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Superrings and Supergroups

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Chapter 1

Introduction

1.1 An example

We start with an example. Let M be a smooth manifold, or an algebraic variety. Then the bundle of differential forms generates a bundle of algebras. If ω_1 is a p-form and ω_2 is a q-form, then $\omega_1 \wedge \omega_2$ is a $(p+q)$-form. Thus we can assemble all differential forms into an algebra, and we obtain a bundle $\mathcal{A}(M)$ of algebras over M. Clearly, for any open set $U \subset M$, the sections of $\mathcal{A}(M)$ over U do not form a commutative algebra, but a $\mathbb{Z}_2$-graded commutative algebra: if ω_1 is a p-form and ω_2 is a q-form, then $\omega_1 \wedge \omega_2 = (-1)^{pq}\omega_2 \wedge \omega_1$. The algebra of sections of $\mathcal{A}(M)$ over U thus naturally splits into a direct sum of even rank differential forms and odd rank differential forms. In other words, we have constructed a sheaf $\mathcal{A}_M$ of $\mathbb{Z}_2$-graded commutative algebras on M.

In this thesis we will develop the algebraic machinery to deal with spaces that come with sheaves of $\mathbb{Z}_2$-graded commutative algebras. Such algebras we will call superalgebras.

1.2 Motivation

It is well-known that the elementary particles from the standard model come into two kinds: bosons and fermions. Already a long time ago in 1925, it was noted by Pauli [1] that on the level of quantum mechanics these fermions have to be treated in a rather unusual way, namely by using anticommuting variables (be aware, this is a slight twist of history). That is, fermions were to be described by sections of algebra bundles such that if η, η' are two such sections, then $\eta\eta' = -\eta'\eta$. Since then, many things have changed in physics, but the fermionic variables pursue to exist. Even more, in modern theories as the minimal supersymmetric standard model, or in super string theory, or in M-theory, F-theory, ..., the role of fermionic variables has gained increased interest and importance.

With the venue of mirror symmetry (see [2] for an introduction with a historical overview and references), varieties with additional noncommuting variables became more and more interesting. Even from a purely mathematical point of view, the idea of making spaces with noncommuting coordinates has become popular and seems to make it possible to get deep mathematical results.

We will not even try to give a historical overview of the history of the subject. Partially this is due to the fact that in the Russian literature anticommuting variables were already used before the texts were translated in English. Therefore there is some debate on who was first. The interested reader is referred to [3–7] for historical notes, remarks and lists of references.

The theory of supermanifolds has already been an object of focus in many publications, for example, see [5, 6, 8–14] for a (incomplete) selection of expositions. However, mainly the category of smooth manifolds was considered. Many authors note that much algebra can be translated to get a superversion of a theorem of commutative algebra. However, little steps are taken to fully develop a theory for superalgebras parallel to the theory of commutative algebras. In this thesis we will be very explicit and present all details: a partial goal is to get an overview of which theorems still hold, when replacing commutative by $\mathbb{Z}_2$-graded commutative. For example, it is already clear that the theorem of Cartier, which states that in characteristic zero, any commutative Hopf algebra is reduced, will not hold for a super Hopf algebra.

1.3 Plan

The idea of the thesis was to generalize the Cayley map, which sets up a birational equivalence between a reductive algebraic group and its Lie algebra, to supermanifolds. Then rather quickly we stumbled over the question what a supergroup should be, and how we should view its Lie superalgebra. On the one hand Lie superalgebras are vector spaces over a field with an algebraic structure, on the other hand, in the literature one views supergroups more in a functorial way. Therefore, the connection between a supergroup and its Lie superalgebra cannot be simply a kind of differentiation. The next obstruction was to find enough generalizations of commutative algebra and algebraic geometry to treat supergroups and supervarieties in a satisfying way. This then turned out to be most of the work. Many authors already dealt with generalizations of commutative algebra to the realm of superrings and superalgebras, but mainly on an ad-hoc basis and sometimes even wrongly or unsatisfactorily. Therefore the plan of the dissertation changed more or less to the following task: give structure to commutative algebra for superrings.

In chapter two we deal with super vector spaces, which are in fact no more than vector spaces with a $\mathbb{Z}_2$-grading, and shortly discuss Lie superalgebras. In chapter three we give a fast introduction to the most elementary objects, like superrings and their modules. Prime ideals in superrings are the focus of chapter four, which will be used extensively when we discuss localization and completion of superrings in chapter five, where we also shortly discuss superschemes. We return back to modules of superrings in chapter six and discuss more general notions that can be treated after having developed the machinery in the preceding chapters. In chapter seven we give a rudimentary scheme for dimension theory of superrings. These first chapters are an attempt to try to generalize results of commutative algebra to superrings. We have used and followed the standard works on commutative algebra as for example [15–19].

In chapter eight we return to the question we started with: the relationship with the Lie superalgebra of a supergroup. In the ninth chapter we come to discuss representations of supergroups. For that we need some more knowledge on coalgebras and comodules and their generalizations to the super case. The presentation in chapters eight and nine closely follows the books [20–22]. In the final chapter, we get to the starting point of our quest: we define a Cayley map for supergroups. In order to do so, we first need to address the question what a rational map is for superschemes and group functors.

1.4 Notation and conventions

A note on notation: We fix a field k for the rest of the paper. We assume that the characteristic of k is zero, but most of the claims hold for nonzero characteristic as well. We write $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and denote the elements of $\mathbb{Z}_2$ by $\bar{0}, \bar{1}$. When $x \in \mathbb{Z}_2$ then $(-1)^x$ is 1 if $x = \bar{0}$ and -1 if $x = \bar{1}$. We use $i, j, \dots$ both for indices that take values in $\mathbb{Z}_2$ as for indices that take values in $\mathbb{Z}$. From the context it will be clear what kind of index it is. If A and B are two sets, we write $A - B$ for the set of elements that are in A but not in B. For the concatenation of maps between super vector spaces we use the convention that the symbol $\otimes$ (tensor product) binds stronger than the symbol $\circ$ (concatenation). The end of an example is indicated by the symbol $\triangle$. The end of a proof is indicated by the symbol $\square$.

On nomenclature: When we have defined an object or property and used the prefix super, then afterwards the prefix super will often be omitted. Important exceptions are superring, superalgebra and super vector spaces, which will always be denoted superring, superalgebra and super vector spaces. As an example, the 'super dimension of a superring' will often be denoted 'the dimension of a superring'. Note that the word 'commutative ring' is thus never used to indicate a supercommutative superring. Giving names to objects in 'supermathematics' is for a great deal a matter of taste and we are not aware of any fix rules. Therefore, some inconsistencies in nomenclature seem unavoidable: super Hopf algebra on the one hand, but Lie superalgebra on the other hand, superring as one construction, and super coalgebra as an alternative. When it comes to choosing nomenclature, we have taken seemingly settled conventions and esthetics as guidance.

On occasion we need Zorn's lemma. In some cases we have spelled out how to use the lemma, especially in the first few chapters. In many cases however, we only indicate that the lemma of Zorn is used and do not give the details. The justification lies in the fact that all applications of Zorn's lemma are very similar.

We distinguish betweem homomorphisms and morphisms. Morphisms are the arrows of the category the objects live in. Hence we will speak of group morphisms instead of group homomorphisms. Homomorphisms are only used for super vector spaces and supermodules and need not preserve the $\mathbb{Z}_2$-grading.

We frequently use categorical language and assume the reader has some familiarity with concepts as initial object, terminal object, universal properties, monomorphism, epimorphism, and so on. We refer to [23–25] for explanations on these matters in case our explanation is not sufficient or missing.

Chapter 2

Super vector spaces

In this chapter we define the most basic notions of super mathematics, the super vector spaces. We will then discuss the category of super vector spaces, come to notions as supermatrices, superdeterminant and supertrace. Then we will introduce Lie superalgebras and briefly discuss some classification issues of these. In this chapter we restrict to finite-dimensional (super) vector spaces.

2.1 Super vector spaces

A super vector space over k is a $\mathbb{Z}_2$-graded vector space over k and we write $V = V_{\bar{0}} \oplus V_{\bar{1}}$. The elements of $V_{\bar{0}}$ and $V_{\bar{1}}$ are called even respectively odd. A homogeneous element is an element that is even or odd. For a homogeneous element v we write $|v|$ for the parity; if $v \in V_{\bar{0}}$ (resp. $V_{\bar{1}}$) we have $|v| = \bar{0}$ (resp. $\bar{1}$). A morphism of super vector spaces is a parity preserving map. We write $\mathrm{Hom}(V, W)$ for all k-linear morphisms from V to W. The field k itself is viewed as a super vector space with zero odd part.

For a super vector space V we say that a linear subspace $U \subset V$ is a sub super vector space if U is $\mathbb{Z}_2$-graded, that is, if $U = (U \cap V_{\bar{0}}) \oplus (U \cap V_{\bar{1}})$. In this case, U itself is a super vector space and the inclusion $U \to V$ is a morphism of super vector spaces. The quotient V/U is then a well-defined super vector space with the $\mathbb{Z}_2$-grading $(V/U)_{\bar{0}} = V_{\bar{0}}/U_{\bar{0}}$ and $(V/U)_{\bar{1}} = V_{\bar{1}}/U_{\bar{1}}$.

The category **sVec** of super vector spaces over k is abelian: Cokernels and kernels are automatically $\mathbb{Z}_2$-graded since the morphisms are $\mathbb{Z}_2$-graded. Direct sums and direct products are constructed as for ordinary vector spaces, but with parity preserving maps. The direct sum $V \oplus W$ is $\mathbb{Z}_2$-graded as

$$(V \oplus W)_i = V_i \oplus W_i\,, \quad i \in \mathbb{Z}_2\,. \tag{2.1}$$

The direct product $V \times W$ is given the $\mathbb{Z}_2$-grading $(V \times W)_i = V_i \times W_i$. It is easy to check that the inclusions $V \to V \oplus W$, $W \to V \oplus W$ and the projections $V \oplus W \to V$, $V \oplus W \to W$ preserve the $\mathbb{Z}_2$-grading. The tensor product $V \otimes W$ exists and is $\mathbb{Z}_2$-graded with

$$(V \otimes W)_i = \bigoplus_{j+k=i} V_j \otimes W_k\,, \quad i \in \mathbb{Z}_2\,. \tag{2.2}$$

The dual of a super vector space V is denoted V^* and has the natural $\mathbb{Z}_2$-grading $\omega \in (V^*)_i \Leftrightarrow |\omega(v)| = |v| + i = \bar{0}$.

We have an inner hom-functor: we denote $\underline{\mathrm{Hom}}(V, W)$ the vector space of all k-linear maps from V to W. The space $\underline{\mathrm{Hom}}(V, W)$ is $\mathbb{Z}_2$-graded; the even maps preserve parity, the odd maps change parity. The functor $(V,W) \mapsto \underline{\mathrm{Hom}}(V,W)$ is an endo-bi-functor on the category of super vector spaces. We have

$$\underline{\mathrm{Hom}}(V,W) \cong V^* \otimes W\,. \tag{2.3}$$

We define the functor $\Pi : \mathbf{sVec} \to \mathbf{sVec}$ by putting $(\Pi V)_{\bar 0} = V_{\bar 1}$, $(\Pi V)_{\bar 1} = V_{\bar 0}$ and on morphisms $f : V \to W$ we put $(\Pi f) : v \mapsto f(v)$, where we view v as an element of ΠV and $f(v)$ as an element of ΠW. The functor Π is sometimes called the parity swapping functor. It is easy to see that for any morphism $f : V \to W$ we have $\mathrm{Ker}\,\Pi f \cong \Pi\mathrm{Ker}\, f$ and $\mathrm{Coker}\,\Pi f \cong \Pi\mathrm{Coker}\, f$.

Remark 2.1.1. Using the isomorphism in the category **sVec** given by

$$V \otimes W \to W \otimes V\,, \quad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v\,, \tag{2.4}$$

most of multilinear algebra can be treated from a categorical point of view, see for example for a proof of the Birkhoff–De Witt theorem along these lines in [6].

Let V be a super vector space such that the dimension of V as a vector space is finite. Then we define the super dimension of V to be the pair $p|q$ where p is the dimension of $V_{\bar 0}$ over k and q is the dimension of $V_{\bar 1}$ over k. We will often say dimension of V, when it is clear that V is a super vector space to denote the super dimension of V. The super vector space $k^{p|q}$ is the super vector space with even part k^p and odd part k^q. Choosing a basis of homogeneous elements in V we get an isomorphism $V \cong k^{p|q}$ for some p and q. A standard basis of a super vector space is a basis of homogeneous elements $e_1, \ldots, e_r$, such that in the ordering the even elements precede the odd elements.

We can write any morphism $f : k^{p|q} \to k^{r|s}$ as a block matrix of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \tag{2.5}$$

where A is an $r \times p$-matrix and D an $s \times q$-matrix. Any element of $\underline{\mathrm{Hom}}(V,W)_{\bar 1}$ can be represented by a block matrix of the form

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \tag{2.6}$$

where B is an $r \times q$-matrix and C is an $s \times p$-matrix. The most interesting case, which is also the one we will need later on, is the case where $p = r$ and $q = s$. We write $\mathrm{Mat}_{p|q}(k)$ for the set of all the matrices that represent elements of $\underline{\mathrm{Hom}}(k^{p|q}, k^{p|q})$. The set $\mathrm{Mat}_{p|q}(k)$ is a vector space in the obvious way and is $\mathbb{Z}_2$-graded in the sense discussed above; that is we decompose each $(p+q) \times (p+q)$-matrix as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}_{\bar 0} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix}_{\bar 1} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \tag{2.7}$$

where A is a $p \times p$-matrix, B is a $p \times q$-matrix, C is a $q \times p$-matrix and D is a $q \times q$-matrix. Hence $\mathrm{Mat}_{p|q}(k)$ becomes a super vector space.

Remark 2.1.2. The super vector space $\mathrm{Mat}_{p|q}(k)$ is also an algebra, where multiplication is ordinary matrix multiplication. It is not hard to check that the product of two even matrices is again even, that the product of an even and an odd matrix is odd and that the product of two odd matrices is even. Therefore, as we will see later, $\mathrm{Mat}_{p|q}(k)$ is a superalgebra. However, it is not supercommutative.

We define the supertrace as the map str : $\mathrm{Mat}_{p|q}(k) \to k$ given by

$$\mathrm{str} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \mathrm{tr}A - \mathrm{tr}D\,, \tag{2.8}$$

where tr denotes the ordinary trace. The supertrace is independent of the basis chosen since any two bases are related by an element of $\mathrm{Mat}_{p|q}(k)_{\bar{0}}$. One easily shows that for homogeneous X and Y in $\mathrm{Mat}_{p|q}(k)$ we have

$$\mathrm{str}(XY) = (-1)^{|X||Y|}\mathrm{str}(YX)\,. \tag{2.9}$$

The set $\mathrm{Mat}_{p|q}(k)^*_{\bar{0}}$ of invertible even elements of $\mathrm{Mat}_{p|q}(k)$ forms an algebraic group (in particular it is a variety over k) and we can define a map $b : \mathrm{Mat}_{p|q}(k)^*_{\bar{0}} \to k$ as follows:

$$b : \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mapsto \frac{\det A}{\det D}\,. \tag{2.10}$$

The function b satisfies $b(XY) = b(X)b(Y)$ and is thus a group morphism. It is interesting to note that the induced map on the tangent spaces is the restriction of str to $\mathrm{Mat}_{p|q}(k)_{\bar{0}}$. In chapter 3 we will see that this correspondence can be generalized to the case when the matrix entries take values in a commutative superring. The map b is then called the Berezinian.

We define the super transpose X^{ST} of an element $X \in \mathrm{Mat}_{p|q}(k)$ as follows. First we write X with respect to a standard basis in block form as

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\,, \tag{2.11}$$

with A a $p \times p$-matrix and D a $q \times q$-matrix and then put

$$X^{ST} = \begin{pmatrix} A^T & C^T \\ -B^T & D^T \end{pmatrix}\,, \tag{2.12}$$

where ^{T} denotes the ordinary transpose. One easily shows that

$$(XY)^{ST} = (-1)^{|X||Y|}Y^{ST}X^{ST}\,, \quad \mathrm{str}X^{ST} = \mathrm{str}X\,. \tag{2.13}$$

Remark 2.1.3. For the remainder of the paper, if we use parity assignments in formulas, we mean that the formula holds as given for homogeneous elements and is extended to arbitrary elements by linearity. If we write subscripts $\bar{0}, \bar{1}$ we mean a decomposition into even and odd parts. Thus for example, if V is a super vector space and $v \in V$ then we write $v = v_{\bar{0}} + v_{\bar{1}}$, where $v_{\bar{0}} \in V_{\bar{0}}$ and $v_{\bar{1}} \in V_{\bar{1}}$. In addition, when we decompose matrices in block form, this will always be done with respect to a standard basis.

2.2 Lie superalgebras

Definition 2.2.1. *A Lie superalgebra is a super vector space $\mathfrak{g}$ together with an operation $[,]$: $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ that preserves the $\mathbb{Z}_2$-grading and satisfies:*

(i) $[x, y] + (-1)^{|x||y|}[y, x] = 0$,

(ii) $(-1)^{|x||z|}[[x, y], z] + (-1)^{|y||x|}[[y, z], x] + (-1)^{|y||z|}[[z, x], y] = 0$.

The requirement (ii) is often called the super Jacobi identity The operation $[,]$ we call the Lie bracket, although some people prefer the name super Lie bracket. A morphism of Lie superalgebras is a super vector space morphism that preserves the Lie bracket; if $\mathfrak{g}$ and $\mathfrak{h}$ are Lie superalgebras, a morphism is a linear $\mathbb{Z}_2$-grading preserving map $f : \mathfrak{g} \to \mathfrak{h}$ satisfying $[f(x), f(y)] = f([x, y])$.

Suppose that $\mathfrak{g}$ is a Lie superalgebra. Then from the definition it follows that $\mathfrak{g}_{\bar{0}}$ is an ordinary Lie algebra and that $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-representation.

Now we discuss some basic examples. In fact, we have already seen some examples of Lie superalgebras: The super vector space $\mathrm{Mat}_{p|q}(k)$ can be equipped with a Lie superalgebra structure, by defining

$$[X, Y] = XY - (-1)^{|X||Y|} YX\,. \tag{2.14}$$

It is easily checked that this makes $\mathrm{Mat}_{p|q}(k)$ into a Lie superalgebra and the obtained Lie superalgebra is denoted $\mathfrak{gl}_{p|q}(k)$. In this case the Lie bracket is also called the super commutator. For any super vector space V we write $\mathfrak{gl}_V$ for the Lie superalgebra of all linear maps $V \to V$ equipped with the commutator. Another example is the kernel of the map $\mathrm{str} : \mathrm{Mat}_{p|q}(k) \to k$; if $X, Y \in \mathrm{Mat}_{p|q}(k)$ have super trace zero, then so does $XY - (-1)^{|X||Y|} YX$ by eqn.(2.9). Thus with the same Lie bracket as $\mathfrak{gl}_{p|q}$ we can make the super vector space of all supertrace zero $(p+q) \times (p+q)$-matrices into a Lie superalgebra, which is denoted by $\mathfrak{sl}_{p|q}(k)$. We now give an example, which we haven't seen yet. Consider the $(p+2q) \times (p+2q)$-matrix Ω defined by

$$\Omega = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & J_q \end{pmatrix}\,, \qquad J_q = \begin{pmatrix} 0 & -\mathbb{1}_q \\ \mathbb{1}_q & 0 \end{pmatrix}\,, \tag{2.15}$$

where for any natural number m, $\mathbb{1}_m$ denotes the $m \times m$ identity matrix. We now define the orthosymplectic Lie superalgebra $\mathfrak{osp}_{p|2q}(k)$ as the super vector space of $(p+2q) \times (p+2q)$-matrices X satisfying $X^{ST}\Omega + \Omega X = 0$ and with the Lie bracket the super commutator of $\mathrm{Mat}_{p|2q}(k)$. The inclusion $\mathfrak{osp}_{p|2q}(k) \to \mathfrak{sl}_{p|2q}(k)$ is a Lie superalgebra morphism.

Many notions that are defined for Lie algebras can also be defined for Lie superalgebras and many well-known results for Lie algebras apply as well to Lie superalgebras. In the next paragraph we explain the classification of the simple Lie superalgebras. We do not prove any result, but refer to the literature where the proof can be found.

An ideal in a Lie superalgebra $\mathfrak{g}$ is a sub super vector space $L \subset \mathfrak{g}$ such that $[\mathfrak{g}, L] \subset L$, or equivalently $[L, \mathfrak{g}] \subset L$. In particular, an ideal is a Lie sub superalgebra. We say a Lie superalgebra is solvable if the following series terminates in a finite number of steps:

$$\mathfrak{g}^0 = \mathfrak{g}\,, \qquad \mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k]\,. \tag{2.16}$$

A Lie superalgebra is called nilpotent if the following series terminates in a finite number of steps:

$$\mathfrak{g}^0 = \mathfrak{g}\,, \qquad \mathfrak{g}^{k+1} = [\mathfrak{g}, \mathfrak{g}^k]\,. \tag{2.17}$$

A Lie superalgebra is called semi-simple if it contains no nontrivial solvable ideals.

There exists a version of Engel's theorem for Lie superalgebras:

Lemma 2.2.2. *Let $\mathfrak{g}$ be a sub Lie superalgebra of $\mathfrak{gl}_V$ and suppose all elements of $\mathfrak{g}$ are nilpotent operators on V, then there is a nonzero $v \in V$ with $\mathfrak{g} \cdot v = 0$.*

By considering homogeneous elements in $\mathfrak{g}$, we see that we can take v to be homogeneous. The proof of lemma 2.2.2 proceeds as in the case for ordinary Lie algebras, see for example [26, 27].

A representation of a Lie superalgebra $\mathfrak{g}$ is a Lie superalgebra morphism $\rho : \mathfrak{g} \to \mathfrak{gl}_V$ for some super vector space V. In that case we call V a $\mathfrak{g}$-module. A $\mathfrak{g}$-submodule is a sub super vector space

$W \subset V$ such that $\rho(\mathfrak{g})W \subset W$. The representation ρ is completely reducible if for any submodule W there is a complement to W in V that is also a submodule. We say a representation is irreducible if there are no nontrivial submodules. We note that Ado's theorem holds for Lie superalgebra; any finite-dimensional Lie superalgebra can be embedded into $\mathfrak{gl}_V$ for some finite-dimensional super vector space V when the characteristic of k is not 2 [28].

The adjoint representation is given by $x \mapsto \mathrm{ad}_x \in \mathfrak{gl}(\mathfrak{g})$, where ad_x is the linear map sending $y \in \mathfrak{g}$ to $[x, y]$. We call a Lie superalgebra simple if $\mathfrak{g}$ is irreducible as $\mathfrak{g}$-module with the adjoint representation. Equivalently, $\mathfrak{g}$ contains no nontrivial ideals. We present a criterion for simplicity of a Lie superalgebra:

Lemma 2.2.3. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a simple Lie superalgebra. Then*

(i) The representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is faithful.

(ii) $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{0}}$.

Conversely, if (i) and (ii) hold and the representation of $\mathfrak{g}_{\bar{0}}$ in $\mathfrak{g}_{\bar{1}}$ is irreducible, then $\mathfrak{g}$ is simple.

The proof of lemma 2.2.3 can be found in [29]. There is a variation of Schur's lemma for Lie superalgebras, for details see for example [29–31]:

Lemma 2.2.4. *Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a super vector space over an algebraically closed field, let $\mathfrak{g}$ be a simple sub Lie superalgebra of $\mathfrak{gl}_V$ and define $C(\mathfrak{g}) = \{a \in \mathfrak{gl}_V | [a, x] = 0\,, \forall\, x \in \mathfrak{g}\}$. Then we have*

(i) $C(\mathfrak{g})$ consists of all multiples of the identity,

(ii) and when $\mathrm{dim} V_{\bar{0}} = \mathrm{dim} V_{\bar{1}}$ then $C(\mathfrak{g})$ consists of the subalgebra of $\mathfrak{gl}_V$ generated by the identity and some linear operator that interchanges $V_{\bar{0}}$ and $V_{\bar{1}}$.

The Cartan–Killing form is the bilinear form $(x, y) \mapsto \mathrm{str}(\mathrm{ad}_x \circ \mathrm{ad}_y)$. In general, it does not hold that the Cartan–Killing form is nondegenerate for semisimple Lie superalgebras. In the classification of simple Lie superalgebras, there appear simple Lie superalgebras that have a Cartan–Killing form that is identically zero. In fact, if $\mathfrak{g}$ is simple, then either the Cartan–Killing form is nondegenerate, or is identically zero.

We call a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ of classical type if (i) $\mathfrak{g}$ is simple and (ii) the representation of $\mathfrak{g}_{\bar{0}}$ in $\mathfrak{g}_{\bar{1}}$ is completely reducible. The classical Lie superalgebras are classified by Kac [32] (also see for example [5,29–31,33–35]). All finite-dimensional simple Lie superalgebras over $\mathbb{C}$ are isomorphic to one of the following:

(i) $A(m, n) = \mathfrak{sl}_{m+1|n+1}(\mathbb{C})$, possibly with substraction of the center if $m = n$.

(ii) $B(m, n) = \mathfrak{osp}_{2m+1,2n}(\mathbb{C})$.

(iii) $C(n) = \mathfrak{osp}_{2,2n}(\mathbb{C})$.

(iv) $D(m, n) = \mathfrak{osp}_{2m,2n}(\mathbb{C})$, with $m > 1$.

(v) $D(2, 1; \alpha)$ for some $\alpha \neq 0, -1$ in $\mathbb{C}$, for a description see [29, 31].

(vi) $F(4)$, for a description see [29, 31].

(vii) $G(3)$, for a description see [29, 31].

(viii) $P(n)$ for $n > 2$; this Lie superalgebra is defined as all $(n+n)\times(n+n)$-matrices in $\mathrm{Mat}_{n|n}(\mathbb{C})$ of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, \quad \text{with} \;\; \operatorname{tr} a = 0\,, b^T = b\,, c^T = -c^T\,. \tag{2.18}$$

(ix) $Q(n)$; this Lie superalgebra is defined as follows. We first define the sub Lie superalgebra $\widetilde{Q(n)}$ of $\mathfrak{gl}_{n+1|n+1}(\mathbb{C})$ formed by $(n+1)\times(n+1)$-matrices of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\,, \quad \text{with} \;\; \operatorname{tr} b = 0\,. \tag{2.19}$$

Then we put $Q(n) = \widetilde{Q(n)}/\mathbb{C}\mathbb{1}_{n+1}$.

The Lie superalgebras $F(4)$, $G(3)$ and $D(2,1;\alpha)$ are rather mysterious exceptional Lie superalgebras. The most remarkable feature of $D(2,1;\alpha)$ is that this is a continuous series of simple Lie superalgebras, a counterpart of which does not exist in the world of Lie algebras. For the classification of real simple Lie superalgebras we refer to [36, 37].

Remark 2.2.5. Lie superalgebras made their first appearance in mathematics in the works of Fröhlicher & Nijenhuis [38], Gerstenhaber [39,40], Nijenhuis & Richardson [41] and Milnor & Moore [42] around 1960 in the context of deformation theory, topology and cohomology theories. Hopf and Steenrod used commutative superalgebras for different algebraic structures in cohomology groups. A little later around 1970 physicists discovered Lie superalgebras independently in their investigations on supersymmetry; pioneers were Gol'fand and Likhtman with their important paper [43], Miyazawa [44], Volkov and Akulov [45], and Wess and Zumino [46, 47].

For more historical notes and references, see for example [4, 5, 48, 49].

Chapter 3

Basics of superrings and supermodules

In this chapter we introduce the concepts of superrings and supermodules. The chapter will serve as a basis for the following chapters. Since important tools such as localization and completion, but also important results on prime ideals, will be discussed in the following chapters, some notions are only shortly discussed and await a further treatment later. Many proofs are similar to the proofs from ordinary commutative algebra. For completeness and ease of reading, we incorporated them in the present text.

3.1 Superrings and superalgebras

Definition 3.1.1. *A superring A is a $\mathbb{Z}_2$-graded ring $A = A_{\bar{0}} \oplus A_{\bar{1}}$ such that the product map $A \times A \to A$ satisfies $A_i A_j \subset A_{i+j}$. A morphism of superrings is a $\mathbb{Z}_2$-grading preserving morphism of rings. The elements of $A_{\bar{0}}$ are called even, the elements of $A_{\bar{1}}$ are called odd and an element that is either even or odd is said to be homogeneous.*

We always assume that a superring has a multiplicative unit element 1 and is associative. Morphisms $f : A \to B$ map the unit element of A to the unit element of B. Furthermore we only consider commutative superrings; we call a superring commutative if

$$ab - (-1)^{|a||b|} ba = 0\,, \qquad \text{for all } a, b \in A_{\bar{0}} \cup A_{\bar{1}}\,. \tag{3.1}$$

Note that in particular $a^2 = 0$ if $a \in A_{\bar{1}}$. We write **sRng** for the category of superrings that are associative, unital and commutative.

Lemma 3.1.2. *Let A be a (not necessarily commutative or associative) superring with unit element, then $1 \in A_{\bar{0}}$.*

Proof. Write $1 = e_{\bar{0}} + e_{\bar{1}}$ for the decomposition of 1 into homogeneous components. We have $e_{\bar{0}} 1 = e_{\bar{0}}$ and thus $e_{\bar{0}} = (e_{\bar{0}})^2 + e_{\bar{0}} e_{\bar{1}}$, but $e_{\bar{0}} e_{\bar{1}}$ is odd whereas $e_{\bar{0}}$ and $e_{\bar{0}} e_{\bar{0}}$ are even. Hence $e_{\bar{0}} e_{\bar{1}} = 0$, from which it follows that $e_{\bar{1}} = 1 e_{\bar{1}} = e_{\bar{0}} e_{\bar{1}} + e_{\bar{1}} e_{\bar{1}} = e_{\bar{1}} e_{\bar{1}}$. But since $e_{\bar{1}}$ is odd and $e_{\bar{1}} e_{\bar{1}}$ is even, we must have $e_{\bar{1}} = 0$ and $1 = e_{\bar{0}} \in A_{\bar{0}}$. ∎

Lemma 3.1.3. *Let A be a superring, then any idempotent lies in $A_{\bar{0}}$.*

Proof. Let $e_{\bar 0}+e_{\bar 1}$ be an idempotent, then $(e_{\bar 0})^2 = e_{\bar 0}$ and $2e_{\bar 0}e_{\bar 1} = e_{\bar 1}$. Multiplying the last equation with $e_{\bar 0}$ we get $2(e_{\bar 0})^2e_{\bar 1} = 2e_{\bar 0}e_{\bar 1} = e_{\bar 0}e_{\bar 1}$, hence $e_{\bar 1} = 2e_{\bar 0}e_{\bar 1} = 0$. □

A sub superring I of A, such that for all $a \in A$ and $x \in I$ we have $xa \in A$ and $ax \in A$, is called an ideal of A. A $\mathbb{Z}_2$-graded ideal I of A is a subring of A such that (1) $ax \in I$ for all $x \in I, a \in A$ and (2) $I = (I \cap A_{\bar 0}) \oplus (I \cap A_{\bar 1})$. Point (2) means that if x lies in I, then also the homogeneous components. It then follows that for all $x \in I, a \in A$ we have $xa \in I$. Given a $\mathbb{Z}_2$-graded ideal I in A, we define the quotient to be the superring $A/I = (A_{\bar 0}/I_{\bar 0}) \oplus (A_{\bar 1}/I_{\bar 1})$. If S is a set of homogeneous elements, we write (S) for the ideal generated by the elements of S. Thus (S) contains all elements of the form $\sum_{m \in M} a_m s_m$ with $a_m \in A$ and $s_m \in S$ and M a finite set. If S consists of elements $f_1, \dots, f_r$ we write $(S) = (f_1, \dots, f_r)$.

Every superring A comes with a canonical ideal J_A, which is defined as the ideal generated by the odd elements - and is thus automatically $\mathbb{Z}_2$-graded. The quotient A/J_A is called the body and denoted $\bar{A}$ and the image of $a \in A$ under the projection $A \to A/J_A$ is denoted $\bar{a}$. Any element $x \in J_A$ is a finite sum $\sum_{i=1}^m a_i b_i$ with $a_i \in A$ and $b_i \in A_{\bar 1}$. Then $x^{m+1} = 0$ and thus J_A consists of nilpotent elements.

The set of nilpotent elements of A is a $\mathbb{Z}_2$-graded ideal; it is an ideal since the sum of two nilpotents is nilpotent and since the product of any element with a nilpotent is again nilpotent and it is $\mathbb{Z}_2$-graded since all elements of $A_{\bar 1}$ are nilpotent. We denote the ideal of nilpotent elements of A by $\mathrm{Nilrad}(A)$ and call it the nilradical of A. Clearly we have $J_A \subset \mathrm{Nilrad}(A)$. When $x \in \mathrm{Nilrad}(A)$, then $1-x$ is invertible with the inverse given by $1+x+x^2+\dots+x^n$ with n so large that $x^{n+1}=0$. If a superring A is such that $\bar{A}$ contains no nilpotents, or equivalently if $\mathrm{Nilrad}(A) = J_A$, we call A reduced. If $\bar{A}$ is an integral domain, we call A a super domain.

For an element $a \in A$ we define $\mathrm{Ann}(a)$ to be the ideal of all elements $b \in A$ such that $ba = 0$ and we call $\mathrm{Ann}(a)$ the annihilator of a. If a is not $\mathbb{Z}_2$-graded, it is not guaranteed that $\mathrm{Ann}(a)$ is a $\mathbb{Z}_2$-graded ideal; if a is homogeneous, then $\mathrm{Ann}(a)$ is $\mathbb{Z}_2$-graded. We therefore avoid the use of the notation $\mathrm{Ann}(a)$ for inhomogeneous a. The elements of $\mathrm{Ann}(a)$ are, by the usual abuse of language, also called annihilators of a.

A zerodivisor is a nonzero element x in A such that there exists a nonzero $y \in A$ with $xy = 0$. The set of zerodivisors do not form an ideal in general. Let us describe the set D of zerodivisors in a superring A. It is clear that all odd elements are in D, $A_{\bar 1} \subset D$. If x is any even element of D, and θ is any odd element, then we claim that $x+\theta$ is in D. Indeed, suppose $xy = 0$ for some nonzero y. Then we can take y to be homogeneous. If $y\theta = 0$, then $y(x+\theta) = 0$ and $x+\theta \in D$. If $y\theta \neq 0$, then $y\theta(x+\theta) = 0$, and again $x+\theta \in D$.

Now let $x+\theta$ be any element in D, then $(x+\theta)(y+\eta) = 0$ for some nonzero $y+\eta$. Written out this means $xy+\theta\eta = 0$ and $x\eta + y\theta = 0$. If $\theta\eta = 0$, then $x \in D$. If $\theta\eta \neq 0$, then $x(\theta\eta) = 0$ implies that $x \in D$. Hence, in any case, $x \in D$. Therefore we conclude that the set D of zerodivisors of a superring A is the set of elements $x+\theta$, with x even and θ odd, such that not both x and θ are zero and such that when x is nonzero, it is the annihilator of some homogeneous element of A and θ is arbitrary. If $x = 0$, then θ is any nonzero odd element.

Rehearsing the discussion in the preceding paragraph we obtain:

Corollary 3.1.4. *The set D of zerodivisors is given by*

$$D = \bigcup_{z \neq 0 \text{ homogeneous}} \mathrm{Ann}(z)\,.$$

Proof. For any homogeneous element in D we are done. If $x+\theta \in D$ is not homogeneous, then x annihilates an even element z. If $\theta z = 0$, then $x+\theta$ annihilates z, and if $\theta z \neq 0$, then $x+\theta$ annihilates $z\theta$. But both z and $z\theta$ are even. □

Definition 3.1.5. *We call an ideal* $\mathfrak{m}$ *of a superring* A *a maximal ideal if* $\mathfrak{m}$ *is not properly contained in any other ideal and* $\mathfrak{m}$ *is properly contained in* A*.*

We have not included the requirement that a maximal ideal is $\mathbb{Z}_2$-graded in the definition, since it follows that any maximal ideal is $\mathbb{Z}_2$-graded:

Lemma 3.1.6. *Let* A *be a superring, then any maximal ideal is automatically* $\mathbb{Z}_2$*-graded and contains* J_A*.*

Proof. Let $\mathfrak{m}$ be a maximal ideal and $a \in A_{\bar{1}}$ and consider the ideal $\mathfrak{m}'$ generated by a and $\mathfrak{m}$. If $a \notin \mathfrak{m}$, then $\mathfrak{m}$ is properly contained in $\mathfrak{m}'$ and thus $\mathfrak{m}' = A$ and it follows that there are $m \in \mathfrak{m}$ and $b \in A$ such that $m + ba = 1$, but $1 - ba$ is invertible. Hence $\mathfrak{m} = A$, which is a contradiction to $\mathfrak{m}$ being a maximal ideal. Hence a maximal ideal contains all odd elements and thus J_A. □

As the maximal ideals in a superring contain all the odd elements, the quotient $A/\mathfrak{m}$ does behave as if A were a commutative ring:

Lemma 3.1.7. *Let* A *be a superring. An ideal* $\mathfrak{m}$ *of* A *is a maximal ideal of* A *if and only if* $A/\mathfrak{m}$ *if a field.*

Proof. If $A/\mathfrak{m}$ is a field, then $A_{\bar{1}} \subset \mathfrak{m}$. If $\mathfrak{m} \subset \mathfrak{m}'$ then every element m' of $\mathfrak{m}' - \mathfrak{m}$ has an invertible image in $A/\mathfrak{m}$ and we may assume m' to be even. This means $am' + m = 1$ for some $a \in A_{\bar{0}}$ and $m \in \mathfrak{m}_{\bar{0}}$ implying $\mathfrak{m}' = A$. On the other hand, if $\mathfrak{m}$ is maximal and $x \in A/\mathfrak{m}$ is nonzero then choose an even preimage y of x in A. The ideal generated by y and $\mathfrak{m}$ equals A and thus there are $a \in A_{\bar{0}}$ and $m \in \mathfrak{m}_{\bar{0}}$, such that $ay + m = 1$ and thus x is invertible. □

We already noted that all elements of the form $1 + y$ with $y \in J_A$ are invertible. The following lemma characterizes the invertible elements:

Lemma 3.1.8. *Let* A *be a superring, then the following are equivalent:* (i) $a \in A$ *is invertible (has a left and a right inverse),* (ii) $a_{\bar{0}}$ *is invertible in* $A_{\bar{0}}$*,* (iii) $\bar{a}$ *is invertible in* $\bar{A}$*.*

Proof. Let a in A be invertible and let $y \in A$ be an inverse; $ya = ay = 1$ (since A is associative, if an element has a right inverse and a left inverse then they are equal). Then applying the projection $A \to \bar{A}$ it is clear that $\bar{a}$ is invertible in $\bar{A}$. From $ay = 1$ we have $a_{\bar{1}}y_{\bar{0}} + a_{\bar{0}}y_{\bar{1}} = 0$ and $a_{\bar{0}}y_{\bar{0}} = 1 - w$, with $w = a_{\bar{1}}y_{\bar{1}}$. The element w is nilpotent, and hence $1 - w$ is invertible and the inverse lies in $A_{\bar{0}}$ and thus $a_{\bar{0}}y_{\bar{0}}(1 - w)^{-1} = 1$, showing that $a_{\bar{0}}$ is invertible in $A_{\bar{0}}$. Conversely, consider $a \in A$. If $a_{\bar{0}}$ is invertible in $A_{\bar{0}}$, then there is $b \in A_{\bar{0}}$ such that $a_{\bar{0}}b = ba_{\bar{0}} = 1$ and thus $ab = 1 + w$ with $w = a_{\bar{1}}b$ a nilpotent element and thus $1 + w$ is invertible. If $\bar{a}$ is invertible in $\bar{A}$, then there is $b \in A$ such that $\bar{a}\bar{b} = \bar{b}\bar{a} = 1$ and thus $ab = 1 - w$ and $ba = 1 - w'$ with $w, w' \in J_A$ and thus a has a left and a right inverse. □

Proposition 3.1.9. *Let* A *be a superring and let* $\mathfrak{m}$ *be a* $\mathbb{Z}_2$*-graded ideal in* A *of the form* $\mathfrak{m} = \mathfrak{m}_{\bar{0}} \oplus A_{\bar{1}}$*. Then* $\mathfrak{m}$ *is a maximal ideal in* A *if and only if* $\mathfrak{m}_{\bar{0}}$ *is a maximal ideal in* $A_{\bar{0}}$*.*

Proof. The quotient $A/\mathfrak{m}$ is a field if and only if $A_{\bar{0}}/\mathfrak{m}_{\bar{0}}$ is a field, since $A/\mathfrak{m} \cong A_{\bar{0}}/\mathfrak{m}_{\bar{0}}$. □

We now show that the projection $A \to \bar{A}$ can be seen as a functor and we give an adjoint to this functor. Let **Rng** denote the category of commutative, associative rings with unit. Define the functor $\mathcal{S} : \mathbf{Rng} \to \mathbf{sRng}$ on objects as $R \mapsto \mathcal{S}(R)$, with $\mathcal{S}(R)_{\bar{0}} = R$ and $\mathcal{S}(R)_{\bar{1}} = 0$. On morphisms $f : R \to R'$ we put $\mathcal{S}(f) : r \mapsto f(r)$. Thus $\mathcal{S}$ does nothing more than considering the objects as superrings with zero odd part.

On the other hand, there are two obvious ways to make a commutative ring from a superring A; we can take $\bar{A}$ and $A_{\bar{0}}$. As we will see, the choice $\bar{A}$ is the more natural of the two. If $f : A \to B$ is a morphism of superrings then $f(J_A) \subset J_B$ and hence we can define $\bar{f} : \bar{A} \to \bar{B}$ to be the induced morphism with $\bar{f}(\bar{a}) = \overline{f(a)}$ for all $a \in A$. Thus the diagram below commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\pi_A"'] & B \arrow[d, "\pi_B"] \\ \bar{A} \arrow[r, "\bar{f}"] & \bar{B} \end{tikzcd} , \tag{3.2}$$

where $\pi_A : A \to A$ and $\pi_B : B \to \bar{B}$ are the canonical projections. We write $\mathcal{B}$ for the functor $\mathcal{B}$: **sRng** $\to$ **Rng** that assigns to each superalgebra A the body $\bar{A}$ and each morphism f the induced morphism $\bar{f}$.

Proposition 3.1.10. *The functor $\mathcal{S}$:* **Rng** $\to$ **sRng** *is right-adjoint to the functor $\mathcal{B}$:* **sRng** $\to$ **Rng**.

Proof. Let A be a superring and R a commutative ring. Then $f \in \mathrm{Hom}_{\mathbf{sRng}}(A, \mathcal{S}(R))$ has to factor over J_A since all odd elements of A need to be mapped to zero. Thus there is a unique morphism $\bar{f} : \bar{A} \to R$ such that $f = \bar{f} \circ \pi$, which we can view as a morphism in **Rng**. Conversely, given a morphism in $g \in \mathrm{Hom}_{\mathbf{Rng}}(\mathcal{B}(A), R)$, then by composition with the projection $A \to \bar{A} = \mathcal{B}(A)$ we obtain a morphism from A to $\mathcal{S}(R)$ in **sRng**. Hence $\mathrm{Hom}_{\mathbf{sRng}}(A, S(R)) \cong \mathrm{Hom}_{\mathbf{Rng}}(\mathcal{B}(A), R)$. Using the commutativity of diagram (3.2), naturality is obvious. $\square$

Definition 3.1.11. *A superalgebra over k is a super vector space over k with a k-bilinear map $A \otimes A \to A$ such that the image of $A_i \otimes A_j$ lies in A_{i+j}.*

A superalgebra over k is thus a superring with the extra structure of being a super vector space over k with a compatible $\mathbb{Z}_2$-grading; that is, the $\mathbb{Z}_2$-grading as a superring and as a super vector space coincide. A morphism of superalgebras is a morphism of superrings that is k-linear. We denote **sAlg** the category of superalgebras over k. The notion of a body carries over to superalgebras. We call a superalgebra commutative if it is commutative as a superring. Unless otherwise specified, all the superalgebras that we consider are commutative, associative and have a unit element 1. The tensor product $A \otimes_k B$ of superalgebras A and B is as a super vector space defined as in the category of super vector spaces and equipped with the product $a \otimes b \cdot a' \otimes b' = (-1)^{|a'||b|} aa' \otimes bb'$.

The polynomial superalgebra (over k) in n even variables X_i, $1 \leq i \leq n$, and m odd variables Θ_α, $1 \leq \alpha \leq m$, is defined to be the algebra over k generated by the X_i and Θ_α subject to the relations $X_i X_j = X_j X_i$, $X_i \Theta_\alpha = \Theta_\alpha X_i$ and $\Theta_\alpha \Theta_\beta = -\Theta_\beta \Theta_\alpha$ for all i, j, α, β. This algebra is denoted $k[X_1, \ldots, X_n | \Theta_1, \ldots, \Theta_m]$. We have $\overline{k[X_1, \ldots, X_n | \Theta_1, \ldots, \Theta_m]} \cong k[X_1, \ldots, X_n]$.

Let V be a super vector space over k and let $T(V) = \bigoplus_{k \geq 0} V^{\otimes k}$ be its tensor superalgebra. As a superalgebra $T(V)$ is generated by 1 and all $v \in V$. The multiplication is defined by the tensor product: the product $v \cdot w$ for $v \in V^{\otimes n}$ and $w \in V^{\otimes m}$ is $v \otimes w$. We call I_V the $\mathbb{Z}_2$-graded ideal generated by all elements of the form $v \otimes w - (-1)^{|v||w|} w \otimes v$, where v, w run over all homogeneous elements of V. We define the symmetric superalgebra over V as the quotient $S(V) = T(V)/I_V$. Furthermore, we call $k[V] = S(V^*)$ the polynomial superalgebra of V. If V is finite-dimensional, then there is a noncanonical isomorphism $k[V] \cong S(V)$ and $k[V]$ is a polynomial superalgebra.

We say a superalgebra A is finitely generated if there exist finitely many homogeneous elements $a_1, \ldots, a_t$ such that any element x in A can be expressed as a polynomial

$$x = \sum_{i_1, \ldots, i_t} c_{i_1 \cdots i_t} a_1^{i_1} \cdots a_t^{i_t} , \tag{3.3}$$

such that only finitely many coefficients $c_{i_1 \cdots i_t}$ are nonzero. In other words, a superalgebra A is finitely generated if and only if there exists a surjective morphism of superalgebras $P \to A$ where P is a polynomial superalgebra.

3.2 Supermodules

On occasion it is convenient to use the notion of $\mathbb{Z}_2$-graded abelian groups. We call an abelian group G a $\mathbb{Z}_2$-graded abelian group if G is a direct sum $G_{\bar{0}} \oplus G_{\bar{1}}$. The elements of $G_{\bar{0}}$ are labeled as even elements whereas the elements of $G_{\bar{1}}$ are labeled as odd elements.

