots,0}_{j+1}, k_{j+2},\dots,k_{n-1}) \in O^{j+1}(n-1)$, and if $k_{j+2} = 1$ then $(\underbrace{0,\dots,0}_{j+1}, k_{j+2},\dots,k_{n-1}) = (\underbrace{0,\dots,0}_{j+1}, 1, \underbrace{0,\dots,0}_{r}, k_{j+r+3},\dots,k_{n-1}) \in A^{j+1,r}(n-1)$ for some $r \in \{0,1,\dots,n-5-j\}$. Namely, $j+1 \ge 1$ implies $k_{n-1} = 0$ in both cases. So, as in (i), we can establish a bijection between the sets $A^{0,j}(n)$ and $O^{j+1}(n-1) \cup \bigcup_{r=0}^{n-5-j} A^{j+1,r}(n-1)$ (since the union is disjoint).

(iii) Let $(\underbrace{0,\dots,0}_{j}, k_{j+1},\dots,k_{n-1},0) \in O^j(n)$ where $j > 0$. This element is the second element in the class $R(\underbrace{0,\dots,0}_{j-1}, 1, k_{j+1}-1,\dots,k_{n-1})$ and

$(\underbrace{0,\dots,0}_{j-1},1,k_{j+1}-1,\dots,k_{n-1}) \in A^{j-1,0}(n-1)$, since $k_j > 1$ implies $k_{n-1} = 0$. ■

We have the following theorem for iterative computing of the partition numbers $p(n)$.

Theorem 2.3 *For each positive integer n,*

$$p(n) = 1 + \sum_{k=1}^{n-1} \sum_{i\geq 0,\ j\geq 0} a^{i,j}(n-k), \tag{5}$$

where the following equalities hold:

$$\begin{aligned} a^{i,j}(n) &= a^{i-1,j+1}(n-1), \ \ i>0, \ \ j\geq 0, \\ a^{0,j}(n) &= a^{j,0}(n-2) + \sum_{r\geq 0} a^{j+1,r}(n-1). \end{aligned}$$

Proof. Note that $a(n) = |\bigcup_{i\geq 0, j\geq 0} A^{i,j}(n)|$ and, since the union is disjoint, $a(n) = \sum_{i\geq 0, j\geq 0} a^{i,j}(n)$. The numbers $a(n)$ can be computed iteratively by Lemma 2.4. Several applications of Corollary 2.3 give

$$\begin{aligned} p(n) &= p(n-1) + \sum_{i\geq 0, j\geq 0} a^{i,j}(n-1) \\ &= p(n-2) + \sum_{i\geq 0, j\geq 0} a^{i,j}(n-2) + \sum_{i\geq 0, j\geq 0} a^{i,j}(n-1) = \cdots = \\ &= p(1) + \sum_{i\geq 0, j\geq 0} a^{i,j}(1) + \ldots + \sum_{i\geq 0, j\geq 0} a^{i,j}(n-2) + \sum_{i\geq 0, j\geq 0} a^{i,j}(n-1) \\ &= 1 + \sum_{k=1}^{n-1} \sum_{i\geq 0, j\geq 0} a^{i,j}(n-k). \end{aligned}$$

■

3. An explicit formula for $p(n)$

We will use the results of the previous section to obtain an explicit formula for the partition numbers $p(n)$. For that aim we will make a profound analysis of the numbers $a^{i,j}(n)$.

Lemma 3.1. *The following equalities hold:*

$$(i) \qquad a^{i,j}(1) = \begin{cases} 1, & i=j=0 \\ 0, & otherwise. \end{cases}$$

$$(ii) \qquad a^{i,j}(2) = 0, \quad for\ each\ i\ and\ j\ such\ that\ (i,j) \neq (1,0).$$

$$(iii) \quad a^{n-1,0}(n) = 1, \quad for\ each\ \ n > 0.$$

$$(iv) \qquad a^{i,j}(n) = \begin{cases} a^{0,j+i}(n-i), & i+j \leq n-3 \\ 0, & i+j > n-3,\ (i,j) \neq (n-1,0), \end{cases}$$

for each $n \geq 3,\ i \geq 1,\ j \geq 0$.

$$(v) \qquad a^{0,j}(n) = a^{j,0}(n-2) + \sum_{r=1}^{n-2j-5} a^{0,j+r}(n-j-2),$$

for each $n \geq 3,\ j \geq 0$.