Definition 3.2.1. *Let A be a superring and let M be a $\mathbb{Z}_2$-graded abelian group $M = \bigoplus_{i\in\mathbb{Z}_2} M_i$. We call M a left A-module if M is a left A-module in the usual sense with the additional requirement that the structure morphism $l : A \times M \to M$ satisfies $A_i \times M_j \subset M_{i+j}$.*

We almost always write am for $l(a, m)$, except when clearness is at risk. A right A-module is defined in a similar way; again the only difference from the usual concept of a right A-module is that the structure morphism $r : M \times A \to M$ respects the $\mathbb{Z}_2$-grading. For a commutative superring A, every left A-module M with structure morphism $l : A \times M \to M$ admits a canonical right A-module structure. We define the structure morphism $r : M \times A \to M$ by

$$r(m, a) = (-1)^{|a||m|} l(a, m)\,, \quad m \in M\,, a \in A\,. \tag{3.4}$$

When we define the right action of A as in eqn.(3.4), it commutes with the left action: $r(l(a, m), b) = l(a, r(m, b))$ for all $a, b \in A$ and $m \in M$. Therefore we can unambiguously write amb for $l(a, r(m, b))$. If M is a left A-module equipped with the compatible right action of A just described, we call M an A-module. A submodule N of an A-module M is a submodule N of M in the usual sense with the requirement $N = (N \cap M_{\bar{0}}) \oplus (N \cap M_{\bar{1}})$, that is, if $n \in N$ then the homogeneous components of n also lie in N. If N is a submodule of M, the quotient module M/N is defined by $(M/N)_i = M_i/N_i$ for $i = \bar{0}, \bar{1}$ and the right action is given by $r(m \bmod N, a) = ma \bmod N$.

Let M be an A-module. We call a proper submodule $N \subset M$ a maximal submodule if the only submodule of M that properly contains N is M itself. A necessary condition that N is a maximal submodule of M is that either $N_{\bar{0}} = M_{\bar{0}}$ or $N_{\bar{1}} = M_{\bar{1}}$. Hence M/N has either a trivial even part, or a trivial odd part, which implies that $A_{\bar{1}}$ has to act trivially on M/N and thus M/N is in a natural way a $\bar{A}$-module.

We define morphisms of A-modules to be parity-preserving maps that commute with the action of A. Since a morphism f preserves parity, when f commutes with the right action of A it also commutes with the left action. (In section 3.7 and in chapter 6 we will also consider odd A-linear maps; then we define A-linearity as commuting with the right action of A.) We call two A-modules M and N isomorphic if there are morphisms of A-modules $f : M \to N$ and $g : N \to M$ such that $f \circ g = \mathrm{id}_N$ and $g \circ f = \mathrm{id}_M$. One easily checks that a morphism $f : M \to N$ is an isomorphism if and only if $\mathrm{Ker}\, f = 0$ and $f(M) = N$.

Direct sums and direct products of A-modules are defined in the usual way; for two A-modules M and N

$$(M \oplus N)_i = M_i \oplus N_i\,, \quad (M \times N)_i = M_i \times N_i\,. \tag{3.5}$$

For the tensor product we have to be a bit more careful. As usual, the tensor product can be defined by its universal property, see for example [50]. We construct the module $M \otimes_A N$ as follows: let $M \boxtimes N$ be the abelian group generated freely by all pairs $m \otimes n$, where m and n run over all homogeneous elements of M respectively N. Then we put a $\mathbb{Z}_2$-grading on this group by saying that an element $m \otimes n$ is even if m, n are both even or both odd and $m \otimes n$ is odd if m is odd and n even or m is even and n is odd. Then $M \boxtimes N = (M \boxtimes N)_{\bar{0}} \oplus (M \boxtimes N)_{\bar{1}}$. We make $M \boxtimes N$ into an A-module by defining the right action $r(m \otimes n, a) = m \otimes (na)$. Next we consider the submodule $R_{M,N} \subset M \boxtimes N$ generated by all homogeneous elements of the form $ma \otimes n - m \otimes an$, $(m+m') \otimes (n+n') - m \otimes n - m \otimes n' - m' \otimes n - m' \otimes n'$ where m, n, m', n' run over all homogeneous elements and a over all homogeneous $a \in A$. The resulting quotient A-module $M \boxtimes N / R_{M,N}$ we call $M \otimes_A N$. One easily verifies that $M \otimes_A N$ has the usual universal property (see for example [15,50]).

The body module of an A-module M is defined to be the quotient $\overline{M} = M/J_A M$ and we write $\bar{m}$ for the image of $m \in M$ in $\overline{M}$. The body module is in a natural way an $\bar{A}$-module, as the action of A factors over J_A; $\bar{a}\bar{m} = \overline{am}$ for $a \in A, m \in M$. In particular, the body $\bar{A}$ is both an A-module and an $\bar{A}$-module.

Lemma 3.2.2. *Let A be a superring and M an A-module, then $\overline{M} \cong \bar{A} \otimes_A M$, where $\bar{A}$ is viewed as an A-module.*

Proof. Consider the maps $f : \bar{A} \otimes_A M \to \overline{M}$ sending $\bar{a} \otimes m$ to $\overline{am}$ and $g : \overline{M} \to \bar{A} \otimes_A M$ sending $\bar{m} \to 1 \otimes m$. The maps f, g are well-defined morphisms and inverse to each other. □

Definition 3.2.3. *Let A be a superring. We define the parity-swapping functor Π in the category of A-modules by $(\Pi M)_{\bar{0}} = M_{\bar{1}}$ and $(\Pi M)_{\bar{1}} = M_{\bar{0}}$. The action of A on ΠM is such that the right action of A on ΠM coincides with the right action of A on M. On morphisms $f : M \to N$ we define Πf as the same morphism from ΠM to ΠN as abelian groups. Thus Π exchanges the labels 'even' and 'odd' for the elements of M.*

Definition 3.2.4. *Let A be a superring and M an A-module. The module M is finitely generated if it is generated as a module by a finite number of homogeneous elements. An ideal in A is finitely generated if it is finitely generated by homogeneous elements as an A-module. If $f : A \to B$ is a morphism of superrings, B becomes an A-module and then we call B an A-superalgebra. We say that B is finitely generated as an A-superalgebra, if there is a finite number of homogeneous elements $b_1, \ldots, b_t$ in B such that each element of B can be written as a polynomial in the b_i with coefficients in A.*

Remark 3.2.5. If A is a superring and I is a $\mathbb{Z}_2$-graded ideal that is generated by inhomogeneous elements $x_1, \ldots, x_n$, then I is also generated by the homogeneous components and thus I is finitely generated. However, if we define an ideal I to be the ideal generated by inhomogeneous elements, then it is not guaranteed that I is $\mathbb{Z}_2$-graded.

The notion of annihilator of an element of a superring carries over to modules. Let M be any module over a superring A. For any $m \in M$ we define $\mathrm{Ann}(m)$ to be the set of all $a \in A$ such that $am = 0$, that is, the left action of a maps m to zero. If $am = 0$ we say that a annihilates m. When m is homogeneous, one easily sees that $\mathrm{Ann}(m)$ is a $\mathbb{Z}_2$-graded ideal in A. The next example shows that if m is not homogeneous, then $\mathrm{Ann}(m)$ might not be $\mathbb{Z}_2$-graded.

Example 3.2.6. Let $A = k[X|\Theta_1, \Theta_2]$ and consider the A-module $A/(X^2 + \Theta_1\Theta_2)$. The element $a = X(X + \Theta_2)$ annihilates $m = X - \Theta_1$ but X^2 and $X\Theta_2$ do not annihilate m. Hence $\mathrm{Ann}(m)$ need not be $\mathbb{Z}_2$-graded if m is not homogeneous. △

For a submodule $N \subset M$ we write $\mathrm{Ann}(N)$ for the $\mathbb{Z}_2$-graded ideal of elements a A such that $aN = 0$. Equivalently, $\mathrm{Ann}(N) = \cap_{n \in N} \mathrm{Ann}(n)$ where the intersection goes over all elements of N. Indeed, $a \in A$ annihilates all homogeneous elements of N if and only if a annihilates all elements of N.

Let $f : A \to B$ be a morphism of superrings. The map f turns B into an A-module so that B is an A-superalgebra. If B is a finitely generated as an A-module, we call the morphism f finite.

Let J_A and J_B be the canonical ideals of A and B respectively. Then $f(J_A) \subset J_B$ and thus there is an induced morphism of commutative rings $\bar{f} : \bar{A} \to \bar{B}$. It is obvious that when f is finite, then so is $\bar{f}$. The following lemma states that under mild assumptions the converse holds as well:

Lemma 3.2.7. *Let $f : A \to B$ be a morphism of superrings such that B is a finitely generated A-superalgebra and such that the induced morphism of commutative rings $\bar{f} : \bar{A} \to \bar{B}$ is finite. Then f is finite.*

Proof. By assumption there are even elements $x_1, \ldots, x_p$ and odd elements $\eta_1, \ldots, \eta_q$ in B and a surjective morphism of superrings

$$\hat{f} : A[X_1, \ldots, X_p | H_1, \ldots, H_q] \to B\,, \tag{3.6}$$

with $\hat{f}(X_i) = x_i$ and $\hat{f}(H_j) = \eta_j$ and where the X_i are even variables and the H_j are odd variables. The assumptions also ensure that the induced morphism $\bar{A}[X_1, \ldots, X_p] \to \bar{B}$ is surjective and that there is a positive integer N, such that every element $\bar{b}$ of $\bar{B}$ is the image of a polynomial $g(X_1, \ldots, X_p)$ with coefficients in $\bar{A}$ and degree less than N. We now claim that B is generated as an A-module by all elements of the form g, $g\eta_j$, $g\eta_{j_1}\eta_{j_2}$, …, $g\eta_1\eta_2\cdots\eta_q$, where g runs over all monomials of degree less than N. Since there are only finitely many such monomials, the lemma is then proved.

Let us denote $A[X]_N$ for all polynomials in $X_1, \ldots, X_p$ with coefficients in A and degree less than N. If b is any element in B, then there is an element $\tilde{b} \in f(A[X]_N)$ such that $b - \tilde{b} \in J_B$. Thus we can write

$$b - \tilde{b} = \sum_j b_j \eta_j\,. \tag{3.7}$$

But also for each b_j there is an element $\tilde{b}_j \in f(A[X]_N)$ such that $b_j - \tilde{b}_j \in J_B$. Hence we can write

$$b = \tilde{b} + \sum_j \tilde{b}_j \eta_j + \sum_{j,k} b_{jk}\eta_j\eta_k\,. \tag{3.8}$$

We can repeat the procedure till we reach an expression

$$b = \tilde{b} + \sum_j \tilde{b}_j \eta_j + \sum_{j,k} \tilde{b}_{jk}\eta_j\eta_k + \ldots + \tilde{b}_{12\cdots q}\eta_1\eta_2\cdots\eta_q\,, \tag{3.9}$$

with all $\tilde{b}_j \in f(A[X]_N)$. This proves the claim. □

3.3 Noetherian superrings

Proposition 3.3.1. *Let A be a superring, then the following are equivalent:*

(i) *Each $\mathbb{Z}_2$-graded ideal of A is finitely generated.*

(ii) *Each ascending chain $I_0 \subset I_1 \subset I_2 \subset \ldots$ of $\mathbb{Z}_2$-graded ideals in A is stationary, that is, there is an integer n such that $I_n = I_{n+1} = I_{n+2} = \ldots$.*

(iii) *Every nonempty set of $\mathbb{Z}_2$-graded ideals contains a maximal element.*

Proof. $(i) \Rightarrow (ii)$: Let $I_0 \subset I_1 \subset I_2 \subset \ldots$ be an ascending chain of $\mathbb{Z}_2$-graded ideals in A. Consider the ideal $I = \cup_k I_k$, which is $\mathbb{Z}_2$-graded and thus has to be finitely generated. Let m be an integer such that I_m contains all the (homogeneous) generators of I. Then $I_l = I_m$ for all $l \geq m$.

$(ii) \Rightarrow (iii)$: Let S be a nonempty set of $\mathbb{Z}_2$-graded ideals in A that has no maximal elements. Since S is nonempty we can find I_0 in S. Since I_0 cannot be maximal we can find I_1 in S that properly contains I_0. Repeating the procedure we find a chain $I_0 \subset I_1 \subset \ldots$ of proper inclusions going on indefinitely, contradicting the assumption that each chain of ideals is stationary.

$(iii) \Rightarrow (i)$: Let I be a $\mathbb{Z}_2$-graded ideal of A. Let S be the set of finitely generated $\mathbb{Z}_2$-graded ideals of A that are contained in I. Then all the $\mathbb{Z}_2$-graded ideals that are generated by finitely many homogeneous elements of I are in S, and thus S contains at least the zero ideal and is therefore not empty. Hence S contains a maximal element I_{max}. If $I \neq I_{max}$ then there is a homogeneous element $x \in I$ that does not lie in I_{max}. The ideal generated by I_{max} and x is finitely generated, contained in I and properly contains I_{max}, contradictory to the choice of I_{max}. Hence there is no such $x \in I - I_{max}$ and $I = I_{max}$ and I is finitely generated. □

Definition 3.3.2. *A superring satisfying any of the three equivalent conditions of proposition 3.3.1 is called a Noetherian superring.*

Proposition 3.3.3. *Let A and B be superrings and let $f : A \to B$ be a surjective morphism. If A is Noetherian, then so is B.*

Proof. For any $\mathbb{Z}_2$-graded ideal I of B consider its inverse image $f^{-1}(I)$ in A, which is a $\mathbb{Z}_2$-graded ideal and hence finitely generated. The images of generators of $f^{-1}(I)$ generate I as an ideal. □

By considering the body of a superring as a superring with zero odd part we immediately get:

Corollary 3.3.4. *When A is a Noetherian superring, then the body $\bar{A}$ is a Noetherian ring.*

The converse of corollary 3.3.4 is not true. A counter example is given by the superalgebras considered by DeWitt and Rogers [7, 10, 51]. Consider the superalgebra A over k defined by $A = k[(\theta_i)_{i \in I\!N}]$. The ideal J_A generated by the odd elements is clearly not finitely generated, whereas $A/J_A \cong k$ is Noetherian.

Proposition 3.3.5. *If A is a Noetherian superring and J is the $\mathbb{Z}_2$-graded ideal generated by the odd elements, then there are finitely many odd elements that generate J. Furthermore, if S is a set of homogeneous elements that generate J, then J is already generated by the odd elements of S.*

Proof. Let J be generated by $\theta_1, \ldots, \theta_s$, which we may assume to be homogeneous. Assume θ_1 to be even, then $\theta_1 \in (A_{\bar{1}})^2$ and it follows that θ_1 is a quadratic expression in the θ_i: $\theta_1 = a + \theta_1 b$ where a, b are linear combinations of the θ_i for $i \neq 1$. Reiteration gives

$$\theta_1 = \sum_{k=1}^{r} ab^k + \theta_1 b^{r+1}\,, \tag{3.10}$$

and since b is nilpotent we see that in fact $\theta_2, \ldots, \theta_s$ generate J. Hence we can remove all even generators of J by this procedure leaving only the odd ones. □

Proposition 3.3.6. *Let A be a Noetherian superring, then $A_{\bar{0}}$ is Noetherian.*

Proof. Let I be any ideal of $A_{\bar{0}}$, J the $\mathbb{Z}_2$-graded ideal in A generated by the odd elements and I' the $\mathbb{Z}_2$-graded ideal in A generated by I. First, we claim that $I' \cap A_{\bar{0}} = (I')_{\bar{0}} = I$. Indeed, for if $x \in (I')_{\bar{0}}$, then $x = \sum r_i f_i$ where $f_i \in I$ and the r_i we may assume to be homogeneous, hence $x \in A_{\bar{0}} I \subset I$. Thus $(I')_{\bar{0}}$ is contained in I. On the other hand, the inclusion $I \subset (I')_{\bar{0}}$ follows from the definition of I'.

I' is generated by a finite number of even elements a_i and a finite number of odd elements b_i. If $x \in I$, then x is an even element of I' and hence we have $x = \sum x_i a_i + \sum y_i b_i$, where $y_i \in A_{\bar{1}} \subset J$.

The $\mathbb{Z}_2$-graded ideal J is generated by a finite number of generators θ_i, which by proposition 3.3.5 can be taken to lie in $A_{\bar{1}}$. But then we see that the set consisting of all elements a_i and $\theta_k b_l$ generate I and lie in $(I')_{\bar{0}} = I$. □

Proposition 3.3.7. *Let R be a commutative Noetherian ring. Then the superring $R[X_1, \ldots, X_n|\Theta_1, \ldots, \Theta_m]$ is Noetherian.*

Proof. In view of the result for commutative rings of the form $R[X_1, \ldots, X_n]$ (see for example [15, 19, 50]) it suffices to show that if A is a Noetherian superring, then $A[\theta]$ with θ an odd variable is a Noetherian superring. Let I be a $\mathbb{Z}_2$-graded ideal in $A[\theta]$ and define

$$I_1 = \{a \in A | \exists b \in A \text{ such that } b + a\theta \in I\} \ . \tag{3.11}$$

It follows that I_1 is a $\mathbb{Z}_2$-graded ideal in A. Hence there are homogeneous generators $t_1, \ldots, t_k \in A$ of I_1. For every t_i we select homogeneous $y_i = c_i + t_i\theta \in I$ where $c_i \in A$ is homogeneous. Let K be the $\mathbb{Z}_2$-graded ideal in $A[\theta]$ generated by the elements $y_1, \ldots, y_k$ and let

$$I_0 = \{a \in A | \exists b \in K \text{ such that } a + b \in I\} \ . \tag{3.12}$$

Then I_0 is a $\mathbb{Z}_2$-graded ideal in A and hence there are homogeneous generators $r_1, \ldots, r_l \in A$ of I_0. Since $K \subset I$ we have $I_0 \subset I \cap A$. Let J be the ideal in $A[\theta]$ generated by the y_i and the r_i. Then we clearly have $J \subset I$. Let $u = x + y\theta \in I$. Since $y \in I_1$ there is u' in K given by $u' = \sum_i y_i d_i$ for some $d_i \in A$ with $u - u' \in A$. Hence $u - u' \in I_0$, but then $u = (u - u') + u'$ is an element of J; we conclude that $I = J$. □

An immediate result of propositions 3.3.3 and 3.3.7 is that any superring that is finitely generated over a commutative ring, is Noetherian.

We call an A-module M Noetherian if one of the following properties holds:

(i) Each $\mathbb{Z}_2$-graded submodule N of M is finitely generated.

(ii) Each ascending chain $M_0 \subset M_1 \subset M_2 \subset \ldots$ of $\mathbb{Z}_2$-graded submodules is stationary.

(iii) Every nonempty subset of $\mathbb{Z}_2$-graded submodules of M has a maximal element.

The proof that the properties (i)-(iii) are equivalent is the same as in proposition 3.3.1. A superring is Noetherian if it is Noetherian as a module over itself. If $f : M \to N$ is a surjective morphism, then by the same reasoning as in proposition 3.3.3 the module N is Noetherian when M is Noetherian. Furthermore, when M is a Noetherian A-module and N is a submodule of M, then N is also a Noetherian A-module; any submodule of N is a submodule of M, and thus finitely generated.

Proposition 3.3.8. *Let A be a Noetherian superring and M a finitely generated A-module, then M is a Noetherian A-module.*

Proof. Suppose M is generated by $m_1, \ldots, m_r$. We use induction on r. For $r = 1$ the module M is isomorphic to $A/\mathfrak{a}$ or $\Pi(A/\mathfrak{a})$ for some ideal $\mathfrak{a}$ in A. Each $\mathbb{Z}_2$-graded submodule of M then corresponds to a $\mathbb{Z}_2$-graded ideal in A and hence M is Noetherian. If $r > 1$ and N a submodule of M, consider the image N' of N in M/Am_1, which is a $\mathbb{Z}_2$-graded module generated by $r - 1$ elements. Thus there are elements $x_1, \ldots, x_k$ in N, such that their images in N' generate N'. The A-module Am_1 is Noetherian and hence we can assume that $N \cap Am_1$ is generated by $y_1, \ldots, y_l$. Take $n \in N$, then there are $a_i \in A$ such that $n - \sum a_i x_i$ goes to zero in N', and hence $n - \sum a_i x_i$ lies in $N \cap Am_1$, so that we can write it as an A-linear combination of the y_j. It follows that n is an A-linear combination of the y_j and the x_i. □

Later in section 6.4 we will have more to say on finitely generated modules of a Noetherian superring.

3.4 Artinian superrings

Noetherian modules satisfy the ascending chain condition: any ascending chain of submodules becomes stationary after a finite number of steps. The descending chain condition requires from a module that any descending chain of submodules becomes stationary after a finite number of terms. A module that satisfies the descending chain condition is said to be Artinian. In this section we present the basics on Artinian modules. When we have dealt with localization and prime ideals we return to Artinian modules again in section 5.2.

Let A be a superring and let M be an A-module. We say that M is a simple module if M does not contain any nontrivial submodules, that is, the only submodules are 0 and M itself. There is a certain duality between maximal submodules and simple modules: if N is a submodule of M, then M/N is simple if and only if N is maximal.

For superrings, a simple module has the peculiar property that it is either even or odd and hence $A_{\bar{1}}$ acts by zero: $J_A M = 0$. If we pick an element $m \in M$ then $A \cdot m \subset M$ and hence either $m = 0$, or $A \cdot m = M$. Thus M is generated by one element and hence $M \cong A/\mathfrak{m}$ or $M \cong \Pi A/\mathfrak{m}$ for some ideal $\mathfrak{m}$, which clearly has to be a maximal ideal. In particular, M is an $\bar{A}$-module and since the image of $\mathfrak{m}$ in $\bar{A}$ is a maximal ideal, M is simple as an $\bar{A}$-module. Conversely, if M is a simple $\bar{A}$-module, it is of the form $\bar{A}/\bar{\mathfrak{m}}$ for some maximal ideal $\bar{\mathfrak{m}}$. We make M into an A-module through the canonical projection $A \to \bar{A}$. Then M is a simple A-module isomorphic to $A/\mathfrak{m}$, where $\mathfrak{m}$ is the inverse image of $\bar{\mathfrak{m}}$ under the canonical projection. We thus have proved the following statement:

Proposition 3.4.1. *There is a one-to-one correspondence between the simple A-modules and simple $\bar{A}$-modules.*

We call a superring Artinian if it satisfies the descending chain condition on $\mathbb{Z}_2$-graded ideals; in other words, a superring A is Artinian if any sequence

$$I_0 \supset I_1 \supset I_2 \supset \dots , \tag{3.13}$$

stabilizes after a finite number of terms. We call a module over a superring Artinian if it satisfies the descending chain condition on $\mathbb{Z}_2$-graded submodules. Equivalently, each nonempty set of submodules contains a minimal element. Thus a superring is Artinian if and only of it is Artinian when considered as a module over itself. If a superring A is Artinian, then so is ΠA as an A-module.

Proposition 3.4.2. *Let M be an Artinian A-module, then all submodules of M are Artinian and all quotients of M are Artinian.*

Proof. If $N \subset M$, then any chain in N is a chain in M, hence stabilizes. If we have a chain in M/N, we can find a chain of preimages of the projection $M \to M/N$ in M. This chain maps surjective onto the given chain and terminates as M is Artinian. Thus the chain in M/N also terminates. □

Proposition 3.4.3. *Let A be a superring. Let M be an A-module, and N a submodule of M. If N and M/N are Artinian, so is M.*

Proof. Let $M_1 \supset M_2 \supset \dots$ be any chain of submodules of A. Consider the chain of images of M_i under the projection $M \to M/N$. Then this chain stabilizes. Hence there is an integer k such that $M_i \bmod N = M_j \bmod N$ for all $i, j \geq k$. Consider the chain $M_1 \cap N \supset M_2 \cap N \supset \dots$. Then there is an integer l such that $M_i \cap N = M_j \cap N$ for all $i, j \geq l$. This implies that $M_i = M_j$ for all i, j greater than or equal to the maximum of k, l. Indeed, let $j \geq i$ be both larger than the maximum of k and l, so that $M_i \supset M_j$ and suppose $m_i \in M_i$. Then $m_i = m_j + n$ for some $m_j \in M_j$ and $n \in N$. We see that $n \in M_i \cap N$, hence $n \in M_j \cap N$ and thus $m_i \in M_j$. □

Corollary 3.4.4. *If M and N are two Artinian modules, so is their direct sum $M \oplus N$.*

Proof. Apply proposition 3.4.3 to the modules $E = M \oplus N$ and N; $E/N \cong M$ is Artinian and N is Artinian. Hence E is Artinian. □

Corollary 3.4.5. *Let A be an Artinian superring and M a finitely generated A-module. Then M is an Artinian A-module.*

Proof. If M is finitely generated, then M is a quotient of a finite direct sum of copies of A and ΠA. Hence the result follows from proposition 3.4.2 and corollary 3.4.4. □

We call a descending chain $M = M_0 \supset M_1 \supset \ldots$ of submodules in M a composition series if M_i/M_{i+1} is a simple module. We define the length of a module to be the minimal length of a composition series. If there are no finite composition series we say the module has infinite length. We denote the length of M by $l(M)$.

Lemma 3.4.6. *Let M be a A-module and N a proper submodule of M. Then $l(N) < l(M)$.*

Proof. We look at the intersections of N with composition series of M. Take any composition series of M:

$$M = M_0 \supset M_1 \supset \ldots \supset M_n = 0\,. \tag{3.14}$$

We have

$$\frac{N \cap M_i}{N \cap M_{i+1}} \cong \frac{N \cap M_i + M_{i+1}}{M_{i+1}} \subset \frac{M_i}{M_{i+1}}\,. \tag{3.15}$$

Hence either the left-hand side is zero, or the left-hand side is simple. In the first case $N \cap M_i \cong N \cap M_{i+1}$, and in the second case $N \cap M_i + M_{i+1} = M_i$. In any case we can delete the redundant terms in the series $M \cap N = M_0 \cap N \supset M_1 \cap N \supset \ldots \supset M_n \cap N$ to get a composition series of $M \cap N$ of length $\leq n$. Suppose equality holds, then for all i we have $N \cap M_i + M_{i+1} = M_i$, so $N \cap M_n = M_n$ implying $M_n \subset N$. And for $n-1$ we see $N \cap M_{n-1} + M_n = M_{n-1}$ and thus also $M_{n-1} \subset N$. Continuing the process we arrive at $M_0 \subset N$, which contradicts the assumption that N is a proper submodule. Hence any composition series of M gives rise to a shorter composition series of N. □

Lemma 3.4.7. *Let M be a module with length $l(M)$. If we have a chain of submodules $M = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_r$ then $r \leq l(M)$.*

Proof. We use induction on the length of M. If $l(M) = 0$ then the statement is trivial. For $l(M) = 1$ we see that M is simple. Hence any chain of submodules consists of one term. Now suppose $l(M) \geq 1$, then we use lemma 3.4.6 and the induction hypothesis to derive that $r - 1 \leq l(M_1) \leq l(M) - 1$. Hence $r \leq l(M)$ □

Corollary 3.4.8. *All composition series of M have the same length $l(M)$.*

Proof. By lemma 3.4.7 we see all chains have length smaller or equal $l(M)$. But by the very definition of the length of M, $l(M)$ is the minimal length of a composition. Hence no composition series can satisfy the strict inequality. □

Theorem 3.4.9. *A module M has a finite composition series if and only if it is Noetherian and Artinian.*

Proof. Suppose M has length $l(M) < \infty$, then all chains have length less than $l(M)$. This proves that M is Noetherian and Artinian. Conversely, suppose M is Artinian and Noetherian. Then as M is Noetherian, we can choose a maximal submodule M_1. M_1 itself is also Noetherian and hence we find a proper maximal submodule M_2. We now observe that M/M_1 is simple, since else M_1 would not be maximal. Similarly M_1/M_2 is simple. We continue the process to find a composition series. This composition series is finite since M is assumed to be Artinian. □

Corollary 3.4.10. *If a superalgebra over a field k is finite-dimensional, it is Artinian and Noetherian.*

Proof. In any chain with proper inclusions, the dimension has to change at every step. Thus M has a finite composition series. □

3.5 Split superrings

In this section we discuss some superrings with a particular simple form. Because of their simplicity they are an easy testing ground for several concepts that will be discussed later. The split superrings admit a geometric intuition: they can be seen as the ring of functions of an ordinary variety in $\mathbb{A}^n$ with some additional noncommutative structure.

Definition 3.5.1. *We say a superring A has a split body if there is a morphism of superrings $\sigma : \bar{A} \to A$, such that $\overline{\sigma(x)} = x$ for all $x \in \bar{A}$; in this definition $\bar{A}$ is considered a superring with trivial odd part.*

Definition 3.5.1 can be rephrased by saying that the exact sequence $0 \to J_A \to A \to \bar{A} \to 0$ splits. We call the morphism σ the splitting morphism. From the definition it follows that a superring A is a split superring if and only if A contains a commutative ring that is isomorphic to $\bar{A}$ and that maps surjectively to $\bar{A}$ under the projection $A \to \bar{A}$. Hence the following lemma follows immediately.

Lemma 3.5.2. *Let B be a commutative ring and let A be the superring given by $A = B[\theta_1, \ldots, \theta_m]/I$ where the θ_i are odd variables and where I is a $\mathbb{Z}_2$-graded ideal contained in J_A. Then A has a split body.*

An example of a superring with a split body is given by the superring associated to the 'supersphere':

$$A = \frac{k[X_1, X_2, X_3|\theta_1, \theta_2, \theta_3]}{(\sum_{i=1}^3 X_i^2, \sum_{i=1}^3 X_i\theta_i)}\,. \tag{3.16}$$

An example of a superring that does not have a split body is given by

$$A = \frac{k[X|\theta_1, \theta_2]}{(X^2 + \theta_1\theta_2)}\,, \tag{3.17}$$

where the body is given by $k[X]/(X^2)$.

Proposition 3.5.3. *Let B be a superring and let A be the superring given by $A = B[\theta_1, \ldots, \theta_m]/I$ where the θ_i are odd variables and where I is a $\mathbb{Z}_2$-graded ideal contained in $(\theta_1, \ldots, \theta_m)$. Then if B has a split body then A has a split body.*

Proof. Clearly, $(\theta_1, \ldots, \theta_m) \subset J_A$ and thus $\bar{B} = \bar{A}$ and since B is a subalgebra containing $\bar{B}$ as a subalgebra. □

Proposition 3.5.4. *Let A be a superring with a split body and splitting morphism $\sigma : \bar{A} \to A$ and suppose $f : A \to B$ is a surjective morphism of superrings with kernel I and denote $\bar{I}$ the image of I in $\bar{A}$. If $\sigma(\bar{I}) \subset I$, then B has a split body.*

Proof. We have an induced surjective ring morphism $\bar{f} : \bar{A} \to \bar{B}$ with kernel $\bar{I}$. For $\bar{b} \in \bar{B}$ we can find $\bar{a}$ with $\bar{f}(\bar{a}) = \bar{b}$. We define $\tau(\bar{b}) = f \circ \sigma(\bar{a})$; then $\tau(\bar{b})$ is independent of the choice of $\bar{a}$ since $\sigma(\bar{I}) \subset I$. Furthermore $\overline{\tau(\bar{b})} = \overline{f \circ \sigma(\bar{a})} = \bar{f}\overline{\sigma(\bar{a})} = \bar{f}(\bar{a}) = \bar{b}$ and hence τ is a splitting morphism. ☐

3.6 Grassmann envelopes

Given a super vector space V over k and a superalgebra A over k, we can consider $V(A) = V \otimes_k A$. Then $V(A)$ is an A-module, and although we will define free modules not until section 3.7, it is not too hard to see that $V(A)$ is a free A-module. One calls $V(A)$ the Grassmann envelope of the first kind. The even part of $V(A)$ is an $A_{\bar{0}}$-module and is called the Grassmann envelope of the second kind. These two constructions play an important role in the theory of Lie superalgebras associated to algebraic supergroups in chapter 8. The name Grassmann envelop was dubbed by Felix Berezin [9], one of the pioneers in the area of super mathematics.

If V is a Lie superalgebra, then $V \otimes A$ is also a Lie superalgebra and $(V \otimes A)_{\bar{0}}$ is a Lie algebra with an $A_{\bar{0}}$-module structure. Later, in subsection 8.6.4 we loosen up the definition of Lie algebra to be a module over a commutative ring R together with a Lie bracket $[,]$, satisfying the usual conditions of being R-linear, $[x, x] = 0$ and the Jacobi identity. Then we can say that $A \mapsto (V \otimes A)_{\bar{0}}$ is a functor from the category of superalgebras to the category of Lie algebras.

Let **C** be some category that admits a faithful embedding into the category of super vector spaces. We take a heuristic approach and use the liberty to specify a posteriori what further properties we require **C** to have. To a super vector space V we associate a functor $T_V : \mathbf{sAlg} \to \mathbf{C}$ as follows: On the objects we put $T_V : A \mapsto (V \otimes A)_{\bar{0}}$; we thus need that **C** is such that it allows that the objects $(V \otimes A)_{\bar{0}}$ are $A_{\bar{0}}$-modules in a functorial way. On morphisms $\phi : A \to B$ the functor T_V acts on the second factor; $T_V(\phi) : v \otimes a \mapsto v \otimes \phi(a)$. We write $T_V \otimes T_W$ for the functor that maps A to $(V \otimes_k A)_{\bar{0}} \otimes_{A_{\bar{0}}} (W \otimes_k A)_{\bar{0}}$ and a morphism $f : A \to B$ is mapped to the morphism that sends $v \otimes a \otimes_{A_{\bar{0}}} w \otimes a'$ to $v \otimes f(a) \otimes_{A_{\bar{0}}} w \otimes f(a')$. Note that $T_{V \otimes W} \neq T_V \otimes T_W$. We remark that there is an important difference between $\otimes_{A_{\bar{0}}}$ and $\otimes_A$. So is $\theta \otimes_A \theta = 0$ but $\theta \otimes_{A_{\bar{0}}} \theta \neq 0$ in general for odd $\theta \in A$. The definition of $\bigotimes_{i \in I} T_{V_i}$ is immediate for finite sets I. If V is a Lie superalgebra, the objects $(V \otimes A)_{\bar{0}}$ are Lie algebras in a natural way and hence we can take **C** to be the category **LieAlg** of Lie algebras, as described above. The following result is due to Deligne and Morgan [6]:

Proposition 3.6.1.

(a) There is a one-to-one correspondence between the natural transformations $\bigotimes_{i \in I} T_{V_i} \to T_W$ and super vector space morphisms $f : \bigotimes_{i \in I} V_i \to W$.

(b) If all objects $T_V(A)$ are functorially Lie algebras over $A_{\bar{0}}$, then the vector space V is a Lie superalgebra.

Proof. (a) We first do the proof for $I = \{1\}$. We use the functoriality and apply the functors to $A = k$ and $A' = k[\theta]$ to get maps $V_{\bar{0}} \to W_{\bar{0}}$ and $V_{\bar{1}} \to W_{\bar{1}}$. Let $f : V \to W$ be the map defined in this way. Consider now a general superalgebra A and consider the element $v \otimes a \in (V \otimes A)_{\bar{0}}$. If a is even, we can use the $A_{\bar{0}}$-linearity to obtain $v \otimes a = v \otimes 1 \cdot a \mapsto f(v) \otimes a$, and if a is odd we consider the morphism $k[\theta] \to A$ given by $\theta \mapsto a$.

$$\begin{tikzcd} T_V(k[\theta]) \arrow[r] \arrow[d] & T_W(k[\theta]) \arrow[d] \\ T_V(A) \arrow[r] & T_W(A) \end{tikzcd} . \tag{3.18}$$

Since $T_V \to T_W$ is a natural transformation, the diagram (3.18) we obtain $v \otimes \theta \mapsto v \otimes a \mapsto f(v) \otimes a$. So indeed the natural transformation agrees with the given map f. On the other hand, a morphism f of super vector spaces induces exactly the same natural transformation.

For more general index sets I the proof is rather similar. Let us first do existence and uniqueness. Write the natural transformation as $\varphi(A)$ for each superalgebra A. If we choose $A = k$, we get a map $f : \bigotimes_{i \in I} V_{i,\bar{0}} \to W_{\bar{0}}$. To specify a map on the tensor product of $V_{i,\bar{0}}$ for $i \notin J$ and $V_{i,\bar{1}}$ for $i \in J$ for some subset $J \subset I$ we use $A = k[(\theta_j)_{j \in J}]$. We take $v_i \in V_{i,\bar{0}}$ for $i \notin J$ and $v_i \in V_{i,\bar{1}}$ for $i \in J$, we write $w_i = v_i$ for $i \notin I$ and $w_i = v_i \theta_i$ for $i \in J$ and determine f from

$$\varphi(k[(\theta_j)_{j \in J}])(\otimes_i w_i) = (-1)^{N(N-1)/2} f(\otimes v_i) \Pi_{i \in J} \theta_i , \quad \text{where } N = \#J . \tag{3.19}$$

The expression (3.19) is dictated by the fact that under the morphism $\theta_i \to 0$ the left-hand side vanishes, hence also the right-hand side. Therefore the right-hand side is a product of the θ_i with $i \in J$. Since the expression (3.19) fixes f, the morphism f is unique and furthermore, given φ we define f in this way. Let now f be defined in this way and consider a general superalgebra A. We have a similar commutative diagram as in part (a). Given an element $\otimes w_i$, where w_i is in $V_{\bar{0}} \otimes A_{\bar{0}}$ for $i \notin J$ for a subset $J \subset I$ and $w_i \in V_{\bar{1}} \otimes A_{\bar{1}}$ for $i \in J$, we can use $A_{\bar{0}}$-linearity to choose $w_i = v_i \otimes 1 \cong v_i$ with $v_i \in V_{\bar{0}}$ in the first case. For the second case we write $w_i = v_i \otimes a_i$ and choose the morphism $k[(\theta_j)_{j \in J}] \to A$ given by $\theta_j \to a_j$. The same commutative diagram as for the case $I = \{1\}$ concludes the proof of the general case.

(b) The rules that determine the Lie algebra structure are given by maps of the form treated in (a) satisfying the axioms of a Lie superalgebra. □

In fact, Deligne and Morgan proved the theorem for a more general case. However, stating the theorem in the general case is a rather difficult task; in fact, to state the theorem is more difficult than to prove it. We therefore refer the reader to [6].

3.7 Free modules and supermatrices

We consider a fixed superring A. A free A-module can be characterized as usual by a universal property: Let S be a set that is a disjoint union of two sets $S_{\bar{0}}$ and $S_{\bar{1}}$; $S = S_{\bar{0}} \cup S_{\bar{1}}$ and $S_{\bar{0}} \cap S_{\bar{1}} = 0$. A free A-module on S is an A-module F_S together with a map $u : S \to F_S$ with $u(S_i) \subset (F_S)_i$ for $i = \bar{0}, \bar{1}$ such that if M is any A-module and $f : S \to M$ is a map of S to M such that $f(S_i) \subset M_i$, then there is a unique morphism of A-modules $v : F_S \to M$ with $f = v \circ u$. From the universal property it follows that F_S is unique up to isomorphism. The construction of F_S is as follows: for each $x \in S_{\bar{0}}$ we take a copy of A and put $u(x)$ to be the unit element in A, for $y \in S_{\bar{1}}$ we take a copy of ΠA and map y to $1 \in \Pi A$. We thus have $F_S = (\bigoplus_{x \in S_{\bar{0}}} A) \oplus (\bigoplus_{y \in S_{\bar{1}}} \Pi A)$ and $u(x) = 1$ in the corresponding factor of A or ΠA; then F_S has the required universal property. We are mainly interested in the case that S is a finite set.

If $|S_{\bar{0}}| = p$ and $|S_{\bar{1}}| = q$ we write $F_S = A^{p|q}$ for the free A-module on S. Thus $A^{p|q} = (\bigoplus_{i=1}^{p} A) \oplus (\bigoplus_{j=1}^{q} \Pi A)$. We call $p|q$ the rank of the module $A^{p|q}$. If S is not finite, we say that the free module on S has infinite rank. By the following lemma the definition of rank makes sense:

Lemma 3.7.1. *The rank of a free A-module on a finite set is well-defined; that is, if $A^{p|q} \cong A^{r|s}$, then $p = q$ and $r = s$.*

Proof. Let $K = A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. Then it is easy to see that $p|q$ is the dimension of the super vector space $A^{p|q} \otimes_A K \cong K^{p|q}$. Thus $K^{p|q} \cong K^{r|s}$, but then $K^p \cong K^r$ and $K^q \cong K^s$ as vector spaces, from which the lemma follows. □

Lemma 3.7.2. *If M is a free A-module of rank $p|q$, then $\overline{M}$ is a free $\bar{A}$-module of rank $p + q$.*

Proof. Follows from the isomorphisms $A \otimes_A \bar{A} \cong \bar{A}$ and $\Pi A \otimes_A \bar{A} \cong \bar{A}$, which are isomorphism of $\bar{A}$-modules since $\bar{A}$-modules don't have a definite parity. We thus have $A^{p|q} \otimes_A \bar{A} \cong \bigoplus_{i=1}^{p+q} \bar{A}$. □

For the rest of this section we assume that A is a superalgebra over k. The goal below is to define supermatrices with entries in A. We consider maps from $A^{p|q}$ to $A^{r|s}$ that preserve sums and commute with the right action of A: $\varphi(ma) = \varphi(m)a$ for all $m \in A^{p|q}$. The set of all such maps we denote $\underline{\mathrm{Hom}}_A(A^{p|q}, A^{r|s})$. We call an element of $\underline{\mathrm{Hom}}_A(A^{p|q}, A^{r|s})$ even if it preserves the $\mathbb{Z}_2$-grading and odd if it reverses the $\mathbb{Z}_2$-grading. It is easy to see that then $\underline{\mathrm{Hom}}_A(A^{p|q}, A^{r|s})$ is a $\mathbb{Z}_2$-graded abelian group.

Since $A^{p|q} \cong k^{p|q} \otimes_k A$, any morphism $F \in \underline{\mathrm{Hom}}_A(A^{p|q}, A^{r|s})$ should be an A-linear sum of $(r + s) \times (p + q)$-matrices with entries in k. For convenience we write $M = A^{p|q}$ and $N = A^{r|s}$. Assume $m_1, \dots, m_{p+q}$ are generators for M and $n_1, \dots, n_{r+s}$ are generators for N. We can always arrange the generators in the standard way, by which we mean that $m_1, \dots, m_p$ are even and $m_{p+1}, \dots, m_{p+q}$ are odd. (We thus also arrange that $n_1, \dots, n_r$ are even and $n_{r+1}, \dots, n_{r+s}$ are odd.)

Given any A-linear map $F \in \underline{\mathrm{Hom}}_A(M, N)$ we define an $(r + s) \times (p + q)$-matrix (F_{ij}) with entries in A by $F(m_i) = \sum_j n_j F_{ji}$. Let $L = A^{u|v}$ be another free A-module with standard basis $l_1, \dots, l_{u+v}$ and let $G : N \to L$ be an element of $\underline{\mathrm{Hom}}_A(N, L)$ that can be represented by a $(u + v) \times (r + s)$-matrix with entries in A given by $G(n_j) = \sum_k l_k G_{kj}$. It is not too hard to see that then $(G \circ F)(m_i) = \sum_{k,j} l_k G_{kj} F_{ji}$ so that $(G \circ F)_{ki} = \sum_j G_{kj} F_{ji}$. Note that it is crucial in this definition that F and G commute with the right action of A. We can decompose the matrix (F_{ij}) in block-form as

$$F = \begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{pmatrix} , \tag{3.20}$$

where F_{00} is of size $r \times p$, F_{01} is of size $r \times q$, F_{10} is of size $s \times p$ and F_{11} is of size $s \times q$. In the sequel, when we decompose a matrix into block matrices, we always mean the block-form as in (3.20). If the map F is even, then the entries of F_{00} and F_{11} are even elements of A, whereas the entries of F_{01} and F_{10} are odd elements of A. When F is an odd homomorphism, then all entries have the opposite parity.

We now focus on the case where $M = N$. We denote $\mathrm{Mat}_{p|q}(A)$ the set of $(p + q) \times (p + q)$-matrices with entries in A. From the above discussion there is a one-to-one correspondence between $\underline{\mathrm{Hom}}_A(M, M)$ and $\mathrm{Mat}_{p|q}(A)$. We make $\mathrm{Mat}_{p|q}(A)$ into a $\mathbb{Z}_2$-graded abelian group by saying that a matrix (F_{ij}) is even (resp. odd) when it is even (resp. odd) as an element of $\underline{\mathrm{Hom}}_A(M, M)$. We make $\mathrm{Mat}_{p|q}(A)$ into an A-module by defining for $a \in A$ the action $(F_{ij})a = ((F \circ a)_{ij})$, where a is identified with the morphism $m \mapsto am$. On the basis elements m_i we have

$$(Fa)(m_i) = F(m_i)a(-1)^{|m_i||a|} = \sum_j m_j F_{ji} a(-1)^{|m_i||a|} . \tag{3.21}$$

We see that $(F_{ji})a$ is given by the matrix with entries $F_{ji}a(-1)^{|i||a|}$, where we used the short-hand $|i| = |m_i|$, which we also use below. Using the law of matrix multiplication $(FG)_{ij} = \sum_k F_{ik} G_{kj}$

we make $\mathrm{Mat}_{p|q}(A)$ into an associative, unital, noncommutative A-superalgebra. We denote the unit matrix by $\mathbb{1}$.

Let $X \in \mathrm{Mat}_{p|q}(A)$, then $\bar{X}$ is the matrix obtained by applying the projection to the body on each entry: in components $(\bar{X})_{ij} = (\overline{X_{ij}})$. Suppose that X is such that $\bar{X} = 0$, then for each entry X_{ij} there is an integer n_{ij} such that $X_{ij}^{n_{ij}} = 0$. Thus $X^N = 0$, where $N = \sum_{ij} n_{ij}$. From this observation the proof of the following lemma is easy.

Lemma 3.7.3. *A matrix $X \in \mathrm{Mat}_{p|q}(A)$ is invertible if and only if $\bar{X}$ is invertible. If X is even and of the form*

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3.22}$$

then X is invertible if and only if A and D are invertible, that is, if and only if $\det A$ and $\det D$ are invertible in $A_{\bar{0}}$.

Proof. If X is invertible, there is a matrix Y with $XY = YX = \mathbb{1}$, applying the body projection on both sides, we see $\bar{X}$ is invertible. If $\bar{X}$ is invertible, there is $Y \in \mathrm{Mat}_{p|q}(A)$ with $\overline{XY} - \mathbb{1} = \overline{YX} - \mathbb{1} = 0$. Hence we have $YX = \mathbb{1} - N$ and$XY = \mathbb{1} - N'$, where N and N' are nilpotent matrices, so that $\mathbb{1} - N$ and $\mathbb{1} - N'$ are invertible. Thus X has a left- and a right inverse. Since $\mathrm{Mat}_{p|q}(A)$ is associative, the left- and right inverse coincide.

If X is even and of the form as stated, it follows that when $\bar{X}$ is invertible, then so are $\bar{A}$ and $\bar{D}$. Using the same argument, we see that also A and D are invertible. Since A and D only contain even elements, we can apply the determinant rule. ☐

Definition 3.7.4. *We define the supertrace* str *of a supermatrix by*

$$\mathrm{str}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{tr}A - \mathrm{tr}D\,. \tag{3.23}$$

Definition 3.7.5. *We define the supertranspose X^{ST} of a supermatrix X as follows. For even supermatrices we define the supertranspose by*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{ST} = \begin{pmatrix} A^T & -C^T \\ B^T & D^T \end{pmatrix}, \tag{3.24}$$

and for odd supermatrices we define the supertranspose by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{ST} = \begin{pmatrix} A^T & C^T \\ -B^T & D^T \end{pmatrix}, \tag{3.25}$$

where the superscript T denotes the ordinary transpose.