P r o o f. (i), (ii) and (iii) follow by definition. (iv) follows after i applications of Lemma 2.4(i). By Lemma 2.4(ii), Lemma 2.4(iii) and Lemma 2.4(i) we have:

$$\begin{aligned} a^{0,j}(n) &= a^{j,0}(n-2) + \sum_{r=0}^{n-j-5} a^{j+1,r}(n-1) \\ &= a^{j,0}(n-2) + \sum_{r=0}^{n-j-6} a^{j,r+1}(n-2) \\ &= a^{j,0}(n-2) + \sum_{r=1}^{n-j-5} a^{j,r}(n-2), \end{aligned}$$

and by (iv) of this lemma we have:

$$a^{0,j}(n) = a^{j,0}(n-2) + \sum_{r=1}^{n-j-5} a^{0,j+r}(n-j-2).$$

Now, the equality (v) follows from the fact that $a^{0,j+r}(n-j-2) = 0$ when $r > n-2j-5$. ∎

It can be concluded that the numbers $a^{0,j}(n)$ deserve a special attention. We consider them in the sequel.

Lemma 3.2. *For each $n \geq 3,\ j \geq 0$,*

$$a^{0,j}(n) = \begin{cases} 0, & j > n-3 \\ 1, & j = n-3 \\ 0, & \frac{n-5}{2} < j < n-3 \\ t > 0, & j \leq \frac{n-5}{2}. \end{cases} \tag{1}$$

P r o o f. By definition, $a^{0,j}(n) = 0$ for $j > n-3$.

The sum that appears in the equality of Lemma 3.1(v) do not exists when $\dfrac{n-5}{2} \leq j \leq n-3$, so in that case we have

$$a^{0,j}(n) = a^{j,0}(n-2).$$

If $j = n-3$ then by definition $a^{n-3,0}(n-2) = 1$, and if $\frac{n-5}{2} < j < n-3$ then $a^{j,0}(n-2) = a^{0,j}(n-2-j) = 0$ by Lemma 3.1(iv), because $0+j > (n-j-2)-3$.

If $j \leq \dfrac{n-5}{2}$, by Lemma 3.1(v) and by (1) we have:

$$a^{0,j}(n) \geq \sum_{r=0}^{n-5-2j} a^{0,j+r}(n-j-2) \geq$$
$$\geq a^{0,j+(n-5-2j)}(n-j-2) = a^{0,(n-j-2)-3}(n-j-2) = 1. \qquad \blacksquare$$

Denote by I the Boolean function with range $\{0,1\}$, i.e., $I(\alpha) = 0$ if α is a false sentence and $I(\alpha) = 1$ if α is a true sentence. Now, from Lemma 3.1 and Lemma 3.2 we have the following.

Lemma 3.3 *Let $n \geq 3$. Then for each $j \geq 0$ we have*

$$a^{0,j}(n) = I(j+3=n) + I\left(\frac{n-5}{2} \geq j\right) + \sum_{r=0}^{\lfloor \frac{n-3j-7}{2} \rfloor} a^{0,r+j}(n-j-2). \tag{2}$$

P r o o f. By Lemma 3.2, i.e., by the equality (1), we have to consider the following cases:

Case $j > n-3$: Then $I(j+3=n) = I(\dfrac{n-5}{2} \geq j) = 0$ and $n-3j-7 < 0$.

Case $j = n-3$: Then $I(j+3=n) = 1$, $I(\dfrac{n-5}{2} \geq j) = 0$ and $n-3j-7 < 0$.

Case $\dfrac{n-5}{2} < j < n-3$: Then $I(j+3=n) = I(\dfrac{n-5}{2} \geq j) = 0$ and $n-3j-7<0$.

Case $j \leq \frac{n-5}{2}$: Then $I(j+3=n)=0$, $I(\dfrac{n-5}{2} \geq j) = 1$ and from Lemma 3.1(v) and Lemma 3.1(iv) we obtain

$$\begin{aligned} a^{0,j}(n) &= a^{j,0}(n-2) + \sum_{r=1}^{n-2j-5} a^{0,j+r}(n-j-2) \\ &= a^{0,j}(n-j-2) + \sum_{r=1}^{n-2j-5} a^{0,j+r}(n-j-2) \\ &= \sum_{r=0}^{n-2j-5} a^{0,j+r}(n-j-2). \end{aligned}$$

By Lemma 3.2 we have $a^{0,j+r}(n-j-2)=0$ for $\dfrac{n-3j-7}{2} < r < n-2j-5$ and that implies, again by Lemma 3.2,

$$\begin{aligned} a^{0,j}(n) &= \sum_{r=0}^{\lfloor\frac{n-3j-7}{2}\rfloor} a^{0,r+j}(n-j-2) + a^{0,n-j-5}(n-j-2) \\ &= 1 + \sum_{r=0}^{\lfloor\frac{n-3j-7}{2}\rfloor} a^{0,r+j}(n-j-2). \end{aligned}$$

■

Rewrite the equality (2) by putting $r=i_1$. Then:

(3) $$a^{0,j}(n) = I(j+3=n) + I(\frac{n-5}{2} \geq j) + \sum_{i_1=0}^{\lfloor\frac{n-3j-7}{2}\rfloor} a^{0,j+i_1}(n-j-2).$$

Note that the sum in the equality (3) exists only for $\dfrac{n-7}{3} \geq j$. Let us assume it is true. Then we apply the equality (3) on the terms $a^{0,i_1+j}(n-j-2)$ in (3), which means that we replace n by $n-j-2$ and j by $j+i_1$. We obtain the following equality, where the double sum exists only for $\frac{n-9}{4} \geq j$:

$$a^{0,j}(n) = I(j+3=n) + I(\frac{n-5}{2} \geq j) + \sum_{i_1=0}^{\lfloor \frac{n-3j-7}{2} \rfloor} 1 + \\ + \sum_{i_1=0}^{\lfloor \frac{n-4j-9}{3} \rfloor} \sum_{i_2=0}^{\lfloor \frac{n-4j-3i_1-9}{2} \rfloor} a^{0,j+i_1+i_2}(n-2j-i_1-4). \tag{4}$$

Namely:

$$a^{0,j}(n) = I(j+3=n) + I(\frac{n-5}{2} \geq j) +$$

$$\sum_{i_1=0}^{\lfloor \frac{n-3j-7}{2} \rfloor} \Bigg(I(j+i_1+3=n-j-2) + I(\frac{n-j-2-5}{2} \geq j+i_1)$$

$$+ \sum_{i_2=0}^{\lfloor \frac{n-j-2-3(i_1+j)-7}{2} \rfloor} a^{0,j+i_1+i_2}(n-j-2-j-i_1-2) \Bigg)$$

$$= I(j+3=n) + I(\frac{n-5}{2} \geq j) +$$

$$\sum_{i_1=0}^{\lfloor \frac{n-3j-7}{2} \rfloor} I(2j+i_1+5=n) + \sum_{i_1=0}^{\lfloor \frac{n-3j-7}{2} \rfloor} I(\frac{n-3j-7}{2} \geq i_1) +$$

$$\sum_{i_1=0}^{\lfloor \frac{n-3j-7}{2} \rfloor} \sum_{i_2=0}^{\lfloor \frac{n-4j-3i_1-9}{2} \rfloor} a^{0,j+i_1+i_2}(n-2j-i_1-4).$$

Since $i_1 \leq \frac{n-3j-7}{2}$ and $n-3j-7 \geq 0$, we have $2j+i_1+5 < n$. Thus, we can conclude that $I(2j+i_1+5=n) = 0$ and $I(\frac{n-3j-7}{2} \geq i_1) = 1$.