Definition 3.7.5 is compatible with the earlier definition we gave of the supertranspose in equation (2.12). We observe that if x and y are $p\times q$- and $q\times r$-matrices respectively with only odd elements as entries, then $(xy)^T = -y^T x^T$. Using this observation and the definitions, the proof of the following lemma can be done by a straightforward calculation:

Lemma 3.7.6. *Let X and Y be two matrices in $\mathrm{Mat}_{p|q}(A)$ for some superring A. Then $\mathrm{str}(XY) = (-1)^{|X||Y|}\mathrm{str}(YX)$, $(XY)^{ST} = (-1)^{|X||Y|}Y^{ST}X^{ST}$ and $\mathrm{str}X^{ST} = \mathrm{str}X$.*

We give $A \otimes_k \mathrm{Mat}_{p|q}(k)$ a superalgebra structure in the usual way: if $a, b \in A$ and $X, Y \in \mathrm{Mat}_{p|q}(k)$ then $a \otimes X \cdot b \otimes Y = (-1)^{|X||b|} ab \otimes XY$. We now give an explicit isomorphism of algebras $A \otimes_k \mathrm{Mat}_{p|q}(k) \to \mathrm{Mat}_{p|q}(A)$. This isomorphism explains why some sign changes can appear when one passes from $A \otimes_k \mathrm{Mat}_{p|q}(k)$ to $\mathrm{Mat}_{p|q}(A)$. Some authors wave this sign change away and simply redefine - for example - the notion of supertransposition. We denote E_{ij} the matrix in $\mathrm{Mat}_{p|q}(k)$ that has a 1 on the (i,j)th place and is zero elsewhere. We have $|E_{ij}| = |i| + |j|$. We write δ_{ij} for the Kronecker delta.

Lemma 3.7.7. *Let A be a superalgebra over k. The k-linear map $\varphi : A \otimes \mathrm{Mat}_{p|q}(k) \to \mathrm{Mat}_{p|q}(A)$ that sends $a \otimes_k E_{ij}$ to the matrix with entries $(\varphi(a \otimes_k E_{ij}))_{mn} = (-1)^{|i||a|} a\delta_{im}\delta_{jn}$ is an algebra isomorphism.*

Proof. By definition, the map sends sums to sums so we have to check that φ preserves products. We have

$$a \otimes_k E_{ij} \cdot b \otimes_k E_{kl} = (-1)^{(|i|+|j|)|b|} ab \otimes \delta_{jk} E_{il}\,, \tag{3.26}$$

and applying φ we obtain a matrix with (m,n)-entry

$$(-1)^{|a||i|+|b||j|}(ab\delta_{jk})\delta_{im}\delta_{ln}\,. \tag{3.27}$$

On the other hand

$$\begin{aligned}\sum_p (\varphi(a \otimes_k E_{ij}))_{mp}(\varphi(b \otimes_k E_{kl}))_{pn} &= \sum_p (-1)^{|a||m|+|b||p|} ab\delta_{im}\delta_{jp}\delta_{kp}\delta_{ln} \\ &= (-1)^{|a||i|+|b||j|} ab\delta_{im}\delta_{jk}\delta_{ln}\,.\end{aligned} \tag{3.28}$$

Clearly, the map φ is surjective and injective. □

We define the functor $\mathrm{GL}_{p|q}$ from the category of superrings **sRng** to the category of groups **Grp** as follows: to each superring A we assign the group of invertible even elements of $\mathrm{Mat}_{p|q}(A)$ and to each morphism of superrings $f : A \to B$ we assign the map that sends a matrix $(X_{ij}) \in \mathrm{GL}_{p|q}(A)$ to the matrix $(f(X_{ij})) \in \mathrm{GL}_{p|q}(B)$ - that is, it works on each matrix entry. Since an algebra morphism maps invertible elements to invertible elements, the matrix $(f(X_{ij})) \in \mathrm{GL}_{p|q}(B)$ is indeed invertible by lemma 3.7.3.

Definition 3.7.8. *For an invertible even supermatrix X we define the superdeterminant* $\mathrm{Ber}\, X$ *by the formula*

$$\mathrm{Ber}\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \frac{\det(A - BD^{-1}C)}{\det D}\,. \tag{3.29}$$

The notation is in honor of Berezin and therefore the superdeterminant is often called the Berezinian.

It is easy to see that $\mathrm{Ber}\, X^{ST} = \mathrm{Ber}\, X$. The following lemma is proved in [9, 11]:

Lemma 3.7.9. *For two elements X and Y of $\mathrm{GL}_{p|q}(A)$ we have* $\mathrm{Ber}\,(XY) = \mathrm{Ber}\, X\, \mathrm{Ber}\, Y$.

The lemma states that we have a natural transformation $\mathrm{GL}_{p|q} \to \mathrm{GL}_{1|0}$.

Chapter 4

Primes and primaries

In this chapter we study the notion of a prime ideal more profoundly. We define primary ideals and consider primary decompositions.

4.1 Properties of prime ideals

Definition 4.1.1. *Let A be a superring. We call an ideal $\mathfrak{p}$ of A a prime ideal if $\mathfrak{p}$ is properly contained in A and $pq \in \mathfrak{p}$ implies that $p \in \mathfrak{p}$ or $q \in \mathfrak{p}$.*

Due to the defining property, a prime ideal is always $\mathbb{Z}_2$-graded; if $p \in \mathfrak{p}$ then $p_{\bar{1}} \in \mathfrak{p}$ since $(p_{\bar{1}})^2 = 0 \in \mathfrak{p}$. Hence all prime ideals of A contain J_A. In order to check that an ideal is prime, we only need to check the definition 4.1.1 for the homogeneous elements by the following lemma.

Lemma 4.1.2. *Let A be a superring and $\mathfrak{p}$ be a properly contained ideal of A. Then for $\mathfrak{p}$ to be prime it is necessary and sufficient that for all homogeneous elements $p, q \in A$ it follows from $pq \in A$ that $p \in \mathfrak{p}$ or $q \in \mathfrak{p}$.*

Proof. The necessity is clear. To proof the sufficiency, let p, q be arbitrary elements of A with $pq \in \mathfrak{p}$. Since $p_{\bar{1}}$ and $q_{\bar{1}}$ square to zero and $0 \in \mathfrak{p}$, we must have $p_{\bar{1}} \in \mathfrak{p}$ and $q_{\bar{1}} \in \mathfrak{p}$. Hence $p_{\bar{0}}q_{\bar{0}} \in \mathfrak{p}$ and thus $p_{\bar{0}} \in \mathfrak{p}$ or $q_{\bar{0}} \in \mathfrak{p}$. Therefore we conclude $p \in A$ or $q \in A$. ☐

We can equivalently define a prime ideal as an ideal $\mathfrak{p}$ of A such that $A/\mathfrak{p}$ is an integral domain with $0 \neq 1$. It follows that A is a super domain if and only if $\mathrm{Nilrad}(A)$ is a prime ideal.

Proposition 4.1.3. *Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ be a set of $\mathbb{Z}_2$-graded ideals in a superring A. If $\mathfrak{p}$ is a prime ideal of A that contains the product $\mathfrak{a}_1 \cdots \mathfrak{a}_r$, then $\mathfrak{p}$ contains at least one of the $\mathfrak{a}_i$.*

Proof. By induction it is sufficient to consider the case $r = 2$. If $\mathfrak{p}$ does not contain $\mathfrak{a}_1$, consider a homogeneous element $a \in \mathfrak{a}_1$ that does not lie in $\mathfrak{p}$. Then for each $a' \in \mathfrak{a}_2$ the element aa' lies in $\mathfrak{p}$ hence $a' \in \mathfrak{p}$ and thus $\mathfrak{a}_2 \subset \mathfrak{p}$. ☐

A slight variation of proposition 4.1.3 involving the intersection instead of the product is given in the following lemma.

Lemma 4.1.4. *Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals and let $\mathfrak{p}$ be a prime ideal such that $\mathfrak{p} = \cap_i \mathfrak{a}_i$. Then $\mathfrak{p} = \mathfrak{a}_i$ for some i.*

Proof. Suppose $\mathfrak{p} \not\supset \mathfrak{a}_i$ for all i. Then there are homogeneous $x_i \in \mathfrak{a}_i$ such that $x_i \neq \mathfrak{p}$ for all i. But $\prod_i x_i$ lies in the intersection and thus in $\mathfrak{p}$; then one of the x_i should have been in $\mathfrak{p}$ already. Thus we obtain a contradiction. Hence $\mathfrak{p}$ contains at least one of the $\mathfrak{a}_i$, say $\mathfrak{p} \supset \mathfrak{a}_1$. Since $\mathfrak{p} = \cap_i \mathfrak{a}_i$ we have furthermore that $\mathfrak{p} \subset \mathfrak{a}_i$ for all i, hence $\mathfrak{a}_1 \subset \mathfrak{p} \subset \mathfrak{a}_1$. □

Proposition 4.1.5. *Suppose A is a superring and $\mathfrak{m}$ is a maximal ideal of A. If for some integer $n \geq 1$ there is a prime ideal $\mathfrak{p}$ of A that contains $\mathfrak{m}^r$, then $\mathfrak{p} = \mathfrak{m}$.*

Proof. We apply proposition 4.1.3 to the product $\mathfrak{a}_i = \mathfrak{m}^i$ and deduce that $\mathfrak{m} \subset \mathfrak{p}$. Since $\mathfrak{m}$ is maximal, we cannot have a proper inclusion and thus $\mathfrak{p} = \mathfrak{m}$. □

Lemma 4.1.6. *Let A and B be superrings and $f : A \to B$ a morphism of superrings. If $\mathfrak{p}$ is a prime ideal in B, then the inverse image $f^{-1}(\mathfrak{p})$ is a prime ideal of B.*

Proof. We give two proofs, as the result is of great importance. (i): If p, q are elements such that $f(q) \notin \mathfrak{p}$ and $f(pq) \in \mathfrak{p}$ then $f(p) \in \mathfrak{p}$ and thus $p \in f^{-1}(\mathfrak{p})$. ($ii$): It is clear that $f^{-1}(\mathfrak{p})$ is a $\mathbb{Z}_2$-graded ideal in A and that the induced morphism $A/f^{-1}(\mathfrak{p}) \to B/\mathfrak{p}$ is injective. Since $B/\mathfrak{p}$ is an integral domain and $A/f^{-1}(\mathfrak{p})$ is isomorphic to a subring of $B/\mathfrak{p}$, it is an integral domain as well. □

Lemma 4.1.7. *Let A be a superring and $\mathfrak{a}$ a $\mathbb{Z}_2$-graded ideal in A. Then there is a one-to-one correspondence between the prime ideals in A that contain $\mathfrak{a}$ and the prime ideals in $A/\mathfrak{a}$. This correspondence preserves inclusions and hence there is a one-to-one correspondence between the maximal ideals in A that contain $\mathfrak{a}$ and the maximal ideals in $A/\mathfrak{a}$.*

Proof. Suppose $\mathfrak{p}$ is a prime ideal in A that contains $\mathfrak{a}$. Since the projection $\pi : A \to A/\mathfrak{a}$ is surjective, one easily checks that $\pi(\mathfrak{p})$ is a prime ideal in $A/\mathfrak{a}$. The ideal $\pi^{-1}(\pi(\mathfrak{p}))$ is a prime ideal by lemma 4.1.6. Suppose $x \in \pi^{-1}(\pi(\mathfrak{p}))$, then $\pi(x) = \pi(y)$ for some $y \in \mathfrak{p}$. Hence $x - y \in \mathfrak{a} \subset \mathfrak{p}$ and therefore $x \in \mathfrak{p}$. It follows that $\mathfrak{p} = \pi^{-1}(\pi(\mathfrak{p})$.

Conversely, suppose $\mathfrak{b}$ is a prime ideal in $A/\mathfrak{a}$. Then by lemma 4.1.6 $\pi^{-1}(\mathfrak{b})$ is a prime ideal in A and clearly it contains $\mathfrak{a}$. Furthermore, we have $\pi(\pi^{-1}(\mathfrak{b})) = \mathfrak{b}$, so that the correspondence holds.

Clearly, if $\mathfrak{p} \subset \mathfrak{p}'$ are two prime ideals in A, then $\pi(\mathfrak{p}) \subset \pi(\mathfrak{p}')$. Thus the correspondence preserves inclusions. □

An immediate application to the ideal generated by the odd elements gives:

Lemma 4.1.8. *Let A be superring, then there is a one-to-one correspondence between the prime ideals in A and the prime ideals in A/J_A. This correspondence is inclusion preserving and thus there is a one-to-one correspondence between the maximal ideals in A and the maximal ideals in A/J_A.*

Proof. We set $\mathfrak{a} = J_A$ in lemma 4.1.7 and note that all prime ideals contain J_A. □

Lemma 4.1.9. *Let A be a superring. Then* (1) *any prime ideal $\mathfrak{p}$ is of the form $\mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus A_{\bar{1}}$, where $\mathfrak{p}_{\bar{0}}$ is a prime ideal of the commutative ring $A_{\bar{0}}$, and* (2) *any maximal ideal $\mathfrak{m}$ is of the form $\mathfrak{m} = \mathfrak{m}_{\bar{0}} \oplus A_{\bar{1}}$, where $\mathfrak{m}_{\bar{0}}$ is a maximal ideal of the commutative ring $A_{\bar{0}}$.*

Proof. It is clear that $A_{\bar{1}}$ is contained in any prime (resp. maximal) ideal. For any ideal $\mathfrak{p}$ of A containing $A_{\bar{1}}$, $\mathfrak{p}_{\bar{0}}$ is an ideal in $A_{\bar{0}}$ containing $J_A \cap A_{\bar{0}} = (A_{\bar{1}})^2$ and

$$A/\mathfrak{p} \cong A_{\bar{0}}/\mathfrak{p}_{\bar{0}} \,. \tag{4.1}$$

Thus if $\mathfrak{p}$ is a prime (resp. maximal) ideal of A, then $\mathfrak{p}_{\bar{0}}$ is a prime (resp. maximal) ideal of $A_{\bar{0}}$. Conversely, if $\mathfrak{p}_{\bar{0}}$ is a prime (resp. maximal) ideal of $A_{\bar{0}}$, then $\mathfrak{p}_{\bar{0}} \oplus A_{\bar{1}}$ is a prime (resp. maximal) ideal of A. □

Corollary 4.1.10. *Let A be a superring, then the intersection of all prime ideals is the nilradical of A.*

Proof. If $a \in \mathrm{Nilrad}(A)$, then a lies in every prime ideal. Conversely, let I be the intersection of the prime ideals in A, then $\bar{I} = I \bmod J_A$ is the intersection of all prime ideals of A. Thus if a lies in I, then $\bar{a}$ lies in $\bar{I}$ and there exists an integer n such that $\bar{a}^n = 0$, and thus $a^n \in J_A$. But then a^n is nilpotent, hence a is nilpotent. □

Corollary 4.1.11. *Let A be a superring and M an A-module with finite length $l(M)$. Then M has length $l(M)$ as an $A_{\bar{0}}$-module.*

Proof. Let $M = M_0 \supset M_1 \supset \ldots \supset M_n$ with $n = l(M)$ be a decomposition series of M as an A-module. We know that all decomposition series of an $A_{\bar{0}}$-module with finite length have the same length - this is the commutative counterpart of corollary 3.4.8. Dropping the $\mathbb{Z}_2$-parity, we have

$$M_i/M_{i+1} \cong A/\mathfrak{m} \cong A_{\bar{0}}/\mathfrak{m}_{\bar{0}}\,, \tag{4.2}$$

for some maximal ideal $\mathfrak{m} = \mathfrak{m}_{\bar{0}} \oplus A_{\bar{1}}$ in A. By lemma 4.1.9 $\mathfrak{m}_{\bar{0}}$ is a maximal ideal in $A_{\bar{0}}$ and hence the series (4.2) is also a composition series of M as an $A_{\bar{0}}$-module. Thus, M has finite length as an $A_{\bar{0}}$-module and the length is $n = l(M)$. □

Definition 4.1.12. *Let A be a superring and I a $\mathbb{Z}_2$-graded ideal in A. The radical of I is the ideal $\sqrt{I}$ defined by the set of all elements $r \in A$ such that $r^n \in I$ for some positive integer n.*

Suppose $r \in \sqrt{I}$, then $r = r_{\bar{0}} + r_{\bar{1}}$ and $r^n = r_{\bar{0}}^n + n r_{\bar{0}}^{n-1} r_{\bar{1}}$. Thus if I is $\mathbb{Z}_2$-graded, then $r_{\bar{0}}^n \in I$. Hence $r_{\bar{0}} \in \sqrt{I}$, so that also $r_{\bar{1}} \in \sqrt{I}$. We therefore conclude that the radical ideal of a $\mathbb{Z}_2$-graded ideal is $\mathbb{Z}_2$-graded. By lemma 4.1.7 we can equivalently characterize the radical of a $\mathbb{Z}_2$-graded ideal I as follows: $a \in \sqrt{I}$ if and only if $a \bmod I \in \mathrm{Nilrad}(A/I)$.

Using corollary 4.1.10 and we obtain

Lemma 4.1.13. *Let I be a $\mathbb{Z}_2$-graded ideal in a superring A. Then the radical of I is the intersection of all prime ideals in A containing I.*

Proof. The nilradical of A/I is given by the intersection of all prime ideals in A/I. The preimage of the nilradical of A/I under the projection $A \to A/I$ is precisely $\sqrt{I}$ and the preimages of the prime ideals in A/I are by lemma 4.1.7 the prime ideals in A that contain I. Thus the radical of I is the intersection of the prime ideals containing I. □

Let X and Y be algebraic sets in $\mathbb{A}^n$ for some n and suppose X is described by a finite set of polynomial equations $f_i = 0$ and Y by a finite set of polynomial equations $g_j = 0$. Let I be the reduced ideal describing X and let J be the reduced ideal describing Y; $I = \sqrt{(f_i)}$ and $J = \sqrt{(g_j)}$. The union $X \cup Y$ is on the one hand defined by $I \cap J$ and on the other hand by the equations $f_i g_j$, that is, by IJ. One is thus lead to conclude that at least over an algebraically closed field we have $\sqrt{IJ} = \sqrt{I \cap J}$. However, we can prove this in general and even for superrings:

Lemma 4.1.14. *Let I, J be ideals in any superring A. Then $\sqrt{IJ} = \sqrt{I \cap J}$.*

Proof. Since $IJ \subset I \cap J$, any prime ideal containing $I \cap J$ also contains IJ. Let $\mathfrak{p}$ be any prime ideal containing IJ. Suppose $\mathfrak{p}$ does not contain $I \cap J$, then there is an x lying both in I and in J but not in $\mathfrak{p}$. Then $x^2 \in IJ$ and thus $x^2 \in \mathfrak{p}$, so we conclude $x \in \mathfrak{p}$. Thus we obtain a contradiction. Hence if a prime ideal contains IJ it also contains $I \cap J$. Thus the sets $A = \{\mathfrak{p} | \mathfrak{p} \supset IJ\,, \mathfrak{p}\,\mathrm{prime}\}$ and $B = \{\mathfrak{p} | \mathfrak{p} \supset I \cap J\,, \mathfrak{p}\,\mathrm{prime}\}$ are equal and hence also the intersection over the elements of A equals the intersection over the elements of B. By lemma 4.1.13 it then follows that $\sqrt{IJ} = \sqrt{I \cap J}$. □

One also uses the term radical ideal for an ideal I satisfying $\sqrt{I} = I$. It follows directly from the definition of prime ideals that they are radical, although one is also lead to this conclusion by lemma 4.1.13 as any prime ideal contains itself.

Lemma 4.1.15. *Let A be a superring and let I be the intersection $\cap_{\mathfrak{m} \text{ max. ideals}} \mathfrak{m}$ of all maximal ideals of A. Then $x \in I$ if and only if for all $a \in A$ the element $1 - ax$ is invertible.*

Proof. If $x \in I$ and if $1 - ax$ is not invertible, then $1 - (ax)_{\bar{0}}$ is not invertible. The $\mathbb{Z}_2$-graded ideal $(1 - (ax)_{\bar{0}})$ is properly contained in A and thus is contained in some maximal ideal $\mathfrak{m}$, meaning that there is $m \in \mathfrak{m}$ such that $1 - (ax)_{\bar{0}} = m$, but then $1 \in \mathfrak{m}$ since $(ax)_{\bar{0}} \in \mathfrak{m}$. Conversely, if $1 - ax$ is invertible for all $a \in A$ and $\mathfrak{m}$ is a maximal ideal not containing x, then $\mathfrak{m}$ does not contain $x_{\bar{0}}$. The $\mathbb{Z}_2$-graded ideal $(\mathfrak{m}, x_{\bar{0}})$ equals A, implying that there is $a \in A$ and $m \in \mathfrak{m}$ such that $1 = ax_{\bar{0}} + m$. But as $x_{\bar{1}} \in \mathfrak{m}$ we see there exists $a \in A$ such that $1 = ax + m$ for some $m \in \mathfrak{m}$, and thus $1 - ax = m$. The latter identity implies that m is invertible and thus $\mathfrak{m} = A$, which is a contradiction. □

The ideal $I = \cap_{\mathfrak{m} \text{ max. ideals}} \mathfrak{m}$ is called the Jacobson radical of A. Since each nilpotent of A is contained in any maximal ideal, the Jacobson radical contains all nilpotents. If M is a simple A-module, then M must be isomorphic to $A/\mathfrak{m}$ or $\Pi A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. Hence the Jacobson radical annihilates any simple module. Conversely, if I is an ideal such that I annihilates any simple module, then I is contained in the Jacobson radical. The Jacobson radical is the largest ideal that annihilates any simple module. One calls an ideal that annihilates a simple module a primitive ideal. It is easy to see that all primitive ideals are $\mathbb{Z}_2$-graded. We conclude that the Jacobson radical is the intersection of all primitive ideals.

Definition 4.1.16. *Let A be a superring. We call a prime ideal $\mathfrak{p}$ of A minimal if for any prime ideal $\mathfrak{q}$ of A the inclusion $\mathfrak{q} \subset \mathfrak{p}$ implies $\mathfrak{q} = \mathfrak{p}$.*

Let $\mathfrak{a}$ be any $\mathbb{Z}_2$-graded ideal in A. Then we say that a prime ideal $\mathfrak{p}$ is minimal over $\mathfrak{a}$ if $\mathfrak{p}$ contains $\mathfrak{a}$ and for any prime ideal $\mathfrak{q}$ the inclusion $\mathfrak{a} \subset \mathfrak{q} \subset \mathfrak{p}$ implies $\mathfrak{p} = \mathfrak{q}$. Equivalently, a prime ideal $\mathfrak{p}$ is minimal over $\mathfrak{a}$ if and only if the image of $\mathfrak{p}$ in $A/\mathfrak{a}$ is a minimal prime. Thus a minimal prime is a prime ideal minimal over the zero ideal. If a prime ideal $\mathfrak{q}$ contains $\mathfrak{a}$, we also say that $\mathfrak{q}$ lies over $\mathfrak{a}$.

Lemma 4.1.17. *Let A be a superring. A prime ideal $\mathfrak{p}$ of A is minimal if and only if the prime ideal $\bar{\mathfrak{p}}$ of $\bar{A}$ is minimal.*

Proof. Follows from lemma 4.1.8 as the correspondence between the prime ideals of A and the prime ideals of $\bar{A}$ preserves inclusions. □

Proposition 4.1.18. *Let A be a Noetherian superring and $\mathfrak{a}$ an ideal in A, then there are only finitely many prime ideals over $\mathfrak{a}$.*

Proof. Suppose the statement fails. Consider the set S of ideals $\mathfrak{b}$ for which there are not finitely many minimal prime ideals over $\mathfrak{b}$ and suppose $S \neq \emptyset$. Since A is Noetherian, there is a maximal element $\mathfrak{b}$ in S. Clearly, $\mathfrak{b}$ cannot be prime, since then there is only one minimal prime over $\mathfrak{b}$, namely $\mathfrak{b}$ itself. Hence there are $f, g \in A - \mathfrak{b}$ with $fg \in \mathfrak{b}$. The ideals $\mathfrak{b} + (g)$ and $\mathfrak{b} + (f)$ both properly contain $\mathfrak{b}$ and are not equal to A, since $\mathfrak{b}$ is contained in some maximal ideal $\mathfrak{m}$, and thus $f \in \mathfrak{m}$ or $g \in \mathfrak{m}$. If $\mathfrak{p}$ is a prime ideal over $\mathfrak{b}$, then it contains f or g, say f; then $\mathfrak{p}$ is a prime ideal over $\mathfrak{b} + (f)$. Therefore $\mathfrak{p}$ is a minimal prime over $\mathfrak{b}$ if and only if it is a minimal prime over $\mathfrak{b} + (g)$ or $\mathfrak{b} + (f)$. But there are finitely many minimal primes over $\mathfrak{b} + (f)$ and $\mathfrak{b} + (g)$, and then also finitely many minimal primes over $\mathfrak{b}$. But that is a contradiction. □

We immediately obtain by taking $\mathfrak{a} = 0$:

Corollary 4.1.19. *Let A be a Noetherian superring. Then there are only finitely many minimal primes.*

Recall that a super domain is a superring A such that the body $\bar{A}$ is an integral domain. A commutative ring is an integral domain if and only if 0 is a prime ideal. In that case, 0 is the only minimal prime. For superrings the analogue is the following:

Lemma 4.1.20. *A superring A is a super domain if and only if J_A is the only minimal prime.*

Proof. A superring A is a super domain if and only if J_A is a prime ideal. The ideal J_A is a prime ideal if and only if J_A is a minimal prime. □

We now define local superrings and give two characterizations. Later we will often use local superrings and use the different characterizations. As in the commutative case, local superrings have many favorable properties. When we have discussed localization in chapter 5 we can often reduce a problem to the case where the superring is local.

Definition 4.1.21. *We call a superring a local superring if there is one unique maximal ideal.*

Lemma 4.1.22. *A superring A is a local superring if and only if all the non-invertible elements of A form an ideal.*

Proof. If A is a local superring with maximal ideal $\mathfrak{m}$ and x is not invertible, then $x_{\bar{0}}$ is not invertible and $x_{\bar{1}}$ lies in $\mathfrak{m}$. The ideal generated by $x_{\bar{0}}$ is contained in $\mathfrak{m}$ by maximality and uniqueness of $\mathfrak{m}$; hence $x \in \mathfrak{m}$. Thus all non invertible elements lie in $\mathfrak{m}$; the invertible elements cannot lie in $\mathfrak{m}$. On the other hand, if all non invertible elements form an ideal I, then this ideal is automatically the largest ideal since an element outside I is contained only in the trivial ideal A. □

The lemma gives a way to prove that a superring is local; if we have a candidate for the unique maximal ideal and we can show that all elements not in that ideal are invertible we are done. In the sequel we will often use this without mentioning.

Lemma 4.1.23. *A superring A is a local superring with maximal ideal $\mathfrak{m}$ if and only if $\bar{A}$ is a local ring with maximal ideal $\bar{\mathfrak{m}}$.*

Proof. Assume A is a local superring with maximal ideal $\mathfrak{m}$. Then $\bar{\mathfrak{m}}$ is a maximal ideal of $\bar{A}$. If $x \in \bar{A}$ does not lie in $\mathfrak{m}$, then there is an element $y \in A - \mathfrak{m}$ with $\bar{y} = x$. Then y is invertible and hence x. Conversely, if for a superring A the body $\bar{A}$ is a local ring with maximal ideal $\bar{\mathfrak{m}}$, then $\mathfrak{m} = \pi^{-1}(\bar{\mathfrak{m}})$ is a maximal ideal of A, where $\pi : A \to \bar{A}$ is the canonical projection onto the body. If $x \in A - \mathfrak{m}$, then $\bar{x}$ lies outside $\bar{\mathfrak{m}}$ and thus is invertible, and hence x is invertible. □

Lemma 4.1.24. *Let A be local superring with maximal ideal $\mathfrak{m}$. Then $A_{\bar{0}}$ is a local ring with maximal ideal $\mathfrak{m}_{\bar{0}}$.*

Proof. By proposition 3.1.9 there is a one-to-one correspondence between the maximal ideals in $A_{\bar{0}}$ and the maximal ideals in A. Thus A has one unique maximal ideal if and only if $A_{\bar{0}}$ has one unique maximal ideal. □

The following lemmas describe some properties of morphisms between local superrings.

Lemma 4.1.25. *Let A, B be local rings with maximal ideals $\mathfrak{m}, \mathfrak{n}$ respectively and let $\varphi : A \to B$ be a morphism. Then $\varphi^{-1}(\mathfrak{n}) \subset \mathfrak{m}$.*

Proof. Let $x \notin \mathfrak{m}$, then x is invertible and hence $\varphi(x)$ is invertible and thus $\varphi(x) \notin \mathfrak{n}$. □

Definition 4.1.26. *We call a morphism $\varphi : A \to B$ between local rings A, B with maximal ideals $\mathfrak{m}, \mathfrak{n}$ respectively a local morphism if $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$, or equivalently $\mathfrak{m} \subset \varphi^{-1}(\mathfrak{n})$, which is again equivalent to $\varphi(\mathfrak{m}) \subset \mathfrak{n}$.*

Lemma 4.1.27. *Let A, B be local rings with maximal ideals $\mathfrak{m}, \mathfrak{n}$ respectively and let $\varphi : A \to B$ be a morphism, then $\mathrm{Ker}(\varphi) \subset \mathfrak{m}$.*

Proof. Let $x \notin \mathfrak{m}$, then x is invertible and hence $\varphi(x)$ is invertible and hence not zero. □

In the commutative setting a well-known theorem due to Cohen [52] states that a ring is Noetherian if and only if all prime ideals are finitely generated. This reduces the problem of checking whether a ring is Noetherian to proving that all prime ideals are finitely generated. One can prove the generalization of Cohen's structure theorem for superrings, as we show in the next proposition:

Proposition 4.1.28. *A superring A is Noetherian if and only if all prime ideals are finitely generated.*

Proof. We follow more or less the proof of [19]. The necessity is clear. To prove sufficiency we show that if the set $\mathcal{S}$ of proper $\mathbb{Z}_2$-graded ideals that are not finitely generated is nonempty, it contains a maximal element, which is a prime ideal.

We order $\mathcal{S}$ by inclusion. Consider a totally ordered subset $\mathcal{P}$ of $\mathcal{S}$. The object $\mathcal{I} = \cup_{I \in \mathcal{P}} I$ is a $\mathbb{Z}_2$-graded proper ideal that is not a finitely generated ideal; if $\mathcal{I}$ were not finitely generated, there would be an ideal in $\mathcal{P}$ that contains all generators and hence is finitely generated. Thus $\mathcal{I}$ is an upper bound of $\mathcal{P}$. By Zorn's lemma, $\mathcal{S}$ contains a maximal element $\mathfrak{m} \in S$.

Suppose $\mathfrak{m}$ were not a prime ideal. Then there are homogeneous $a, b \in A - \mathfrak{m}$ and $ab \in \mathfrak{m}$. Consider the ideal $\mathfrak{m}'$ generated by $\mathfrak{m}$ and a and consider the ideal $\mathfrak{m}''$ of all elements $x \in A$ such that $ax \in \mathfrak{m}$. Since a is homogeneous, $\mathfrak{m}'$ and $\mathfrak{m}''$ are $\mathbb{Z}_2$-graded ideals. Also, $a \in \mathfrak{m}'$ and $b \in \mathfrak{m}''$ so that both contain $\mathfrak{m}$ properly. If $\mathfrak{m}' = A$, then there are $x \in A$ and $m \in \mathfrak{m}$ with $xa + m = 1$ but then $b = (xa + m)b \in \mathfrak{m}$, contradicting $b \notin \mathfrak{m}$. Hence $\mathfrak{m}'$ is finitely generated. If $\mathfrak{m}'' = A$ then $a = 1a \in \mathfrak{m}$, which is also impossible by assumption and thus $\mathfrak{m}''$ is finitely generated as well. Let $\{u_i + av_i\}_{1 \leq i \leq r}$ be homogeneous generators for $\mathfrak{m}'$ with $u_i \in \mathfrak{m}$ and $v_i \in A$ and let $\{w_j\}_{1 \leq j \leq s}$ be homogeneous generators for $\mathfrak{m}''$. Let $x \in \mathfrak{m}$, then since $\mathfrak{m} \subset \mathfrak{m}'$ there are $x_i \in A$ with

$$x = \sum_{i=1}^{n} (u_i + av_i)x_i = \sum_{i=1}^{n} u_i x_i + a \sum_{i=1}^{n} v_i x_i \,. \tag{4.3}$$

Since the sum $\sum_i v_i x_i$ lies in $\mathfrak{m}''$ the set $\{u_i\}_{1 \leq i \leq r} \cup \{aw_j\}_{1 \leq j \leq s}$ generates $\mathfrak{m}$. Hence $\mathfrak{m} \notin \mathcal{S}$, which is a contradiction. Therefore $\mathfrak{m}$ must be a prime ideal. □

Theorem 4.1.29. *Let A be an Artinian superalgebra, then A is Noetherian and all prime ideals are maximal and there are only finitely many maximal ideals.*

Proof. Let S be the set of ideals that are a product of maximal ideals. Since A is Artinian we can find a minimal element $J \in S$. Since J is minimal we have $\mathfrak{m} \supset J\mathfrak{m} = J$ for all maximal ideals $\mathfrak{m}$. Hence J is contained in the Jacobson radical. Also $J^2 \subset J$ is a product of maximal ideals, hence $J^2 = J$. Suppose $J \neq 0$, then we can choose an ideal I that is minimal among the ideals that do not annihilate J. Then we have $(IJ)J = IJ^2 = IJ \neq 0$ and thus $IJ \subset I$ and therefore $IJ = I$. Since I does not annihilate J, there is a homogeneous x in I with $xJ \neq 0$. But then we must have $(x) = I$ and there must be an $j \in J$ with $x = xj$ and hence $(1 - j)x = 0$. But j is contained the Jacobson radical and thus $1 - j$ is invertible. Hence $x = 0$. But then the assumption $J \neq 0$ is false; thus $J = 0$.

We conclude that we can write 0 as a product of maximal ideals, and by Artinianity thus $0 = \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_t$ for some maximal ideals $\mathfrak{m}_i$. For each i we have that $V_i = \mathfrak{m}_1 \cdots \mathfrak{m}_i/\mathfrak{m}_1 \cdots \mathfrak{m}_{i+1}$ is a super vector space over $A/\mathfrak{m}_{i+1}$. We now wish to show that the dimension of such super vector spaces is finite. Any chain of subspaces in V_i corresponds to a chain of ideals in A containing $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{i+1}$. The chain in A terminates and hence the chain in V_i too; hence the dimension of V_i is finite. Putting the composition series together (really concatenation) we see that A has a finite length and is Noetherian.

Suppose that $\mathfrak{p}$ is a prime ideal in A. Then $\mathfrak{p} \supset 0 = \mathfrak{m}_1 \cdots \mathfrak{m}_t$. By proposition 4.1.3 $\mathfrak{p}$ contains one of the maximal ideals $\mathfrak{m}_i$ and thus $\mathfrak{m}_i = \mathfrak{p}$. Hence there are only finitely many prime ideals. □

A superring that has only finitely many maximal ideals is called a semilocal superring - some people prefer the name quasi-local. Theorem 4.1.29 then states that Artinian superrings are semilocal. We will not go into the theory of semilocal superrings.

4.2 Primary ideals and primary decompositions

Definition 4.2.1. *We call an ideal $\mathfrak{q}$ of A a primary ideal if $\mathfrak{q}$ is $\mathbb{Z}_2$-graded, $\mathfrak{q}$ is properly contained in A and if $x \notin \mathfrak{q}$ and $y \notin \mathfrak{q}$ but $xy \in \mathfrak{q}$ then we have $x^r \in \mathfrak{q}$ and $y^s \in \mathfrak{q}$ for some natural numbers r and s.*

Or equivalently

Definition 4.2.2. *We call an ideal $\mathfrak{q}$ of A a primary ideal if $\mathfrak{q}$ is $\mathbb{Z}_2$-graded, and $A/\mathfrak{q}$ is a nonzero superring such that all zerodivisors are nilpotent.*

We need to impose that a primary ideal is $\mathbb{Z}_2$-graded, as the following example shows:

Example 4.2.3. Take $A = k[x|\vartheta]$ und $\mathfrak{q} = (x^2 + 2x\vartheta, x^3)$. Then A is in fact commutative and we can safely consider the quotient $A' = A/\mathfrak{q}$, which is a ring (no longer a superring). As k-vector space we have $A' = k \oplus k\bar{x} \oplus k\bar{\vartheta} + k\bar{x}\bar{\vartheta}$, where a bar denotes the image in A'. We easily see that $a \in A'$ is either a unit or nilpotent. Hence $\mathfrak{q}$ is primary in the ordinary sense, but not $\mathbb{Z}_2$-graded, since $x^2 \notin \mathfrak{q}$. △

Proposition 4.2.4. *A $\mathbb{Z}_2$-graded ideal $\mathfrak{q}$ is primary if and only if for all homogeneous $x, y \in A - \mathfrak{q}$ with $xy \in \mathfrak{q}$, the images of x and y in $A/\mathfrak{q}$ are nilpotent.*

Proof. Clearly the condition is necessary. To prove sufficiency, let $x, y \in A - \mathfrak{q}$ with $xy \in \mathfrak{q}$ and suppose $\mathfrak{q}$ satisfies the condition stated in the proposition. Then since $\mathfrak{q}$ is $\mathbb{Z}_2$-graded $x_{\bar{0}}y_{\bar{0}} + x_{\bar{1}}y_{\bar{1}} \in \mathfrak{q} \subset \sqrt{\mathfrak{q}}$. Since $x_{\bar{1}}y_{\bar{1}} \in \sqrt{\mathfrak{q}}$ we have $x_{\bar{0}}y_{\bar{0}} \in \sqrt{\mathfrak{q}}$, but then $(x_{\bar{0}})^N(y_{\bar{0}})^N \in \mathfrak{q}$ and thus $x_{\bar{0}}$ and $y_{\bar{0}}$ are nilpotent in $A/\mathfrak{q}$ and thus in $\sqrt{\mathfrak{q}}$. But then $x_{\bar{0}}, y_{\bar{0}}, x_{\bar{1}}, y_{\bar{1}}$ are all in $\sqrt{\mathfrak{q}}$ and thus also $x_{\bar{0}} + x_{\bar{1}}$ and $y_{\bar{0}} + y_{\bar{1}}$. Thus x and y are nilpotent in $A/\mathfrak{q}$. Hence $\mathfrak{q}$ is primary. □

Proposition 4.2.5. *Let $\mathfrak{q}$ be a primary ideal. Then $\sqrt{\mathfrak{q}}$ is the smallest prime ideal containing $\mathfrak{q}$.*

Proof. It suffices to prove that $\sqrt{\mathfrak{q}}$ is a prime ideal. Further note that $A_{\bar{1}} \subset \sqrt{\mathfrak{q}}$. It thus suffices to show that when the product of two even elements x and y lies in $\sqrt{\mathfrak{q}}$ then so does x or y. So assume x, y are even and $xy \in \sqrt{\mathfrak{q}}$. Then $x^n y^n \in \mathfrak{q}$ for some integer n. Thus if $x^n \notin \mathfrak{q}$ and $y^n \notin \mathfrak{q}$ then y^{mn} and x^{mn} lie in $\mathfrak{q}$ for some m. But then x and y are elements of $\sqrt{q}$. □

Lemma 4.2.5 allows us to make the following definition:

Definition 4.2.6. *When we write $\mathfrak{p} = \sqrt{\mathfrak{q}}$ for a primary ideal $\mathfrak{q}$, we say that $\mathfrak{q}$ is $\mathfrak{p}$-primary.*

The following lemmas are straightforward generalizations of standard results in commutative algebra. We follow the presentation of [16].

Proposition 4.2.7. *Let A be a superring and $\mathfrak{a}$ an ideal in A such that $\sqrt{\mathfrak{a}}$ is a maximal ideal. Then $\mathfrak{a}$ is a primary ideal.*

Proof. Let $\mathfrak{m} = \sqrt{\mathfrak{a}}$ and let $\bar{\mathfrak{m}}$ be the image of $\mathfrak{m}$ under the projection $A \to A/\mathfrak{a}$. Clearly all elements of $\bar{\mathfrak{m}}$ are nilpotent. If $y \bmod \mathfrak{a}$ is not in $\mathfrak{m}$, then $y \notin \mathfrak{m}$. Hence there is $z \in A$ with $yz = 1 + m$ for some $m \in \mathfrak{m}$. Thus $(y \bmod \mathfrak{a})(z \bmod \mathfrak{a}) = 1 + u$ with u nilpotent. But then $x \bmod \mathfrak{a}$ is invertible in $A/\mathfrak{a}$. Therefore, all elements of $A/\mathfrak{a}$ are either nilpotent or invertible and thus any zerodivisor is nilpotent. □

Lemma 4.2.8. *Let $\mathfrak{q}_i$ be $\mathfrak{p}$-primary ideals for $1 \leq i \leq n$. Then $\mathfrak{q} = \bigcap_{i=1}^n \mathfrak{q}_i$ is also $\mathfrak{p}$-primary.*

Proof. First we note that $\sqrt{\bigcap_{i=1}^n \mathfrak{q}_i} = \bigcap_{i=1}^n \sqrt{\mathfrak{q}_i} = \mathfrak{p}$. Thus we have to prove that $\mathfrak{q}$ is a primary ideal. Suppose x, y are homogeneous elements of A that do not lie in $\mathfrak{a}$ but xy lies in $\mathfrak{a}$. Then $x \notin \mathfrak{q}_k$, $y \notin \mathfrak{q}_l$ for some k, l but $xy \in \mathfrak{q}_i$ for all i. If $k \neq l$ then y lies in $\sqrt{\mathfrak{q}_k} = \mathfrak{p}$ and x lies in $\sqrt{\mathfrak{q}_l} = \mathfrak{p}$. If $k = l$ then x and y lie in $\sqrt{\mathfrak{q}_k} = \mathfrak{p}$. Hence in any case there are positive integers m, n such that x^m lies in $\mathfrak{q}_i$ for all i and y^n lies all $\mathfrak{q}_i$ for all i. □

For a $\mathbb{Z}_2$-graded ideal $\mathfrak{a}$ in a superring A and a homogeneous element $a \in A$ we write $(\mathfrak{a} : a)$ for the ideal consisting of those elements $b \in A$ such that $ba \in \mathfrak{a}$. One easily checks that $(\mathfrak{a} : a)$ is a $\mathbb{Z}_2$-graded ideal. The elements of $(0 : a)$ are the annihilators of a. Clearly if $a \in \mathfrak{a}$ then $(\mathfrak{a} : a) = A$.

Lemma 4.2.9. *Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal and $x \in A$ homogeneous. (i) If $x \in \mathfrak{q}$ then $(\mathfrak{q} : x) = A$. (ii) If $x \notin \mathfrak{q}$ then $(\mathfrak{q} : x)$ is $\mathfrak{p}$-primary. (iii) If $x \notin \mathfrak{p}$ then $(\mathfrak{q} : x) = \mathfrak{q}$.*

Proof. (i) is obvious. (ii): Suppose $y \in (\mathfrak{q} : x)$, then $xy \in \mathfrak{p}$. As $x \notin \mathfrak{q}$ then we must have $y \in \mathfrak{p}$. Thus $\mathfrak{q} \subset (\mathfrak{q} : x) \subset \mathfrak{p}$ and thus by taking radicals we see $\sqrt{(\mathfrak{q} : x)} = \mathfrak{p}$. Now we need to prove that $(\mathfrak{q} : x)$ is a primary ideal. Suppose yz are homogeneous, not in $(\mathfrak{q} : x)$ but $yz \in (\mathfrak{q} : x)$. Then it follows that $xyz \in \mathfrak{q}$. By assumption $xy \notin \mathfrak{q}$ and $xz \notin \mathfrak{q}$, so that we must have $y \in \mathfrak{p}$ and $z \in \mathfrak{p}$. (iii) If $x \notin \mathfrak{p}$, then $xy = 0$ has no solutions in $A/\mathfrak{q}$ other than $y = 0$. Thus $(\mathfrak{q} : x) = \mathfrak{q}$ in this case. □

4.3 Primary decompositions

In this section all superrings are assumed to be Noetherian. An ideal $\mathfrak{a}$ in A is said to be irreducible if for any decomposition $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$, where all three are $\mathbb{Z}_2$-graded, it follows that $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} = \mathfrak{c}$. Using the assumption that the superring is Noetherian we obtain:

Lemma 4.3.1. *Every $\mathbb{Z}_2$-graded ideal is a finite intersection of irreducible $\mathbb{Z}_2$-graded ideals.*

Proof. Suppose the set S of $\mathbb{Z}_2$-graded ideals that are not an intersection of irreducible $\mathbb{Z}_2$-graded ideals is nonempty. Since A is Noetherian, S has a maximal element $\mathfrak{a}$. But $\mathfrak{a}$ cannot be irreducible, hence $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ where $\mathfrak{b}$ and $\mathfrak{c}$ properly contain $\mathfrak{a}$ and are $\mathbb{Z}_2$-graded. Hence $\mathfrak{b}$ and $\mathfrak{c}$ are finite intersections of irreducible $\mathbb{Z}_2$-graded ideals, and hence also $\mathfrak{a}$, contradicting $\mathfrak{a} \in S$. Thus the assumption $S \neq \emptyset$ is false. □

Lemma 4.3.2. *Every irreducible $\mathbb{Z}_2$-graded ideal is a primary ideal.*

Proof. Let $\mathfrak{q}$ be an irreducible $\mathbb{Z}_2$-graded ideal. We claim that we only need to prove that if the zero ideal in $A' = A/\mathfrak{q}$ is irreducible, it is primary. To prove the claim: Since $\mathfrak{q}$ is irreducible, the zero ideal in A' is irreducible. If x, y are in A, but not in $\mathfrak{q}$, such that xy lies in $\mathfrak{q}$, then x and y are nilpotent in A' as the zero ideal in A' is primary. Thus $\mathfrak{q}$ is primary. So we proceed to prove that if the zero ideal is irreducible, it is primary.

Let $xy = 0$ and suppose that $y \neq 0$ and x, y are homogeneous (this we can do using corollary 3.1.4). Then consider the chain $\mathrm{Ann}(x) \subset \mathrm{Ann}(x^2) \subset \ldots$. This chain becomes stationary and thus there is an n such that $\mathrm{Ann}(x^n) = \mathrm{Ann}(x^{n+1}) = \ldots$. We claim that then $(x^n) \cap (y) = 0$: if $a \in (y)$ then $ax = 0$ and if furthermore $a = bx^n$ then $bx^{n+1} = 0$. But then $b \in \mathrm{Ann}(x^{n+1}) = \mathrm{Ann}(x^n)$ and thus $a = 0$. Since 0 is assumed to be irreducible and $y \neq 0$, we must have x^n=0. This shows that the zero ideal is primary. □

Let $\mathfrak{a}$ be a $\mathbb{Z}_2$-graded ideal in A. Then we call a primary decomposition of $\mathfrak{a}$ a decomposition $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_k$, where the $\mathfrak{q}_i$ are primary ideals.