The sum $\sum_{i_2=0}^{\lfloor \frac{n-4j-3i_1-9}{2} \rfloor} a^{0,j+i_1+i_2}(n-2j-i_1-4)$ exists only for $n-4j-3i_1-9 \geq 0$, implying $i_1 \leq \frac{n-4j-9}{3}$. Hence, the equality (4) follows.

If we continue that way we can prove the next lemma. There, and further on, we use the following notation:

$$q_{n,j,i_1,\dots,i_{p-1}}(l,p) = \lfloor \frac{n-(l+2)j-(2l+5)-\sum_{s=1}^{p-1}(l+2-s)i_s}{l+2-p} \rfloor$$

for $l=0,1,2,3,\dots,\ p=1,2,3,\dots$. For fixed values of $n,\ j,\ i_1,\ \dots,\ i_{p-1}$ we simply write $q(l,p)$ instead of $q_{n,j,i_1,\dots,i_{p-1}}(l,p)$.

Lemma 3.4 *If* $n \geq (k+2)j+2k+5$ *and* $k>0$, *then*

$$\begin{aligned} a^{0,j}(n) &= 1+\sum_{l=1}^{k-1}\sum_{i_1=0}^{q(l,1)}\cdots\sum_{i_l=0}^{q(l,l)}1+ \\ &+\sum_{i_1=0}^{q(k,1)}\cdots\sum_{i_k=0}^{q(k,k)}a^{0,j+i_1+\dots+i_k}\Big(n-k(j+2)-\sum_{s=1}^{k-1}(k-s)i_s\Big). \end{aligned} \tag{5}$$

P r o o f. We use induction on k. For $k=1$ the equality (5) is the equality (3) and for $k=2$ the equality (5) is the equality (4). Let the equality (5) be true for some k and let $n \geq (k+3)j+2(k+1)+5$. Now we apply the equality (3) to the terms $a^{0,j+i_1+\dots+i_k}\Big(n-k(j+2)-\sum_{s=1}^{k-1}(k-s)i_s\Big)$. We obtain

$$\begin{aligned} &a^{0,j+i_1+\dots+i_k}\Big(n-k(j+2)-\textstyle\sum_{s=1}^{k-1}(k-s)i_s\Big) = \\ &= I\Big(j+i_1+\dots+i_k+3 = n-k(j+2)-\textstyle\sum_{s=1}^{k-1}(k-s)i_s\Big)+ \\ &+I\Big(\tfrac{n-k(j+2)-\sum_{s=1}^{k-1}(k-s)i_s-5}{2} \geq j+i_1+\dots+i_k\Big)+ \\ &+\sum_{i_{k+1}=0}^{u}a^{0,j+i_1+\dots+i_k+i_{k+1}}\Big(n-k(j+2)-\textstyle\sum_{s=1}^{k-1}(k-s)i_s-(j+i_1+\dots+i_k)-2\Big) \end{aligned}$$

where

$$u = \lfloor \tfrac{n-k(j+2)-\sum_{s=1}^{k-1}(k-s)i_s-3(j+i_1+\dots+i_k)-7}{2} \rfloor = q(k+1,k+1)$$

and

$$\begin{aligned} &n-k(j+2)-\textstyle\sum_{s=1}^{k-1}(k-s)i_s-(j+i_1+\dots+i_k)-2 = \\ &= n-(k+1)(j+2)-\textstyle\sum_{s=1}^{k}(k+1-s)i_s. \end{aligned}$$

The equality (5) holds under the assumption $n \geq (k+2)j+2k+5$, and it implies that the values $q(k,p) \geq 0$ at least for $i_1=i_2=\dots=i_p=0$ ($p=1,2,\dots,k$). Hence, the sums $\sum_{i_p=0}^{q(k,p)}$ in (5) exist for every $p=1,2,\dots,k$.