Corollary 4.3.3. *Every $\mathbb{Z}_2$-graded ideal has a primary decomposition.*

Proof. From the above two lemmas we see that any ideal is a finite intersection of irreducible ideals, and thus a finite intersection of primary ideals. □

We now ask whether a primary decomposition is unique in a certain sense. Of course, we then should require that for all i we have $\mathfrak{q} \not\supseteq \cap_{i \neq j} \mathfrak{q}_j$. Furthermore, if some of the $\mathfrak{q}_i$ have the same radical ideal $\mathfrak{p}$, then their intersection is also a $\mathfrak{p}$-primary ideal. Hence, we are led to the following definition of a minimal decomposition:

Definition 4.3.4. *Let $\mathfrak{a}$ be an ideal. We call a decomposition $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$ minimal if all the radicals $\sqrt{\mathfrak{q}_i}$ are distinct and for all i we have $\mathfrak{q} \not\supseteq \cap_{i \neq j} \mathfrak{q}_j$.*

Our first result about uniqueness is captured in the following proposition:

Proposition 4.3.5. *Let $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$ be a minimal primary decomposition and write $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$. Then the $\mathfrak{p}_i$ are independent of the decomposition.*

Proof. Let $x \in A$ be homogeneous, then $(\mathfrak{a} : x) = (\cap_i \mathfrak{q}_i : x) = \cap_i (\mathfrak{q}_i : x)$. Hence by lemma 4.2.9 we have $\sqrt{\mathfrak{a} : x} = \cap_i \sqrt{\mathfrak{q}_i : x} = \cap_{i : x \neq \mathfrak{q}_i} \mathfrak{p}_i$. Since the decomposition is minimal, we can find for all i an homogeneous x such that $x \neq \mathfrak{q}_i$ but $x \in \mathfrak{q}_j$ for all $j \neq i$. Then $\sqrt{\mathfrak{a} : x} = \mathfrak{p}_i$. So suppose $\sqrt{\mathfrak{a} : x}$ is any prime ideal $\mathfrak{p}$, then $\mathfrak{p} = \cap_{i : x \neq \mathfrak{q}_i} \mathfrak{p}_i$. Using lemma 4.1.4 we find that $\mathfrak{p}$ is one of the $\mathfrak{p}_i$. Hence the $\mathfrak{p}_i$ in any minimal primary decomposition are the prime ideals occurring in the set

$$\left\{ \sqrt{\mathfrak{a} : x} \,|\, x \in A,\, x \,\text{homogeneous} \right\} ,$$

which proves the claim. □

Corollary 4.3.6. *The prime ideals occurring as the radicals of the primary ideals in a primary decomposition of $\mathfrak{a}$ are precisely the prime ideals that occur as annihilators of homogeneous elements of $A/\mathfrak{a}$.*

Proof. Let x in A be homogeneous. Consider $\bar{x}$ the image of x in $A/\mathfrak{a}$. Then the annihilator of $\bar{x}$ is $(\mathfrak{a} : x)$. But the prime ideals of the primary decomposition in $\mathfrak{a}$ are precisely the prime ideals $\mathfrak{p}$ for which there exists a homogeneous $x \in A$ with $\mathfrak{p} = (\mathfrak{a} : x)$. □

Definition 4.3.7. *Let $\mathfrak{a}$ be a $\mathbb{Z}_2$-graded ideal in A and let $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$ be a minimal primary decomposition of $\mathfrak{a}$. Then the prime ideals $\sqrt{\mathfrak{q}_i}$ are called the primes associated to $\mathfrak{a}$. The minimal primes along the $\sqrt{\mathfrak{q}_i}$ (that is, $\sqrt{\mathfrak{q}_i}$ is minimal if it is not contained in another $\sqrt{\mathfrak{q}_j}$) are called the minimal or isolated primes belonging to $\mathfrak{a}$. Those that are not minimal are called the embedded prime ideals.*

Remark 4.3.8. The names isolated, minimal and embedded come from algebraic geometry: an ideal $\mathfrak{a}$ gives rise to a variety X. The primary ideals in a primary decomposition of $\mathfrak{a}$ describe the irreducible components and their subvarieties. Thus the minimal primes are then the radical ideals describing the maximal irreducible subvarieties, i.e., the irreducible components of X.

Proposition 4.3.9. *Let $\mathfrak{a}$ be any $\mathbb{Z}_2$-graded ideal in A. Then any prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$ also contains a minimal prime associated to $\mathfrak{a}$. Hence the minimal primes associated to $\mathfrak{a}$ are precisely the minimal ones among the prime ideals containing $\mathfrak{a}$.*

Proof. Let $\mathfrak{p}$ be a prime ideal containing $\mathfrak{a}$. Then $\sqrt{\mathfrak{p}} = \mathfrak{p} \supset \sqrt{\mathfrak{a}} = \cap \mathfrak{p}_i$, where the $\mathfrak{p}_i$ are the prime ideals associated to $\mathfrak{a}$. In fact, the intersection only goes over the minimal primes associated to $\mathfrak{a}$. By the same reasoning as in the proof of lemma 4.1.4 we see $\mathfrak{p}$ must contain one of the minimal $\mathfrak{p}_i$. Since $\mathfrak{a} \subset \sqrt{\mathfrak{a}} = \cap \{\mathfrak{p} \,|\, \mathfrak{p} \text{ is a minimal prime associated to } \mathfrak{a}\}$. Hence $\mathfrak{a}$ is contained in all of its minimal primes. This proves the claim. □

We already characterized the radical ideal of an ideal $\mathfrak{a}$ as the intersection of all prime ideals containing $\mathfrak{a}$ in lemma 4.1.13. The intersection only needs to be taken over those prime ideals containing $\mathfrak{a}$ that are minimal over $\mathfrak{a}$, which are only finitely many in Noetherian superrings. The following corollary to proposition 4.3.9 relates this to the minimal primes associated to $\mathfrak{a}$:

Corollary 4.3.10. *The radical ideal of an ideal $\mathfrak{a}$ is the intersection of the minimal primes associated to $\mathfrak{a}$.*

Remark 4.3.11. Thus we have another proof of the fact that any $\mathbb{Z}_2$-graded ideal in a Noetherian superring only has a finite number of minimal primes among the primes containing it. Taking the ideal to be the zero ideal, we see that a Noetherian superring only has finitely many minimal prime ideals. The geometric statement is that any affine variety only has a finite number of irreducible components.

Proposition 4.3.12. *Let $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$ be a primary decomposition of $\mathfrak{a}$ and let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ be the primes associated to $\mathfrak{a}$. Then the union of all these prime ideals is given by*

$$\cup_i \mathfrak{p}_i = \{x \in A \text{ homogeneous} \,|\, (\mathfrak{a} : x) \neq \mathfrak{a}\} \,. \tag{4.4}$$

Proof. The set D of zerodivisors of a superring is given by (see 3.1.4)

$$D = \bigcup_{x \neq 0 \text{ homogeneous}} \operatorname{Ann}(x) \,,$$

hence $D = \sqrt{D} = \cup_{x \neq 0} \sqrt{\operatorname{Ann}(x)}$. The annihilator of an element x is given by $\operatorname{Ann}(x) = (0 : x)$. The set E of elements that form the zerodivisors modulo $\mathfrak{a}$ is thus

$$E = \cup_{x \notin \mathfrak{a}} \sqrt{(\mathfrak{a} : x)} \,.$$

From lemma 4.2.9 it follows that $\sqrt{(\mathfrak{a} : x)}$ is the intersection of the prime ideals $\mathfrak{p}_j$ such that $x \notin \mathfrak{q}_i$. Thus for $x \notin \mathfrak{a}$ we see that the ideal $(\mathfrak{a} : x)$ is contained in some $\mathfrak{p}_j$. Hence $E \subset \cup_j \mathfrak{p}_i$. On the other hand, each $\mathfrak{p}_i$ is of the form $\sqrt{(\mathfrak{a} : x)}$, and thus $\mathfrak{p}_i$ is contained in E. □

Corollary 4.3.13. *The set of zerodivisors of A is the union of the prime ideals associated to 0.*

Chapter 5

Localization and completion

In this chapter we discuss two important notions that will be used frequently in the sequel: completion and localization. The two constructions do not differ much from their counterparts in commutative algebra. The purpose of this chapter is then to see which properties of localizations and completions that are familiar for commutative rings do hold for superrings as well. In section 5.2 we apply the knowledge of completions to say more on Artinian superrings and in section 5.3 we specialize to a class of superrings, namely, to quotients of Grassmann algebras. In section 5.4 we start with a rudimentary version of the theory of superschemes. In the sections starting from 5.5 we discuss completions of superrings. In order to introduce and to work with completions we have to know some properties of filtered superrings and associated graded superrings. As an application of the results on filtrations and graded superrings, we discuss the Artin–Rees lemma for superrings in section 5.7. In section 5.12 we discuss the structure theorem of Cohen for superrings.

5.1 Localization

Let A be a superring. A multiplicative set in A is a set $S \subset A$ such that $1 \in S$ and for all $a, b \in S$ the product ab also lies in S. As in general $\mathbb{Z}_2$-graded rings, to have a suitable notion of localization we need a multiplicative set that consists of homogeneous elements only. However, in superrings the odd elements square to zero and when $0 \in S$ the localized rings are zero. We therefore consider only those multiplicative sets of a superring A that lie in the even part $A_{\bar{0}}$.

Let A be a superring and let S be a multiplicative set in $A_{\bar{0}}$. We define a superring $S^{-1}A$, called the localization of A at S, as follows: as a set $S^{-1}A$ is defined to be quotient $S \times A/\sim$, with $(s, a) \sim (s', a')$ if and only if there exists $z \in S$ with $z(as' - a's) = 0$. We write a/s or $\frac{a}{s}$ for the equivalence class of (s, a). The multiplication is defined by $a/s \cdot a'/s' = aa'/ss'$ and addition by

$$\frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'}\,, \tag{5.1}$$

where it is easily verified that the right-hand side is independent of the chosen representatives. The unit element is $1/1$ and the $\mathbb{Z}_2$-grading is given by $a/s \in (S^{-1}A)_j$ if and only if $a \in A_j$, so that $(S^{-1}A)_{\bar{0}} \cap (S^{-1}A)_{\bar{1}} = 0$ and hence $(S^{-1}A) = (S^{-1}A)_{\bar{0}} \oplus (S^{-1}A)_{\bar{1}}$. Note that $0 \in S$ if and only if $S^{-1}A = 0$. We denote i_S the canonical morphism $A \to S^{-1}A$ that maps $a \in A$ to $a/1$. If S is the set $\{f^n\}_{n \geq 0}$ for some $f \in A_{\bar{0}}$ then we write $S^{-1}A = A_f$.

Proposition 5.1.1. *Let A be a superring and S a multiplicative set in $A_{\bar{0}}$ with canonical morphism $i_S : A \to S^{-1}A$. The localization of A at S has the following universal property: The morphism*

$i_S : A \to S^{-1}A$ maps elements of S to invertible elements in $S^{-1}A$ and if $f : A \to B$ is any morphism of superrings such that the elements of $f(S)$ are invertible in B, then there exists a unique morphism of superrings $f' : S^{-1}A \to B$ with $f' \circ i_S = f$.

Proof. That all elements in $i_S(S)$ are invertible is obvious. Suppose we are given a map $f : A \to B$. For any $s \in S$ we have $1 = f'(s/s) = f'(s/1)f'(1/s) = f'(i_S(s))f'(1/s) = f(s)f'(1/s)$ so that $f'(1/s) = f(s)^{-1}$. We conclude that f' is uniquely determined and given by $f'(a/s) = f'(i_S(a))f(s)^{-1} = f(a)f(s)^{-1}$. □

Hence $S^{-1}A$ has the usual universal property; the pair $(i_S, S^{-1}A)$ with the properties announced in proposition 5.1.1 is therefore determined up to isomorphism. The following two propositions are straightforward and only rely on the universal property. Also see [18] for the commutative case.

Proposition 5.1.2. *Let A and B be superrings and let S be a multiplicative set in $A_{\bar{0}}$ and T a multiplicative set in $B_{\bar{0}}$. Let $f : A \to B$ be a morphism with $f(S) \subset T$. Then there exists a unique morphism $f' : S^{-1}A \to T^{-1}B$ such that the following diagram commutes:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ S^{-1}A \arrow[r, "f'"'] & T^{-1}B \end{tikzcd} , \qquad (5.2)$$

where the vertical arrows represent the canonical morphisms. If furthermore we have $f(S) = T$, then f' is injective (resp. surjective) when f' is injective (resp. surjective).

Proof. Applying the universal property of proposition 5.1.1 to the composite $A \to B \to T^{-1}B$ gives a morphism $f' : S^{-1}A \to T^{-1}B$ such that diagram (5.2) commutes. The morphism f' maps a/s to $f(a)/f(s)$. Suppose $f(S) = T$ and $f(a/s) = 0$, then since $f(S) = T$ there is $s' \in S$ with $f(s')f(a) = 0$ but then a/s is already zero. Surjectivity is clear. □

Proposition 5.1.3. *Let A be a superring, and let S and T be two multiplicative sets in $A_{\bar{0}}$. Let $i_S : A \to S^{-1}A$ be the canonical morphism and let $T' = i(T)$, then we have a canonical isomorphism $(ST)^{-1}A \cong (T')^{-1}(S^{-1}A)$.*

Proof. If $f : A \to B$ is a morphism such that the set $f(ST)$ consists of invertible elements in B, then also the set $f(S)$ consists of invertible elements and hence there exists a unique morphism $f' : S^{-1}A \to B$ with $f' \circ i_S = f$. The elements $f'(T')$ are invertible in B and hence there exists a unique morphism $f'' : T'^{-1}(S^{-1}A) \to B$ with $f'' \circ i_{T'} = f'$, where $i_{T'} : S^{-1}A \to T'^{-1}(S^{-1}A)$ is the canonical morphism. We put $i = i_{T'} \circ i_S$, so that $f = f'' \circ i$. Clearly if g is such that $f = g \circ i$, then $(g \circ i_{T'}) \circ i_S = f$ and hence $g \circ i_{T'} = f'$. It follows that $g = f''$. Hence $T'^{-1}(S^{-1}A)$ has the same universal property as $(ST)^{-1}A$ and therefore $(ST)^{-1}A \cong T'^{-1}(S^{-1}A)$ □

Localizing commutes with going to the body of a superalgebra in the following sense:

Proposition 5.1.4. *Let A be a superring, S a multiplicative set in $A_{\bar{0}}$ and let $\bar{S}$ be the image of S under the projection $A \to \bar{A}$. Then $\overline{S^{-1}A} \cong \bar{S}^{-1}\bar{A}$.*

Proof. The $\mathbb{Z}_2$-graded ideal in $S^{-1}A$ generated by the odd elements of $S^{-1}A$ is $J_A \cdot S^{-1}A$. Consider the canonical morphism $p : S^{-1}A \to \overline{S^{-1}A}$ and the morphism $g : \overline{S^{-1}A} \to \bar{S}^{-1}\bar{A}$ given by $a/s \bmod J_A \cdot S^{-1}A \mapsto \bar{a}/\bar{s}$. Clearly, the morphism g is surjective. Suppose that $a/s \bmod J_A \cdot S^{-1}A$

is mapped to zero by g, then there is $s' \in S$ such that $\bar{s}'\bar{a} = 0$. Therefore $s'a \in J_A$, which implies $a/s = as'/ss' \in J_A \cdot S^{-1}A$. Alternatively, one can argue using lemma 5.1.2: we take $f : A \to B$ to be the projection $\pi : A \to \bar{A}$ and $T = \pi(S)$. The kernel of $\pi' : S^{-1}A \to \bar{S}^{-1}\bar{A}$ is then found to be the $\mathbb{Z}_2$-graded ideal generated by the nilpotent elements of $S^{-1}A$. ☐

We now investigate the relation between the $\mathbb{Z}_2$-graded ideals of $S^{-1}A$ and the $\mathbb{Z}_2$-graded ideals of A. We use this to prove that the localizations of Noetherian rings are Noetherian.

Lemma 5.1.5. *Let A be a superring, S a multiplicative set in $A_{\bar{0}}$ and $i : S \to S^{-1}A$ the canonical morphism. If I is a $\mathbb{Z}_2$-graded ideal in $S^{-1}A$, then $i^{-1}(I) \cdot S^{-1}A = I$.*

Proof. Clearly $i^{-1}(I) \cdot S^{-1}A \subset I$. Conversely, if $a/s \in I$ then $a \in i^{-1}(I)$ and thus it follows that $a/s \in i^{-1}(I) \cdot S^{-1}A$. ☐

Using lemma 5.1.5 the proof of the following corollary is trivial:

Corollary 5.1.6. *The map $\Psi : I \mapsto i^{-1}(I)$ is an injective map from the set of $\mathbb{Z}_2$-graded ideals in $S^{-1}A$ to the set of $\mathbb{Z}_2$-graded ideals in A. Furthermore the map Ψ preserves inclusions.*

Corollary 5.1.7. *Let A be a Noetherian superring and S a multiplicative set in $A_{\bar{0}}$. Then $S^{-1}A$ is Noetherian.*

Proof. If I is a $\mathbb{Z}_2$-graded ideal in $S^{-1}A$, then $I = i^{-1}(I) \cdot S^{-1}A$ and $i^{-1}(I)$ is finitely generated. ☐

Proposition 5.1.8. *Let A be a superring, S a multiplicative set in $A_{\bar{0}}$ and $i : A \to S^{-1}A$ the canonical morphism. There is a one-to-one inclusion preserving correspondence between the prime ideals in $S^{-1}A$ and the prime ideals in A not meeting S.*

Proof. By lemma 4.1.6, if $\mathfrak{p}$ is a prime ideal in $S^{-1}A$ then $i^{-1}(\mathfrak{p})$ is a prime ideal in A. For a $\mathbb{Z}_2$-graded ideal I in A let I' denote the $\mathbb{Z}_2$-graded ideal generated by $i(I)$ in $S^{-1}A$. Let $\mathfrak{p}$ be a prime ideal in A not meeting S and suppose that $\mathfrak{p}'$ is not proper - that is $\mathfrak{p}' = S^{-1}A$. Then there exists $p \in \mathfrak{p}$ and $s \in S$ with $p/s = 1$ and thus there exists $s' \in S$ with $s'(p - s) = 0$, which is impossible since $\mathfrak{p} \cap S = \emptyset$. If $\frac{a}{s}, \frac{a'}{s'} \in S^{-1}A$ with $\frac{aa'}{ss'} \in \mathfrak{p}'$, then there is $p \in \mathfrak{p}$ and $z \in S$ with $\frac{aa'}{ss'} = \frac{p}{z}$ and hence there is $z' \in S$ with $aa'zz' \in \mathfrak{p}$ and hence a or a' in $\mathfrak{p}$. Hence $\mathfrak{p}'$ is a prime ideal.

If $\mathfrak{p}$ is a prime ideal in A not meeting S then $\mathfrak{p} \subset i^{-1}(\mathfrak{p}')$. If $x \in i^{-1}(\mathfrak{p}')$ then $x/1 \in \mathfrak{p}'$ and hence there is $p \in \mathfrak{p}$ and $s \in S$ with $p/s = x/1$ and thus there is $s' \in S$ with $s'p = s'sx$ and thus $x \in \mathfrak{p}$. Thus $\mathfrak{p} = i^{-1}(\mathfrak{p}')$ and by lemma 5.1.5 for any prime ideal $\mathfrak{q}$ in $S^{-1}A$ we have $(i^{-1}(\mathfrak{q}))'$. It is clear that the correspondence between the prime ideals in $S^{-1}A$ and the prime ideals in A not meeting S is inclusion preserving. ☐

For any prime ideal $\mathfrak{p}$ in a superring A, the set $S = A - \mathfrak{p}$ is multiplicative. In this case we use the notation $S^{-1}A = A_{\mathfrak{p}}$.

Proposition 5.1.9. *Let A be a superring and $\mathfrak{p}$ a prime ideal in A, then $A_{\mathfrak{p}}$ is a local ring with maximal ideal the $\mathbb{Z}_2$-graded ideal generated by the image of $\mathfrak{p}$ in $A_{\mathfrak{p}}$.*

Proof. Denote $\mathfrak{p}'$ the $\mathbb{Z}_2$-graded ideal in $A_{\mathfrak{p}}$ generated by the image $\mathfrak{p}$ in $A_{\mathfrak{p}}$. Any element not in $\mathfrak{p}'$ is invertible and $\mathfrak{p}' \neq A_{\mathfrak{p}}$. ☐

Proposition 5.1.10. *Let A be a superring, $\mathfrak{p}$ a prime ideal in A. Denote $\bar{\mathfrak{p}}$ the image of $\mathfrak{p}$ in $\bar{A}$, then $\overline{A_{\mathfrak{p}}} \cong (\bar{A})_{\bar{\mathfrak{p}}}$*

Proof. Immediate from proposition 5.1.4. ☐

Consider a prime ideal $\mathfrak{p} \subset A$. There is a relation between the localizations A_f, where f runs over the homogeneous elements that are not contained in $\mathfrak{p}$, and the localization $A_{\mathfrak{p}}$. In order to see this relation we need to define a direct system. Suppose that $f \in \sqrt{(g)}$ for some even $f, g \in A$, then $f^n = ag$ for some integer n and some even a. Hence we have an induced morphism $\psi_{g,f} : A_g \to A_f$ given by $\psi_{g,f} : b/g^s \mapsto ba^s/f^{ns}$. The induced morphism does not depend on the choice of n and a. One checks, that if $f \in \sqrt{(g)}$ and $g \in \sqrt{(f)}$, then the maps $\psi_{f,g} : A_f \to A_g$ and $\psi_{g,f} : A_g \to A_f$ are each others inverse. We define the direct system $(\psi_{g,f} : A_g \to A_f; f \in \sqrt{(g)})_{f,g \notin \mathfrak{p}}$ to be the category of all localizations A_f, where $f \notin \mathfrak{p}$ and $f \in A_{\bar{0}}$, and with morphisms the maps $\psi_{g,f} : A_g \to A_f$, whenever $f \in \sqrt{(g)}$. We define the (direct) limit of this direct system as the direct product superring $\coprod_{f \notin \mathfrak{p}} A_f$ modded out by the following equivalence relation: we call $a/f \in A_f$ equivalent to $a'/f' \in A_{f'}$ if there is an element $g \in A - \mathfrak{p}$ with $g \in \sqrt{(f)} \cap \sqrt{(f')}$ such that $\psi_{f,g}(a/f) = \psi_{f',g}(a'/f')$ in A_g. One easily checks that this defines a superring, which we denote $\varinjlim_{f \notin \mathfrak{p}} A_f$. Furthermore, there are canonical insertions $i_f : A_f \to A_{\mathfrak{p}}$, which send a/f in A_f to the equivalence class of a/f in $\varinjlim_{f \notin \mathfrak{p}} A_f$. The insertions i_f are morphisms of A-modules.

Lemma 5.1.11. *Let A be a superring, $\mathfrak{p}$ a prime ideal. The superring $\varinjlim_{f \notin \mathfrak{p}} A_f$ is characterized uniquely by the following universal property: The canonical insertions $i_f : A_f \to \varinjlim_{f \notin \mathfrak{p}} A_f$ commute with the maps $\psi_{f,g} : A_f \to A_g$, for $g \in \sqrt{(f)}$, that is, $i_g \circ \psi_{f,g} = i_f$. If Z is any A-module together with morphisms of A-modules $j_f : A_f \to Z$ with $j_g \circ \psi_{f,g} = j_f$, then there is a unique morphism of A-modules $\psi : \varinjlim_{f \notin \mathfrak{p}} A_f \to Z$ such that the following diagram commutes for all even $f, g \in A - \mathfrak{p}$ with $g \in \sqrt{(f)}$:*

$$
\begin{tikzcd}
A_f \arrow[rr, "\psi_{f,g}"'] \arrow[dr, "i_f"] \arrow[ddr, "j_f"'] & & A_g \arrow[dl, "i_f"'] \arrow[ddl, "f_g"] \\
& \varinjlim\limits_{f \notin \mathfrak{p}} A_f \arrow[d, dashed, "\psi"'] & \\
& Z &
\end{tikzcd}
\quad . \tag{5.3}
$$

Proof. That for all even $f, g \in A - \mathfrak{p}$ with $g \in \sqrt{(f)}$ we have $i_g \circ \psi_{f,g} = i_f$ is obvious. To get the morphism from $\varinjlim_{f \notin \mathfrak{p}} A_f$ to Z we note that the image of any equivalence class of a/f is completely determined by the image of a/f under j_f in Z. When we denote $[a/f]$ for the equivalence class of a/f in $\varinjlim_{f \notin \mathfrak{p}} A_f$ we thus have $\psi([a/f]) = j_f(a/f)$. As the j_f commute with the maps $\psi_{f,g}$ the morphism ψ is well-defined. □

Proposition 5.1.12. *We have the following isomorphism of A-modules:*

$$\varinjlim_{f \notin \mathfrak{p}} A_f \cong A_{\mathfrak{p}} \, . \tag{5.4}$$

where the limit goes over all even $f \in A$ that are not in $\mathfrak{p}$.

Proof. The isomorphism is given by the map φ that sends the equivalence class of a/f^r in $\varinjlim_{f \notin \mathfrak{p}} A_f$ to the element a/f^r in $A_{\mathfrak{p}}$. The map φ is clearly surjective, so we show injectivity. Suppose a/f^r

is in the kernel of φ, then there is some even $g \in A - \mathfrak{p}$ such that $ga = 0$. In particular, $f^r g^r a = 0$. But then the element $ag^r/f^r g^r = 0$ as an element of A_{fg}, which shows that a/f^r is equivalent to zero in $\varinjlim_{f \notin \mathfrak{p}} A_f$. □

For a given superring A, we denote S_A the set of elements a in A such that $\bar{a} \neq 0$, where $\bar{a}$ denotes the image of a in the body $\bar{A}$. The set S_A is a multiplicative set if and only if J_A is a prime ideal. In that case we define $\mathrm{Frac}(A)$ to be $S_A^{-1}A = A_{J_A}$ and call it the superring of fractions of A. The superring $\mathrm{Frac}(A)$ is a local superring and $\overline{\mathrm{Frac}(A)}$ is isomorphic to the field of fractions of the integral domain $\bar{A}$. In particular, the body of $\mathrm{Frac}(A)$ is a field.

Let A be a superring and let M be an A-module. For a multiplicative set S in $A_{\bar{0}}$ we construct an A-module $S^{-1}M$ as follows: we consider on the product $S \times M$ the equivalence relation $(s, m) \sim (s', m')$ if and only if there is $z \in S$ with $z(sm - s'm) = 0$ in M. We let m/s or $\frac{m}{s}$ denote the equivalence class of (s, m) and define addition by

$$\frac{m}{s} + \frac{m'}{s'} = \frac{ms' + m's}{ss'}\,, \tag{5.5}$$

where the right-hand side is independent of the choice of representatives. The action of A is defined by $a \cdot m/s = (am)/s$. We denote $j_S : M \to S^{-1}M$ the canonical morphism sending m to $m/1$. We call $S^{-1}M$ the localization of M at S.

Lemma 5.1.13. *Let A be a superring, S a multiplicative set in $A_{\bar{0}}$ and M an A-module. Then $S^{-1}A \otimes_A M \cong S^{-1}M$ as A-modules.*

Proof. Define $f : S^{-1}A \otimes_A M \to S^{-1}M$ and $g : S^{-1}M \to S^{-1}A \otimes_A M$ by

$$f : \frac{a}{s} \otimes m \mapsto \frac{am}{s}\,, \quad g : \frac{m}{s} \mapsto \frac{1}{s} \otimes m\,. \tag{5.6}$$

Then $f \circ g = \mathrm{id}_{S^{-1}M}$ and $g \circ f = \mathrm{id}_{S^{-1}A \otimes_A M}$. Clearly f and g preserve the parity and commute with the (left and right) action of A. □

The module $S^{-1}M$ also has a universal property:

Proposition 5.1.14. *Let M be an A-module, S a multiplicative set in $A_{\bar{0}}$ and $j_S : M \to S^{-1}M$ the canonical morphism. Then the A-module $S^{-1}M$ has the following universal property: For all $s \in S$ the A-linear map $l_s : S^{-1}M \to S^{-1}M$ given by $l_s : x \mapsto sx$ is invertible. For any A-module N such that for all $s \in S$ the linear map l_s is invertible and $f : M \to N$ a morphism of A-modules, there exists a unique morphism $f' : S^{-1}M \to N$ such that $f' \circ j_S = f$.*

Proof. Suppose we are given a module N such that for all $s \in S$ the A-linear map l_s is invertible and a morphism $f : M \to N$. Existence of the morphism $f' : S^{-1}M \to N$ follows when we define $f'(m/s) = (l_s)^{-1} f(m)$ and uniqueness follows from $f(m) = f'(j_S(m)) = f'(s \cdot m/s) = s \cdot f'(m/s) = l_s(f'(m/s))$. □

The linear map $l_s : S^{-1}M \to S^{-1}M$ given by $l_s : x \mapsto sx$ is called the linear homothety along s, also see [18]. Using proposition 5.1.14 we immediately obtain:

Corollary 5.1.15. *Let A be a superring, M an A-module and S a multiplicative set in $A_{\bar{0}}$ such that $l_s : m \mapsto sm$ is invertible for all $s \in S$, then $S^{-1}M \cong M$. In particular, when we denote $i_S : A \to S^{-1}A$ the canonical morphism, then if $S' \subset i_S(S)$ we have $S'^{-1}(S^{-1}A) \cong S^{-1}A$.*

We also have the analogue of proposition 5.1.3:

Proposition 5.1.16. *Let A be a superring and S and T multiplicative sets in $A_{\bar{0}}$. Denote $T' = i_S(T)$ the image of T in $S^{-1}A$. For any A-module M we have $(ST)^{-1}M \cong T'^{-1}(S^{-1}M)$.*

Proof. The proof is exactly the same as the proof of proposition 5.1.3. ☐

Proposition 5.1.17. *Let A be a superring and let M', M, M'' be A-modules. The sequence of A-modules*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0 \quad , \tag{5.7}$$

with morphisms f and g is exact if and only if the sequence

$$0 \longrightarrow S^{-1}M' \xrightarrow{f'} S^{-1}M \xrightarrow{g'} S^{-1}M'' \longrightarrow 0 \tag{5.8}$$

is exact for all multiplicative sets S in $A_{\bar{0}}$, where f' and g' are the induced morphisms as in proposition 5.1.2.

Proof. For the 'if'-part, put $S = \{1\}$. To prove the 'only if'-part: Suppose that for $m' \in M'$, $f'(m'/s) = 0$, then there is $s' \in S$ with $f(s'm') = s'f(m') = 0$ and thus $s'm' = 0$ and $m'/s = 0$. Therefore f' is injective. Clearly $g' \circ f' = 0$ and g' is surjective. If for $m \in M$, $g'(m/s) = 0$ there is $s' \in S$ with $s'm = f(m')$ for some $m' \in M'$. But then $m/s = f(m')/ss'$. This proves that $\mathrm{Im}\,(f') = \mathrm{Ker}\,(g')$. ☐

Corollary 5.1.18. *If N is an A-submodule of M, then we can identify $S^{-1}N$ with an A-submodule of $S^{-1}M$ and $S^{-1}(M/N)$ with $S^{-1}M/S^{-1}N$. Furthermore, if $f : M \to N$ is a morphism of A-modules and $f' : S^{-1}N \to S^{-1}M$ the induced morphism, then* $\mathrm{Ker}\, f' = S^{-1}\mathrm{Ker}\, f$ *and* $\mathrm{Coker}\, f' = S^{-1}\mathrm{Coker}\, f$.

If $\mathfrak{p}$ is a prime ideal in a superring A, then for any A-module M we write $M_{\mathfrak{p}}$ for $S^{-1}M$, where $S = A - \mathfrak{p}$.

Lemma 5.1.19. *Let A be a superring and let $\mathfrak{m}$ and $\mathfrak{m}'$ be maximal ideals in A, then $(A/\mathfrak{m})_{\mathfrak{m}'}$ is the zero-module if $\mathfrak{m} \neq \mathfrak{m}'$ and is isomorphic to $A/\mathfrak{m}$ if $\mathfrak{m} = \mathfrak{m}'$. In particular, $A/\mathfrak{m} \cong A_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}$.*

Proof. If $\mathfrak{m} = \mathfrak{m}'$ then all elements in $A - \mathfrak{m}$ are invertible modulo $\mathfrak{m}$. Hence the first part follows from corollary 5.1.15. If $\mathfrak{m} \neq \mathfrak{m}'$, then there is an even $m \in \mathfrak{m}$ not lying in $\mathfrak{m}'$. Consider the morphism that sends $a/s \in A_{\mathfrak{m}'}$ to ma/ms in $\mathfrak{m}_{\mathfrak{m}'}$. This map is injective and surjective. But then it follows that $\mathfrak{m}_{\mathfrak{m}'} \cong A_{\mathfrak{m}'}$. Thus $(A/\mathfrak{m})_{\mathfrak{m}'} \cong A_{\mathfrak{m}'}/\mathfrak{m}_{\mathfrak{m}'} = 0$. The final statement then follows from localizing the exact sequence $0 \to \mathfrak{m} \to A \to A/\mathfrak{m} \to 0$ at $A - \mathfrak{m}$. ☐

Proposition 5.1.20. *Let A be a Noetherian reduced superring, that is, $\bar{A}$ contains no nilpotent elements. Then there is a localization $S^{-1}A$ of A such that $\overline{S^{-1}A}$ is an integral domain.*

Proof. Since A is Noetherian, there are only finitely many minimal primes. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the set of minimal primes of A, and let $\bar{\mathfrak{p}}_1, \ldots, \bar{\mathfrak{p}}_n$ bet the corresponding set of minimal primes in A/J. If $n = 1$, then $\mathfrak{p}_1 \supset J$ and $\mathfrak{p}_1$ is the nilradical, which contains all nilpotents. Since A/J is reduced, all nilpotents lie in J and hence $\mathfrak{p}_1 = J$. So A/J has a unique minimal prime ideal 0 and thus A/J is an integral domain. Indeed, $\bar{\mathfrak{p}}_1$ is the nilradical ideal of $\bar{A}$. But then $\bar{\mathfrak{p}}_1 = 0$ and $\bar{A}$ is an integral domain.

Now let $n > 1$. By assumption, we have $\cap_{i=1}^{n}\mathfrak{p}_i = J$. It follows that if $x \in \mathfrak{p}_i$ but $x \notin \mathfrak{p}_j$ for $i \neq j$, then $\bar{x} \neq 0$. By minimality of the $\mathfrak{p}_i$, for each $i > 1$ we can find an even x_i in $\mathfrak{p}_i$, with $x_i \notin \mathfrak{p}_1$ and thus $\bar{x}_i \neq 0$. The product $x = x_2x_3 \cdots x_n$ lies in the intersection $\cap_{i>1}\mathfrak{p}_i$, but not in $\mathfrak{p}_1$ (hence not nilpotent) and it is even. But then by proposition 5.1.8 A_x is a superring with a unique minimal prime ideal $\mathfrak{p}_1A_x$. Thus if J' is the $\mathbb{Z}_2$-graded ideal generated by the odd elements of A_x, then A_x/J' has a unique minimal prime, which is the nilradical of A_x/J'. By proposition 5.1.4 $A_x/J' \cong (A/J)_{\bar{x}}$. Suppose $\bar{a}/\bar{x}^s \in (A/J)_{\bar{x}}$ is nilpotent; then there are integers m, n such that $\bar{x}^m \bar{a}^n = 0$ in A/J and hence $\bar{x}^m \bar{a}$ is nilpotent, but since A/J is reduced $\bar{x}^m \bar{a} = 0$ and thus $\bar{a}/\bar{x}^s = 0$. Hence $0 = \mathrm{Nilrad}(A_x/J')$ is the unique minimal prime and thus A_x/J is an integral domain. □

We end this section with a useful lemma that tells us that two modules are isomorphic if all localizations at maximal ideals are isomorphic.

Lemma 5.1.21. *A morphism $\varphi : M \to N$ is injective (resp. surjective) if and only if all induced morphisms $\varphi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ are injective (resp. surjective), where $\mathfrak{m}$ ranges over all the maximal ideals of A.*

Proof. First, suppose m goes to zero in $M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$. When m goes to zero in $M_{\mathfrak{m}}$, both its homogeneous parts go to zero and we may assume that m is homogeneous. The annihilator of m is contained in no maximal ideal and hence equals A and thus $m = 0$. We obtain that $M = 0$ if and only if $M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$.

We have $\mathrm{Ker}\,(\varphi_{\mathfrak{m}}) = (\mathrm{Ker}\,\varphi)_{\mathfrak{m}}$, hence if $\mathrm{Ker}\,(\varphi_{\mathfrak{m}}) = 0$ for all maximal ideals $\mathfrak{m}$ then $\mathrm{Ker}\,\varphi = 0$. We also have $(\mathrm{Coker}\,\varphi_{\mathfrak{m}}) = (\mathrm{Coker}\,\varphi)_{\mathfrak{m}}$. Hence $\mathrm{Coker}\,\varphi = 0$ if and only if $(\mathrm{Coker}\,\varphi)_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$. □

5.2 Application to Artinian superrings

We now apply the results of the previous section to get more insight into Artinian superrings.

Proposition 5.2.1. *If M is an R-module of finite length $l(M) = n$ and $\mathfrak{m}$ and $\mathfrak{n}$ are different maximal ideals in R, then $(M_{\mathfrak{m}})_{\mathfrak{n}} = 0$*

Proof. We take a composition series of M and localize at $\mathfrak{m}$:

$$M_{\mathfrak{m}} = (M_0)_{\mathfrak{m}} \supset (M_1)_{\mathfrak{m}} \supset \ldots \supset (M_n)_{\mathfrak{m}} = 0\,. \tag{5.9}$$

We know that the quotients M_i/M_{i+1} are simple, and hence by lemma 5.1.19 after localizing with respect to $\mathfrak{m}$ some terms become equal: if $\mathfrak{m} \neq \mathrm{Ann}(M_i/M_{i+1})$ then $(M_i)_{\mathfrak{m}} = (M_{i+1})_{\mathfrak{m}}$. Hence we get a composition series for $M_{\mathfrak{m}} = M'_0 \supset M'_1 \ldots \supset M'_r = 0$ and $r \leq n$ by deleting the redundant and for all i we have $\mathrm{Ann}(M'_i/M'_{i+1}) = \mathfrak{m}$. If we localize this series with respect to $\mathfrak{n}$ we get a composition series for $(M_{\mathfrak{m}})_{\mathfrak{n}}$ with all quotients zero; all terms in eqn.(5.9) are equal and thus zero. □

Using proposition 5.2.1 we immediately obtain:

Corollary 5.2.2. *The length of $M_{\mathfrak{m}}$ is the number of times that $R/\mathfrak{m}$ or $\Pi(R/\mathfrak{m})$ appears in any composition series of M; hence the number of times $R/\mathfrak{m}$ or $\Pi(R/\mathfrak{m})$ appears in a composition series is independent of the composition series.*

Corollary 5.2.3. *If M has finite length the maps $M \to M_{\mathfrak{m}}$ for a maximal ideal $\mathfrak{m}$ in A make together an isomorphism*

$$M \cong \oplus_{\mathfrak{m}\,\text{max.ideal}} M_{\mathfrak{m}}\,. \tag{5.10}$$

Proof. By lemma 5.1.21 it is sufficient to check that $M_{\mathfrak{n}} \cong (\oplus_{\mathfrak{m}\,\text{max.ideal}} M_{\mathfrak{m}})_{\mathfrak{n}}$ for all maximal ideals $\mathfrak{n}$. But M has finite length and thus $(M_{\mathfrak{m}})_{\mathfrak{n}} \cong M_{\mathfrak{m}}$ if $\mathfrak{m} = \mathfrak{n}$ and zero otherwise. But then each morphism $M_{\mathfrak{n}} \to (\oplus_{\mathfrak{m}\,\text{max.ideal}} M_{\mathfrak{m}})_{\mathfrak{n}} \cong M_{\mathfrak{n}}$ is the identity, and thus an isomorphism. □

Corollary 5.2.4. *Any Artinian superring A is a finite direct product of local Artinian rings.*

Proof. As A-modules we have $A = \oplus_{\mathfrak{m}} A_{\mathfrak{m}}$ where the sum is over all maximal ideals and by theorem 4.1.29 there are only finitely many summands. The direct product algebra $\prod_{\mathfrak{m}} A_{\mathfrak{m}}$ is as A-module just the direct sum. All maps $A \to A_{\mathfrak{m}}$ are superring morphisms, and hence the map $A \to \prod_{\mathfrak{m}} A_{\mathfrak{m}}$ is a superring morphism. □

Lemma 5.2.5. *Let A be a superring with canonical ideal J_A. If J_A is Artinian as an A-module and $\bar{A}$ is an Artinian algebra, then A is an Artinian superring.*

Proof. Since $\bar{A}$ is an Artinian algebra, $\bar{A}$ is an Artinian A-module, where the action of $a \in A$ on $\bar{b} \in \bar{A}$ is given by $(a, \bar{b}) \mapsto \bar{a}\bar{b}$. Given a composition series of $\bar{A}$ and J_A we can concatenate them to get a composition series of A. □

Proposition 5.2.6. *If A is a superalgebra that is a finite-dimensional super vector space over k, then A is a finite product of local superalgebras A_i and the elements of the maximal ideals of the A_i are nilpotent.*

Proof. Since A is finite-dimensional, it is Artinian and by theorem 4.1.29 there are only finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$. By corollary 5.2.4 A is a direct product of local superalgebras. For every maximal ideal $\mathfrak{m}_i$ of A we have a component A_i and the maximal ideal of A_i is the restriction of $\mathfrak{m}_i$ to A_i. Hence every prime ideal in A_i coincides with the maximal ideal in A_i. But the nilradical is the intersection of the prime ideals, hence the maximal ideal in A_i coincides with the nilradical. □

5.3 Geometric superalgebras

We consider superalgebras over a fixed field k. We call a superalgebra A properly geometric if $\bar{A} = k$. If $\bar{A}$ is a field, then $\bar{A}$ is an extension of k and in case $\bar{A}$ is a proper extension of k, we call A geometric.

Note that all properly geometric superalgebras over k have split bodies. All superrings that have a trivial odd part have a split body. If A is a superring with a split body, then A has a sub superring isomorphic to $\bar{A}$, since the splitting morphism is injective. Thus a geometric split superalgebra A is a quotient of $\bar{A}[\theta_i; i \in I]$ for some index set I.

Lemma 5.3.1. *Let A be a finitely generated superalgebra over k. If A is geometric, then $\bar{A}$ is an algebraic extension of k.*

Proof. $\bar{A}$ is a finitely generated k-algebra and it is a field. Thus we are in the situation of the weak Nullstellensatz, which states that $\bar{A}$ is algebraic over k (see for example [16, 17, 53]). □

The Grassmann algebras over k are defined by $\Lambda_n = k[\theta_1, \ldots, \theta_n]$, where θ_j are odd variables. The Grassmann algebras have a few remarkable properties: (*i*) Λ_n has only one prime ideal J_n, which is the $\mathbb{Z}_2$-graded ideal generated by the odd elements. (*ii*) $\Lambda_{J_n} \cong \Lambda_n$. (*iii*) Λ_n is properly geometric. (*iv*) Λ_n is finitely generated. (*v*) Λ_n has a split body. (*vi*) Λ_n is finite-dimensional. If a superalgebra satisfies the properties (*iii*) and (*v*), then it is a quotient of a Grassmann algebra by the following lemma:

Lemma 5.3.2. *Let A be a finitely generated properly geometric superalgebra over k. Then A is a quotient of a Grassmann algebra.*

Proof. The body of A is isomorphic to the field k, which is Artinian. Since A is finitely generated, there are a finite number of odd generators $\eta_1, \ldots, \eta_n$. Given $x \in A$, then $x - \bar{x} = \sum r_i \eta_i$ and $r_i - \bar{r}_i = \sum r_{ij}\eta_j$, $r_{ij} - \bar{r}_{ij} = \sum r_{ijk}\eta_k$ and so on. The process terminates since there are finitely many η_i. Hence x can be expressed in terms of the η_i with k-coefficients. □

Corollary 5.3.3. *If A is an Artinian properly geometric superalgebra, then A is a quotient of a Grassmann algebra. In particular, any finite-dimensional properly geometric superalgebra is a quotient of a Grassmann algebra.*

We note that if a superalgebra A is geometric, there is only one prime ideal, namely J_A. The elements of J_A are nilpotent so that any element not in J_A is invertible. Hence localizing A at J_A gives $A_{J_A} \cong A$.

Lemma 5.3.4. *Let A be a finite-dimensional superalgebra over k and suppose $\bar{A} \cong k \times \cdots \times k$, then A is a direct product of quotients of Grassmann algebras.*

Proof. By proposition 5.2.6 we know that A is a product algebra of the localizations at different maximal ideals of A; $A \cong A_1 \times \cdots \times A_r$. The A_i are local and the elements of the maximal ideals in A_i are nilpotent. Taking the body gives $\bar{A} \cong \bar{A}_1 \times \cdots \times \bar{A}_r \cong k \times \cdots \times k$ and assume there are s copies of k. The projection to the body preserves idempotents, and hence the unit element e_i of A_i is a sum of the basis idempotents f_j. But then $r \leq s$. On the other hand, $\bar{A}_1 \times \cdots \times \bar{A}_r$ contains r independent idempotents, hence $r \geq s$ and thus $r = s$. Clearly, $k \subset \bar{A}_i$ for all i, and if $\bar{A}_i$ is contained in a product $k \times \cdots \times k$, then $\bar{A}_i$ must be contained in one of the factors; otherwise the 1 of $\bar{A}_i$ cannot be a zerodivisor of the 1's of the all the other A_j, $j \neq i$. □

Proposition 5.3.5. *Let A be a Noetherian geometric superring and let M be a finitely generated A-module. Then $\overline{M}$ is a finite-dimensional super vector space over $\bar{A}$, say of dimension $p|q$, and if $e_1, \ldots, e_{p+q}$ is a standard basis of $\overline{M}$, then the preimages of the e_i in M generate M.*

Note that with a standard basis we mean that for $1 \leq i \leq p$ the e_i are even and for $p+1 \leq j \leq p+q$ the e_j are odd.

Proof. We denote the preimages of the e_i in M by the same symbol. For $x \in M$ we can find $a_i \in A$ such that $x - \sum_i a_i e_i \in JM$. Call N the submodule of M generated by the e_i, so that $M = N + J_A M$. Then it follows that $J_A(M/N) = M/N$. Since each element of J_A is nilpotent and J_A is finitely generated, there exists an integer s such that $(J_A)^s = 0$. Combining these observations we conclude that $M/N = (J_A)^s M/N = 0$. □

5.4 Superschemes

5.4.1 The affine superscheme

Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a superring. We know there is an inclusion preserving one-to-one correspondence between the prime ideals in A and the prime ideals in $\bar{A}$. Now consider $A_{\bar{0}}$ as a commutative ring. It has an ideal $A_{\bar{1}}^2$, which is contained in all prime ideals. Furthermore, we have $\bar{A} \cong A_{\bar{0}}/A_{\bar{1}}^2$ so that we conclude that there is an inclusion preserving one-to-one correspondence between the prime ideals of $A_{\bar{0}}$ and A. We use this fact to associate a topological space to each superring. This topological space we equip with a sheaf of superrings. The result is called an affine superscheme and a general superscheme has to look locally like an affine superscheme. The presentation below is very similar to the usual expositions of the construction of the spectrum of a commutative ring. Therefore the discussion will be rather short and some proofs are omitted. All omitted proofs for the commutative case can be found in the textbooks [54, 55] and can be copied almost literally for the super case.

The topological space

For any commutative ring R, we denote $\mathrm{Spec}(R)$ the topological space of prime ideals. The topology is defined by prescribing the closed sets. The closed sets of $\mathrm{Spec}(R)$ are the sets

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supset \mathfrak{a}\} \,, \tag{5.11}$$

where $\mathfrak{a}$ is any ideal in R. If $\mathfrak{a}$ is generated by a single element $f \in R$, we also write $V(f)$ for $V(\mathfrak{a})$. One easily checks that $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b})$ and $V(\sum_i \mathfrak{a}_i) = \bigcap_i V(\mathfrak{a}_i)$, so that the closed sets $V(\mathfrak{a})$ indeed define a topology. For any $f \in R$ we define the principal open set $D(f)$ by

$$D(f) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid f \notin \mathfrak{p}\} \,. \tag{5.12}$$

Equivalently, we can define $D(f)$ as the complement of $V(f)$. If J is any ideal in R that is contained in the nilradical of R, then the projection $\pi : R \to R/J$ induces a continuous map

$$\pi' : \mathrm{Spec}(R/J) \to \mathrm{Spec}(R) \,, \quad \pi'(\mathfrak{p}) = \pi^{-1}(\mathfrak{p}). \tag{5.13}$$

As J is contained in the nilradical, the map π' is a bijection and one easily checks that its inverse, which sends $\mathfrak{q} \in \mathrm{Spec}(R)$ to $\mathfrak{q} \bmod J \in \mathrm{Spec}(R/J)$, is continuous. Hence π' is in fact a homeomorphism. Thus for any superring A there is a homeomorphism of topological spaces:

$$\mathrm{Spec}(\bar{A}) \cong \mathrm{Spec}(A_{\bar{0}}) \,, \tag{5.14}$$

induced by the projection $A_{\bar{0}} \to \bar{A} \cong A_{\bar{0}}/A_{\bar{1}}^2$. To any superring A we now associate the topological space $\mathrm{Spec}(A_{\bar{0}})$ and call it the spectrum of A. For convenience, we mostly work with the description of the spectrum of A as the topological space of prime ideals of $A_{\bar{0}}$, although describing the spectrum as the prime ideals of $\bar{A}$ is equivalent. The following lemma gives some properties of the principal open sets on $\mathrm{Spec}(A_{\bar{0}})$:

Lemma 5.4.1. *Let A be a superring, then we have:*

(i) *If $F = \{f_i \mid i \in I\}$ is a set of elements of $A_{\bar{0}}$, then $\bigcup_{i \in I} D(f_i) = \mathrm{Spec}(A_{\bar{0}})$ if and only if the ideal generated by the f_i is A. Thus, the principal open sets $D(f_i)$ cover $\mathrm{Spec}(A_{\bar{0}})$ if and only if there are finitely many $f_{i_1}, \ldots, f_{i_k}$ in F and finitely many $a_1, \ldots, a_k \in A$ such that $1 = \sum_{j=1}^k a_j f_{i_j}$.*

(ii) *$D(\mathfrak{a}) \subset D(\mathfrak{b})$ if and only if $V(\mathfrak{b}) \subset V(\mathfrak{a})$ if and only if $\sqrt{\mathfrak{a}} \subset \sqrt{\mathfrak{b}}$. In particular, $D(f) \subset D(g)$ if and only if $f \in \sqrt{g}$.*

(iii) *$D(f) \cap D(g) = D(fg)$.*

(iv) *If $D(f) = D(g)$, then $A_f \cong A_g$.*

(v) *The principal open sets form a basis of the topology.*

Proof. (i): The functions f_i generate $A_{\bar{0}}$ in $A_{\bar{0}}$ if and only if they generate A in A. Hence we are back in the commutative case. (ii): The first equivalence is trivial. For the second; $V(\mathfrak{b}) \subset V(\mathfrak{a})$ if and only if any prime ideal that contains $\mathfrak{b}$ also contains $\mathfrak{a}$. But $\sqrt{\mathfrak{a}}$ is the intersection of the prime ideals that contain $\mathfrak{a}$. (iii): A prime ideal $\mathfrak{p}$ does not contain f and g if and only if it does not contain fg. (iv): By (ii) there are $a, b \in A$ and integers m, n such that $f^n = ag$ and $g^m = bf$. We have a natural morphism $A_f \to A_g$ that maps c/f^s to cb^s/g^{ms} and a natural morphism $A_g \to A_f$ that maps d/g^t to da^t/f^{nt}. Since $f^{mn} = a^m bf$ and $ab^n g = g^{mn}$, these maps are isomorphisms. (v): Any open set U is the complement of a set $V(\mathfrak{a})$ for some $\mathbb{Z}_2$-graded ideal in A. This complement is not empty if there exists an even $a \in \mathfrak{a}$ that is not contained in some prime ideal $\mathfrak{p}$. Then $D(a) \subset U$. ∎

The sheaf

We now equip the topological space $\mathrm{Spec}(A_{\bar{0}})$ with a sheaf $\mathcal{O}$ of superrings, such that the stalks are local rings. In this way, $\mathrm{Spec}(A_{\bar{0}})$ becomes a locally superringed space. First we describe some properties that we want the sheaf to have. Then we define the sheaf and state as a proposition that the sheaf indeed has the desired properties.

The stalk at $\mathfrak{p}$ of the sheaf $\mathcal{O}$ is defined as the inductive limit $\varinjlim_{\mathfrak{p}\in U} \mathcal{O}(U)$, denoted $\mathcal{O}_\mathfrak{p}$ and we want it to be $A_\mathfrak{p}$. This is in analogy with the case for commutative rings. Also, if we have a morphism of superrings $\psi : A \to B$, then for any prime ideal $\mathfrak{q}$ of B, the preimage $\psi^{-1}(\mathfrak{p})$ is a prime ideal of A and we have an induced morphism $\psi_\mathfrak{p} : A_{\psi^{-1}(\mathfrak{p})} \to B_\mathfrak{p}$.

On the principal open sets $D(f)$ we want that $\mathcal{O}(D(f)) = A_f$. We thus only allow even f for the principal open sets. If $D(f) \subset D(g)$ then $f^n = ag$ for some $a \in A$, from which we conclude that there is a natural morphism $A_g \mapsto A_f$ given by

$$\frac{b}{g^m} \mapsto \frac{a^m b}{f^{mn}}\,. \tag{5.15}$$

The map $A_g \mapsto A_f$ is well-defined and does not depend on the choice of the exponent n and the element a. If we have inclusions $D(f) \subset D(g) \subset D(h)$, then one easily checks that the following diagram commutes:

$$\begin{tikzcd} A_h \arrow[rr] \arrow[dr] & & A_f \\ & A_g \arrow[ur] & \end{tikzcd}\,. \tag{5.16}$$

Furthermore, if $D(f) = D(g)$, then by lemma 5.4.1 we have $A_f \cong A_g$. The assignment $D(f) \mapsto A_f$ therefore defines a presheaf on the principal open sets. The following lemma shows that the assignment in fact gives a sheaf on the principal open sets:

Lemma 5.4.2. *Suppose $D(f) = D(f_1) \cup \ldots \cup D(f_k)$.*

(i) If $s \in A_f$ goes to zero for all maps $A_f \to A_{f_i}$, then $s = 0$ in A_f.

(ii) If a set of elements $s_i \in A_{f_i}$ is given, such that s_i and s_j have the same image in $A_{f_i f_j}$ for all $1 \le i, j \le k$, then there is an element $s \in A_f$ that has image s_i in A_{f_i} for all i.

Proof. As $D(f) \cong \mathrm{Spec}(A_f)$ it is sufficient to prove the lemma for $f = 1$. For (i): s goes to zero in A_{f_i} if and only if there is an integer N such that $f_i^N s = 0$ for all i. But as the $D(f_i)$ form a cover, there is a relation $1 = f_1 a_1 + \ldots + f_k a_k$. But then $s = 1s = (f_1 a_1 + \ldots + f_k a_k)^{Nk} s = 0$. For (ii): We can write $s_i = a_i / f_i^n$ for some n and some $a_i \in A$. That the images of s_i and s_j agree in $A_{f_i f_j}$ means that $a_i f_j^n / (f_i f_j)^n = a_j f_i^n / (f_i f_j)^n$. But then there is an integer m such that

$$(f_i f_j)^m \Big(a_i f_j^n - a_j f_i^n \Big) = 0\,, \qquad \text{for all } i, j\,. \tag{5.17}$$

We define $b_i = a_i f_i^m$ and write $s_i = b_i / f_i^m$. Then eqn.(5.17) reads $b_i f_j^{n+m} = b_j f_i^{n+m}$. From $\mathrm{Spec}(A_{\bar{0}}) = \cup D(f_i) = \cup D(f_i^{n+m})$ we infer that there is a relation $1 = \sum_i c_i f_i^{n+m}$ for some even c_i. Define $s = \sum_i c_i b_i$, then as $s f_i^{n+m} = b_i$ we see that the image of s in A_{f_i} is s_i. ☐

With these preliminaries we can give the general definition of the sheaf on $\mathrm{Spec}(A_{\bar{0}})$. To check that it is really a sheaf is then further an exercise in dealing with sheaves and can be found in the textbooks [55] and [54]. On an arbitrary open set $U \subset \mathrm{Spec}(A_{\bar{0}})$ we define $\mathcal{O}(U)$ as follows: $\mathcal{O}(U)$ is the superring of all functions $s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$, $s : \mathfrak{p} \mapsto s_{\mathfrak{p}}$, such that $s_{\mathfrak{p}} \in A_{\mathfrak{p}}$ and for all $\mathfrak{p}$ in U there exists an open neighborhood V of $\mathfrak{p}$ and elements $a \in A$ and $f \in A_{\bar{0}}$, with $f \notin \mathfrak{q}$ for all $\mathfrak{q} \in V$, such that $s_{\mathfrak{q}}$ equals the image of a/f in $A_{\mathfrak{q}}$ for all $\mathfrak{q} \in V$. As the principal sets form a basis of the topology, we may in fact always assume that $V = D(f)$. We call the sheaf $\mathcal{O}$ associated to A the structure sheaf of $\mathrm{Spec}(A)$. We now state that the given definition of the structure sheaf $\mathcal{O}$ has the required properties:

Proposition 5.4.3. *Let A be a superring and let $\mathcal{O}$ be the sheaf of superrings on* $\mathrm{Spec}(A_{\bar{0}})$ *defined above. Then*

(i) On the principal open sets we have $\mathcal{O}(D(f)) = A_f$.

(ii) The stalk at $\mathfrak{p}$ is $A_{\mathfrak{p}}$.

Proof. We only indicate the proof and refer for the details to [54,55]. (i) is proved in 5.4.2(ii). For (ii): the principal open sets are a basis for the topology and hence in calculating the inductive limit $\varinjlim_{\mathfrak{p} \in U} \mathcal{O}(U)$ we can take the limit over the principal open sets. If $f \in \sqrt{g}$, we have a morphism $A_g \to A_f$, which we use to construct the direct system $(A_g \to A_f : f, g \notin \mathfrak{p}$, and $f \in \sqrt{g})$; the limit of this direct system is then the stalk at $\mathfrak{p}$. But the map

$$\varphi : \varinjlim_{f \notin \mathfrak{p}} A_f \to A_{\mathfrak{p}} \,, \qquad \frac{a}{f} \mapsto \frac{a}{f} \in A_{\mathfrak{p}} \,, \tag{5.18}$$

is an isomorphism of superrings by proposition 5.1.12. □

From proposition 5.4.3 we see that the stalks are local rings, and the global sections are the elements of A.

If $\mathfrak{p}$ is a prime ideal in A and s is a section that is defined in some open neighborhood U of $\mathfrak{p}$, then we write $s_{\mathfrak{p}}$ for the image of s in $A_{\mathfrak{p}}$. The 'function value' of s in $\mathfrak{p}$ is defined as the image of $s_{\mathfrak{p}}$ in $A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$, and is denoted $s(\mathfrak{p})$. All odd sections thus have zero value at any point, but as sections they are not zero.

5.4.2 The general superscheme

To define the category of superschemes is then done in a similar way as the category of schemes is constructed. We define a locally superringed space to be a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of superrings on X such that the stalks $\mathcal{O}_{X,x}$ are local superrings for all $x \in X$. A morphism of locally superringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair (φ, ψ) with $\varphi : X \to Y$ a continuous map and ψ a set of morphisms $\psi_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$ for all open sets $U \subset Y$ satisfying the following two conditions: (i) The ψ_U are compatible with restrictions, that is, for all inclusions $V \subset U \subset Y$ the following diagram commutes

$$\begin{tikzcd} \mathcal{O}_Y(U) \arrow[r, "\psi_U"] \arrow[d] & \mathcal{O}_X(\varphi^{-1}(U)) \arrow[d] \\ \mathcal{O}_Y(V) \arrow[r, "\psi_V"] & \mathcal{O}_X(\varphi^{-1}(V)) \end{tikzcd} \quad . \tag{5.19}$$

In diagram (5.19) the vertical arrows are the restriction maps. (ii): Because of the compatibility of the morphisms ψ_U, there is an induced morphism on the stalks $\psi_x : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$, which we require to be a local morphism. That is, if $\mathfrak{m}_{\varphi(x)}$ is the maximal ideal of $\mathcal{O}_{Y,\varphi(x)}$ and $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$ then $\psi_x(\mathfrak{m}_{\varphi(x)}) \subset \mathfrak{m}_x$, or equivalently, $\psi_x^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{\varphi(x)}$.

Definition 5.4.4. *Let X be a topological space with a sheaf $\mathcal{O}_X$ of local superrings. We say that $(X, \mathcal{O}_X)$ is a superscheme if for each point $x \in X$ there is an open neighborhood U of x such that the topological space U equipped with the restriction of the sheaf $\mathcal{O}_X$ to U is isomorphic as a locally superringed space to an affine superscheme* $\mathrm{Spec}(A_{\bar{0}})$ *with its sheaf of superrings. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are two superschemes then a morphism of superschemes is a morphism $f = (\varphi, \psi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally superringed spaces.*

It follows from the definition that for a superring A, the topological space $X = \mathrm{Spec}(A_{\bar{0}})$ equipped with the sheaf of superrings $\mathcal{O} : D(f) \mapsto A_f$ defines a superscheme. We denote Ssch the category of superschemes. For a superring A, we also write $\mathrm{Spec}(A)$ for the superscheme constructed from A.

Proposition 5.4.5. *Let $(X, \mathcal{O}_X)$ be a superscheme and let A be a superring. Denote $(Y, \mathcal{O}_Y)$ the superscheme associated to the spectrum of A. Then the natural morphism*

$$\Gamma : \mathrm{Hom}_{\mathrm{Ssch}}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)) \to \mathrm{Hom}_{\mathbf{sRng}}(A, \mathcal{O}_X(X)) \tag{5.20}$$

that sends a morphism of superschemes $f = (\varphi, \psi)$ to the morphism of superrings $\psi_Y : A \to \mathcal{O}_X(X)$, is a bijection.

Proof. First we show that Γ is injective. Let $f = (\varphi, \psi)$ be a morphism and let $\Gamma(f) : A \mapsto \mathcal{O}_X(X)$ be the map on global sections. For any $x \in X$ the image $\varphi(x)$ is recovered by

$$\begin{aligned} \varphi(x) &= \left\{ a \in A \mid a_{\varphi(x)} \in \mathfrak{m}_{\varphi(x)} \right\} \\ &= \left\{ a \in A \mid \psi_x(a_{\varphi(x)}) \in \mathfrak{m}_x \right\} \\ &= \left\{ a \in A \mid (\psi_X(a))_x \in \mathfrak{m}_x \right\} . \end{aligned} \tag{5.21}$$

Indeed, the first equality follows from the fact that for any prime ideal $\mathfrak{p} \subset A$ the kernel of the map $A \to A_\mathfrak{p} \to A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$ is precisely $\mathfrak{p}$. The second equality follows from the fact that $\psi_\mathfrak{p}$ is a local morphism. The third equality follows since ψ is compatible with restrictions to any open subset in X, so that the following diagram commutes

$$\begin{tikzcd} A = \mathcal{O}_Y(Y) \arrow[r, "\psi_X"] \arrow[d] & \mathcal{O}_X(X) \arrow[d] \\ A_{\varphi(x)} = \mathcal{O}_{Y,f(x)} \arrow[r, "\psi_x"'] & \mathcal{O}_{X,x} \end{tikzcd} . \tag{5.22}$$

Thus the continuous map $\varphi : X \to Y$ is uniquely determined by $\psi_X = \Gamma(f)$. To show that the maps ψ_U for open sets $U \subset Y$ are uniquely determined by ψ_X as well, we only need to check this on a basis of the topology, which is given by the principal open sets $D(g)$, for an even element g of A. Consider the following diagram:

$$\begin{tikzcd} A = \mathcal{O}_Y(Y) \arrow[r, "\psi_X"] \arrow[d] & \mathcal{O}_X(X) \arrow[d] \\ A_g = \mathcal{O}_{Y,D(g)} \arrow[r, dashed, "\psi_{D(g)}"'] & \mathcal{O}_{X,\varphi^{-1}(D(g))} \end{tikzcd} . \tag{5.23}$$

The map $\psi_{D(g)}$ is the unique morphism $A_g \to \mathcal{O}_X(\varphi^{-1}(D(g)))$ that makes the diagram commutative. Indeed, the image of the elements in the multiplicative set generated by g under the composition $A \to \mathcal{O}_X(X) \to \mathcal{O}_X(\varphi^{-1}(D(g)))$ is contained in the set of invertible elements of $\mathcal{O}_X(\varphi^{-1}(D(g)))$. But then there is one unique morphism $A_g \mapsto \mathcal{O}_X(\varphi^{-1}(D(g)))$ such that the diagram commutes by the universal property of the localized superring A_g (see proposition 5.1.1). Thus the map Γ is injective.

To prove surjectivity of Γ, we first assume $X = \mathrm{Spec}(B)$ for some superring B and that a morphism of superrings $\chi : A \to B$ is given. As alluded before, this induces a morphism $\chi_\mathfrak{p} : A_{\chi^{-1}(\mathfrak{p})} \to B_\mathfrak{p}$ for any prime ideal $\mathfrak{p} \subset B$. By the above, we know that there is only one possible continuous map $\varphi : X \to Y$, which is easily seen to be $\varphi(\mathfrak{p}) = \chi^{-1}(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \subset B$. The map φ is then continuous since for any principal open subset $D(g) \subset Y$ we have

$$\begin{aligned} \varphi^{-1}(D(g)) &= \{\mathfrak{p} \subset B \,, \text{prime ideal} \mid g \notin \chi^{-1}(\mathfrak{p})\} \\ &= \{\mathfrak{p} \subset B \,, \text{prime ideal} \mid \chi(g) \notin \mathfrak{p}\} \\ &= D(\chi(g))\,. \end{aligned} \tag{5.24}$$

As the $D(g)$ are a basis, φ is continuous. Since $\varphi^{-1}(D(g)) = D(\chi(g))$ we immediately see that on the principal open subsets we can define $\psi_{D(g)}$ by the morphism

$$\psi_{D(g)} = \chi_g : A_g \to B_{\chi(g)}\,, \qquad \frac{a}{g^r} \mapsto \frac{\chi(a)}{\chi(g)^r}\,. \tag{5.25}$$

The map $\psi_{D(g)}$ is then compatible with restrictions: Let $D(h) \subset D(g)$, then $h^n = vg$ for some $v \in A_{\bar{0}}$ and $\chi(h)^n = \chi(v)\chi(g)$. Thus we have maps $A_g \to A_h$, sending a/g^r to av^r/h^{nr} and $B_{\chi(g)} \to B_{\chi(h)}$ sending $b/\chi(g)^r$ to $b\chi(v)^r/\chi(h)^{nr}$. Then the following diagram commutes:

$$\begin{tikzcd} A_g \arrow[r, "\psi_{D(g)}"] \arrow[d] & B_{\chi(g)} \arrow[d] \\ A_h \arrow[r, "\psi_{D(h)}"] & B_{\chi(h)} \end{tikzcd}\;. \tag{5.26}$$

For any prime ideal $\mathfrak{p}$ in B, we then consider the compositions $A_g \to B_{\chi(g)} \to B_\mathfrak{p}$, where g runs over all the even elements of A with $\chi(g) \notin \mathfrak{p}$. As the diagram (5.26) commutes, there is a unique morphism $A_{\chi^{-1}(\mathfrak{p})} \to B_\mathfrak{p}$, which is the induced morphism on the stalks $\psi_\mathfrak{p}$. We see that $\psi_\mathfrak{p}$ maps a/g to $\chi(a)/\chi(g)$ for any $g \notin \psi^{-1}(\mathfrak{p})$. Thus ψ coincides with the natural morphism $\chi_\mathfrak{p} : A_{\chi^{-1}(\mathfrak{p})} \to B_\mathfrak{p}$ and we see that $\psi_\mathfrak{p}$ is local. The maps ψ_U for open $U \subset Y$ thus combine with φ to give a morphism $f = (\varphi, \psi)$ of superschemes and on the global sections we see that $\psi_X = \chi$.

Assuming surjectivity of Γ for affine superschemes, let $(X, \mathcal{O}_X)$ be any superscheme. We can cover X by affine superscheme $(X_\alpha, \mathcal{O}_{X_\alpha})$, where $\mathcal{O}_{X_\alpha}$ is the restriction of $\mathcal{O}_X$ to X_α. For any $\chi : A \to \mathcal{O}_X(X)$ define χ_α as the composition of χ with the restriction morphism $\mathcal{O}_X(X) \to \mathcal{O}_{X_\alpha}(X_\alpha) = \mathcal{O}_X(X_\alpha)$. Then there are morphisms $f_\alpha = (\varphi_\alpha, \psi_\alpha)$ with $\psi_\alpha(X_\alpha) = \chi_\alpha$. The following diagram commutes due to the definition of χ_α:

$$\begin{tikzcd} A \arrow[rr, "\chi_\alpha"] \arrow[dr, "\psi"] \arrow[dd, "\chi_\beta"'] & & \mathcal{O}_X(X_\alpha) \arrow[dd, "r_{\alpha,\alpha\beta}"] \\ & \mathcal{O}_X(X) \arrow[ur, "r_\alpha"'] \arrow[dl, "r_\beta"] & \\ \mathcal{O}_X(X_\beta) \arrow[rr, "r_{\beta,\alpha\beta}"'] & & \mathcal{O}_X(X_\alpha \cap X_\beta) \end{tikzcd} , \tag{5.27}$$

where r_α, r_β, $r_{\alpha,\alpha\beta}$ and $r_{\beta,\alpha\beta}$ are the restriction morphisms. Hence, we have two morphisms of superschemes f_α, f_β from $(X_\alpha \cap X_\beta, \mathcal{O}_{X_\alpha \cap X_\beta})$ to $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ that give the same morphism of superrings $A \to \mathcal{O}_X(X_\alpha \cap X_\beta)$. But the map Γ is injective and therefore on the intersections $X_\alpha \cap X_\beta$ the morphisms f_α agree. We can thus glue the f_α together to form a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. □

The following two corollaries are immediate:

Corollary 5.4.6. *The object* $\mathrm{Spec}(\mathbb{Z})$ *is terminal in the category of superschemes.*

Corollary 5.4.7. *The category of affine superschemes is contravariant equivalent to the category of superrings.*

An important example of an affine superscheme is the space $\mathbb{A}_k^{n|m}$, which is defined as $\mathrm{Spec}(A_{n|m})$ where $A_{n|m} = k[x_1, \dots, x_n | \vartheta_1, \dots, \vartheta_m]$.

Another way of formulating proposition 5.4.5 is as follows:

Theorem 5.4.8. *Let* $(X, \mathcal{O}_X)$ *be any affine superscheme and let* A *be a superring. Any morphism of superschemes* $X \to \mathrm{Spec}(A)$ *factors over* $\mathrm{Spec}(\mathcal{O}_X(X))$*:*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \mathrm{Spec}(A) \\ & \mathrm{Spec}(\mathcal{O}_X(X)) \arrow[ur] & \end{tikzcd} . \tag{5.28}$$

Remark 5.4.9. Having developed the notion of superschemes so far, one can go on and define closed sub superschemes, closed immersions, open sub superschemes, and so on. We do not follow this route, since this is merely scheme- or sheaf-theoretic and has less to do with special properties of superrings. We refer to [54–56] for the development of these scheme-theoretic notions in the usual algebro-geometric setting. We content ourselves with indicating that fibred products exist in the category of superschemes. The scheme-theoretic notions that we will need are defined on the way. We follow [55].

Definition 5.4.10. *Let* $S = (Z, \mathcal{O}_Z)$ *be a superscheme. We say a superscheme* $T = (X, \mathcal{O}_X)$ *is a superscheme over* S *if there is a morphism of superschemes* $f : T \to S$*. We call* f *the structural morphism. If* $T' = (Y, \mathcal{O}_Y)$ *is another superscheme over* S *with structural morphism* $f' : T' \to S$*, we define a morphism as a morphism of superschemes* $g : T \to T'$ *such that* $f' \circ g = f$*.*

Note that the superschemes over a fixed superscheme S form a category. We want to prove the existence of the fibred product in this category, which is defined as follows:

Definition 5.4.11. *Let $T = (X, \mathcal{O}_X)$ and $T' = (Y, \mathcal{O}_Y)$ be superschemes over S with structural morphisms $f : T \to S$ and $f' : T' \to S$, then the fibred product of T and T' over S is a superscheme W over S with structural morphism $g : W \to S$ and morphisms $p : W \to T$ and $p' : W \to T'$, such that the following diagram commutes*

$$\begin{tikzcd} W \arrow[r, "p"] \arrow[d, "p'"'] \arrow[dr, "g"] & T \arrow[d, "f"] \\ T' \arrow[r, "f'"'] & S \end{tikzcd}, \tag{5.29}$$

and if Z is any superscheme over S with maps $q : Z \to T$ and $q' : Z \to T'$ such that $f \circ q = f' \circ q'$, then there is a unique morphism $h : Z \to W$ such that $p \circ h = q$ and $p' \circ h = q'$. We call p and p' the projections on to X and X' respectively.

Remark 5.4.12. If $T = (X, \mathcal{O}_X)$ is a superscheme over S with structural morphism $f : T \to S$, we will often simply write X for T and omit the mention of the structural morphism. If X, Y are superschemes over S, we write $X \times_S Y$ for the fibred product of X and Y over S.

If U is an open subspace of X, then the restriction of the structure sheaf of X to U makes U in a superscheme, such that for all $u \in U$ the stalk of $\mathcal{O}_U$ at u is the same as the stalk of $\mathcal{O}_X$ at u. We call U an open sub superscheme. The inclusion of U in X defines a morphism of superschemes, which, as one easily verifies, is a monomorphism in the category of superschemes. Note that by a monomorphism is meant a morphism $h : X \to Y$ such that if $f, g : W \to X$ are two morphisms such that $h \circ f = h \circ g$ then $f = g$. A word on notation: if $f : X \to Y$ is a morphism of superschemes and V is an open subset of Y, then with $f^{-1}(V)$ we mean the superscheme defined by the topological space $f^{-1}(V)$ with the structure sheaf given by the restriction of the structure sheaf of X to $f^{-1}(V)$.

Lemma 5.4.13. *If $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ are superschemes over $S = \mathrm{Spec}(C)$ then the fibred product $X \times_S Y$ exists and is isomorphic to $\mathrm{Spec}(A \otimes_C B)$.*

Proof. This follows from the (dual) universal property of the tensor product in the category of C-modules and theorem 5.4.8. □

If U is an open sub superscheme of a superscheme X over S, then U is also a superscheme over S, where the structural morphism $U \to S$ is the restriction of the structural morphism $X \to S$ to U.

Lemma 5.4.14. *Suppose that the fibred product $X \times_S Y$ exists and write $p : X \times_S Y \to X$ for the projection to X. If U is an open subset of X, then $p^{-1}(U)$ is isomorphic to $U \times_S Y$.*

Proof. The proof is purely diagram manipulating: Write $x : X \to S$ and $y : Y \to S$ for the structural morphisms, the restriction of x to U we also denote by x. Suppose $f : Z \to U$ and $g : Z \to Y$ are morphisms such that $x \circ f = y \circ g$. Then there is a unique morphism $h : Z \to X \times_S Y$, such that $p \circ h = f$. Hence $h(Z) \subset p^{-1}(U)$, the morphism h factors over $h' : Z \to p^{-1}(U)$ as $h = i \circ h'$, where $i : p^{-1}(U) \to X \times_S Y$ is the canonical injection. As i is a monomorphism, the morphism h' is unique. Hence $p^{-1}(U)$ has the required universal property of the fibred product. □

Lemma 5.4.15. *Let X and Y be superschemes over S. Let $\{X_i\}$ be a set of open sub superschemes of X such that the X_i cover X. If all fibred products $X_i \times_S Y$ exist, then $X \times_S Y$ exists.*

Proof. Denote $p_i : X_i \times_S Y \to X_i$ the projection to X_i. Write $U_{ij} = p_i^{-1}(X_i \cap X_j)$, then U_{ij} and U_{ji} are by the previous lemma 5.4.14 the fibred product isomorphic to $(X_i \cap X_j) \times_S Y$. Hence we have an isomorphism $U_{ij} \cong U_{ji}$ and we write $\phi_{ij} : U_{ij} \to U_{ji}$ for the isomorphism. By the same reasoning, the open sub superschemes $p_i^{-1}(X_i \cap X_j \cap X_k)$, $p_j^{-1}(X_i \cap X_j \cap X_k)$ and $p_k^{-1}(X_i \cap X_j \cap X_k)$ are isomorphic. It follows by the uniqueness of the isomorphisms that the restrictions of ϕ_{ik} and of $\phi_{jk} \circ \phi_{ij}$ to $U_{ij} \cap U_{ik}$ are the same. Hence we can glue the $X_i \times_S Y$ together to a superscheme W (see for example [56, Corollary I.14]). We have morphisms $q : W \to Y$ and $p : W \to X$ by gluing the projections $X_i \times_S Y \to Y$ and $X_i \times_S Y \to X_i$ together.

Now suppose $f : Z \to X$ and $g : Z \to Y$ are morphisms such that $x \circ f = y \circ g$, where $x : X \to S$ and $y : Y \to S$ are the structural morphisms. We write $Z_i = f^{-1}(X_i)$ and find unique morphisms $h_i : Z_i \to X_i \times_S Y$. By construction of W, the h_i glue together to a morphism $h : Z \to W$. The morphism h is uniquely determined by the h_i. Hence W has the required universal property. □

Theorem 5.4.16. *Let X and Y be superschemes over S. Then the fibred product $X \times_S Y$ exists.*

Proof. Let $\{X_i\}$ and $\{Y_a\}$ be affine coverings of X and Y respectively. If S is affine, we can apply lemma 5.4.15 to conclude that $X \times_S Y_a$ exists, and thus we apply lemma 5.4.15 again to conclude that $X \times_S Y$ exists.

Now suppose S is arbitrary. Cover S by open affine sub superschemes $\{S_i\}$. Let $X_i = x^{-1}(S_i)$ and $Y_i = y^{-1}(S_i)$ where $x : X \to S$ and $y : Y \to S$ are the structural morphisms. We know that the fibred products $X_i \times_{S_i} Y_i$ exist. One easily sees that $X_i \times_{S_i} Y_i \cong X_i \times_S Y$. Applying again lemma 5.4.15 proves the theorem. □

5.4.3 The underlying scheme

Let X be a superscheme with structure sheaf $\mathcal{O}_X$. We define the underlying scheme $\overline{X}$ of X to have the same underlying topological space as X, but with the sheaf such that the stalk at $x \in X$ is $\mathcal{O}_{\overline{X},x} = \overline{\mathcal{O}_{X,x}}$. Consider the presheaf defined by the assignment $U \mapsto \overline{\mathcal{O}_X(U)}$ for each open set U. This is indeed a presheaf: By the commutativity of diagram (3.2) of section 3.1 we have for each open inclusion $U \subset V \subset X$ a unique induced restriction $res' : \overline{\mathcal{O}_X(V)} \to \overline{\mathcal{O}_X(U)}$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{O}_X(V) \arrow[r, "res"] \arrow[d] & \mathcal{O}_X(U) \arrow[d] \\ \overline{\mathcal{O}_X(V)} \arrow[r, "res'"'] & \overline{\mathcal{O}_X(U)} \end{tikzcd} , \qquad (5.30)$$

where the horizontal arrows are the restrictions and the vertical arrows are the projections to the body. Thus we have a presheaf and the stalk at $x \in X$ of this presheaf is indeed $\overline{\mathcal{O}_{X,x}}$. If the sheaf $\mathcal{O}_{X,x}$ consists of split superrings, one easily verifies that this presheaf is a sheaf. In the general case, we define the sheaf $\mathcal{O}_{\bar{X}}$ to be the sheafification of this presheaf. By construction we have a morphism $\overline{X} \to X$ that 'embeds' the underlying scheme into the superscheme X.

Theorem 5.4.17. *Let $f : X \to Y$ be a morphism of superschemes and let $\overline{X}$ and $\overline{Y}$ be the underlying schemes of X and Y respectively. Then there is a unique morphism of schemes $\bar{f} : \overline{X} \to$*

$\overline{Y}$ *such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \overline{X} \arrow[u] \arrow[r, "\bar{f}"'] & \overline{Y} \arrow[u] \end{tikzcd} . \tag{5.31}$$

Proof. Let $f : X \to Y$ denote the morphism of topological spaces and let $\phi_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ be the morphism on sections over U for any open $U \subset Y$. Then we have an induced morphism on the stalks $\varphi_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. We define $\bar{f}$ as follows: as a morphism of topological spaces, we take the same as f. On the sections we define $\bar{\phi}_U$ via the stalks. For each $x \in X$ and $a \in \mathcal{O}_{Y,f(x)}$ we define $\bar{\varphi}_x(\bar{a}) = \overline{\varphi_x(a)}$. This is required by the commutativity of 5.31 and fixes the morphisms $\bar{\phi}_U : \overline{\mathcal{O}_Y}(U) \to \overline{\mathcal{O}_X}(f^{-1}(U))$. Indeed, if $\bar{\tau} \in \overline{\mathcal{O}_Y}(U)$ for some open set $U \subset Y$, we need to define $\bar{\phi}_U(\bar{\tau})$ as the function that sends $\mathfrak{p} \in f^{-1}(U)$ to $\bar{\varphi}_{\mathfrak{p}}(\bar{\tau}_{f(\mathfrak{p})})$. In this way we automatically have compatibility with restrictions and thus we have a morphism of ringed spaces. Now let $\mathfrak{m}_x$ and $\mathfrak{m}_{f(x)}$ be the maximal ideals of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,f(x)}$ respectively, then by definition of $\bar{\varphi}_x$ we have $\bar{\varphi}_x(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$. □

5.4.4 Projective superschemes

We say a superring A is a $\mathbb{Z}$-graded superring if A is a direct sum $A = \bigoplus_{i\geq 0} A_i$ of abelian groups such that $(A_i)_{\bar{0}} = A_{\bar{0}} \cap A_i$, $(A_i)_{\bar{1}} = A_{\bar{1}} \cap A_i$ and the multiplication map satisfies $A_iA_j \subset A_{i+j}$. We say an element a of A is homogeneous if it lies in some $A_{i,\bar{0}}$ or $A_{i,\bar{1}}$. We call an element $\mathbb{Z}$-homogeneous if it lies in some A_i. We write $\deg(a)$ for the $\mathbb{Z}$-degree of a $\mathbb{Z}$-homogeneous element of A. An ideal $\mathfrak{a} \subset A$ is called homogeneous if for any element $a \in \mathfrak{a}$ also all its homogeneous components $a_{i,\bar{0}} \in A_{i,\bar{0}}$ and $a_{i,\bar{1}} \in A_{i,\bar{1}}$ lie in $\mathfrak{a}$. Intersection, sum and product of homogeneous ideals are again homogeneous and an ideal is homogeneous if and only if it can be generated by homogeneous elements. A homogeneous ideal $\mathfrak{p}$ is prime if and only if for any homogeneous $a, a' \in A$ that are not in $\mathfrak{p}$ also $aa' \notin \mathfrak{p}$. We denote A_+ the homogeneous ideal given by $A_+ = \bigoplus_{i\geq 1} A_i$.

We want to associate a topological space to a $\mathbb{Z}$-graded superring. Following for example [55] we define $\mathrm{Proj}(A)$ to be the set of homogeneous prime ideals in A that do not contain A_+. We give $\mathrm{Proj}(A)$ the topology defined by the closed sets $V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Proj}(A) \mid \mathfrak{p} \supset \mathfrak{a}\}$, where $\mathfrak{a}$ is any homogeneous ideal in A. As in the affine case, one easily checks that this indeed defines a topology. For any even homogeneous $f \in A_+$ we define the principal open subset $D_+(f)$ to be the subset of $\mathrm{Proj}(A)$ of the homogeneous prime ideals $\mathfrak{p}$ that do not contain f. As in the affine case, the set $D_+(f)$ is the complement of $V(f)$ and the set of all $D_+(f)$, where f runs over all homogeneous even elements of A_+, forms a basis of the topology. Indeed, if $V(\mathfrak{a})$ is a closed subset, we choose $a \in \mathfrak{a}_{\bar{0}} \cap A_+$, then $D_+(a)$ lies in the complement of $V(\mathfrak{a})$.

Let $\{f_i \in A_+ \cap A_{\bar{0}} \mid i \in I\}$ be a set of even $\mathbb{Z}$-homogeneous elements in A_+ such that the f_i generate A_+. Then we have $\mathrm{Proj}(A) = \bigcup_{i\in I} D_+(f_i)$, since if $\mathfrak{p}$ is a homogeneous prime ideal not in the union, it contains all the f_i and thus A_+, which is impossible. The converse need not hold. A counterexample is provided by the commutative $\mathbb{Z}$-graded ring $A = k[x, y]/(y^2)$, where $A_0 = k$ and the $\mathbb{Z}$-degrees of x and y are both 1. Then $\mathrm{Proj}(A)$ contains only the element $\mathfrak{p} = (y)$: Suppose a homogeneous prime ideal $\mathfrak{p}$ contains $f = x^m + \lambda x^{m-1}y$ for some $\lambda \in k$. Since $y \in \mathfrak{p}$ it follows that $x^m \in \mathfrak{p}$. But then we need that $x \in \mathfrak{p}$ and it follows that $A_+ = (x, y) \subset \mathfrak{p}$, which is impossible by the definition of $\mathrm{Proj}(A)$. But then $D_+(x)$ is an open subset of $\mathrm{Proj}(A)$ that covers $\mathrm{Proj}(A)$. However,

x does not generate A_+. The following lemma does give a sufficient and necessary condition for a set of even $\mathbb{Z}$-homogeneous elements to give rise to a covering of $\mathrm{Proj}(A)$:

Lemma 5.4.18. *Let $\{f_i \mid i \in I\}$ be a set of even $\mathbb{Z}$-homogeneous elements in A_+. Then $\bigcup_{i\in I} D_+(f_i) = \mathrm{Proj}(A)$ if and only if the radical ideal of the ideal $(f_i : i \in I)$ contains A_+.*

Proof. Suppose that the radical ideal of $(f_i : i \in I)$ contains A_+. Then any homogeneous prime ideal $\mathfrak{p}$ not contained in the union $\bigcup_{i\in I} D_+(f_i)$ must contain $(f_i : i \in I)$ and therefore also the radical of $(f_i : i \in I)$. But then $\mathfrak{p} \supset A_+$, and thus $\mathfrak{p}$ does not correspond to a point in $\mathrm{Proj}(A)$. For the converse, suppose that the radical ideal of $(f_i : i \in I)$ does not contain A_+. Then there is a homogeneous a such that no power of a lies in $(f_i : i \in I)$. Consider the set Ω of homogeneous ideals that contain $(f_i : i \in I)$ but do not contain any power of a. Then Ω is not empty as $(f_i : i \in I) \in \Omega$ and Ω can be partially ordered by inclusion. If $\{I_\alpha\}$ is some totally ordered subset of Ω, then $\bigcup_\alpha I_\alpha$ is homogeneous, contains $(f_i : i \in I)$ and does not contain any power of a. By Zorn, there is a maximal element $\mathfrak{m} \in \Omega$. Suppose, that there are $x, y \notin \mathfrak{m}$ that are homogeneous and that $xy \in \mathfrak{m}$. Then the homogeneous ideals $(\mathfrak{m}, x)$ and $(\mathfrak{m}, y)$ both properly contain $\mathfrak{m}$ and thus contain some power of a. There are thus $r, s, r', s' \in A$ and $m, m' \in \mathfrak{m}$ such that $a^k = (rx + sm)$ and $a^l = (r'y + s'm')$ for some positive integers k, l. It follows then that $a^{k+r} = (rx + sm)(r'y + s'm') \in \mathfrak{m}$, which is a contradiction; hence no such x and y exist. Therefore $\mathfrak{m}$ is a prime ideal. Since $a \notin \mathfrak{m}$, $\mathfrak{m}$ does not contain A_+. But then $\mathfrak{m}$ corresponds to a point in $\mathrm{Proj}(A)$ not contained in the union $\bigcup_{i\in I} D_+(f_i)$. Hence the $D_+(f_i)$ do not cover $\mathrm{Proj}(A)$. $\square$

If S is a multiplicative set in A that only contains even $\mathbb{Z}$-homogeneous elements, then the localization $S^{-1}A$ has a natural $\mathbb{Z}$-grading and is again a superring: For homogeneous a we define the $\mathbb{Z}$-grading of a/s to be the $\mathbb{Z}$-degree of a minus the $\mathbb{Z}$-degree of s and we call a/s even (resp. odd) if a is even (resp. odd). The sub superring of all elements of $\mathbb{Z}$-degree zero we denote $(S^{-1}A)_0$. This is again a superring and all elements are of the form a/s for some $\mathbb{Z}$-homogeneous $a \in A$ with $\mathbb{Z}$-degree equal to the $\mathbb{Z}$-degree of s. In the case where $S \subset A_0$, one easily sees that $(S^{-1}A)_0$ is naturally isomorphic to the superring obtained by localizing A_0 with respect to S. If S is the multiplicative set generated by an even $\mathbb{Z}$-homogeneous element f, then we write $A_{(f)}$ for $(S^{-1}A)_0$.

Lemma 5.4.19. *Let S be a multiplicative set of A that only contains even $\mathbb{Z}$-homogeneous elements. Let T' be a multiplicative set inside $(S^{-1}A)_0$ that only contains even elements. Consider the set T of even elements $t \in A$ such that $t/s \in T'$ for some $s \in S$. Then T is a multiplicative set inside A that only contains even $\mathbb{Z}$-homogeneous elements and*

$$(T')^{-1}(S^{-1}A)_0 \cong ((TS)^{-1}A)_0 \,. \tag{5.32}$$

Proof. We denote the elements of $(T')^{-1}(S^{-1}A)_0$ by $(a/s, t/z)$, where $a/s \in (S^{-1}A)_0$ and $t/z \in T'$. All elements of $(T')^{-1}(S^{-1}A)_0$ are of the form $(a/s, t/z)$, with $\deg(a) = \deg(s)$, $\deg(z) = \deg(z)$ and $t \in T$.

We define a map $\varphi : (T')^{-1}(S^{-1}A)_0 \to ((TS)^{-1}A)_0$ by

$$\varphi : (a/s, t/z) \mapsto \frac{az}{st} \,. \tag{5.33}$$

It is easily checked that φ is a well-defined map and preserves the $\mathbb{Z}_2$-grading. Furthermore, we see that $\varphi(x) + \varphi(y) = \varphi(x + y)$ and $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in (T')^{-1}(S^{-1}A)_0$. Therefore φ is a morphism of superrings. Suppose $\varphi((a/s, t/z)) = 0$, then there are $s' \in S$ and $t' \in T$ so that $s't'az = 0$ in A. But $(a/s, t/z)$ is zero in $(T')^{-1}(S^{-1}A)_0$ if and only if there are s'' and t'' in S and T respectively, such that $s''t''a = 0$. It follows that $\mathrm{Ker}(\varphi) = 0$. To prove surjectivity, let $\frac{a}{st} \in ((ST)^{-1}(A))_0$ be given with $s \in S$ and $t \in T$. It follows that $\deg(a) = \deg(s) + \deg(t)$.

As $t \in T$, there is an s' with $\deg(s') = \deg(t)$ such that $s'/t \in T'$. The element $(a/ss', t/s')$ of $(T')^{-1}(S^{-1}A)_0$ is thus well-defined and

$$\varphi : (a/ss', t/s') \mapsto \frac{as'}{tss'} \sim \frac{a}{st}\,. \tag{5.34}$$

□

Lemma 5.4.20. *Let f be an even $\mathbb{Z}$-homogeneous element of nonzero $\mathbb{Z}$-degree. Then the map $\varphi : D_+(f) \to \mathrm{Spec}(A_{(f)})$ given by $\varphi(\mathfrak{p}) = (\mathfrak{p}A_f)_0$ is a homeomorphism.*

Proof. We define an inverse morphism as follows: suppose $\mathfrak{q}$ is a prime ideal in $A_{(f)}$, define $\psi(\mathfrak{q})$ as the set of all elements that are sums of $\mathbb{Z}$-homogeneous elements y for which there exist integers m, n with $y^m/f^n \in \mathfrak{q}$. Then $\psi(\mathfrak{q})$ is by construction homogeneous, does not contain f and contains all odd elements. We thus need to show that $\psi(\mathfrak{q})$ is an ideal and is prime. Suppose $x, y \in \psi(\mathfrak{q})$ and x, y are homogeneous. If x, y do not have the same $\mathbb{Z}$-degree, then by definition $x + y$ lies in $\psi(\mathfrak{q})$. If x and y have the same $\mathbb{Z}$-degree and the integers a, b, c, d are such that x^a/f^b and y^c/f^d lie in $\mathfrak{q}$, then it follows that $ad = bc$. Thus $x^{ac}/f^{bc} \in \mathfrak{q}$ and $y^{ac}/f^{bc} \in \mathfrak{q}$, from which we deduce $(x+y)^{2ac}/f^{2bc} \in \mathfrak{q}$ so that $x + y \in \psi(\mathfrak{q})$. If $a \in A$ is homogeneous and x is a homogeneous element in $\psi(\mathfrak{q})$ with $x^m/f^n \in \mathfrak{q}$, then $(rx)^{m\deg(f)}/f^{n\deg(r)} \in \mathfrak{q}$, from which we conclude that $rx \in \psi(\mathfrak{q})$. Now suppose $x, y \in A$ are homogeneous such that $xy \in \psi(\mathfrak{q})$, so that there are positive integers m, n with $z = x^n y^n/f^m \in \mathfrak{q}$. Then $n\deg(x) + n\deg(y) = m\deg(f)$ and raising z to the power $\deg(f)$ gives

$$\frac{x^{n\deg(f)}}{f^{n\deg(x)}}\frac{y^{n\deg(f)}}{f^{n\deg(y)}} \in \mathfrak{q}\,, \tag{5.35}$$

from which we conclude that $\psi(\mathfrak{q})$ is a prime ideal. It is furthermore straightforward to check that $\psi(\varphi(\mathfrak{p})) = \mathfrak{p}$ for any homogeneous prime ideal $\mathfrak{p}$ in $D_+(f)$ and that $\varphi(\psi(\mathfrak{q})) = \mathfrak{q}$ for any prime ideal $\mathfrak{q}$ in $A_{(f)}$.

One easily verifies that $\mathfrak{p} \in D_+(f)$ contains the homogeneous even element $g \in A$ if and only if $\varphi(\mathfrak{p})$ contains $h = g^{\deg(f)}/f^{\deg(g)} \in A_{(f)}$. Thus $\varphi(V(g) \cap D_+(f)) \subset V(h)$ and $\varphi^{-1}(V(h)) \subset V(g) \cap D_+(f)$. But then we must have equalities and thus φ is continuous and sends open sets to open sets. Since we already proved φ is a bijection, φ is a homeomorphism. □

Corollary 5.4.21. *Let f be an even $\mathbb{Z}$-homogeneous element of nonzero $\mathbb{Z}$-degree. There is a one-to-one correspondence between the homogeneous prime ideals of A_f and the prime ideals of $A_{(f)}$.*

Proof. There is a one-to-one correspondence between the prime ideals in A not containing f and the prime ideals in A_f. It is easily seen that this correspondence preserves the $\mathbb{Z}$-grading. □

For any prime ideal $\mathfrak{p}$ not containing A_+, consider the multiplicative set S of all even $\mathbb{Z}$-homogeneous elements of A that are not in $\mathfrak{p}$. Then the localization $S^{-1}A$ is a $\mathbb{Z}$-graded superring and contains a sub superring of elements of $\mathbb{Z}$-degree zero. We define $A_{(\mathfrak{p})}$ to be this sub superring of elements of $\mathbb{Z}$-degree zero. It is easily seen that this again defines a superring. We then obtain a sheaf $\mathcal{O}$ on $\mathrm{Proj}(A)$ as follows: for any open set $U \subset \mathrm{Proj}(A)$ we define $\mathcal{O}(U)$ to be the superring of all functions $s : U \to A_{(\mathfrak{p})}$, such that the image $s_\mathfrak{p}$ of $\mathfrak{p} \in U$ under s lies in $A_{(\mathfrak{p})}$ and such that for all $\mathfrak{p} \in U$ there is an open neighborhood $V \subset U$ containing $\mathfrak{p}$ for which there are $\mathbb{Z}$-homogeneous $a \in A$ and $f \in A_{\bar{0}}$ of the same $\mathbb{Z}$-degree and $f \notin \mathfrak{q}$ for all $\mathfrak{q} \in V$, such that $s_\mathfrak{q}$ coincides with the image of a/f in $A_{(\mathfrak{q})}$ for all $\mathfrak{q} \in V$. We call $\mathcal{O}$ the structure sheaf of $\mathrm{Proj}(A)$. We use the notation $(\mathcal{O}, \mathrm{Proj}(A))$ for the topological space $\mathrm{Proj}(A)$ equipped with the sheaf of superrings $\mathcal{O}$ that we just defined.