If we suppose that $j+i_1+\cdots+i_k+3=n-k(j+2)-\sum_{s=1}^{k-1}(k-s)i_s$ then, by using the inequalities $i_p \le q(k,p)$ for $p=1,2,\ldots,k$, we will obtain $n \le j+3$, contrary to $n \ge (k+2)j+2k+5$. Hence, we have $I\left(j+i_1+\cdots+i_k+3=n-k(j+2)-\sum_{s=1}^{k-1}(k-s)i_s\right)=0$. On the other side, $i_k \le q(k,k)$ implies $I\left(\frac{n-k(j+2)-\sum_{s=1}^{k-1}(k-s)i_s-5}{2} \ge j+i_1+\cdots+i_k\right)=1$.

We obtain

$$a^{0,j}(n)=1+\sum_{l=1}^{k-1}\sum_{i_1=0}^{q(l,1)}\cdots\sum_{i_l=0}^{q(l,l)}1+\sum_{i_1=0}^{q(k,1)}\cdots\sum_{i_k=0}^{q(k,k)}1+$$

$$+\sum_{i_1=0}^{q(k,1)}\cdots\sum_{i_k=0}^{q(k,k)}\sum_{i_{k+1}=0}^{q(k+1,k+1)}a^{0,j+i_1+\cdots+i_k+i_{k+1}}\Big(n-(k+1)(j+2)-$$

$$-\sum_{s=1}^{k}(k+1-s)i_s\Big)=$$

$$=1+\sum_{l=1}^{k}\sum_{i_1=0}^{q(l,1)}\cdots\sum_{i_l=0}^{q(l,l)}1+\sum_{i_1=0}^{q(k,1)}\sum_{i_2=0}^{q(k,2)}\cdots$$

$$\cdots\sum_{i_k=0}^{q(k,k)}\sum_{i_{k+1}=0}^{q(k+1,k+1)}a^{0,j+i_1+\cdots+i_{k+1}}\Big(n-(k+1)(j+2)-\sum_{s=1}^{k}(k+1-s)i_s\Big).$$

The assumption $n \ge (k+3)j+2(k+1)+5$ implies $q(k+1,k+1) \ge 0$ (at least for $i_1=\cdots=i_k=0$). We conclude from $q(k+1,k+1) \ge 0$ that $i_k \le q(k+1,k)$ $(< q(k,k))$ and, continuing that way, we conclude that $i_p \le q(k+1,p)$ $(< q(k,p))$ for each $p=1,2\ldots,k$. ∎

Corollary 3.1. *If* $(k+3)j+2k+7 > n \ge (k+2)j+2k+5$ *and* $k \ge 0$, *then*

$$a^{0,j}(n)=1+\sum_{l=1}^{k}\sum_{i_1=0}^{q(l,1)}\cdots\sum_{i_l=0}^{q(l,l)}1.$$

Proof. If $k>0$ the statement follows from the proof of Lemma 3.4. If $k=0$ then $3j+7>n\ge 2j+5$ and the statement follows by equality (3). ∎

Let n and j be given. If there is a $k \ge 0$ such that $(k+3)j+2k+7 > n \ge (k+2)j+2k+5$ then $k=\lfloor\frac{n-2j-5}{j+2}\rfloor$. Now, by Lemma 3.2 and Corollary 3.1 we have the following lemma.