Lemma 5.4.22. *Let $\mathfrak{p} \in \mathrm{Proj}(A)$, then the stalk of the sheaf $\mathcal{O}$ at $\mathfrak{p}$ is $A_{(\mathfrak{p})}$.*

Proof. The stalk can be constructed as follows: Consider the set of all pairs (s, U), where U is an open neighborhood of $\mathfrak{p}$ and $s \in \mathcal{O}(U)$. We define the pairs (s, U) and (t, V) to be equivalent if there is an open subset W contained in U and V such that the restrictions of s and t to W are equivalent as elements in $\mathcal{O}(W)$. The equivalence classes of the pairs (s, U) can then be equipped with the structure of a superring and the superring obtained in this way is the stalk $\mathcal{O}_{\mathfrak{p}}$.

By the very definition of the sheaf $\mathcal{O}$ on $\mathrm{Spec}(A)$ we can assume that for any element $(s, U) \in \mathcal{O}_{\mathfrak{p}}$ the open set U is so large that s is given by a/g for some $g \notin \mathfrak{p}$ and $a \in A$ with equal degree. Then there is an obvious morphism $\mathcal{O}_{\mathfrak{p}} \to A_{(\mathfrak{p})}$ that sends $(a/g, U)$ to the element a/g in $A_{(\mathfrak{p})}$. This implies that the map is well-defined. Clearly, it is surjective. To check injectivity, we assume (s, U) is mapped to zero. We may then assume $s = a/g$ for some $g \notin \mathfrak{p}$. By assumption there is an even element $h \in \mathfrak{p}$ such that $ha = 0$. But then the section s vanishes on $U \cap D_+(h)$ and hence (s, U) is zero in $\mathcal{O}_{\mathfrak{p}}$. □

Proposition 5.4.23. *Let f be an even homogeneous element of A_+. Then the restriction of the structure sheaf $\mathcal{O}$ of A to $D_+(f)$ makes $D(f)$ into an affine superscheme, which is isomorphic to* $\mathrm{Spec}(A_{(f)})$.

Proof. By lemma 5.4.20 the topological spaces $D_+(f)$ and $\mathrm{Spec}(A_{(f)})$ are homeomorphic with the homeomorphism $\varphi : D_+(f) \to \mathrm{Spec}(A_{(f)})$ defined by $\varphi(\mathfrak{p}) = (\mathfrak{p}A_f)_0$. From the proof of the same lemma we know that $\varphi(D_+(f) \cap D_+(g)) = D(g^{\deg(f)}/f^{\deg(g)})$.

Call $(X, \mathcal{O}_X)$ the superringed space $D_+(f)$ with the sheaf $\mathcal{O}_{\mathrm{Proj}(A)}|_{D_+(f)}$ and $(Y, \mathcal{O}_Y)$ the superscheme of $\mathrm{Spec}(A_{(f)})$. Call S the multiplicative set of even $\mathbb{Z}$-homogeneous in A_f not contained in $\mathfrak{p}A_f$ for some homogeneous prime ideal $\mathfrak{p}$ in $D_+(f)$. By lemma 5.4.19 we have $\mathcal{O}_{Y,\varphi(\mathfrak{p})} = (S^{-1}A_f)_0$, which is again isomorphic to $A_{(\mathfrak{p})}$ since $f \in S$. Therefore $\mathcal{O}_{X,\mathfrak{p}} \cong \mathcal{O}_{Y,\varphi(\mathfrak{p})}$ as superrings. But since the stalks are local rings, the isomorphism must be a local morphism. Explicitly, the morphism is given by $\chi_{\mathfrak{p}} : \mathcal{O}_{Y,\varphi(\mathfrak{p})} \to \mathcal{O}_{X,\mathfrak{p}}$, $\chi_{\mathfrak{p}} : (a/f^r, g/f^t) \mapsto af^t/gf^r$.

A principal open set $D(g/f^t) \subset Y$ is the spectrum of $A_{(f,g)}$ and thus we have an induced morphism $\chi_{D(g/f^t)} : \mathcal{O}_Y(D(g/f^t)) \to \mathcal{O}_X(D_+(g) \cap D_+(f))$ given by $(a/f^r, g/f^t) \mapsto af^t/gf^r$. But then there is only one way to extend this to a morphism of sheaves. Suppose U is an open set of Y and s is a section over U. Then we define $\chi_U(s)$ to be the morphism that sends $\mathfrak{p} \in \varphi^{-1}(U)$ to $\chi_{\mathfrak{p}}(s_{\varphi(\mathfrak{p})})$. Then $\chi_U(s)$ is indeed a section of $\mathcal{O}_X(U)$ since if on $V \subset U$ the section s is given by $s_{\mathfrak{q}} = (a/f^r, g/f^t) \in \mathcal{O}_{Y,\mathfrak{q}}$ then we have $\chi_U(s) : \mathfrak{p} \mapsto af^t/gf^r \in \mathcal{O}_{X,\mathfrak{p}}$ for all $\mathfrak{p} \in \varphi^{-1}(U)$. It is then easily seen that the maps χ_U are compatible with restrictions and that the induced morphism on the stalks is precisely the local morphism $\chi_{\mathfrak{p}}$. As $\chi_{\mathfrak{p}}$ is an isomorphism, we have an isomorphism of sheaves $\mathcal{O}_X \cong \mathcal{O}_Y$ and the proposition is proved. □

Corollary 5.4.24. *For any $\mathbb{Z}$-graded superring,* $\mathrm{Proj}(A)$, *together with its structure sheaf $\mathcal{O}$ of superrings, is a superscheme.*

We define the projective superspace $\mathbb{P}_k^{n|m}$ to be $\mathrm{Proj}(A_{n+1|m})$, where $A_{n+1|m} = k[x_0, \ldots, x_n \mid \vartheta_1, \ldots, \vartheta_m]$. The elements x_i and ϑ_α we give $\mathbb{Z}$-degree 1. The elements $x_0, \ldots, x_n$ define an ideal whose radical is A_+, and thus the subsets $D_+(x_i)$ provide a cover. One easily sees that $D_+(x_i) \cong \mathrm{Spec}(A_{n|m}) = \mathbb{A}_k^{n|m}$.

5.5 Completion

Let A be a superring. We define a filtration of A to be a set of $\mathbb{Z}_2$-graded ideals $F = \{\mathfrak{a}_k\}_{k \geq 0}$ with

$$F : \quad A = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \mathfrak{a}_3 \supset \cdots , \tag{5.36}$$

with $\mathfrak{a}_i\mathfrak{a}_j \subset \mathfrak{a}_{i+j}$. The F-associated graded superring is given by

$$\mathrm{gr}_F(A) = A/\mathfrak{a}_1 \oplus \mathfrak{a}_1/\mathfrak{a}_2 \oplus \mathfrak{a}_2/\mathfrak{a}_3 \oplus \cdots . \tag{5.37}$$

Addition is done componentwise and the rule for multiplication is given as follows: consider the two elements $x = x' \bmod \mathfrak{a}_{m+1} \in \mathfrak{a}_m/\mathfrak{a}_{m+1}$ and $y = y' \bmod \mathfrak{a}_{k+1} \in \mathfrak{a}_k/\mathfrak{a}_{k+1}$ in $\mathrm{gr}_F(A)$, then we put $xy = x'y' \bmod \mathfrak{a}_{m+k+1}$. Since the $\mathfrak{a}_i$ are assumed to be $\mathbb{Z}_2$-graded, so is the A-module $\mathfrak{a}_k$ and hence also $\mathfrak{a}_k/\mathfrak{a}_{k+1}$; this induces a componentwise $\mathbb{Z}_2$-grading on $\mathrm{gr}_F(A)$: $\mathrm{gr}_F(A)_i = (A/\mathfrak{a}_1)_i \oplus (\mathfrak{a}_1/\mathfrak{a}_2)_i \oplus \cdots$, where $i \in \mathbb{Z}_2$. A particular example of a filtration of A is obtained by taking $\mathfrak{a}_i = \mathfrak{a}^i$ for some fixed $\mathbb{Z}_2$-graded ideal $\mathfrak{a}$. We then call the filtration given by

$$A := \mathfrak{a}^0 \supset \mathfrak{a} \supset \mathfrak{a}^2 \supset \mathfrak{a}^3 \supset \cdots , \tag{5.38}$$

the $\mathfrak{a}$-adic filtration. The associated graded superring is then denoted $\mathrm{gr}_{\mathfrak{a}}(A)$.

We define an inverse system of A-modules to be a set of A-modules M_i, where i runs over some directed partially ordered index set I, together with morphisms of A-modules $f_{ij} : M_i \to M_j$ for $i \geq j$ satisfying $f_{jk} \circ f_{ij} = f_{ik}$ and $f_{ii} = \mathrm{id}_{M_i}$. Recall that a partially ordered index set I is directed if for any two $i, j \in I$ there is $k \in I$ with $i, j \geq k$. We write $(f_{ij} : M_i \to M_j)_I$ for the inverse system. If I is clear from the context it will be omitted. To an inverse system $X = (f_{ij} : M_i \to M_j)_I$ we associate a category of cones. A cone over X is defined as an A-module C, called the apex of the cone, together with maps $c_j : C \to M_j$ for all $j \in I$ satisfying $f_{ij} \circ c_i = c_j$ for all $i, j \in I$. We then write $(C, c_i)_I$ for the cone with apex C and morphisms $c_i : C \to M_i$. A morphism of cones $(C, c_i)_I \to (D, d_i)_I$ is a morphism of A-modules $g : C \to D$ such that $c_i \circ g = d_i$. We write $\mathrm{Cone}(X)$ for the category of cones associated to X. See the figure in equation (5.39) for a sketch of the situation:

$$\begin{tikzcd}
\cdots & M_j \arrow[l] & & M_i \arrow[ll, "f_{ij}"'] & \cdots \arrow[l] \\
& & C \arrow[ul, "c_j"] \arrow[ur, "c_i"'] & & \\
& & D \arrow[u, "g"] \arrow[uul, "d_j"] \arrow[uur, "d_i"'] & &
\end{tikzcd} . \tag{5.39}$$

We call the terminal object in the category of cones associated to X the (inverse) limit of the inverse system X. The terminal object consists of an A-module X_∞ and a set of morphisms $x_i : X_\infty \to M_i$. We write $X_\infty = \varprojlim M_i$ and we call the morphisms x_i the projections from the limit to the inverse system. The object X_∞ is thus characterized by the following universal property: there are morphisms $x_i : X_\infty \to M_i$ satisfying $f_{ij} \circ x_i = x_j$ and if C is an A-module together with morphisms $c_i : C \to M_i$ satisfying $f_{ij} \circ c_i = c_j$, then there is a unique morphism of A-modules $h : X_\infty \to C$ such that $c_i = x_i \circ h$. By the universal property, or equivalently by being a terminal object in some category, the A-module X_∞ is determined up to isomorphism.

A filtration $F = \{\mathfrak{a}_k\}_{k \geq 0}$ in a superring A gives rise to an inverse system $X_F := (A/\mathfrak{a}_i \to A/\mathfrak{a}_j : i \geq j)$: we have natural morphisms $A/\mathfrak{a}_i \to A/\mathfrak{a}_j$ if $i \geq j$ and if $i \geq j \geq k$, then the composite of $A/\mathfrak{a}_i \to A/\mathfrak{a}_j$ and $A/\mathfrak{a}_j \to A/\mathfrak{a}_k$ equals the morphism $A/\mathfrak{a}_i \to A/\mathfrak{a}_k$. The limit of this inverse system is called the completion of A with respect to F and is denoted $\hat{A} = \varprojlim A/\mathfrak{a}_i$. If the filtration of F is clear from the context, we simply call $\hat{A}$ the completion. An explicit construction of $\hat{A}$ is as follows: We take the subset of $\prod_i (A/\mathfrak{a}_i)$ that contains all the elements

$(a_i \bmod \mathfrak{a}_i)_i = (a_1 \bmod \mathfrak{a}_1, a_2 \bmod \mathfrak{a}_2, \ldots)$ such that $a_i \equiv a_j \bmod \mathfrak{a}_j$ for all $i \geq j$. We can turn this subset into a superring by defining the addition and the multiplication pointwise and the $\mathbb{Z}_2$-grading we define by the rule that $(a_i \bmod \mathfrak{a}_i)_i$ is even (resp. odd) when all $a_i \bmod \mathfrak{a}_i$ are even (resp. odd) elements in $A/\mathfrak{a}_i$. The superring obtained in this way has the universal property of $\hat{A}$. Often when we write $(a_i \bmod \mathfrak{a}_i)_i$ we refer to this description of the completion $\hat{A}$.

Proposition 5.5.1. *Let A be a superring and $F = \{\mathfrak{a}_k\}_{k\geq 0}$ a filtration of A. Write $\hat{A} = \varprojlim A/\mathfrak{a}_k$. Write p_k for the projections $p_k : \hat{A} \to A/\mathfrak{a}_k$ and denote $\hat{\mathfrak{a}}_k$ be the kernel of p_k. Then $\hat{A}$ has a filtration $\hat{F} : \hat{A} = \hat{\mathfrak{a}}_0 \supset \hat{\mathfrak{a}}_1 \supset \hat{\mathfrak{a}}_2 \supset \cdots$. We have* (1) $\mathfrak{a}_k/\mathfrak{a}_{k+1} \cong \hat{\mathfrak{a}}_k/\hat{\mathfrak{a}}_{k+1}$, (2) $\hat{A}/\hat{\mathfrak{a}}_k \cong A/\mathfrak{a}_k$, (3) $\mathrm{gr}_{\hat{F}}(\hat{A}) \cong \mathrm{gr}_F(A)$, *and* (4) $\varprojlim \hat{A}/\hat{\mathfrak{a}}_k \cong \hat{A}$.

Proof. Most is standard and can be found in for example [15,16]. We have morphisms of A-modules $\varphi_k : \mathfrak{a}_k \to \hat{\mathfrak{a}}_k/\hat{\mathfrak{a}}_{k+1}$ sending $x \in \mathfrak{a}_k$ to $(x \bmod \mathfrak{a}_i)_i \bmod \hat{\mathfrak{a}}_{k+1}$, which are surjective and with kernel $\mathfrak{a}_{k+1}$. The induced maps $f_k : \mathfrak{a}_k/\mathfrak{a}_{k+1} \to \hat{\mathfrak{a}}_k/\hat{\mathfrak{a}}_{k+1}$ are thus isomorphisms and can be combined to form a morphism $\psi : \mathrm{gr}_F(A) \to \mathrm{gr}_{\hat{F}}(\hat{A})$; one easily checks that for homogeneous x and y we have $\psi(xy) = \psi(x)\psi(y)$. This proves (1) and (3). Assertion (2) can be seen by using the explicit construction given above. To prove (4) we note that by (3) the systems $(A/\mathfrak{a}_i \to A/\mathfrak{a}_j : j \leq i)$ and $(\hat{A}/\hat{\mathfrak{a}}_i \to A/\hat{\mathfrak{a}}_j : j \leq i)$ are isomorphic, and hence the categories of cones are isomorphic. Then the terminal objects are isomorphic too. □

Lemma 5.5.2. *Let A be a superring and suppose A is filtered by a filtration $F = \{\mathfrak{a}_i\}_{i\geq 0}$ and by a filtration $G = \{\mathfrak{n}_j\}_{j\geq 0}$. If for all $\mathfrak{a}_i$ there is an $\mathfrak{n}_j$ with $\mathfrak{n}_j \subset \mathfrak{a}_i$ and for all $\mathfrak{n}_k$ there is $\mathfrak{a}_l$ with $\mathfrak{a}_l \subset \mathfrak{n}_k$ then $\varprojlim A/\mathfrak{a}_i \cong \varprojlim A/\mathfrak{n}_j$.*

Proof. Any cone $(Z, f_i : Z \to A/\mathfrak{a}_i)$ over the inverse system $X := (A/\mathfrak{a}_i \to A/\mathfrak{a}_j, i \geq j)$ gives rise to a cone over $Y := (A/\mathfrak{n}_i \to A/\mathfrak{n}_j, i \geq j)$ and vice versa. Thus we get functors $\mathrm{Cone}(X) \to \mathrm{Cone}(Y)$ and $\mathrm{Cone}(Y) \to \mathrm{Cone}(X)$ and these functors are inverse to each other. Hence $\mathrm{Cone}(X)$ and $\mathrm{Cone}(Y)$ are isomorphic and thus the terminal objects are the isomorphic. □

A particular case, which will be of interest later when we prove the Cohen structure theorem in section 5.12, is treated in the following theorem:

Theorem 5.5.3. *Let A be a complete superring with respect to the $\mathfrak{p}$-adic filtration, where $\mathfrak{p}$ is a prime ideal. Then $A_{\bar{0}}$ is complete with respect to the $\mathfrak{p}_{\bar{0}}$-adic filtration.*

Proof. Clearly we have $(\varprojlim A/\mathfrak{p}^i)_{\bar{0}} \cong \varprojlim A_{\bar{0}}/(\mathfrak{p}^i)_{\bar{0}}$. But the right-hand side is isomorphic to $\varprojlim A_{\bar{0}}/(\mathfrak{p}_{\bar{0}})^i$ since $(\mathfrak{p}^{2i})_{\bar{0}} \subset (\mathfrak{p}_{\bar{0}})^i$ and $(\mathfrak{p}_{\bar{0}})^i \subset (\mathfrak{p}^i)_{\bar{0}}$. □

Theorem 5.5.4. *Let A be a Noetherian superring and $F = \{\mathfrak{a}_k\}_{k\geq 0}$ a filtration of A with $\mathfrak{a}_k = I^k$ for some $\mathbb{Z}_2$-graded ideal I. Then $\mathrm{gr}_F(A)$ is Noetherian.*

Proof. Since A is Noetherian I is finitely generated and A/I is Noetherian. But then if $x_1, \ldots, x_p$ are homogeneous generators of I, then the $x_i \bmod I$ generate $\mathrm{gr}_F(A)$ as an A/I-algebra and hence $\mathrm{gr}_F(A)$ is Noetherian. □

If A is a superring with a filtration $F = \{\mathfrak{a}_k\}_{k\geq 0}$ and completion $\hat{A}$ we have a canonical morphism $j : A \to \hat{A}$ mapping $a \in A$ to the element $(a \bmod \mathfrak{a}_1, a \bmod \mathfrak{a}_2, \ldots) = (a \bmod \mathfrak{a}_i)_i$. We call j the canonical insertion of the completion with respect to F.

Lemma 5.5.5. *Let A be a superring with filtration $F = \{\mathfrak{a}_k\}_{k\geq 0}$ and let $\hat{A}$ be the completion $\hat{A} = \varprojlim A/\mathfrak{a}_k$ with the induced filtration $\hat{F} = \{\hat{\mathfrak{a}}_k\}_{k\geq 0}$ and let $j : A \to \hat{A}$ be the canonical insertion. Then:*

(i) $j(\mathfrak{a}_k) \subset \hat{\mathfrak{a}}_k$,

(ii) $j^{-1}(\hat{\mathfrak{a}}_k) = \mathfrak{a}_k$,

(iii) $\mathrm{Ker}\,(j) = \bigcap_{k \geq 0} \mathfrak{a}_k$.

Proof. By construction the morphism that sends $a \in A$ to $a \bmod \hat{\mathfrak{a}}_i$ in $\hat{A}/\hat{\mathfrak{a}}_i$ is surjective by 5.5.1. This proves (i) and (ii). For (iii): if $a \in \mathrm{Ker}\,(j)$ then $a \bmod \mathfrak{a}_i = 0$ for all i, which means $a \in \mathfrak{a}_i$ for all i. □

Definition 5.5.6. *Let A be a superring with filtration $F = \{\mathfrak{a}_k\}_{k \geq 0}$ and let $\hat{A}$ be the completion $\hat{A} = \varprojlim A/\mathfrak{a}_k$ with the induced filtration $\hat{F} = \{\hat{\mathfrak{a}}_k\}_{k \geq 0}$ and let $j : A \to \hat{A}$ be the canonical insertion. We say a superring is Hausdorff if $\mathrm{Ker}\,(j) = 0$. We call a superring F-complete, or complete with respect to F if $j : A \to \hat{A}$ is an isomorphism.*

Corollary 5.5.7. *Let A be a superring with a filtration $F = \{\mathfrak{a}_k\}$ and let $\hat{A}$ be the completion with respect to F. Then $\hat{A}$ is complete and Hausdorff. A complete superring is Hausdorff.*

Proof. The second statement follows from lemma 5.5.5 and the observation that $\bigcap_k \hat{\mathfrak{a}}_k = 0$. By 5.5.1 we know that $\hat{A}$ is isomorphic to the limit of the inverse system $(\hat{A}/\hat{\mathfrak{a}}_i \to \hat{A}/\hat{\mathfrak{a}}_j : i \geq j)$. Denote $q_i : \hat{A} \to \hat{A}/\hat{\mathfrak{a}}_i$ the canonical projection and $p_i : \hat{\hat{A}} = \varprojlim \hat{A}/\hat{\mathfrak{a}}_i$ the morphisms from the limit into the inverse system. One checks that $p_i \circ j = q_i$ and thus by the universal property, there is a unique morphism $k : \varprojlim \hat{A}/\hat{\mathfrak{a}}_i \to \hat{A}$ such that $q_i \circ k = p_i$. Now we apply the universal property again to conclude that $j \circ k$ is the identity on $\varprojlim \hat{A}/\hat{\mathfrak{a}}_i$ and $k \circ j$ is the identity on $\hat{A}$. □

Proposition 5.5.8. *Let A be a ring and suppose $\{\mathfrak{a}_k\}_{k \geq 0}$ is a filtration with $\mathfrak{a}_k = \mathfrak{m}^k$ for some maximal ideal $\mathfrak{m}$ in A. Then $\hat{A} = \varinjlim A/\mathfrak{a}_k$ is a local ring with maximal ideal $\hat{\mathfrak{m}}$.*

Proof. Let $x = (x_i)_i \notin \hat{\mathfrak{m}}$, then $x_1 \neq 0$. But then no x_i is zero. Then thus $x_i \notin \mathfrak{m} \cdot A/\mathfrak{m}^i$, but $A/\mathfrak{m}^i$ is easily seen to be a local ring with maximal ideal $\mathfrak{m} \cdot A/\mathfrak{m}^i$ (see the lemma 5.5.9 below) and thus each x_i is a unit. We write thus y_i for the inverse of x_i in $A/\mathfrak{m}^i$, then from $x_i \equiv x_j \bmod \mathfrak{m}^j$ for $j \leq i$ follows that $y_i \equiv y_j \bmod \mathfrak{m}^j$ for $j \leq i$. □

Lemma 5.5.9. *Let A be a superring with maximal ideal $\mathfrak{m}$ and let $\mathfrak{n}$ be a $\mathbb{Z}_2$-graded ideal such that $\mathfrak{m}^k \subset \mathfrak{n} \subset \mathfrak{m}$ for some integer k. Then $A/\mathfrak{n}$ is a local ring with maximal ideal $\mathfrak{m} \cdot A/\mathfrak{n}$*

Proof. Let $x \notin \mathfrak{m} \cdot A/\mathfrak{n}$, then $x = a \bmod \mathfrak{n}$ for some $a \notin \mathfrak{m}$. Since $\mathfrak{m}$ is maximal, there is $b \in A$ such that $ab = 1 - m$ for some $m \in \mathfrak{m}$. Thus $ab(1 + m + m^2 + \ldots + m^k) = 1 + w$ for some $w \in \mathfrak{m}^k$. But then $b(1 + m + m^2 + \ldots + m^k) \bmod \mathfrak{n}$ is an inverse to x. □

Proposition 5.5.10. *Let A be a superring and $\mathfrak{m}$ a maximal ideal. Then first localizing A with respect to $\mathfrak{m}$ and then completing with respect to the maximal ideal $\mathfrak{m}A_\mathfrak{m}$ of $A_\mathfrak{m}$ yields a result isomorphic to completing A directly with respect to $\mathfrak{m}$.*

Proof. Follows at once from the assertion that $A/\mathfrak{m}^k \cong (A/\mathfrak{m}^k)_\mathfrak{m}$ for all positive integers k. The assertion follows since all elements of $A - \mathfrak{m}$ act as invertible maps on $A/\mathfrak{m}^k$. That is, all homotheties $l_s : A/\mathfrak{m}^k \to A/\mathfrak{m}^k$ along $s \in A - \mathfrak{m}$ are invertible maps. Then the assertion follows from corollary 5.1.15. □

Lemma 5.5.11. *Let A, B be local rings with maximal ideals $\mathfrak{m}, \mathfrak{n}$ respectively and let $\varphi : A \to B$ be a surjective morphism. If A is complete with respect to the $\mathfrak{m}$-adic filtration then $\mathfrak{m} \subset \varphi^{-1}(\mathfrak{n})$.*

Proof. Choose $s \notin \mathfrak{n}$ and assume $s \in \varphi(\mathfrak{m})$. Then $\varphi(x) = s$ for some $x \in \mathfrak{m}$. Since s is invertible, there is $t \in B$ with $st = 1$ and since φ is surjective there is a $y \in A$ with $\varphi(y) = t$. Hence $\varphi(xy) = 1$ and thus $xy = 1 + \eta$ for some $\eta \in \mathrm{Ker}(\varphi) \subset \mathfrak{m}$. But since A is complete, the element $1 + \eta$ is invertible. Thus x is invertible, contradicting $x \in \mathfrak{m}$. Hence if $s \notin \mathfrak{n}$, then $s \notin \varphi(\mathfrak{m})$. Thus $\varphi(\mathfrak{m}) \subset \mathfrak{n}$, which implies $\mathfrak{m} \subset \varphi^{-1}(\mathfrak{n})$. □

Hence we have proved:

Proposition 5.5.12. *Let A, B be local rings with maximal ideals $\mathfrak{m}, \mathfrak{n}$ respectively and assume A is complete with respect to the $\mathfrak{m}$-adic filtration. Then every surjective morphism $\varphi : A \to B$ is local.*

See definition 4.1.26 for the definition of a local morphism.

Corollary 5.5.13. *Let A, B be local rings with maximal ideals $\mathfrak{m}, \mathfrak{n}$ respectively and assume A is complete with respect to the $\mathfrak{m}$-adic filtration. For a surjective morphism $f : A \to B$ we have $f(\mathfrak{m}) = \mathfrak{n}$.*

5.6 Complete rings and convergence

In this section we fix the following notation: A is a complete ring with respect to the filtration $F = \{\mathfrak{a}_k\}_{k\geq 0}$, $\hat{A}$ is the completion $\hat{A} = \varprojlim A/\mathfrak{a}_k$ with the induced filtration $\hat{F} = \{\hat{\mathfrak{a}}_k\}_{k\geq 0}$ and $j : A \to \hat{A}$ is the canonical insertion, which is thus an isomorphism.

Lemma 5.6.1. *We have $j(\mathfrak{a}_k) = \hat{\mathfrak{a}}_k$ and $\bigcap_{k\geq 0} \mathfrak{a}_k = 0$.*

Proof. Follows from lemma 5.5.5 and the fact that j is an isomorphism. □

Definition 5.6.2. *Let $\{a_i\}_{i\geq 0}$ be a sequence of elements of A. We say that the a_i converge to $a \in A$ if for all integers n there is an integer i_n such that $a - a_l \in \mathfrak{a}_n$ whenever $l \geq i_n$. We call the sequence $\{a_i\}_{i\geq 0}$ a Cauchy sequence if for each integer n there is an integer i_n such that $a_k - a_l \in \mathfrak{a}_n$ whenever $k, l \geq i_n$.*

Lemma 5.6.3. *Let $\{a_i\}_{i\geq 0}$ be a sequence and suppose it converges to a and to b, then $a = b$.*

Proof. The sequence $\{0 = a_i - a_i\}_{i\geq 0}$ converges to $a - b$, hence $a - b \in \bigcap_k \mathfrak{a}_k = 0$. Hence $a = b$. □

Due to the lemma we call the unique element of A to which $\{a_i\}_{i\geq 0}$ converges the limit and we write $\lim_i a_i$ for the limit. A sequence that has a limit is called a converging sequence.

Lemma 5.6.4. *Let $\{a_i\}_{i\geq 0}$ and $\{b_i\}_{i\geq 0}$ be sequences converging to a and b respectively. Then the sequences $\{a_i + b_i\}_{i\geq 0}$ and $\{a_i b_i\}_{i\geq 0}$ are also converging and have limits $a + b$ and ab respectively.*

Proof. The proof follows directly from the identities: $(a + b) - (a_i + b_i) = (a - a_i) + (b - b_i)$ and $ab - a_i b_i = (a - a_i)b_i + a(b - b_i)$. □

Proposition 5.6.5. *A sequence $\{a_i\}_{i\geq 0}$ is converging if and only if it is a Cauchy sequence.*

Proof. If a sequence converges then it is a Cauchy sequence since $a_i - a_j = (a - a_j) - (a - a_i)$. Conversely, let $\{a_i\}_{i\geq 0}$ be a Cauchy sequence in $\hat{A}$. Each a_i is (represented by) a sequence $a_i = ((a_i)_k)_k$ with $(a_i)_k \in A/\mathfrak{a}_k$. Let for each integer n be i_n such that $a_i - a_j \in \mathfrak{a}_n$ for all $i, j \geq i_n$. Then the limit of the sequence can be represented by the element $((a_{i_1})_1, (a_{i_2})_2, \ldots)$. □

Corollary 5.6.6. *Suppose $\{a_i\}_{i\geq 0}$ is a sequence such that $a_i \in \mathfrak{a}_i$, then with $b_j = \sum_{i=0}^{j} a_i$ we obtain a converging sequence $\{b_j\}_{j\geq 0}$.*

Proof. Indeed $b_j - b_k \in \mathfrak{a}_j$ for $k \geq j$. □

Definition 5.6.7. *With the premises of corollary 5.6.6, we call* $\sum_{i=0}^{\infty} a_i = \lim_j b_j$.

An easy application of the above is the following result involving the geometric series in complete superrings:

Lemma 5.6.8. *Let a be an element of $\mathfrak{a}_1$, then $1 - a$ is invertible in A.*

Proof. The inverse is given by $\sum_{i=0}^{\infty} a^i$, which is a converging sum as $a^i \in \mathfrak{a}^i \subset \mathfrak{a}_i$. □

5.7 Stable filtrations and the Artin–Rees lemma

This section is devoted to prove the Artin–Rees lemma. To state and prove the result, we don't need completions, but knowledge of filtrations and associated graded superrings. The Artin–Rees lemma is used in section 5.11 to prove that the completion functor, which sends an A-module M to the $\varprojlim A/\mathfrak{a}^i$-module $\varprojlim M/\mathfrak{a}^i M$, is exact.

Definition 5.7.1. *Let A be a superring, $\mathfrak{a}$ a $\mathbb{Z}_2$-graded ideal in A and $F : M = M_0 \supset M_1 \supset M_2 \supset \cdots$ a filtered A-module. We say F is a $\mathfrak{a}$-filtration if $\mathfrak{a}M_n \subset M_{n+1}$. We say that F is $\mathfrak{a}$-stable if there is an integer N such that $\mathfrak{a}M_k = M_{k+1}$ for all $k \geq N$.*

Let M be an A-module with an $\mathfrak{a}$-filtration F. As in section 5.5 we define an associated graded module $\mathrm{gr}_F(M)$ by

$$\mathrm{gr}_F(M) = M/M_1 \oplus M_1/M_2 \oplus \dots . \tag{5.40}$$

The action of $\mathrm{gr}_{\mathfrak{a}}(A)$ on $\mathrm{gr}_F(M)$ is defined as follows: a homogeneous element $a \bmod \mathfrak{a}^{i+1}$ in the $\mathfrak{a}^i/\mathfrak{a}^{i+1}$ summand of $\mathrm{gr}_{\mathfrak{a}}(A)$ sends the homogeneous element $m \bmod M_{j+1}$ in the M_j/M_{j+1}-summand of $\mathrm{gr}_F(M)$ to the element $am \bmod M_{i+j+1}$ in the M_{j+i}/M_{i+j+1}-summand of $\mathrm{gr}_F(M)$. This turns $\mathrm{gr}_F(M)$ in a natural way into a $\mathrm{gr}_{\mathfrak{a}}(A)$-module.

Proposition 5.7.2. *Let $\mathfrak{a}$ be a $\mathbb{Z}_2$-graded ideal of the superring A and let F be an $\mathfrak{a}$-stable filtration of the A-module M; $F = \{M_k\}_{k \geq 0}$, such that all the M_k are finitely generated. Then $\mathrm{gr}_F(M)$ is a finitely generated $\mathrm{gr}_{\mathfrak{a}}(A)$-module.*

Proof. Suppose $\mathfrak{a}M_i = M_{i+1}$ for all $i \geq N$, then $\mathfrak{a}/\mathfrak{a}^2(M_i/M_{i+1}) = M_{i+1}/M_{i+2}$ for all $i \geq N$. We can map the generators of M_j into the M_j/M_{j+1}-summand: taking all these images of the generators of M_j for $j \leq N$ we obtain a set of generators for $\mathrm{gr}_F(M)$. □

Definition 5.7.3. *Let A be a superring, $\mathfrak{a}$ a $\mathbb{Z}_2$-graded ideal. Then we call the blow-up superalgebra of A at $\mathfrak{a}$ the superalgebra $B_{\mathfrak{a}}A$, where*

$$B_{\mathfrak{a}}A := A \oplus \mathfrak{a} \oplus \mathfrak{a}^2 \oplus \ldots \cong A[t\mathfrak{a}] \subset A[t] . \tag{5.41}$$

The addition and $\mathbb{Z}_2$-grading of $B_{\mathfrak{a}}A$ are defined componentwise and the multiplication is defined as follows: for a, b in the $\mathfrak{a}^i$-, respectively $\mathfrak{a}^j$-summand we define ab to be equal to element that is ab in the $\mathfrak{a}^{i+j}$-summand and zero in any other summand. For an A-module we define the blow-up module in $\mathfrak{a}$ to be the $B_{\mathfrak{a}}A$-module

$$B_{\mathfrak{a}}M = M \oplus \mathfrak{a}M \oplus \mathfrak{a}^2 M \oplus \dots . \tag{5.42}$$

The action of $B_{\mathfrak{a}}A$ on $B_{\mathfrak{a}}M$ is given by: an element a in the $\mathfrak{a}^i$-summand of $B_{\mathfrak{a}}A$ maps the element m of the $\mathfrak{a}^j M$-summand of $B_{\mathfrak{a}}M$ to am in the $\mathfrak{a}^{i+j}M$-summand of $B_{\mathfrak{a}}M$. The $\mathbb{Z}_2$-grading and addition are defined pointwise.

Lemma 5.7.4. *If A is Noetherian and $\mathfrak{a}$ a $\mathbb{Z}_2$-graded ideal then $B_\mathfrak{a}A$ is Noetherian.*

Proof. Let $x_1, \dots, x_p$ be a set of homogeneous generators of $\mathfrak{a}$, then $1 \in A$ and $x_i \in \mathfrak{a}$ generate $B_\mathfrak{a}A$ as an A-algebra. Hence $B_\mathfrak{a}A$ is Noetherian. □

Note that $B_\mathfrak{a}A/\mathfrak{a}B_\mathfrak{a}A \cong \mathrm{gr}_\mathfrak{a}(A)$.

Lemma 5.7.5. *Let A be a superring, $\mathfrak{a}$ a $\mathbb{Z}_2$-graded ideal and M a finitely-generated A-module. Let $F : M_0 \supset M_1 \supset \cdots$ be an $\mathfrak{a}$-filtration of M of with all M_k finitely generated. Then, F is $\mathfrak{a}$-stable if and only if the $B_\mathfrak{a}A$-module $B_\mathfrak{a}M := M_0 \oplus M_1 \oplus \dots$ is finitely generated.*

Proof. Define $N_n = \oplus_{i=0}^n M_i$ and $\tilde{M}_n = N_n \oplus \mathfrak{a}M_n \oplus \mathfrak{a}^2 M_n \oplus \cdots$. Each N_n is finitely generated over A and hence for each n, $\tilde{M}_n$ is a finitely generated $B_\mathfrak{a}A$. Since $B_\mathfrak{a}M$ is the union of all the $\tilde{M}_n$, $B_\mathfrak{a}M$ is finitely generated if and only if for a certain k we have $\tilde{M}_k = B_\mathfrak{a}M$, which happens if and only if $M_{m+k} = \mathfrak{a}^m M_k$ for all $m \geq 0$. □

With lemma 5.7.5 it is not difficult to prove the following theorem, which is the Artin–Rees lemma for superrings:

Theorem 5.7.6. *Let A be a Noetherian superring, $\mathfrak{a}$ a $\mathbb{Z}_2$-graded ideal, M a finitely generated A-module and $M' \subset M$ a submodule. When $F : M = M_0 \supset M_1 \supset \cdots$ is an $\mathfrak{a}$-stable filtration, then the induced filtration on M' is $\mathfrak{a}$-stable. That is, there is an integer N such that $\mathfrak{a}^k(M' \cap M_i) = M' \cap M_{i+k}$ for all $i \geq N$.*

Proof. Consider the filtration $F' = \{M_i'\}_{i \geq 0}$ with $M_i = M_i \cap M'$. Then $B_{F'}M' = M_0' \oplus M_1' \oplus \dots \subset M_0 \oplus M_1 \oplus \dots = B_F M$ seen as $B_F A$-modules. Then $B_\mathfrak{a}A$ is a Noetherian ring and F is $\mathfrak{a}$-stable. Hence $B_F M$ is finitely generated by lemma 5.7.5, so that $B_{F'}M'$ is finitely generated and hence F' is $\mathfrak{a}$-stable. □

5.8 Completions of Noetherian superrings

Let A be a superring with a filtration $F = \{\mathfrak{a}_k\}_{k \geq 0}$, and let $\hat{A} = \varprojlim A/\mathfrak{a}_i$ be the completion with respect to F of A. In this section we introduce a map $A \to \mathrm{gr}_F(A)$ that will enable us to prove that $\hat{A}$ is Noetherian when A is Noetherian and $\mathfrak{a}_k = \mathfrak{a}^k$ for some $\mathbb{Z}_2$-graded ideal $\mathfrak{a} \subset A$.

Let $f \in A$ be given and assume there is an integer k such that $f \in \mathfrak{a}_k$ but $f \notin \mathfrak{a}_{k+1}$. In other words, k is the smallest integer such that $f \bmod \mathfrak{a}_{k+1} \neq 0$. Then we define $\mathrm{in}(f)$ as the element of $\mathrm{gr}_F(A)$ lying in the $\mathfrak{a}_k/\mathfrak{a}_{k+1}$-component given by $f \bmod \mathfrak{a}_{k+1}$. If for f no such k exists, or in other words $f \in \bigcap_i \mathfrak{a}_i$, then we put $\mathrm{in}(f) = 0$. In this way we have defined a map $\mathrm{in} : A \to \mathrm{gr}_F(A)$; it is important to note that this is not a morphism, it is only a map of sets.

We define $\deg : A \to \mathbb{N}$ to be the map that sends $f \in A$ to the smallest integer k such that $f \bmod \mathfrak{a}_{k+1} \neq 0$; if no such smallest integer exists, we put $\deg(f) = \infty$. For a homogeneous element g of $\mathrm{gr}_F(A)$ we also use degree to indicate in which summand g lies. We thus have $\deg(\mathrm{in}(f)) = \deg(f)$.

Lemma 5.8.1. *Let f, g be elements of A, then* (1) $\deg(f+g) \geq \min\big(\deg(f), \deg(g)\big)$ *and* (2) $\deg(fg) \geq \deg(f) + \deg(g)$.

Proof. Follows at once from $f \in \mathfrak{a}_k \Rightarrow \deg(f) \geq k$. □

Lemma 5.8.2. *If $f_1, \dots, f_s$ are elements of A with the same degree then $\sum_i \mathrm{in}(f_i) = 0$ or $\sum_i \mathrm{in}(f) = \mathrm{in}(\sum_i f_i)$.*

Proof. Let the degree be k, then $\sum_i f_i \in \mathfrak{a}_k$, hence $\mathrm{in}(\sum_i f_i)$ lies in some $\mathfrak{a}_l/\mathfrak{a}_{l+1}$-summand of $\mathrm{gr}_F(A)$ for $l \geq k$. If $l = k$ then we have $\sum_i \mathrm{in}(f) = \mathrm{in}(\sum_i f_i)$ and if $l > k$ then $\sum_i \mathrm{in}(f) = 0$. □

Lemma 5.8.3. *If f, g are elements of A then* $\mathrm{in}(f)\mathrm{in}(g) = 0$ *or* $\mathrm{in}(f)\mathrm{in}(g) = \mathrm{in}(fg)$.

Proof. Similar as the proof of lemma 5.8.2. □

Proposition 5.8.4. *Let A be a superring that is complete with respect to a filtration $F = \{\mathfrak{a}_k\}_{k\geq 0}$. Let $\mathrm{gr}_F(A)$ be the associated graded superring to F and let I be a $\mathbb{Z}_2$-graded ideal of A and suppose $a_1, \dots, a_s$ are elements of I such that* $\mathrm{in}(a_1), \dots, \mathrm{in}(a_s)$ *generate* $\mathrm{in}(I)$. *Then the a_i generate I.*

Proof. Write $I' = (a_1, \dots, a_s)$. Since A is Hausdorff, there is an integer d such that none of the a_i is contained in $\mathfrak{a}_d$. Let $f \in I$ and let e be the degree of f. Then $\mathrm{in}(f) = \sum G_j \mathrm{in}(a_j)$ where we can take the G_j to be homogeneous and of degree $e - \deg(a_j)$. Choose g_j in A such that $\mathrm{in}(g_j) = G_j$; then we have $\sum_j \mathrm{in}(g_j)\mathrm{in}(a_j) = \mathrm{in}(\sum_j g_j a_j)$, which equals $\mathrm{in}(f)$. It follows that $f - \sum g_j a_j$ lies in $\mathfrak{a}_{e+1}$ and thus we may repeat the procedure till we arrive at a stage where we find $F = f - f'$ with $f' \in I'$ and $F \in \mathfrak{a}_{d+1}$. By the same reasoning we find $F = \sum \mathrm{in}(g_j)\mathrm{in}(a_j)$ for some elements g_j in A. The degrees of the g_j are positive, and thus $g_j \in \mathfrak{a}_{e-d}$. Thus we get at a first step $F - \sum_j g_j^{(1)} a_j \in \mathfrak{a}_{e-d+1}$, and n steps later $F - \sum_{i,j} g_j^{(i)} a_j \in \mathfrak{a}_{e-d+n}$ with the $g_j^{(i)} \in \mathfrak{a}_{e-d+i-1}$. Define $G_j^n = \sum_{i=1}^n g_j^{(i)}$, then the G_j^n build a Cauchy sequence for each j, the limit of which we denote by h_j. Thus $F = \sum h_j a_j$ lies in I' and thus f lies in I'. □

Corollary 5.8.5. *Let A be a Noetherian superring with the filtration $F = \{\mathfrak{a}_k\}_{k\geq 0}$ where $\mathfrak{a}_k = \mathfrak{a}^k$ for some $\mathbb{Z}_2$-graded ideal $\mathfrak{a}$. Then $\hat{A} = \varprojlim A/\mathfrak{a}^k$ is a Noetherian superring.*

Proof. Let I be a $\mathbb{Z}_2$-graded ideal in $\hat{A}$ and consider the $\mathbb{Z}_2$-graded ideal $\mathrm{in}(I)$ in $\mathrm{gr}_{\hat{F}}(\hat{A}) \cong \mathrm{gr}_F(A)$. Since $\mathrm{gr}_F(A)$ is Noetherian, $\mathrm{in}(I)$ is finitely generated and we may assume that $\mathrm{in}(I)$ is generated by homogeneous elements of the form $\mathrm{in}(f)$. Hence there are finitely many $f \in \hat{A}$ such that $\mathrm{in}(f)$ generate $\mathrm{in}(I)$ and thus, by proposition 5.8.4 those f already generate I. □

5.9 Complete filtered pairs

Definition 5.9.1. *Let A be a superring and $\mathfrak{a} \subset A$ a $\mathbb{Z}_2$-graded ideal such that A is complete with respect to the $\mathfrak{a}$-adic filtration. In this situation we call A together with the $\mathfrak{a}$-adic filtration $\{\mathfrak{a}^k\}_{k\geq 0}$ a complete filtered pair (CFP) and denote it by $(A, \{\mathfrak{a}^k\}_{k\geq 0})$. If $(A, \{\mathfrak{a}^k\}_{k\geq 0})$ and $(B, \{\mathfrak{b}^k\}_{k\geq 0})$ are two CFP's, then a morphism of complete filtered pairs is a morphism $\varphi : A \to B$ of superrings such that $\varphi(\mathfrak{a}) \subset \mathfrak{b}$, or equivalently $\varphi(\mathfrak{a})^k \subset \mathfrak{b}^k$ for all k.*

Proposition 5.9.2. *Let $\varphi : (A, \{\mathfrak{m}^k\}_{k\geq 0}) \to (B, \{\mathfrak{n}^k\}_{k\geq 0})$ be a morphism of CFP's, then φ preserves limits.*

Proof. Let $\{r_i\}_{i\geq 0}$ be a sequence in A with limit r. Then choose integers i_n such that $r - r_j \in \mathfrak{m}^n$ if $j \geq i_n$. It follows that $\varphi(r) - \varphi(r_j) \in \mathfrak{n}^n$ whenever $j \geq i_n$. □

Lemma 5.9.3. *Let $(A, \{\mathfrak{m}^k\}_{k\geq 0})$ be a CFP. Let $\{r_i\}_{i\geq 0}$ be a Cauchy sequence in A for which there exists an integer p such that for all i we have $r_i \in \mathfrak{m}^p$, then $\lim_i r_i \in \mathfrak{m}^p$.*

Proof. There exists an integer N such that $(\lim_i r_i) - r_j \in \mathfrak{m}^p$ for all $j \geq N$. But since $r_j \in \mathfrak{m}^p$ for all j, we have $\lim_i r_i \in \mathfrak{m}^p$. □

There exists a useful functor gr from the category of CFP's to the category of superrings that we now describe. If $(A, \{\mathfrak{m}^k\}_{k\geq 0})$ and $(B, \{\mathfrak{n}^k\}_{k\geq 0})$ are CFP's and $\varphi : A \to B$ is a morphism of CFP's then the functor maps the CFP's to their associated graded superrings, so that

$$\mathrm{gr} : (A, \{\mathfrak{m}^k\}_{k\geq 0}) \mapsto \mathrm{gr}_{\mathfrak{m}}(A) = A/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \dots .$$

The functor gr maps the morphism φ to the map $\mathrm{gr}(\varphi)$ that sends the homogeneous element $r \bmod \mathfrak{m}^{k+1}$ to $\varphi(r) \bmod \mathfrak{n}^{k+1}$. It is easily checked that the resulting map $\mathrm{gr}(\varphi)$ is a morphism of superrings.

Proposition 5.9.4. *Suppose $\varphi : (A, \{\mathfrak{m}^k\}_{k\geq 0}) \to (B, \{\mathfrak{n}^k\}_{k\geq 0})$ is a morphism of CFP's. If $\mathrm{gr}(\varphi)$ is a monomorphism, then φ is a monomorphism. If $\mathrm{gr}(\varphi)$ is an epimorphism, then φ is an epimorphism.*

Proof. If $r \in A$ is nonzero, then there is an integer k such that $r \in \mathfrak{m}^k$ but $r \notin \mathfrak{m}^{k+1}$. Thus $\mathrm{in}(r)$ is nonzero and thus $\mathrm{gr}(\varphi)(\mathrm{in}(r))$ is nonzero, which implies $\varphi(r) \bmod \mathfrak{m}^{k+1} \neq 0$.

For the epimorphisms, consider $s \in B$ and assume $\mathrm{gr}(\varphi)$ is surjective. Consider $\mathrm{in}(s) \in \mathrm{gr}_{\mathfrak{n}}(B)$ and assume it lies in the ith component; $\mathrm{in}(s) \in \mathfrak{n}^i/\mathfrak{n}^{i+1}$. Then there is $x = r_1 \bmod \mathfrak{m}^{i+1}$ that maps to $\mathrm{in}(s)$. Hence $s - \varphi(r_1) \in \mathfrak{n}^{i+1}$. Then we find $r_2 \bmod \mathfrak{m}^{i+2}$ that maps to $\mathrm{in}(s - \varphi(r_1))$ and $s - \varphi(r_1) - \varphi(r_2) \in \mathfrak{n}^{i+2}$. We thus find a Cauchy sequence $x_n = \sum_{i=1}^n r_i$ and $s - \varphi(x_n) \in \mathfrak{n}^{i+n}$. Hence $s = \lim_{n\to\infty} \varphi(x_n) = \varphi(\lim_{n\to\infty} x_n)$ and thus φ is surjective. □

Remark 5.9.5. The conclusion of proposition 5.9.4 is more briefly stated by saying that the functor gr reflects epimorphisms and monomorphisms.

Proposition 5.9.6. *Let $(A, \{\mathfrak{m}^k\}_{k\geq 0})$ and $(B, \{\mathfrak{n}^k\}_{k\geq 0})$ be CFP's and suppose φ is a morphism of CFP's such that $\varphi(\mathfrak{m}^k) = \mathfrak{n}^k$ for all k, then $\mathrm{gr}(\varphi)$ is surjective.*

Proof. Let $s \in B$ and suppose that $s \in \mathfrak{n}^k - \mathfrak{n}^{k+1}$. Then there is by assumption $r \in \mathfrak{m}^k - \mathfrak{m}^{k+1}$ such that $\varphi(r) = s$. Hence $\mathrm{gr}(\varphi)(\mathrm{in}(r)) = \mathrm{in}(s)$. □

As a corollary we obtain:

Theorem 5.9.7. *Let $(A, \{\mathfrak{m}^k\}_{k\geq 0})$ and $(B, \{\mathfrak{n}^k\}_{k\geq 0})$ be CFP's and suppose φ is a morphism of CFP's. Then $\mathrm{gr}(\varphi)$ is surjective if and only if $\varphi(\mathfrak{m}^k) = \mathfrak{n}^k$ for all k.*

Proof. The only thing that is left to prove is that if $\mathrm{gr}(\varphi)$ is surjective, then for all $s \in \mathfrak{n}^k$, there is an $r \in \mathfrak{m}^k$ such that $\varphi(r) = s$. Using lemma 5.9.3 and the proof of proposition 5.9.4 this is immediate. □

5.10 Maps from power series rings

Lemma 5.10.1. *Let B be a superalgebra over a commutative ring A, and suppose $\mathfrak{a}$ is a $\mathbb{Z}_2$-graded ideal in B such that B is complete with respect to the $\mathfrak{a}$-adic filtration. Given elements $e_1, \ldots, e_n \in \mathfrak{a}_{\bar{0}}$ and $\eta_1, \ldots, \eta_s \in \mathfrak{a}_{\bar{1}}$, then there exists a unique A-algebra morphism*

$$\varphi : A[[x_1, \ldots, x_n | \vartheta_1, \ldots, \vartheta_s]] \to B\,,$$

such that the x_i are sent to the e_i and the ϑ_α to the η_α.

Proof. Call T the A-algebra $A[[x_1, \ldots, x_n|\vartheta_1, \ldots, \vartheta_s]]$ and call K the $\mathbb{Z}_2$-graded ideal of T generated by $x_1, \ldots, x_n$ and $\vartheta_1, \ldots, \vartheta_s$. Also, call S the A-algebra $A[x_1, \ldots, x_n|\vartheta_1, \ldots, \vartheta_s]$ and L the $\mathbb{Z}_2$-graded ideal of S generated by $x_1, \ldots, x_n$ and $\vartheta_1, \ldots, \vartheta_s$. Then $T/K^t \cong S/L^t$ for all integers t and there is a unique A-algebra morphism from S to $B/\mathfrak{n}^t$ sending x_i to e_i and ϑ_α to η_α. This map factors over T/K^t. But B is the inverse limit of the $B/\mathfrak{n}^t$ and hence there is a unique morphism from T to B sending the x_i and ϑ_α to the e_i and η_α respectively. Since B is complete, the morphism is well-defined, by which we mean in this case that we can write the image of a sequence as a sequence of images, which converges as this sequence is a Cauchy sequence and B is complete. □

Lemma 5.10.2. *With the preliminaries of lemma 5.10.1, if in addition the induced morphism* $A \to B/\mathfrak{n}$ *is an epimorphism and the* e_i, η_α *together generate* $\mathfrak{n}$*, then* $\varphi : A[[x_1, \ldots, x_n|\vartheta_1, \ldots, \vartheta_s]] \to B$ *is an epimorphism.*

Proof. From the assumptions and theorem 5.9.7 it follows that the morphism $\mathrm{gr}(\tilde{A}) \to B$, where $\tilde{A} = A[[x_1, \ldots, x_n|\vartheta_1, \ldots, \vartheta_s]]$, is an epimorphism. But the functor gr reflects epimorphisms. □

Lemma 5.10.3. *With the preliminaries of lemma 5.10.1, if in addition the induced morphism* $\mathrm{gr}(\tilde{A}) \to \mathrm{gr}(B)$*, with* $\tilde{A} = A[[x_1, \ldots, x_n|\vartheta_1, \ldots, \vartheta_s]]$*, is a monomorphism, then* φ *is a monomorphism.*

Proof. The functor gr reflects monomorphisms by proposition 5.9.4. □

5.11 Exactness of inverse limits

Let A be a Noetherian superring with an $\mathfrak{a}$-adic filtration, where $\mathfrak{a}$ is a $\mathbb{Z}_2$-graded ideal, and let $\hat{A}$ be the completion of A with respect to the $\mathfrak{a}$-adic filtration. For any A-module M we define $\hat{M} = \varprojlim M/\mathfrak{a}^iM$; thus $\hat{M}$ is the terminal object in the category of cones over the inverse system $(M/\mathfrak{a}^iM \to M/\mathfrak{a}^jM : i \geq j)$. An explicit construction of $\hat{M}$ can be given along the same lines as in section 5.5.

Suppose $f : M \to N$ is a morphism of A-modules and $p_i : \hat{M} \to M/\mathfrak{a}^iM$ the projections to the inverse system $(M/\mathfrak{a}^iM \to M/\mathfrak{a}^jM : i \geq j)$. Then f induces morphisms $f_i : M/\mathfrak{a}^iM \to N/\mathfrak{a}^iN$ that are defined by $f_i(m \bmod \mathfrak{a}^iM) = f(m) \bmod \mathfrak{a}^iN$. The composites $f_i \circ p_i : \hat{M} \to N/\mathfrak{a}^iN$ form a cone over the inverse system $(N/\mathfrak{a}^iN \to N/\mathfrak{a}^jN : i \geq j)$ with apex $\hat{M}$. Thus there is a unique morphism $\hat{f} : \hat{M} \to \hat{N}$ such that the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & M/\mathfrak{a}^iM \arrow[rr] \arrow[ddd, "f_i"'] & & M/\mathfrak{a}^jM \arrow[r] \arrow[ddd, "f_j"] & \cdots \\
& & \hat{M} \arrow[ul, "p_i"'] \arrow[ur, "p_j"] \arrow[d, dashed, "\hat{f}"] & & \\
& & \hat{N} \arrow[dl, "q_i"'] \arrow[dr, "q_j"] & & \\
\cdots \arrow[r] & N/\mathfrak{a}^iN \arrow[rr] & & N/\mathfrak{a}^jN \arrow[r] & \cdots
\end{tikzcd} , \tag{5.43}$$

and where $q_i : \hat{N} \to N/\mathfrak{a}^iN$ are the projections from the limit $\hat{N}$ to the inverse system. We thus can see completion as a functor that assigns to an A-module M the $\hat{A}$-module $\hat{M}$ and that maps the morphisms $f : M \to N$ to the morphism $\hat{f}$ that we just described. Below we show that the completion functor is exact. We follow the exposition of [15].

Proposition 5.11.1. *Let* A *be a Noetherian superring and let* $\mathfrak{a}$ *be a* $\mathbb{Z}_2$*-graded ideal. If*

$$0 \to M \to N \to P \to 0 \tag{5.44}$$

is an exact sequence of finitely generated A-modules, then the induced sequence

$$0 \to \hat{M} \to \hat{N} \to \hat{P} \to 0 \tag{5.45}$$

is exact.

Proof. Without loss of generality we may assume that $M \subset N$. Call g the morphism $N \to P$.

We first prove surjectivity of $\hat{g}$: let $(p_j \bmod \mathfrak{a}^j P)_j \in \hat{P}$. Choose n_1 such that $g(n_1) \bmod \mathfrak{a}P = p_1 \bmod \mathfrak{a}P$. Next we find $\tilde{n}_2 \in N$ with $g(\tilde{n}_2) \bmod \mathfrak{a}^2 P = p_2 \bmod \mathfrak{a}^2 P$. Then $(\tilde{n}_2 - n_1) \bmod \mathfrak{a}N$ maps to $(p_2 - p_1) \bmod \mathfrak{a}P = 0$. Hence there is $a_2 \in \mathrm{Ker}(g)$ with $n_2 - n_1 + a_2 \in \mathfrak{a}N$. Then define $n_2 = \tilde{n}_2 + a_2$. We have $g(n_2) = g(\tilde{n}_2)$ and $n_2 \equiv n_1 \bmod \mathfrak{a}N$. We can inductively repeat this procedure to find for all p_i an n_i that maps to p_i such that $n_i \equiv n_j \bmod \mathfrak{a}^j N$ for all $j \leq i$.

The next step is to prove that $\hat{M} \cong \varprojlim M/M \cap \mathfrak{a}^i N$. The filtration $N \supset \mathfrak{a}N \supset \mathfrak{a}^2 N \supset \dots$ is $\mathfrak{a}$-stable and N is finitely generated; the filtration $M \supset (M \cap \mathfrak{a}N) \supset (M \cap \mathfrak{a}^2 N) \supset \dots$ is then $\mathfrak{a}$-stable by the Artin–Rees lemma 5.7.6. Hence there is $r \geq 0$ such that for all $k \geq 0$ we have $\mathfrak{a}^k(M \cap \mathfrak{a}^r N) = M \cap \mathfrak{a}^{r+k} N$ and thus $M \cap \mathfrak{a}^{r+k} N \subset \mathfrak{a}^k M$ for all $k \geq 0$. Since in addition the inclusion $\mathfrak{a}^j M \subset M \cap \mathfrak{a}^i N$ holds for all $j \geq i$, we can apply lemma 5.5.2 to conclude the that $\hat{M} \cong \varprojlim M/M \cap \mathfrak{a}^i N$.

Consider how we defined the morphism $\hat{f}$ in the diagram of eqn.(5.43); if all f_i are identically zero, then the zero morphism $0 : \hat{M} \to \hat{N}$ makes the diagram commute and by uniqueness, $\hat{f} = 0$. Thus the composition $\hat{M} \to \hat{N} \to \hat{P}$ is the zero morphism from $\hat{M}$ to $\hat{P}$.

Now assume $n = (n_i \bmod \mathfrak{a}^i N)_i$ goes to zero in $\hat{P}$. But then $g(n_i) \in \mathfrak{a}^i N$ which implies $n_i \in M + \mathfrak{a}^i N$; it follows that $n_i \bmod \mathfrak{a}^i N = m_i \bmod \mathfrak{a}^i N$ for some $m_i \in M$. The m_i fit together to define an element in $\hat{M}$ and we can write $n_i \equiv m_i \bmod \mathfrak{a}^i N$. The element $m = (m_i \bmod M \cap \mathfrak{a}^i N) \in \hat{M}$ goes to $n \in \hat{N}$. The injectivity of $\hat{M} \to \hat{N}$ is obvious: $m = (m_i \bmod M \cap \mathfrak{a}^i N)$ maps to $(m_i \bmod \mathfrak{a}^i N)$ in $\hat{N}$, hence if m goes to zero, then all m_i already lie in $\mathfrak{a}^i N$ and thus in $M \cap \mathfrak{m}^i N$. □

Proposition 5.11.2. *Let A be a Noetherian superring, $\mathfrak{a}$ a $\mathbb{Z}_2$-graded ideal in A, $\hat{A} = \varprojlim A/\mathfrak{a}^i$ the $\mathfrak{a}$-adic completion with respect to $\mathfrak{a}$. When M is a finitely generated A-module, then the natural morphism $\hat{A} \otimes_A M \to \hat{M} := \varprojlim M/\mathfrak{a}^k M$ is an isomorphism.*

Proof. If $M = A$ then the statement is trivially true. If M is a finite direct sum of copies of A it is also easily seen to be true. If M is any general finitely generated A-module, there are finitely generated free A-modules G and F such that the sequence $F \to G \to A \to 0$ is exact; since M is finitely generated, the existence of a surjective morphism $G \to M$, with G a finitely generated free module, is clear and the kernel of this morphism is again a finitely generated module. By the preceding proposition 5.11.1 the horizontal lines of the diagram

$$\begin{tikzcd}
\hat{F} \arrow[r] \arrow[d] & \hat{G} \arrow[r] \arrow[d] & \hat{M} \arrow[r] \arrow[d] & 0 \\
\hat{A} \otimes_A F \arrow[r] & \hat{A} \otimes_A G \arrow[r] & \hat{A} \otimes_A M \arrow[r] & 0
\end{tikzcd} \tag{5.46}$$

are exact and the two first vertical arrows are isomorphisms; hence the right vertical arrow is also an isomorphism. □

Corollary 5.11.3. *Let $f : A \to B$ be a surjective morphism of superrings where A is Noetherian and complete with respect to $\mathfrak{a}$-adic filtration, for some $\mathbb{Z}_2$-graded ideal $\mathfrak{a}$. Define $\mathfrak{b} = f(\mathfrak{a})$ to be the image of $\mathfrak{a}$ in B, which is a $\mathbb{Z}_2$-graded ideal in B. Then B is complete with respect to the $\mathfrak{b}$-adic filtration.*

Proof. From corollary 5.5.13 we have $\mathfrak{b}^k = f(\mathfrak{a}^k)$ and $\mathfrak{a}^k \cdot B = \mathfrak{b}^k$. Let I be the kernel of f, then $\hat{I} = \varprojlim I/\mathfrak{a}^k I \cong \hat{A} \otimes_A I \cong I$ is the kernel of the induced morphism $\hat{f} : \hat{A} \cong A \to \hat{B}$, which is a surjective morphism of superrings. Hence $B \cong A/I \cong \hat{A}/\hat{I} \cong \hat{B}$. □

Corollary 5.11.4. *Let A be a Noetherian superring, $\mathfrak{a}$ a $\mathbb{Z}_2$-graded ideal and $\hat{A}$ the $\mathfrak{a}$-adic completion. Then for any $\mathbb{Z}_2$-graded ideal I in A, the map $\hat{A} \otimes_A I \to \hat{A}$ is an injective morphism of A-modules.*

Proof. Let I be any $\mathbb{Z}_2$-graded ideal of A. The sequence $0 \to I \to A \to A/I \to 0$ is exact. Tensoring with $\hat{A}$ and using that $\hat{I} \cong I \otimes_A \hat{A}$, we see that proposition 5.11.1 implies that the sequence $0 \to \hat{I} \to \hat{A}$ is exact. This proves the corollary. □

Remark 5.11.5. In proposition 6.2.8 we will see that the result of corollary 5.11.4 can be restated as follows: the completion $\hat{A}$ of a Noetherian superring A is a flat A-module. The notion of flatness is defined in definition 6.2.1 in chapter 6.

5.12 Cohen's structure theorem

Proposition 5.12.1. *Let A be a superring and suppose L is a field inside A, then there is an isomorphism $\pi : L \to L'$ of fields where $L' \subset A_{\bar{0}}$.*

Proof. Denote *proj* : $A \to A_{\bar{0}}$ the projection that sends $a_{\bar{0}} + a_{\bar{1}}$ to $a_{\bar{0}}$. The map π is given by restricting *proj* : $A \to A_{\bar{0}}$ to L. The field L' we then take to be the image of $\pi(L) = L'$. □

Proposition 5.12.1 might at first seem unnecessary. However, the next example shows that there are cases where a superring contains a field that does not lie in the even part.

Example 5.12.2. Let $A = k(x)[\theta]$ and consider the k-algebra morphism $k[Y] \to A$ given by $Y \to x + \theta$, then since the kernel is trivial and the image of a nonzero element is a unit, there is a unique morphism $f : k(Y) \to k(x)[\theta]$ making the following diagram commute:

$$\begin{tikzcd} k[Y] \arrow[r] \arrow[d, "i"'] & k(x)[\theta] \\ k(Y) \arrow[ur, "f"'] & \end{tikzcd} \quad . \tag{5.47}$$

Hence the image is a field in A, not lying in $A_{\bar{0}}$. △

Theorem 5.12.3. *Let A be a Noetherian superring that is complete with respect to the $\mathfrak{m}$-adic filtration, where $\mathfrak{m}$ is a maximal ideal, and assume that A contains a field. Then A contains a coefficient field, that is, a field L inside A that is isomorphic to $A/\mathfrak{m}$.*

Proof. By the proposition 3.3.6, lemma 4.1.24 and theorem 5.5.3 the commutative ring $A_{\bar{0}}$ is local, complete and Noetherian. By proposition 5.12.1 we may assume the field is contained in $A_{\bar{0}}$. Hence everything can be analyzed in $A_{\bar{0}}$ and we may apply Cohen's structure theorem [57]; also see for example [15]. □

Theorem 5.12.4. *Let A be a complete local Noetherian superring with maximal ideal $\mathfrak{m}$ and residue class field K. If A contains a field, then $A \cong K[[x_1, \ldots, x_s|\vartheta_1, \ldots, \vartheta_t]]/I$ for some s, t and $\mathbb{Z}_2$-graded ideal I.*

Proof. We may assume the field lies in $A_{\bar{0}}$, hence there is a coefficient field L in $A_{\bar{0}} \subset A$. Thus A is an L-superalgebra. Let $a_1, \ldots, a_s, b_1, \ldots, b_s$ be homogeneous generators of $\mathfrak{m}$ with a_i even and b_l odd. By lemma 5.10.1 there is a morphism $\varphi : K[[x_1, \ldots, x_s|\vartheta_1, \ldots, \vartheta_s]]$, mapping x_i to a_i and ϑ_k to b_k. The morphism φ is an epimorphism by lemma 5.10.2. We can take I to be the kernel of φ. $\square$

Chapter 6

Categories of modules

In this chapter we study some generalities of modules of some superring A. In particular, we define flat, projective and injective modules and give some of their most elementary properties. We give some generalizations of classical results from commutative algebra such as the Hamilton–Cayley theorem, Krull's intersection theorem and Nakayama's lemma. In the final section of this chapter we discuss properties of base-change, that is, we relate the two categories of modules of two different superrings.

6.1 Generalities

In this section we present some general aspects of the category of modules of a fixed superring A.

6.1.1 Internal Hom-functors

We write $\underline{\mathrm{Hom}}_A(M,N)$ for the set of all maps $f : M \to N$ such that $f(m+m') = f(m) + f(m')$ and $f(m \cdot a) = f(m) \cdot a$ for all $m, m' \in M$ and $a \in A$. We refer to the elements of $\underline{\mathrm{Hom}}_A(M,N)$ as homomorphisms. We equip $\underline{\mathrm{Hom}}_A(M,N)$ with the $\mathbb{Z}_2$-grading

$$\underline{\mathrm{Hom}}_A(M,N)_i = \{f \in \underline{\mathrm{Hom}}_A(M,N) \mid f(M_j) \subset N_{i+j}\} \ , \tag{6.1}$$

and with the following action of A:

$$(a \cdot f)(m) = a \cdot (f(m)) \, , \quad (f \cdot a)(m) = (-1)^{|a||m|} f(m) \cdot a \, . \tag{6.2}$$

With this structure $\underline{\mathrm{Hom}}_A(M,N)$ becomes an A-module. Furthermore, we have $\underline{\mathrm{Hom}}_A(M,N)_{\bar{0}} = \mathrm{Hom}_A(M,N)$ and the even homomorphisms, which are the morphisms in the category of A-modules, also commute with the left action of A on M and N. We stress that $\underline{\mathrm{Hom}}_A(M,N)$ contains more than the 'arrows' in the category; hence in commutative diagrams, all maps are assumed to be morphisms, that is, even homomorphisms, unless otherwise specified.

Proposition 6.1.1. *Given two sequences*

$$\begin{array}{ll} (S1) & 0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \\ (S2) & L' \xrightarrow{d} L \xrightarrow{e} L'' \longrightarrow 0 \end{array}$$

of A-modules with morphisms f, g, d, e.

(i) The sequence $(S1)$ *is exact if and only if for all* A*-modules* M *the sequence*

$$(S3) \qquad 0 \longrightarrow \underline{\mathrm{Hom}}_A(M,N') \xrightarrow{\quad f^* \quad} \underline{\mathrm{Hom}}_A(M,N) \xrightarrow{\quad g^* \quad} \underline{\mathrm{Hom}}_A(M,N'')$$

is exact, where $f^*(a) = f \circ a$ *and* $g^*(b) = g \circ b$.

(ii) The sequence $(S2)$ *is exact if and only if for all* A*-modules* M *the sequence*

$$(S4) \qquad 0 \longrightarrow \underline{\mathrm{Hom}}_A(L'',M) \xrightarrow{\quad e_* \quad} \underline{\mathrm{Hom}}_A(L,M) \xrightarrow{\quad d_* \quad} \underline{\mathrm{Hom}}_A(L',M)$$

is exact, where $e_*(a) = a \circ e$ *and* $d_*(b) = b \circ d$.

(iii) The sequence $(S2)$ *is exact if and only if for all* A*-modules* M *the sequence*

$$(S5) \qquad M \otimes_A L' \xrightarrow{\mathrm{id}\otimes d} M \otimes_A L \xrightarrow{\mathrm{id}\otimes e} M_A \otimes L'' \longrightarrow 0$$

is exact.

Proof. (i): Suppose $(S1)$ is exact, then it is clear that f^* is injective and $g^* \circ f^* = 0$. Suppose $b \in \mathrm{Ker} g^*$, then $g(b(m)) = 0$ for all $m \in M$. There exists a unique $n \in N'$ such that $b(m) = f(n)$ and it is easy to see that the morphism h that assigns to each $m \in M$ the unique $n \in N'$ such that $b(m) = f(n)$ is a homomorphism of A-modules with parity $|h| = |b|$. Hence $b = f \circ h$. Suppose $(S3)$ is exact for all M. Using $M = N'$ we find $g \circ f = g_* \circ f_*(\mathrm{id}_{N'}) = 0$. Taking $M = \mathrm{Ker} f$ and $i : \mathrm{Ker} f \to N'$ the canonical injection we get $f^*(i) = 0$ and thus $\mathrm{Ker} f = 0$. When we take $M = \mathrm{Ker} g$ and $j : \mathrm{Ker} g \to N$ the canonical injection, we get $g^*(j) = 0$ and hence $j = f^*(h) = f \circ h$ for some morphism h, since j preserves parity. Hence $\mathrm{Ker} g = \mathrm{Im} j \subset \mathrm{Im}(f)$.

(ii): Suppose $(S2)$ is exact. It is cleat that e_* is injective and that $d_* \circ e_* = 0$. Suppose $b \in \mathrm{Ker} d_*$, then $\mathrm{Ker} e \subset \mathrm{Ker} b$. We define a homomorphism $c : L'' \to M$ as follows: for $x \in L''$ there is $y \in L$ with $e(y) = x$ and with $|x| = |y|$, we then put $c(x) = b(y)$. The map is a well-defined morphism since $\mathrm{Ker} e \subset \mathrm{Ker} b$. Hence $b = c \circ e$. Suppose $(S4)$ is exact for all M. Choose $M = L''$ and apply to $\mathrm{id}_{L''}$ to find $e \circ d = d_* \circ e_*(\mathrm{id}_{L''}) = 0$. When we take $M = L''/\mathrm{Im} e$ and the canonical projection $p : L'' \to L''/\mathrm{Im} e$ we find $e_*(p) = 0$ and hence $p = 0$, which means $L'' = \mathrm{Im} e$. When we choose $M = L/\mathrm{Im} d$ and $q : L \to L/\mathrm{Im} d$ the canonical projection, then $d_*(q) = 0$ and hence $q = e_*(r) = r \circ e$ for some morphism $r : L'' \to L/\mathrm{Im} d$. Hence $\mathrm{Ker} e \subset \mathrm{Ker} q = \mathrm{Im} d$.

(iii) When $(S5)$ is exact for all M, take $M = A$. Conversely, if $(S2)$ is exact, then clearly $\mathrm{id} \otimes e \circ \mathrm{id} \otimes d = 0$ and $\mathrm{id} \otimes e$ is surjective. To prove $\mathrm{Ker}\,\mathrm{id} \otimes e \subset \mathrm{Im}\,\mathrm{id} \otimes d$ we let $X = \mathrm{Im}\,\mathrm{id} \otimes d$ and define a morphism $v : M \otimes_A L'' \to M \otimes_A L/X$ as follows: for $m \otimes x$ in $M \otimes_A L''$ we find $y \in L$ with $e(y) = x$ and put $v(m \otimes x) = m \otimes y \bmod X$. Then v is a well-defined morphism. Denoting by w the morphism $M \otimes_A L/X \to M \otimes_A L''$ induced by $\mathrm{id} \otimes e$, we see that $v \circ w$ is the identity on $M \otimes_A L/X$. Hence w is injective and thus $\mathrm{Ker} e \subset X$. □

Remark 6.1.2. The proof of proposition 6.1.1 also shows that $(S1)$ is exact if and only if for all A-modules M the sequence

$$(S3') \qquad 0 \longrightarrow \mathrm{Hom}_A(M,N') \xrightarrow{\quad f^* \quad} \mathrm{Hom}_A(M,N) \xrightarrow{\quad g^* \quad} \mathrm{Hom}_A(M,N'')$$

is exact and that $(S2)$ is exact if and only if for all A-modules M the sequence

$$(S4') \qquad 0 \longrightarrow \mathrm{Hom}_A(L'',M) \xrightarrow{\quad e_* \quad} \mathrm{Hom}_A(L,M) \xrightarrow{\quad d_* \quad} \mathrm{Hom}_A(L',M)$$

is exact.

For a given A-module we write $\underline{\mathrm{Hom}}_A(M,-)$ for the functor that assigns to each A-module N the A-module $\underline{\mathrm{Hom}}_A(M,N)$ and to any morphism $f : N \to N'$ the morphism f^* defined in proposition 6.1.1. Similarly, we define the functor $\underline{\mathrm{Hom}}_A(-,M)$ that assigns to each A-module N the A-module $\underline{\mathrm{Hom}}_A(N,M)$ and to each morphism $d : N \to N'$ the morphism d_* as defined in proposition 6.1.1. And finally, the functor that assigns to each A-module N the A-module $M \otimes_A N$ and to each morphism $d : N \to N'$ the map $\mathrm{id} \otimes d$ we write as $M \otimes_A -$. Then the result of proposition 6.1.1 entails that $\underline{\mathrm{Hom}}_A(M,-)$ and $\underline{\mathrm{Hom}}(-,M)$ are left-exact, whereas $M \otimes_A -$ is right-exact. The functor $- \otimes_A M$, being defined in the obvious way, is naturally isomorphic to $M \otimes_A -$. Some immediate properties of the $\underline{\mathrm{Hom}}$-functors are:

$$\begin{aligned} \underline{\mathrm{Hom}}_A(M \oplus M', N) &\cong \underline{\mathrm{Hom}}_A(M,N) \oplus \underline{\mathrm{Hom}}_A(M',N)\,, \\ \underline{\mathrm{Hom}}_A(M, N \oplus N') &\cong \underline{\mathrm{Hom}}_A(M,N) \oplus \underline{\mathrm{Hom}}_A(M,N')\,. \end{aligned} \tag{6.3}$$

Proposition 6.1.3. *Let M be an A-module. For all A-modules P and Q we have an isomorphism of A-modules $\alpha_{PQ} : \underline{\mathrm{Hom}}_A(P, \underline{\mathrm{Hom}}_A(M,Q)) \to \underline{\mathrm{Hom}}(P \otimes_A M, Q)$ and if $x : P' \to P$ and $y : Q \to Q'$ are two morphisms of A-modules then the diagram*

$$\begin{tikzcd} \underline{\mathrm{Hom}}_A(P, \underline{\mathrm{Hom}}_A(M,Q)) \arrow[r, "\alpha_{PQ}"] \arrow[d] & \underline{\mathrm{Hom}}(P \otimes_A M, Q) \arrow[d] \\ \underline{\mathrm{Hom}}_A(P', \underline{\mathrm{Hom}}_A(M,Q')) \arrow[r, "\alpha_{P'Q'}"] & \underline{\mathrm{Hom}}(P' \otimes_A M, Q') \end{tikzcd} \tag{6.4}$$

commutes, where the vertical arrows are induced by the $\underline{\mathrm{Hom}}$-functors. In other words, the functor $\underline{\mathrm{Hom}}_A(M,-)$ is left-adjoint to $- \otimes_A M$.

Proof. Given A-modules P and Q, we show $\underline{\mathrm{Hom}}_A(P, \underline{\mathrm{Hom}}_A(M,Q)) \cong \underline{\mathrm{Hom}}(P \otimes_A M, Q)$ for all M. We define $\alpha_{PQ} : \underline{\mathrm{Hom}}_A(P \otimes_A M, Q) \to \underline{\mathrm{Hom}}_A(P, \underline{\mathrm{Hom}}_A(M,Q))$ by $(\alpha_{PQ} f)(p)(m) = f(p \otimes m)$ for all $p \in P$ and $m \in M$. Clearly $|\alpha_{PQ}(f)| = |f|$. The inverse is $\beta_{PQ} : \underline{\mathrm{Hom}}_A(P, \underline{\mathrm{Hom}}_A(M,Q)) \to \underline{\mathrm{Hom}}_A(P \otimes_A M, Q)$ sending g to $g(p \otimes m) = g(p)(m)$, so that β_{PQ} is an even map. The maps α_{PQ} and β_{PQ} are clearly inverse to each other and are morphisms of A-modules; they commute with right action of A. For naturality, the proof is virtually the same as in the non-super case. Observing that in the definition of α_{PQ} and β_{PQ} the order of all symbols stays the same, no signs can enter the calculation. □

6.1.2 Parity swapping

In definition 3.2.3 we introduced the functor Π, mapping an A-module M to an A-module ΠM with reversed parity assignment. We have a canonical morpism $M \to \Pi M$, mapping m in M to m^π, which is the same element as m, but then seen as element of ΠM; for homogeneous elements $|m^\pi| = |m| + 1$. From definition 3.2.3 and the above we immediately have

Lemma 6.1.4. *The canonical morphism $m \mapsto m^\pi$ is an odd homomorphism of A-modules; $(ma)^\pi = m^\pi a$ and $(am)^\pi = (-1)^{|a|} a \cdot m^\pi$.*

We have a canonical isomorphism $\underline{\mathrm{Hom}}_A(A,M) \cong M$ as A-modules, where $f : A \to M$ is mapped to $f(1)$. However, $\underline{\mathrm{Hom}}_A(\Pi A, M)$ is not canonically isomorphic to ΠM since the same map does not commute with the right action of A; of course they are isomorphic as abelian groups. If $f \in \underline{\mathrm{Hom}}_A(M,N)$ has parity $|f|$ then viewing f as a morphism from ΠM to ΠN, f has the same parity, and hence $\underline{\mathrm{Hom}}_A(\Pi M, \Pi N) \cong \underline{\mathrm{Hom}}_A(M,N)$ as $\mathbb{Z}_2$-graded abelian groups, however not as A-modules. Furthermore, $\Pi\underline{\mathrm{Hom}}_A(M,N) \cong \underline{\mathrm{Hom}}_A(\Pi M,N) \cong \underline{\mathrm{Hom}}_A(M,\Pi N)$ as abelian groups. Let us briefly show that the isomorphism $\Pi\underline{\mathrm{Hom}}_A(M,N) \cong \underline{\mathrm{Hom}}_A(M,\Pi N)$ holds in the category of A-modules; $f : M \to \Pi N$ we map to $\tilde{f} : M \to N$ given by $\tilde{f}(m) = f(m)$. Hence as morphisms of abelian groups, f and $\tilde{f}$ are the same. However, if f is an even element of $\underline{\mathrm{Hom}}_A(M,\Pi N)$, then $\tilde{f}$ is odd as an element of $\underline{\mathrm{Hom}}_A(M,N)$; we therefore consider $\tilde{f}$ as an element of $\Pi\underline{\mathrm{Hom}}_A(M,N)$. The assignment $f \mapsto \tilde{f}$ commutes with the right action of A since $\widetilde{f \cdot a}(m) = f \cdot a(m) = (-1)^{|a||m|} f(m) \cdot a = \tilde{f} \cdot a(m)$. The same trick does not work for $\underline{\mathrm{Hom}}_A(\Pi M,N)$ as the same assignment only commutes with the left action of A and not with the right action.

6.1.3 Abelian structure

If $f : M \to N$ is a morphism of A-modules, then f is injective if and only if f is a monomorphism and f is surjective if and only if f is an epimorphism. As usual, the kernel and the cokernel of f are defined by $\mathrm{Ker} f = \{m \in M | f(m) = 0\}$ and $\mathrm{Coker} f = N/f(M)$, which are both $\mathbb{Z}_2$-graded modules since f preserves the degree. The morphism f is injective if and only if $\mathrm{Ker} f = 0$ and f is surjective if and only if $\mathrm{Coker} f = 0$. The kernel and cokernel have the usual universal properties and can equivalently be described as the equalizer respectively coequalizer of f and the zero morphism. The zero morphism is the unique morphism $0 : M \to N$ that sends $m \in M$ to $0 \in N$. The initial object in the category of A-modules is the zero-module 0, which is also the terminal object and therefore the zero object of the category. All these statements are nothing new and are trivial to prove. For more details and an explanation of the category theoretical terms we refer to [23–25].

Lemma 6.1.5. *Let A be a superring. Then the category of A-modules is an abelian category.*

Proof. The only thing that is to prove is that every monomorphism is the kernel of some morphism and that every epimorphism is the cokernel of some morphism. Let $f : M \to N$ be a monomorphism. Now consider the A-module $N/f(M)$ and the projection map $p : N \to M/f(M)$. The kernel of this morphism is precisely $f(M)$. Now let $f : M \to N$ be an epimorphism. Then $N \cong M/\mathrm{Ker} f$ and thus $i : \mathrm{Ker} f \to M$ shows that f is the cokernel of i. □

We have in fact already used that the category of A-modules is abelian, as we have been working with exact sequences already. On several occasions we have also seen that some functors are exact, which only makes sense in abelian categories. In abelian categories a quick way to prove that a certain functor is left- or right-exact is to show the existence of an adjoint functor. This is based on the following observation (which is no special feature of superrings but for completeness we state and prove it):

Proposition 6.1.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Suppose that two functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ are adjoint to each other, where L is left-adjoint to R; there is a natural isomorphism $\mathrm{Hom}_{\mathcal{B}}(L(a), b) \cong \mathrm{Hom}_{\mathcal{A}}(a, R(b))$ for all $\mathcal{A}$-objects a and $\mathcal{B}$-objects b. Then L is right-exact and R is left-exact.*

Proof. We prove that L is right-exact. The proof of the statement about R is done by reversing some arrows.

Suppose

$$0 \longrightarrow a' \longrightarrow a \longrightarrow a'' \longrightarrow 0 \tag{6.5}$$

is an exact sequence. Now apply L and $\mathrm{Hom}_{\mathcal{B}}(-, b)$ for any $\mathcal{B}$-object b to get a (maybe not exact) sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{B}}(L(a'), b) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(L(a), b) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(L(a''), b) \longrightarrow 0 \quad . \tag{6.6}$$

As L is left-adjoint to R we have a commutative diagram of abelian groups

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathcal{B}}(L(a''),b) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{B}}(L(a),b) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{B}}(L(a'),b) \arrow[d] \\ 0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(a'',R(b)) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(a,R(b)) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(a',R(b)) \end{tikzcd} , \tag{6.7}$$

where all the vertical arrows are isomorphisms and the bottom row is exact. Therefore the top row is exact too. But then the sequence (6.6) is exact for all $\mathcal{B}$-objects b. From an analogue of proposition 6.1.1(ii) (and remark 6.1.2) for the category $\mathcal{B}$ it then follows that the sequence

$$L(a') \longrightarrow L(a) \longrightarrow L(a'') \longrightarrow 0 \tag{6.8}$$

is exact. □

We thus immediately obtain that the functor $M \mapsto M \otimes_A N$ is right-exact, by using proposition 6.1.3. Since for any A-modules M and N there is an isomorphism of abelian groups $\mathrm{Hom}_A(M, \Pi N) \cong \mathrm{Hom}_A(\Pi M, N)$ the functor Π is exact.

6.1.4 Body modules

If M is an A-module, then the body module $\overline{M} = M/J_A M$ is in a natural way an $\bar{A}$-module. Given a morphism of A-modules $f : M \to M'$, then we have $f(J_A M) \subset J_A N$ and thus we have a unique morphism $\bar{f}$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[d] \\ \overline{M} \arrow[r, "\bar{f}"] & \overline{N} \end{tikzcd} \tag{6.9}$$

commutes, where the vertical arrows are the canonical projections (also see diagram (3.2)). Hence we have a functor from the category of A-modules to the category of $\bar{A}$-modules. Clearly, the functor is not full, since any morphism of A-modules has to preserve the parity.

There is an adjoint to the functor $M \mapsto \overline{M}$, which is defined as follows. For any $\bar{A}$-module M we define an A-module $\check{M}$ such that as abelian groups we have $\check{M}_{\bar{0}} = M$ and $\check{M}_{\bar{1}} = 0$ and that for any $m \in \check{M}$ we have the A-action $m \cdot a = m \cdot \bar{a}$. One can now almost literally copy the proof of proposition 3.1.10 to obtain:

Lemma 6.1.7. *The functor $M \mapsto \overline{M}$ from the category of A-modules to the category of $\bar{A}$-modules is left-adjoint to the functor $M \to \check{M}$ from the category of $\bar{A}$-modules to the category of A-modules.*

Proof. As indicated one can copy the proof of proposition 3.1.10 or alternatively, one can use the result of proposition 6.5.3 from section 6.5 by taking $f : A \to \bar{A}$ the projection to the body. □

Lemma 6.1.8. *The functor that assigns to each A-module M the $\bar{A}$-module $\overline{M}$ is right-exact.*

Proof. We observe that $\overline{M} \cong M \otimes_A \bar{A}$ as A-modules. A sequence is exact if and only if it is exact as a sequence of abelian groups and hence the statement follows from proposition 6.1.1. An alternative proof is to use proposition 6.1.6 in combination with lemma 6.1.7. □

6.2 Flat modules and projective modules

In this section we present the definitions of flat and projective modules for modules of superrings. The definitions do not differ from their counterparts in commutative algebra. Therefore also most of the basic properties coincide. We show that if M is a projective respectively flat A-module, then the body module $\overline{M}$ is a projective respectively flat $\bar{A}$-module.

Definition 6.2.1. *Let M be an A-module. If $\underline{\mathrm{Hom}}_A(M,-)$ is exact, then we call M projective. If $M \otimes_A -$ is exact, then we call M flat.*

We now first focus on projective modules. The following lemma gives equivalent characterizations of projective modules.

Lemma 6.2.2. *Let P be an A-module. The following are equivalent:*

(i) The functor $\underline{\mathrm{Hom}}_A(P,-)$ is exact.

(ii) For each surjective morphism $f : M \to N$ and any homomorphism $g : P \to N$, there is a homomorphism $h : P \to M$ such that $f \circ h = g$.

(iii) Every exact sequence

$$0 \longrightarrow M \xrightarrow{q} N \xrightarrow{p} P \longrightarrow 0 \tag{6.10}$$

splits; that is, there is a morphism $s : P \to N$ such that $p \circ s = \mathrm{id}_P$.

(iv) P is a direct summand of a free module, which means that there is a free module F and a module Q such that $F \cong P \oplus Q$.

Proof. $(i) \Leftrightarrow (ii)$ is just paraphrasing the definition: the sequence $\underline{\mathrm{Hom}}(P,M) \to \underline{\mathrm{Hom}}(P,N) \to 0$ is exact for all exact sequences $M \to N \to 0$ if and only if P is projective, if and only if the induced morphism $\underline{\mathrm{Hom}}(P,M) \to \underline{\mathrm{Hom}}(P,N)$ is surjective. $(ii) \Rightarrow (iii)$: Apply (ii) to $N = P$. $(iii) \Rightarrow (iv)$: Let F be a free module that maps surjectively on to P, which exists, since we can choose a generator of F for each element of P. We get an exact sequence $0 \to K \to F \to P \to 0$, with K the kernel of the map $p : F \to P$. The sequence splits and hence there exists $s : P \to F$ with $p \circ s = \mathrm{id}_M$. Any $f \in F$ we can write as $f - s \circ p(f) + s \circ p(f)$ and $f - s \circ p(f) \in \mathrm{Ker} p$ and $s \circ p(f) \in \mathrm{Im} s$. If $f \in \mathrm{Im}(s) \cap \mathrm{Ker} f$, then $f = s(x)$ for some $x \in P$ and $0 = p(f) = x$ so that $\mathrm{Ker} p \cap \mathrm{Im} s = 0$ and thus $F = \mathrm{Ker} p \oplus \mathrm{Im} s$. Since s is injective, $\mathrm{Im} s \cong P$ and thus $F \cong \mathrm{Ker} p \oplus P$. $(iv) \Rightarrow (ii)$: Let $F = P \oplus Q$ with F a free module. Suppose we are given a morphism $g : P \to N$ and a surjective morphism $f : M \to N$, we can extend the morphism g to a morphism $g' : F \to N$ by first projecting to P. Since P is a direct summand we have a morphism $s : P \to F$ and a projection $p : F \to P$ such that $p \circ s = \mathrm{id}_M$. For each generator $x \in F$, choose an element $m_x \in M$ such that $f(m_x) = g \circ p(x)$. Call h' the unique morphism $F \to M$ that assigns to x the element $m_x \in M$. Then the diagram

$$\begin{tikzcd} F \arrow[r, "p"] \arrow[d, "h'"] & P \arrow[r] \arrow[d, "g"] & 0 \\ M \arrow[r, "f"] & N \arrow[r] & 0 \end{tikzcd} \tag{6.11}$$

commutes. The map $h = h' \circ s$ satisfies the requirements since for all $x \in P$ we have $f \circ h' \circ s(x) = g \circ p \circ s(x) = g(x)$. $\square$

We immediately obtain from characterization (iv) of lemma 6.2.2 the following class of projective modules:

Corollary 6.2.3. *Any free module is a projective module.*

Theorem 6.2.4. *If P is a projective A-module, then $\bar{P}$ is a projective $\bar{A}$-module.*

Proof. The result follows directly from corollary 6.5.4, which we prove when we discuss base changes, by taking $B = A/J_A$ and f the projection $A \to \bar{A}$. We now present an alternative more direct proof: Let $f : M \to N$ be a surjective morphism of $\bar{A}$-modules. We can view M and N as A-modules as follows $m \cdot a = m \cdot \bar{a}$. Then f is a surjective morphism of A-modules. Now suppose $g : \bar{P} \to N$ is any morphism of $\bar{A}$-modules. We have a morphism of A-modules $P \to \bar{P} \to N$, by concatenating the projection $\pi : P \to \bar{P}$ with g. Then we have a morphism of A-modules $h : P \to M$ such that $f \circ h = g \circ \pi$. By proposition 3.1.10 (or by direct arguments) the morphism h factors over $\bar{P}$, that is, there is a morphism $\bar{h} : \bar{P} \to M$ such that $h = \bar{h} \circ \pi$. $\square$

From the definition and proposition 6.1.1 it follows that the A-module M is flat if and if for any injective morphism $f : N \to N'$ the induced morphism $\mathrm{id} \otimes f : M \otimes_A N \to M \otimes_A N'$ is injective. This characterization of flatness we use to show the following:

Proposition 6.2.5. *A projective module is flat.*

Proof. Let $f : M \to N$ be an injective morphism. If F is a free module on homogeneous generators $(t_i)_{i \in I}$ then $M \otimes_A F \cong \oplus_{i \in I} M$. Clearly the induced morphism $\oplus_{i \in I} M \to \oplus_{i \in I} N$ is injective, and so F is flat. If now P is projective, then there is a free module F such that P is a direct summand of F; that is, there is a surjective morphism $p : F \to P$ and an injective morphism $s : P \to F$ such that $p \circ s = \mathrm{id}_P$. Clearly $p \otimes \mathrm{id}_M : F \otimes_A M \to P \otimes_A M$ is surjective and a left inverse to $s \otimes \mathrm{id}_M : P \otimes_A M \to F \otimes_A M$, and hence $s \otimes \mathrm{id}_M$ is injective. The diagram

$$\begin{tikzcd} 0 \arrow[r] & F \otimes_A M \arrow[r, "\mathrm{id}_F \otimes f"] \arrow[d, "p \otimes \mathrm{id}_M"] & F \otimes_A N \arrow[d, "p \otimes \mathrm{id}_N"] \\ & P \otimes_A M \arrow[r, "\mathrm{id}_P \otimes f"] & P \otimes_A N \end{tikzcd} \tag{6.12}$$

is commutative. Let $t_i \in P$ and $m_i \in M$ be such that $\sum t_i \otimes f(m_i) = 0$. We then have $\sum s(t_i) \otimes f(m_i) = 0$ so that $\sum s(t_i) \otimes m_i$ is in the kernel of $F \otimes_A M \to F \otimes_A N$. Since the upper row of diagram (6.12) is exact we have $\sum s(t_i) \otimes m_i = 0$, but then $\sum t_i \otimes m_i = p(\sum s(t_i) \otimes m_i) = 0$. Thus also the map $\mathrm{id}_P \otimes f$ is injective. $\square$

The fact that any module is a quotient of a free module implies that the category of A-modules has enough projectives. By this we mean that for any A-module M, there is a projective module P and a surjective morphism $P \to M$. Since any free module is projective, we can take P to be the free module on a set of homogeneous generators for M. Now let K be the kernel of the map $P \to M$, then we can find a projective (and even free) A-module P_1, such that P_1 maps surjectively onto K. Thus the following sequence is exact

$$P_1 \longrightarrow P \longrightarrow M \longrightarrow 0 \quad . \tag{6.13}$$

Applying the same reasoning to the kernel of the composite map $P_1 \to P$ and continuing this process, one obtains a projective resolution of M:

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P \longrightarrow M \longrightarrow 0 \quad , \tag{6.14}$$

which is an exact sequence consisting of all projective A-modules, except for M. When all the P_i are in fact free, one calls the projective resolution a free resolution. Using projective resolutions one can define left-derived functors (see for example [15, 50] for a pedagogical treatment). In section 6.3 we will show that the category of A-modules has enough injectives, which then allows the construction of right-derived functors. Without proof (which does hardly deviate from the proof in the commutative case) we mention that as in the commutative case two projective resolutions are homotopic to each other.