Lemma 3.5. *For each $n \geq 1$ and $j \geq 0$, the numbers $a^{0,j}(n)$ can be computed as follows. $a^{0,0}(1) = 1$, $a^{0,j}(1) = 0$ for each $j \geq 1$, $a^{0,j}(2) = 0$ for each $j \geq 0$, and if $n \geq 3$, $j \geq 0$, then*

$$a^{0,j}(n) = I(j+3=n) + I(\frac{n-5}{2} \geq j) + \sum_{l=1}^{\lfloor \frac{n-2j-5}{j+2} \rfloor} \sum_{i_1=0}^{q(l,1)} \cdots \sum_{i_l=0}^{q(l,l)} 1.$$

We have from Theorem 2.3 and Lemma 3.1 the following.

Proposition 3.1.

$$p(n) = n + \sum_{k=1}^{n-3} \sum_{i=0}^{n-k-3} \sum_{j=0}^{n-k-i-3} a^{0,i+j}(n-k-i). \tag{6}$$

P r o o f. By Lemma 3.1 we have

$$a^{i,j}(n-k) = \begin{cases} 1, & i=j=0,\ n-k=1 \\ 0, & (i,j) \neq (1,0),\ n-k=2 \\ 1, & i=1,\ j=0,\ n-k=2 \\ 1, & i=n-k-1,\ j=0 \\ a^{0,i+j}(n-k-1), & i+j \leq n-k-3 \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

We now apply the equality (7) in the formula (5):

$$\begin{aligned} p(n) &= 1 + \sum_{i\geq 0, j\geq 0} a^{i,j}(1) + \sum_{i\geq 0, j\geq 0} a^{i,j}(2) + \sum_{k=1}^{n-3} \sum_{i\geq 0, j\geq 0} a^{i,j}(n-k) \\ &= 1 + a^{0,0}(1) + a^{1,0}(2) + \\ &\qquad + \sum_{k=1}^{n-3} \left(a^{n-k-1,0}(n-k) + \sum_{0\leq i+j\leq n-k-3} a^{i,j}(n-k) \right) \\ &= 3 + \sum_{k=1}^{n-3} a^{n-k-1,0}(n-k) + \sum_{k=1}^{n-3} \sum_{i=0}^{n-k-3} \sum_{j=0}^{n-k-i-3} a^{i,j}(n-k) \\ &= 3 + (n-3) + \sum_{k=1}^{n-3} \sum_{i=0}^{n-k-3} \sum_{j=0}^{n-k-i-3} a^{0,i+j}(n-k-i) \\ &= n + \sum_{k=1}^{n-3} \sum_{i=0}^{n-k-3} \sum_{j=0}^{n-k-i-3} a^{0,i+j}(n-k-i). \end{aligned}$$

■

The formula (6) can be transformed by using Lemma 3.5. Namely, since $n-k-i \geq 3$ in the terms $a^{0,i+j}(n-k-i)$ of (6), we have the following:

$$p(n) = n + S_1(n) + S_2(n) + S_3(n),$$

where

$$S_1(n) = \sum_{k=1}^{n-3} \sum_{i=0}^{n-k-3} \sum_{j=0}^{n-k-i-3} I(j = n-k-2i-3),$$

$$S_2(n) = \sum_{k=1}^{n-3} \sum_{i=0}^{n-k-3} \sum_{j=0}^{n-k-i-3} I(\frac{n-k-3i-5}{2} \geq j),$$

$$S_3(n) = \sum_{k=1}^{n-3} \sum_{i=0}^{n-k-3} \sum_{j=0}^{n-k-i-3} \sum_{l=1}^{\lfloor \frac{n-k-3i-2j-5}{i+j+2} \rfloor} \sum_{i_1=0}^{q(l,1)} \cdots \sum_{i_l=0}^{q(l,l)} 1.$$

The sums $S_1(n)$ and $S_2(n)$ can be simplified as follows.

The inequality $0 \leq j \leq n-k-2i-3$ must hold in $S_1(n)$ and it implies $i \leq \frac{n-k-3}{2}$. Then

$$\begin{aligned} S_1(n) &= \sum_{k=1}^{n-3} \sum_{i=0}^{\lfloor \frac{n-k-3}{2} \rfloor} \sum_{j=0}^{n-k-i-3} I(j = n-k-2i-3) \\ &= \sum_{k=1}^{n-3} \sum_{i=0}^{\lfloor \frac{n-k-3}{2} \rfloor} 1 = \sum_{k=1}^{n-3} \Big(\lfloor \frac{n-k-3}{2} \rfloor + 1 \Big) = \lfloor \frac{(n-2)^2}{4} \rfloor. \end{aligned}$$

Because the inequalities $j \leq \dfrac{n-k-3i-5}{2} < n-k-i-3$ and $0 \leq n-k-3i-5$ must hold in the formula for $S_2(n)$, we have:

$$\begin{aligned}
S_2(n) &= \sum_{k=1}^{n-5}\sum_{i=0}^{\lfloor\frac{n-k-5}{3}\rfloor}\sum_{j=0}^{\lfloor\frac{n-k-3i-5}{2}\rfloor} I(\frac{n-k-3i-5}{2}\geq j)\\
&= \sum_{k=1}^{n-5}\sum_{i=0}^{\lfloor\frac{n-k-5}{3}\rfloor}\sum_{j=0}^{\lfloor\frac{n-k-3i-5}{2}\rfloor} 1 = \sum_{k=1}^{n-5}\sum_{i=0}^{\lfloor\frac{n-k-5}{3}\rfloor}\Big(\lfloor\frac{n-k-3i-5}{2}\rfloor+1\Big)\\
&= \sum_{i=0}^{\lfloor\frac{n-5}{3}\rfloor}\sum_{k=1}^{n-3i-4}\lfloor\frac{n-k-3i-3}{2}\rfloor = \sum_{i=0}^{\lfloor\frac{n-5}{3}\rfloor}\lfloor\Big(\frac{n-3i-4}{2}\Big)^2\rfloor.
\end{aligned}$$

For the sum $S_3(n)$ we have that $n-k-3i-2j-5\geq i+j+2$, and that implies $j\leq\lfloor\frac{n-k-4i-7}{3}\rfloor$, $i\leq\lfloor\frac{n-k-7}{4}\rfloor$, $k\leq n-7$. So,

$$S_3(n)=\sum_{k=1}^{n-7}\sum_{i=0}^{\lfloor\frac{n-k-7}{4}\rfloor}\sum_{j=0}^{\lfloor\frac{n-k-4i-7}{3}\rfloor}\sum_{l=1}^{\lfloor\frac{n-k-3i-2j-5}{i+j+2}\rfloor}\sum_{i_1=0}^{q(l,1)}\cdots\sum_{i_l=0}^{q(l,l)}1.$$

After all previous results we have the following explicit formula for computing the partitions numbers.

Theorem 3.1 *The partition numbers $p(n)$, for each $n\geq 1$, are computable by the formula*

$$p(n)=n+\lfloor\frac{(n-2)^2}{4}\rfloor+\sum_{i=0}^{\lfloor\frac{n-5}{3}\rfloor}\lfloor\frac{(n-3i-4)^2}{4}\rfloor+$$

$$+\sum_{k=1}^{n-7}\sum_{i=0}^{\lfloor\frac{n-k-7}{4}\rfloor}\sum_{j=0}^{\lfloor\frac{n-k-4i-7}{3}\rfloor}\sum_{l=1}^{\lfloor\frac{n-k-3i-2j-5}{i+j+2}\rfloor}\sum_{i_1=0}^{q(l,1)}\cdots\sum_{i_l=0}^{q(l,l)}1,$$

where

$$q(l,t)=\lfloor\frac{n-k-i-(l+2)(i+j)-(2l+5)-\sum_{s=1}^{t-1}(l+2-s)i_s}{l+2-t}\rfloor$$

for $t=1,2,\ldots,l$.

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