Lemma 6.2.6.

(i) If $(M_i)_{i\in I}$ is a family of flat A-modules, then $M = \oplus_{i\in I} M_i$ is flat.

(ii) If M and M' are flat A-modules, then $M \otimes_A M'$ is flat.

Proof. (i): Write $M = \bigoplus_{i\in I} M_i$. Since $(\bigoplus_{i\in I} M_i) \otimes_A N \cong \bigoplus_{i\in I}(M_i \otimes_A N)$ and a morphism $\mathrm{id}_M \otimes f : \bigoplus_{i\in I}(M_i \otimes_A N) \to \bigoplus_{i\in I}(M_i \otimes_A N')$ is injective if and only each of the restrictions $f_i : M_i \otimes_A N \to M_i \otimes_A N'$ is injective, the first claim is obvious. (ii) Let $f : N \to N'$ be an injective morphism. Then $\mathrm{id}_{M'} \otimes f : M' \otimes_A N \to M' \otimes_A N'$ is injective. Tensoring with M gives the result. ◻

Lemma 6.2.7. *Let A be a superring and S a multiplicative set in $A_{\bar{0}}$. Then $S^{-1}A$ is a flat A-module.*

Proof. This follows immediately from proposition 5.1.17. ◻

The following lemma gives equivalent characterizations of flat modules, also see [18, 50]:

Proposition 6.2.8. *Let M be an A-module, then the following are equivalent:*

(i) M is a flat A-module.

(ii) For every exact sequence of A-modules $N' \to N \to N''$ the associated sequence $M \otimes_A N' \to M \otimes_A N \to M \otimes_A N''$ is exact (by convention such a sequence is exact if and only if it is exact at the middle node).

(iii) For all exact sequences

$$0 \longrightarrow K \xrightarrow{\ i\ } L \xrightarrow{\ p\ } M \longrightarrow 0 \quad , \tag{6.15}$$

and for all A-modules N the associated sequence

$$0 \longrightarrow K \otimes_A N \xrightarrow{i\otimes \mathrm{id}_N} L \otimes_A N \xrightarrow{p\otimes \mathrm{id}_N} M \otimes_A N \longrightarrow 0 \tag{6.16}$$

is exact.

(iv) For any $\mathbb{Z}_2$-graded ideal $\mathfrak{a}$ of A the morphism $\mathfrak{a} \otimes_A M \to M$ sending $a \otimes m$ to am, is injective.

Proof. Clearly (i) and (ii) are equivalent by the definition of flatness. $(i) \Rightarrow (ii)$: Assume we have an exact sequence of A-modules

$$N' \xrightarrow{\ f\ } N \xrightarrow{\ g\ } N'' \quad , \tag{6.17}$$

then we have an induced short exact sequence

$$0 \longrightarrow f(N') \longrightarrow N \longrightarrow g(N) \longrightarrow 0 \quad , \tag{6.18}$$

and thus the sequence

$$0 \longrightarrow f(N') \otimes_A M \longrightarrow N \otimes_A M \longrightarrow g(N) \otimes_A M \longrightarrow 0 \tag{6.19}$$

is exact. Now consider the sequence $N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M$ and suppose $x \in \mathrm{Ker}\,(g \otimes \mathrm{id}_M)$. Then by the exactness of the sequence (6.19) we see that $x \in \mathrm{Im}\,(f \otimes \mathrm{id}_M)$, that is, x lies in the image of f. Hence $N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M$ is exact. $(i) \Leftarrow (ii)$: By proposition 6.1.1 the functor $N \mapsto M \otimes_A M$ is right-exact. We thus only need to show that if $0 \to N \to N''$ is exact, then so is $0 \to N \otimes_A M \to N'' \otimes_A M$. We then take $N' = 0$. $(i) \Rightarrow (iv)$: this is immediate. $(i) \Leftarrow (iv)$: We first claim that if $0 \to N \to \bigoplus_{i \in I} A \oplus \bigoplus_{j \in J} \Pi A$ is exact then so is $0 \to N \otimes_A M \to \bigoplus_{i \in I} M \oplus \bigoplus_{j \in J} \Pi M$. Suppose that some element goes to zero in $\bigoplus_{i \in I} M \oplus \bigoplus_{j \in J} \Pi M$, then it already goes to zero in a finite direct sum, and hence we restrict to exact sequences of the form $0 \to N \to A^{p|q}$. We use an induction argument. Let F_1 and F_2 be two free modules such that $F_1 \oplus F_2 = A^{p|q}$ and we may assume by (iv) and exactness of Π that for all exact sequences $0 \to N_i \to F_i$, with $i = 1, 2$ the sequences $0 \to N_i \otimes_A M \to F_i \otimes_A M$ are exact. Now consider any monomorphism $N \to A^{p|q}$ and identify N with a submodule of $A^{p|q}$. Define $N_1 = N \cap F_1$ and $N_2 = N \cap F_2$. Then the following diagram is commutative and has exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & N_1 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_1 \arrow[r] & A^{p|q} \arrow[r] & F_2 &
\end{tikzcd} \quad . \tag{6.20}$$

We tensor with M and obtain the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & N_1 \otimes_A M \arrow[r] \arrow[d] & N \otimes_A M \arrow[r] \arrow[d] & N_2 \otimes_A M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_1 \otimes_A M \arrow[r] & M^{p|q} \arrow[r] & F_2 \otimes_A M &
\end{tikzcd} \quad . \tag{6.21}$$

The lower row is exact since $(F_1 \oplus F_1) \otimes_A M = M^{p|q}$, that is, the lower row splits. This implies that the map $F_1 \otimes_A M \to M^{p|q}$ has a left inverse and thus is injective. The columns of diagram (6.21) are exact by the assumption on F_1 and F_2. Using a diagram-chasing argument one sees that the morphism of the middle column is injective. This proves the claim.

Any module N is a quotient of a free module F. Thus let $0 \to K \to F \to N \to 0$ be a short exact sequence, with F free and K the kernel of the surjective map $p : F \to N$. Now suppose

$j : N' \to N$ is a monomorphism and call $F' = p^{-1}(N') \subset F$, then we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "\mathrm{id}_K"] & F' \arrow[r] \arrow[d, "i"] & N' \arrow[r] \arrow[d, "j"] & 0 \\ 0 \arrow[r] & K \arrow[r] & F \arrow[r] & N \arrow[r] & 0 \end{tikzcd}, \tag{6.22}$$

where $i : F' \to F$ is the canonical injection. Tensoring with M gives

$$\begin{tikzcd} & K \otimes_A M \arrow[r] \arrow[d] & F' \otimes_A M \arrow[r] \arrow[d] & N' \otimes_A M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K \otimes_A M \arrow[r] & F \otimes_A M \arrow[r] & N \otimes_A M \arrow[r] & 0 \end{tikzcd}. \tag{6.23}$$

The first column of diagram (6.23) is the identity map, the second is a monomorphism by the first claim and the by the same reason the bottom row is exact. A diagram-chasing argument shows that the kernel of the last column is trivial. This proves that the functor $N \mapsto N \otimes_A M$ is left-exact, hence exact. And thus M is flat. □

Remark 6.2.9. Note that in the last part of the proof of proposition 6.2.8 no sign of any difference between commutative rings and superrings was seen. The argument relied only on diagrammatics. This is a general feature; as soon as one enters the realm of diagrammatics all differences between commutative rings and superrings disappear.

Proposition 6.2.10. *Let E be a flat A-module, then $\bar{E}$ is a flat $\bar{A}$-module.*

Proof. Suppose $\bar{E}$ is not flat, then there is an injection $i : M \to N$ of $\bar{A}$-modules such that $\mathrm{id}_{\bar{E}} \otimes i : \bar{E} \otimes_{\bar{A}} M \to \bar{E} \otimes_{\bar{A}} N$ has a nontrivial kernel. We can view M and N as (even) A-modules, where J_A acts trivially. Let us denote M^* respectively N^* the abelian group M respectively N seen as A-module. Similarly we write $i^* : M^* \to N^*$ for the induced morphism of A-modules; then i^* is injective. We have a well-defined morphism of abelian groups $\bar{E} \otimes_{\bar{A}} M \to E \otimes_A M^*$ given by $\bar{e} \otimes_{\bar{A}} m \mapsto e \otimes_A m$, which is injective (since it has a left inverse) and surjective and thus an isomorphism. The diagram

$$\begin{tikzcd} E \otimes_A M^* \arrow[r, "\mathrm{id}_E \otimes_A i^*"] \arrow[d] & E \otimes_A N^* \arrow[d] \\ \bar{E} \otimes_{\bar{A}} M \arrow[r, "\mathrm{id}_{\bar{E}} \otimes_{\bar{A}} i"'] & \bar{E} \otimes_{\bar{A}} N \end{tikzcd} \tag{6.24}$$

commutes and the vertical arrows are isomorphisms of abelian groups. But then the upper morphism $\mathrm{id}_E \otimes_A i^*$ has a nontrivial kernel, contradicting the flatness of E. □

6.3 Injective modules

In this section we prove that the category of A-modules has enough injectives, which allows us to construct right-derived functors and injective resolutions.

Definition 6.3.1. *Let A be a superring and I an A-module. We call I an injective module if the functor $\underline{\mathrm{Hom}}_A(-, I)$ is exact.*

There are other characterizations of injective modules as the next lemma shows (see for example [50]):

Lemma 6.3.2. *Let A be a superring and I an A-module. Then I is injective if and only if one of the two following conditions holds:*

(i) For any injective morphism $i : M' \to M$ and morphism $f : M' \to I$, there is a morphism $g : M \to I$ such that $g \circ i = f$.

(ii) Any exact sequence $0 \to I \to M \to M'' \to 0$ splits.

Proof. The functor $\underline{\mathrm{Hom}}_A(-, I)$ is exact if and only if condition (i) is satisfied. Now assume that condition (i) holds and an exact sequence $0 \to I \to M \to M'' \to 0$ is given. Applying (i) to the identity morphism $\mathrm{id}_I : I \to I$ we see that there is a morphism $g : M \to I$ such that the following diagram commutes and the top row is exact:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "\mathrm{id}_I"] & M \arrow[dl, "g"] \\ & I & \end{tikzcd} . \tag{6.25}$$

Thus (ii) holds. Now assume condition (ii) holds. Consider an injective morphism $i : M' \to M$ and a morphism $f : M' \to I$. Define the module N as $N = I \oplus M/K$ where K is the submodule of $I \oplus M$ generated by the elements of the form $(f(x), 0) - (0, i(x))$, where x runs over all homogeneous elements of M'. Then there are morphisms $u : I \to N$ and $v : M \to N$ defined by $u(x) = (x, 0) \bmod K$ and $v(y) = (0, y) \bmod K$. One easily checks that the morphism u is injective so that the following diagram commutes and has exact rows

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "i"] \arrow[d, "f"] & M \arrow[d, "v"] \\ 0 \arrow[r] & I \arrow[r, "u"'] & N \end{tikzcd} . \tag{6.26}$$

Now we apply condition (ii) to the bottom row to find a morphism $g : N \to I$, which can be concatenated with v to get the required map $g \circ v : M \to I$. Hence (i) holds. $\square$

Recall that an abelian group G is called divisible if for all nonzero integers n the map $G \to G$, sending g to ng, is surjective. Furthermore, an abelian group is injective (when we view it as a $\mathbb{Z}$-module) if and only if it is divisible, see for example [15, 50].

For two abelian groups G and H we write $\mathrm{Hom}_{\mathbb{Z}}(G, H)$ for the morphisms of abelian groups $G \to H$. If G is an abelian group and A is a superring, which is also an abelian group, we want to turn $\mathrm{Hom}_{\mathbb{Z}}(A, G)$ into an A-module. As a set of morphisms of abelian groups, $\mathrm{Hom}_{\mathbb{Z}}(A, G)$ is already an abelian group. We give $\mathrm{Hom}_{\mathbb{Z}}(A, G)$ the following $\mathbb{Z}_2$-grading: we call $f \in \mathrm{Hom}_{\mathbb{Z}}(A, G)$ even respectively odd if $f(A_{\bar{1}}) = 0$ respectively $f(A_{\bar{1}}) = 0$. Then we can write $\mathrm{Hom}_{\mathbb{Z}}(A, G) = \mathrm{Hom}_{\mathbb{Z}}(A, G)_{\bar{0}} \oplus \mathrm{Hom}_{\mathbb{Z}}(A, G)_{\bar{1}}$ and $\mathrm{Hom}_{\mathbb{Z}}(A, G)$ is a $\mathbb{Z}_2$-graded abelian group. For any $a \in A$ we define the right action of a on $f \in \mathrm{Hom}_{\mathbb{Z}}(A, G)$ as follows $(f \cdot a)(a') = f(aa')$ for all $a' \in A$. It is easily checked that this turns $\mathrm{Hom}_{\mathbb{Z}}(A, G)$ into an A-module.

Lemma 6.3.3. *Let M be an A-module and G an abelian group. Then we have an isomorphism of abelian groups* $\mathrm{Hom}_{\mathbb{Z}}(M, G) \cong \underline{\mathrm{Hom}}_A(M, \mathrm{Hom}_{\mathbb{Z}}(A, G))$.

Proof. We define a morphism of abelian groups $\alpha : \mathrm{Hom}_{\mathbb{Z}}(M, G) \to \underline{\mathrm{Hom}}_A(M, \mathrm{Hom}_{\mathbb{Z}}(A, G))$ as follows: Let $\psi : M \to G$ be a morphism of abelian groups and $m \in M$, then $\alpha(\psi)(m) : a \mapsto \psi(ma)$. It is easy to verify that $\alpha(\psi)$ satisfies $\alpha(\psi)(m + m') = \alpha(\psi)(m) + \alpha(\psi)(m')$ and $\alpha(\psi)(m) \cdot a = \alpha(\psi)(ma)$ for all $m, m' \in M$ and $a \in A$. Thus $\alpha(\psi) \in \underline{\mathrm{Hom}}_A(M, \mathrm{Hom}_{\mathbb{Z}}(A, G))$. We define a second morphism of abelian groups $\beta : \underline{\mathrm{Hom}}_A(M, \mathrm{Hom}_{\mathbb{Z}}(A, G)) \to \mathrm{Hom}_{\mathbb{Z}}(M, G)$ as follows: For any $\varphi \in \underline{\mathrm{Hom}}_A(M, \mathrm{Hom}_{\mathbb{Z}}(A, G))$ we define the morphism $\beta(\varphi) : M \to G$ by $\beta(\varphi) : m \mapsto \varphi(m)(1)$, which clearly satisfies $\beta(\varphi)(m + m') = \beta(\varphi)(m) + \beta(\varphi)(m')$. The morphisms α and β are morphisms of abelian groups and are inverse to each other. □

Lemma 6.3.4. *Let D be a divisible abelian group. Then the A-module* $\mathrm{Hom}_{\mathbb{Z}}(A, D)$ *is an injective A-module.*

Proof. Let $i : M' \to M$ be an injective morphism of A-modules. Then the induced sequence of abelian groups $\mathrm{Hom}_{\mathbb{Z}}(M, D) \to \mathrm{Hom}_{\mathbb{Z}}(M', D) \to 0$ is exact. Using the isomorphism of lemma 6.3.3 we obtain a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_A(M, \mathrm{Hom}_{\mathbb{Z}}(A, D)) \arrow[r] & \underline{\mathrm{Hom}}_A(M', \mathrm{Hom}_{\mathbb{Z}}(A, D)) \arrow[r] & 0 \\
Hom_{\mathbb{Z}}(M, D) \arrow[r] \arrow[u] & Hom_{\mathbb{Z}}(M', D) \arrow[r] \arrow[u] & 0
\end{tikzcd} , \tag{6.27}$$

of which the bottom row is exact and where the horizontal arrows are the maps $f \mapsto f \circ i$. But then the top row is exact as well, proving that the functor $M \mapsto \underline{\mathrm{Hom}}_A(M, \mathrm{Hom}_{\mathbb{Z}}(A, D))$ is exact. □

As a consequence, we obtain that the category of A-modules has enough injectives:

Theorem 6.3.5. *Let M be an A-module. Then there is an injective A-module I such that M injects into I.*

Proof. Consider the A-module as an abelian group. Since the category of abelian groups has enough injectives there is monomorphism $f : M \to D$ for some divisible group D. Then there is a natural morphism of A-modules $M \to \mathrm{Hom}_{\mathbb{Z}}(A, D)$ given by $m \mapsto f_m$, where $f_m(a) = f(ma)$. One easily checks that $f_{m+m'} = f_m + f_{m'}$ and $f_{ma} = f_m \cdot a$ for all $m, m' \in M$ and $a \in A$. The map $m \mapsto f_m$ is thus a morphism of A-modules. If $f_m = f_{m'}$ then in particular $f_m(1) = f_{m'}(1)$, which implies $m = m'$. Hence $m \mapsto f_m$ is an injective morphism. By lemma 6.3.4 the A-module $\mathrm{Hom}_{\mathbb{Z}}(A, D)$ is injective and thus we have shown that any A-module can be injected into an injective module. □

6.4 Finitely generated and Noetherian modules

In this section we rederive some classical results of commutative algebra for finitely generated modules: we discuss the lemma of Nakayama, the Hamilton–Cayley theorem, the Krull intersection theorem and we start the discussion with paving the way for a result on generic freeness.

Definition 6.4.1. *Let M be an A-module. We say a prime ideal $\mathfrak{p}$ of A is associated to M, when there is a homogeneous $m \in M$ such that* $\mathfrak{p} = \mathrm{Ann}(m)$.

A prime ideal associated to M that is minimal is called a minimal prime of M and the other prime ideals associated to M are called embedded primes. The nomenclature for prime ideals associated to a module is very similar to the nomenclature of prime ideals associated to a primary decomposition of an $\mathbb{Z}_2$-graded ideal. This is not a coincidence and is explained in for instance [15, chapter 3].

Lemma 6.4.2. *Let A be a Noetherian superring. If M is a nonzero A-module, then there are primes associated to M.*

Proof. Since M is nonzero, the set of $\mathbb{Z}_2$-graded ideals $\mathrm{Ann}(m)$, where m runs over the nonzero homogeneous elements is not empty. Since A is Noetherian, there is a maximal element $\mathfrak{p}$. We will show that $\mathfrak{p}$ is prime. Suppose $\mathfrak{p}$ is the annihilator of $m \in M$. Assume there are homogeneous $a, b \in A$ with $b \notin \mathfrak{p}$ and $ab \in \mathfrak{p}$. Then $ab \cdot m = 0$, and thus $a \cdot (bm) = 0$ and $bm \neq 0$. Clearly, $\mathfrak{p} \subset \mathrm{Ann}(bm)$ and by maximality of $\mathfrak{p}$ the $\mathbb{Z}_2$-graded ideal $\mathrm{Ann}(bm)$ must equal $\mathfrak{p}$, so that $a \in \mathrm{Ann}(bm) = \mathfrak{p}$. ☐

Theorem 6.4.3. *Let A be a Noetherian superring. Let M be a finitely generated A-module. Then there exists a filtration*

$$M = M_0 \supset M_1 \supset \ldots \supset M_n \supset M_{n+1} = 0\,. \tag{6.28}$$

of submodules such that M_i/M_{i+1} is isomorphic to either $A/\mathfrak{p}_i$ or $\Pi A/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$.

Proof. Let S be the set of submodules that admit such a decomposition. Then S is not-empty, since it contains 0. Since M is Noetherian, S contains a maximal element N. If $M \neq N$ then $M/N \neq 0$ and there exists a prime ideal associated to M/N. Hence M/N contains a submodule N'/N that is isomorphic to either $A/\mathfrak{p}$ or $\Pi A/\mathfrak{p}$ for some prime $\mathfrak{p}$. Hence N' lies in S and properly contains N. Hence we must have $M = N$. ☐

We apply theorem 6.4.3 to show that under suitable circumstances we can make a module M of a Noetherian superring A free by localizing at some element. The method of Takeuchi [58] only works for reduced rings, since then we can localize as in proposition 5.1.20 to obtain an integral domain and in an integral domain 0 is a prime ideal. By theorem 6.4.3 we know that M admits a decomposition

$$M = M_0 \supset M_1 \supset \ldots \supset M_n \supset M_{n+1} = 0\,, \tag{6.29}$$

such that M_i/M_{i+1} for $0 \leq i \leq n$ is as an A-module isomorphic to $A/\mathfrak{p}_i$ or $\Pi(A/\mathfrak{p}_i)$ where $\mathfrak{p}_i$ is a prime ideal. If we can regroup a few terms in such a filtration such that M_i/M_{i+1} is of the form $A/\mathfrak{a}_i$ (or $\Pi(A/\mathfrak{a}_i)$) where $\mathfrak{a}_i$ is either the zero ideal or a $\mathbb{Z}_2$-graded ideal such that $\overline{\mathfrak{a}_i}$ is nonzero, then we can for each nonzero $\mathfrak{a}_i$ choose an element $a_i \in \mathfrak{a}_{i,\bar{0}}$ with $\bar{a}_i \neq 0$ and if $\mathfrak{a}_i = 0$ we put $a_i = 1$. If $\bar{A}$ is an integral domain, the multiplicative set generated by $a = a_1 \cdots a_n$ does not contain zero and $(A/\mathfrak{a}_i)_a = 0$ if $\mathfrak{a}_i \neq 0$. Hence

$$M_a = (M_a)_0 \supset (M_a)_1 \supset \ldots \supset (M_a)_{k+1} = 0\,, \tag{6.30}$$

with $k \leq n$ and $(M_a)_i/(M_a)_{i+1} \cong A_a$ and then M_a is free. If $\bar{A}$ is not an integral domain, we can first localize A such that $\bar{A}$ is an integral domain (see proposition 5.1.20). We thus have shown:

Theorem 6.4.4. *Let A be a reduced Noetherian superring and M a finitely generated A-module. Then there exists a nonzero even element $a \in A$ such that the localization M_a is a free A_a-module.*

If a superring A is reduced, then the underlying commutative ring has no nilpotents. Then a localization of A can be done such that $\bar{A}$ is an integral domain, which implies that $\mathrm{Spec}(A)$ is irreducible as a topological space. If M is a finitely generated module, then there is a sheaf $\mathcal{M}$ on $\mathrm{Spec}(A)$ such that on the principal open sets $D(f)$ we have $\mathcal{M}(D(f)) = M_f$ (see for example [54,55] for this construction). We say $\mathcal{M}$ is the sheaf associated to M. Theorem 6.4.4 then implies that on an open dense subset the sheaf $\mathcal{M}$ is free. In other words, $\mathcal{M}$ is a locally free sheaf. More generally, we call a superscheme $(X, \mathcal{O}_X)$ Noetherian if it admits an open cover by affine Noetherian superschemes. A sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ for which there are affine open sets $U_i \cong \mathrm{Spec}(A_i)$ such that the restriction of $\mathcal{M}$ to U_i is isomorphic to the sheaf associated to an A_i-module M_i, is called a quasi-coherent sheaf. If all the M_i are finitely generated, we call $\mathcal{M}$ a coherent sheaf. Theorem 6.4.4 then says that under the condition that the A_i are reduced, a coherent sheaf is locally free.

We come to three 'classics' of commutative algebra: Nakayama's lemma, the Hamilton–Cayley theorem and the Krull intersection theorem. Especially the lemma of Nakayama, with which we start below, will prove useful in later sections.

Proposition 6.4.5 (Nakayama's lemma). *Let M be a finitely generated A-module. Suppose that I is a $\mathbb{Z}_2$-graded ideal contained in the Jacobson radical (see for example section 4.1). Then if $IM = M$, then $M = 0$.*

Proof. Let $m_1, \ldots, m_n$ be a set of homogeneous generators for M. Then $m_n \in IM$ and hence we find $a_i \in I$ with $m_n = \sum_i a_i m_i$. Since $1 - a_n$ is invertible by lemma 4.1.15, we can eliminate m_n from the set of generators, and M is generated by $n-1$ elements. So we may assume M is generated by 1 element m. But then $m = am$ for some $a \in I$ and then $(1-a)m = 0$, hence $m = 0$. $\square$

Let N be a submodule of a finitely generated A-module M and I a $\mathbb{Z}_2$-graded ideal contained in the Jacobson radical of A. If we can write M as $M = IM + N$, then it follows that $M = N$ by applying the Nakayama lemma 6.4.5 to the quotient M/N. When A is a local superring with maximal ideal $\mathfrak{m}$ and M is finitely generated, then $\mathfrak{m}M = M$ implies $M = 0$. For a local ring, the Nakayama lemma has an important consequence for projective modules:

Lemma 6.4.6. *Let A be a local superring. Then every finitely generated projective A-module is a free module.*

Proof. Let $m_1, \ldots, m_n$ be a set of homogeneous elements such that their images in $M/\mathfrak{m}M$ are a (standard) basis for the super vector space $M/\mathfrak{m}M$. We then have a morphism $h : A^{p|q} \to M$ for some p and q with $p + q = n$. Let N be the submodule of M generated by the $m_1, \ldots, m_n$. If $m \in M$, then there are $a_i \in A$ such that $m - \sum_i a_i m_i$ goes to zero in $M/\mathfrak{m}M$, which means $m - \sum_i a_i m_i \in \mathfrak{m}M$. We thus conclude that $M = N + \mathfrak{m}M$. The Nakayama lemma implies that $M = N$. Thus the map $A^{p|q} \to M$ is surjective. As M is projective we infer that there is a morphism $s : M \to A^{p|q}$ such that the following diagram commutes and the bottom line is exact:

$$\begin{tikzcd} & M \arrow[dl, "s"'] \arrow[d, "\mathrm{id}_M"] & \\ A^{p|q} \arrow[r, "h"'] & M \arrow[r] & 0 \end{tikzcd} \quad . \tag{6.31}$$

Let e_i be a homogeneous basis for $1 \leq i \leq n$ of A such that $h(e_i) = m_i$. There are elements S_{ij} of A such that $s(m_i) = e_i + \sum_j e_j S_{ji}$. As $h(s(m_i)) = m_i$ we have $\sum_j S_{ij} e_j \in \mathrm{Ker}\, h$. Now suppose $x = \sum_i e_i x_i \in \mathrm{Ker}\, h$, then $0 = \sum_i m_i x_i$ and since the $m_i \bmod \mathfrak{m}$ are linearly independent over $A/\mathfrak{m}$ we must have $x_i \in \mathfrak{m}$. Furthermore, we have

$$0 = s \circ h(x) = \sum_i e_i x_i + \sum_{ij} e_i S_{ij} x_j \, . \tag{6.32}$$

As the e_i are independent we have $x_i = -S_{ij} x_j$. But then $x = -\sum_{ij} e_i S_{ij} x_j \in \mathfrak{m}\mathrm{Ker}\, h$. Hence $\mathrm{Ker}\, h = \mathfrak{m}\mathrm{Ker}\, h$ and thus by Nakayama's lemma 6.4.5 we conclude $\mathrm{Ker}\, h = 0$. $\square$

Theorem 6.4.7 (Hamilton–Cayley). *Let M be a finitely generated A-module. Given a morphism $\varphi : M \to M$ with $\varphi(M) \subset IM$ for some $\mathbb{Z}_2$-graded ideal I in A. There there exists a monic polynomial $p = X^N + a_1X^{N-1} + \ldots + a_N$ in $A_{\bar{0}}[X]$ with $p(\varphi) = 0$ and with $a_i \in I^i$ and where $A_{\bar{0}}[X]$ is the polynomial ring in one variable with coefficients in $A_{\bar{0}}$.*

Proof. If M is finitely generated so is $\overline{M}$. Using the classical version of the Cayley–Hamilton theorem (see for example [15, 50]) we find a polynomial $\bar{p} \in \bar{A}[X]$ of the form

$$\bar{p}(X) = X^n + r_1X^{n-1} + \ldots + r_n\,, \tag{6.33}$$

with $\bar{p}(\bar{\varphi}) = 0$ and $r_i \in \bar{I}^i$. We can construct an element $p \in A_{\bar{0}}[x]$ such that under the projection $A \to \bar{A}$ the polynomial p goes to $\bar{p}$. We write

$$p(X) = X^n + b_1X^{n-1} + \ldots + b_n\,, \tag{6.34}$$

where we can choose the $b_i \in (I_{\bar{0}})^i$. For any $m \in M$ we consider $p(\varphi)(m)$. From $\overline{\varphi^s(m)} = \bar{\varphi}^s(\bar{m})$ for each integer s, it follows that $\overline{p(\varphi)(m)} = 0$. Hence $p(\varphi)M \subset J_AM$. Hence on the generators we can write

$$p(\varphi)(m_i) = \sum_k j_{ik}m_k\,, \quad j_{ik} \in J_A\,. \tag{6.35}$$

Since the j_{ik} are nilpotent, there is a power r such that $p(\varphi)^r = 0$. The polynomial $p(\varphi)^r$ is monic and the coefficients a_i in

$$P(X) = p(X)^r = X^N + a_1X^{N-1} + \ldots + a_N\,, \tag{6.36}$$

are in $I^i_{\bar{0}}$. □

Corollary 6.4.8. *Let A be a superring, I a $\mathbb{Z}_2$-graded ideal in A, M a finitely generated A-module and suppose $a \in A$ is homogeneous with $aM \subset IM$. Then there is an integer $n > 0$ and a homogeneous element $b \in I$ with $(a^n + b)M = 0$. In particular, when $M = IM$, there is an even element $b \in I$ with $(1 + b)M = 0$.*

Proof. A direct application of the Hamilton–Cayley theorem 6.4.7 by taking $\varphi(m) = ma$: then we find that there is a monic polynomial $p(X) = \sum_{i=0}^n X^i a_{n-i}$ in $A_{\bar{0}}$ such that $a_k \in I^k_{\bar{0}}$ and $p(a) = 0$. Therefore $a^n + b$ acts by zero on M, for some homogeneous $b \in I$. The second statement follows by taking $a = 1$. □

When the $\mathbb{Z}_2$-graded ideal I of corollary 6.4.8 lies in the Jacobson radical, then it follows $M = 0$ and thus we get an alternative way of deducing the lemma of Nakayama. Indeed, if I is the Jacobson radical, then the element $1 + b$ is invertible and thus $(1 + b)M = 0$ implies $M = 0$.

We now turn to the preparation that we will use to prove the Krull intersection theorem for superrings. The presentation follows the lines of [19].

Lemma 6.4.9. *Let A be a Noetherian superring, I a $\mathbb{Z}_2$-graded ideal in A. Suppose M is a finitely generated A-module and $N \subset M$ a $\mathbb{Z}_2$-graded submodule. Then there exists a submodule $Q \subset M$ and an integer $n > 0$ such that*

(i) $Q \cap N = IN$ and

(ii) $I^nM \subset Q$.

Proof. Consider the set S of all submodules $N' \subset M$ such that $N' \cap N = IN$. Then $S \neq \emptyset$ and there is thus a maximal element Q. We claim that Q satisfies the properties (i) and (ii) of the lemma. Clearly $Q \cap N = IN$. Since M is finitely generated, to prove (ii) it suffices to show that for all homogeneous $x \in I$, there exists an integer r with $x^r M \subset Q$. Take $x \in I$ homogeneous and consider for each integer $s > 0$ the submodules $(Q : x^s) = \{m \in M \mid x^s m \in Q\}$, then $(Q : x^s) \subset (Q : x^{s+1})$. Since M is Noetherian, there is an r such that for $s \geq r$ from $x^s m \in Q$ follows that $x^r m \in Q$. We claim that $(x^r M + Q) \cap N = IN$. From the inclusion $Q \supset IN$ it follows that $IN \subset (x^r M + Q) \cap N$. On the other hand, if $m = x^r m' + q$ for $m' \in M$, with $q \in Q$, lies in N, then $xm \in IN$. Hence $x^{r+1} m' \in Q$, and thus we see that $m \in Q$. But then $x^r M + Q \subset Q$ by the maximality of Q so that we conclude $x^r M \subset Q$. □

Theorem 6.4.10 (Krull's intersection theorem)**.** *Let A be a Noetherian superring, I a $\mathbb{Z}_2$-graded ideal in A and M a finitely generated A-module. Call $N = \cap_{i \geq 0} I^i M$, then $IN = N$.*

Proof. By lemma 6.4.9 there exists a submodule A of M such that $Q \cap N = IN$ and an integer n such that $I^n M \subset Q$. But then $N \subset I^n M \subset Q$. Hence $N = Q \cap N = IN$. □

We have the following corollaries:

Corollary 6.4.11. *Let A be a Noetherian superring, I a $\mathbb{Z}_2$-graded ideal in A and M a finitely generated A-module. Then we have:*

(i) There is an even $y \in I$ with $(1 + y) \cap_{i \geq 0} I^i M = 0$.

(ii) If in addition I is contained in the Jacobson radical then $\cap_{i \geq 0} I^i M = 0$.

(iii) If A is local with maximal ideal $\mathfrak{m}$ then $\cap_{i \geq 0} \mathfrak{m}^i = 0$.

Proof. By combining theorem 6.4.10 with the second part of corollary 6.4.8 we obtain (i). The second and third part of the corollary are then immediate. □

6.5 Base change

Let A and B be superrings and let $f : A \to B$ be a morphism. In this case we say that B is an A-superalgebra. We will use the map f to get functors that relate the category of A-modules to the category of B-modules.

We define the functor $f_* : \textbf{A-mod} \to \textbf{B-mod}$ as follows: For an A-module M we define $f_*(M) = M \otimes_A B$, which is canonically a right B-module. For a homomorphism of A-modules $u : M \to N$ we define $f_*(u) : f_*(M) \to f_*(N)$ by $f_*(u) = u \otimes \mathrm{id}_B$. The functor f_* is right-exact by proposition 6.1.1. The functor f_* preserves injective morphisms if and only if B is flat as an A-module, where the action of A on B is the one prescribed by f.

We define a functor $f^* : \textbf{B-mod} \to \textbf{A-mod}$ for the same $f : A \to B$ as follows: For each B-module M we let $f^*(M)$ be the A-module, where the right action of $a \in A$ on $m \in M$ is defined by $(m, a) \mapsto mf(a)$. For a morphism $v : M \to N$ of B-modules we put $f^*(v)(m) = v(m)$.

Lemma 6.5.1. *Let A, B be superrings, $f : A \to B$ a morphism and let M and N be A-modules. We have $f_*(M \otimes_A N) \cong f_*(M) \otimes_B f_*(N)$ and $f_*(M \oplus N) = f_*(M) \oplus f_*(N)$.*

Proof. Follows from $(M \otimes_A N) \otimes_A B \cong (M \otimes_A B) \otimes_B (N \otimes_A B)$ where the isomorphisms are given by $\psi : m \otimes n \otimes b \mapsto (m \otimes 1) \otimes (n \otimes b) = (-1)^{|n||b|}(m \otimes b) \otimes (n \otimes 1)$ and the inverse is $\psi^{-1} : (m \otimes b_1) \otimes (n \otimes b_2) \mapsto (-1)^{|n||b_1|} m \otimes n \otimes b_1 b_2$.

For the second part we note that it follows from the definition of the direct sum that $(M \oplus N) \otimes_A B \cong (M \otimes_A B) \oplus (N \otimes_A B)$. □

Lemma 6.5.2. *Let A be a local superalgebra with maximal ideal $\mathfrak{m}$, and let M and N be two finitely generated A-modules. Then $M \otimes_A N = 0$ if and only if M or N is zero.*

Proof. Applying lemma 6.5.1 to the canonical projection $f : A \to B = A/\mathfrak{m}$ we see that $(M \otimes_A B) \otimes_B (N \otimes_A B) = 0$. Both factors are super vector spaces, hence one of them has to vanish. Suppose $M \otimes_A A/\mathfrak{m} = 0$, then $M/\mathfrak{m}M = 0$ and thus $M = \mathfrak{m}M$. By Nakayama's lemma $M = 0$. □

Proposition 6.5.3. *Let A and B be superrings and $f : A \to B$ a morphism. There is a natural isomorphism of $\mathbb{Z}_2$-graded abelian groups $\underline{\mathrm{Hom}}_A(M, f^*(N)) \cong \underline{\mathrm{Hom}}_B(f_*(M), N)$ for $M \in$ **A-mod** and $N \in$ **B-mod**. In other words, f_* is left-adjoint to f^*.*

Proof. Note that $y \in \underline{\mathrm{Hom}}_A(M, f^*(N))$ implies that $y(ma) = y(m)f(a)$. We define a morphism of sets $\alpha : \underline{\mathrm{Hom}}_B(f_*(M), N) \to \underline{\mathrm{Hom}}_A(M, f^*(N))$ by

$$\alpha x(m) = x(m \otimes 1)\,. \tag{6.37}$$

The inverse to α is given by the morphism $\beta : \underline{\mathrm{Hom}}_A(M, f^*(N)) \to \underline{\mathrm{Hom}}_B(f_*(M), N)$ defined by

$$\beta y(m \otimes b) = y(m)b\,. \tag{6.38}$$

The maps α and β are well-defined and preserve the parity. The naturality is shown by a direct application of the definitions. □

Corollary 6.5.4. *Let A and B be superalgebras and $f : A \to B$ a morphism of superalgebras. If P is a projective A-module, then $P \otimes_A B$ is a projective B-module.*

Proof. Suppose $x : K \to L$ is a surjective morphism of B-modules and $y : P \otimes_A B \to L$ is a morphism of B-modules. The induced sequence of A-modules $f^*(K) \to f^*(L) \to 0$ is exact since $f^*(x)(k) = x(k)$; the maps $f^*(x)$ and x are the same as morphisms of abelian groups. We have a morphism $f^*(y) : P \to f^*(L)$ sending $p \in P$ to $y(p \otimes 1)$. Since P is a projective A-module, there is a morphism $h : P \to f^*(K)$ such that $f^* \circ h = f^*(y)$. We define $h_* = f_*(h) : P \otimes_A B \to K$ by $h_*(p \otimes b) = h(p)b$. Then h_* is a well-defined morphism of B-modules. We observe that $x \circ h_*(p \otimes b) = x(h(p))b = (f^*(x) \circ h(p))b = (f^*(y)(p))b = y(p \otimes 1)b = y(p \otimes b)$ and thus $x \circ h_* = y$, which means that $P \otimes_A B$ is projective. □

Let B be an A-superalgebra, so that B is an A-module. Let us write ab for the left action of $a \in A$ on $b \in B$; we thus suppress writing explicitly the morphism $A \to B$. If M and N are A-modules, then $\underline{\mathrm{Hom}}_A(M, N) \otimes_A B$ and $\underline{\mathrm{Hom}}_B(M \otimes_A B, N \otimes_A B)$ are B-modules, and we expect them to be isomorphic. In order to be able to show in theorem 6.5.9 that under suitable circumstances this is indeed so, we need some preliminaries.

If N is an A-module, we write $N^{p|q}$ for the A-module $N \otimes_A A^{p|q} \cong (\bigoplus_{i=1}^p N) \oplus (\bigoplus_{j=1}^q \Pi N)$. For $1 \leq k \leq p+q$ we have a morphism of A-modules $u_k : A \to A^{p|q}$, which maps 1 to the 1 in the kth summand of $A^{p|q}$. In the remainder of this subsection, we denote e_k the image of 1 under u_k; $e_k = u_k(1)$. Each element n of $N^{p|q}$ admits a unique decomposition $n = \sum_{k=1}^{p+q} n_k e_k$.

Definition 6.5.5. *If M is a finitely generated A-module we have an exact sequence*

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\;p\;} M \longrightarrow 0 \quad , \tag{6.39}$$

where F is a finite free module and K is the kernel of the morphism $p : F \to K$. If K is finitely generated, we call M finitely presented. Equivalently, M is finitely presented if and only if there are finite free modules F and G such that the sequence

$$F \longrightarrow G \xrightarrow{\;p\;} M \longrightarrow 0 \tag{6.40}$$

is exact. The exact sequence (6.40) is called a short free resolution of M.

Clearly, the property of being finitely presented is preserved under base extension: if M is a finitely presented A-module, then $M \otimes_A B$ is a finitely presented B-module. Some direct consequences of the definition:

Proposition 6.5.6. *If A is a Noetherian superring and M is a finitely generated A-module, then M is finitely presented.*

Proof. If $p : F \to M$ is a surjective A-module morphism and F finitely generated, then F is a Noetherian A-module, and thus $\mathrm{Ker}p$ is finitely generated. □

Proposition 6.5.7. *If P is a finitely generated projective A-module, then P is finitely presented.*

Proof. If P is projective and finitely generated, then there is a finite free module F and a surjective morphism $p : F \to P$. By lemma 6.2.2 we can write $F \cong P \oplus \mathrm{Ker}p$ and since $\mathrm{Ker}p$ is a quotient of F it is finitely generated. □

For an A-module M there may be several different short free resolutions. The following proposition is not needed to prove theorem 6.5.9 and relates different free resolutions.

Proposition 6.5.8. *Let M be finitely presented and let*

$$\begin{array}{c} F \xrightarrow{q} G \xrightarrow{p} M \longrightarrow 0 \\ \\ F' \xrightarrow{q'} G' \xrightarrow{p'} M \longrightarrow 0 \end{array} \tag{6.41}$$

be two short free resolutions of M. Then there are A-module morphisms $\alpha : F \to F'$ and $\beta : G \to G'$ such that the diagram with exact rows

$$\begin{tikzcd} F \arrow[r, "q"] \arrow[d, "\alpha"'] & G \arrow[r, "p"] \arrow[d, "\beta"'] & M \arrow[r] \arrow[d, "\mathrm{id}_M"'] & 0 \\ F' \arrow[r, "q'"] & G' \arrow[r, "p'"] & M \arrow[r] & 0 \end{tikzcd} \tag{6.42}$$

commutes. Furthermore, if α is surjective, then so is β.

Proof. Let x be a generator in G, then $p(x) = p'(y)$ for some $y \in G'$. We define $\beta : G \to G'$ by $\beta(x) = y$ and extend by A-linearity so that $p = p' \circ \beta$. If x is a generator in F then $p' \circ \beta \circ q(x) = 0$ and hence there is $y \in F'$ with $q'(y) = \beta \circ q(x)$. We define $\alpha : F \to F'$ by $\alpha(x) = y$ and extend by A-linearity. Now suppose α is surjective and $g' \in G'$. Then we can find $g \in G$ such that $p(g) = p'(g')$, so that $\beta(g) - g' = q'(f')$ for some $f' \in F'$. Hence $\beta(g) - g' = q' \circ \alpha(f)$ for some $f \in F$ and thus $\beta(g) - g' = \beta \circ q(f)$, which implies $g' \in \mathrm{Im}\beta$. □

Theorem 6.5.9. *Let A and B be superalgebras and let M and N be A-modules. Suppose we have a superalgebra morphism $f : A \to B$. We have a morphism of B-modules*

$$\alpha : \underline{\mathrm{Hom}}_A(M, N) \otimes_A B \to \underline{\mathrm{Hom}}_B(f_*(M), f_*(N)) \tag{6.43}$$

defined by

$$\alpha(\varphi \otimes_A b)(m \otimes_A b') = (-1)^{|m||b|}\varphi(m) \otimes_A bb' \,. \tag{6.44}$$

If B is a flat A-module and M is finitely presented, then the map α is an isomorphism of B-modules.

Proof. First we show that α is a morphism of B-modules:

$$\alpha(\varphi \otimes b_1 b_2)(m \otimes b_3) = (-1)^{|m|(|b_1|+|b_2|)} \varphi(m) \otimes b_1 b_2 b_3 \,, \tag{6.45}$$

on the one hand and

$$\begin{aligned} ((\alpha(\varphi \otimes b_1)) \cdot b_2)(m \otimes b_3) &= (-1)^{|b_2|(|m|+|b_3|)} \alpha(\varphi \otimes b_1)(m \otimes b_3) \cdot b_2 \\ &= (-1)^{|m|(|b_1|+|b_2|)+|b_2||b_3|} \varphi(m) \otimes b_1 b_3 b_2 \,, \end{aligned} \tag{6.46}$$

which equals (6.45). Furthermore, the map α preserves sums and the parity.

Next, we first verify the isomorphism on the module $M = A$. In this case $\underline{\mathrm{Hom}}_A(A, N) \cong N$, $A \otimes_A B \cong B$ and $\underline{\mathrm{Hom}}_B(B, N \otimes_A B) \cong N \otimes_A B$. It is easy to check that the map α is the identity. Now we deal with $M = A^{p|q}$. We have an isomorphism of abelian groups $\underline{\mathrm{Hom}}_A(A^{p|q}, N) \otimes_A B \cong N^{p|q} \otimes_A B$ and also

$$\begin{aligned} \underline{\mathrm{Hom}}_B(A^{p|q} \otimes_A B, N \otimes_A B) &\cong \underline{\mathrm{Hom}}_B(B^{p|q}, N \otimes_A B) \\ &\cong (N \otimes_A B)^{p|q} \cong N^{p|q} \otimes_A B \,, \end{aligned} \tag{6.47}$$

are isomorphisms of abelian groups. The isomorphisms are however not isomorphisms of B-modules, since the isomorphisms do not commute with the right action of B. The morphism of B-modules induces a map $\hat{\alpha} : N^{p|q} \otimes_A B \to N^{p|q} \otimes_A B$ of abelian groups given by

$$\hat{\alpha} : \sum_j e_j n_j \otimes b \mapsto \sum_j e_j (-1)^{|b||e_j|} n_j \otimes b \,, \tag{6.48}$$

where the e_j are the images of $1 \in A$ in the jth summand of $N^{p|q} \cong N \otimes_A A^{p|q}$. The minus sign in eqn.(6.48) should not surprise us, since we do not have morphisms of modules. However, we see that $\hat{\alpha}$ is an isomorphism of abelian groups, since it squares to the identity and preserves sums. We already verified that α is a morphism of B-modules, hence also for $M = A^{p|q}$ the map α is an isomorphism.

Now we assume that B is a flat A-module and M is finitely presented. Hence there are finite free A-modules G and F and connecting maps δ and ϵ such that the following sequence

$$F \xrightarrow{\ \delta\ } G \xrightarrow{\ \epsilon\ } M \longrightarrow 0 \tag{6.49}$$

is exact. Applying the functors $- \otimes_A B$ and $\underline{\mathrm{Hom}}_B(-, N \otimes_A B)$ we get an exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}_B(f_*(M), f_*(N)) \longrightarrow \underline{\mathrm{Hom}}_B(f_*(G), f_*(N)) \longrightarrow \underline{\mathrm{Hom}}_B(f_*(F), f_*(N)) \quad . \tag{6.50}$$

Similarly we obtain an exact sequence (using flatness of B)

$$0 \longrightarrow \underline{\mathrm{Hom}}_A(M, N) \otimes B \longrightarrow \underline{\mathrm{Hom}}_A(G, N) \otimes B \longrightarrow \underline{\mathrm{Hom}}_A(F, N) \otimes B \quad . \tag{6.51}$$

Writing α_X for the morphism $\alpha_X : \underline{\mathrm{Hom}}_A(X, N) \otimes_A B \to \underline{\mathrm{Hom}}_B(f_*(X), f_*(N))$ as defined in (6.44) and putting $X = F, G, M$ we get a commutative diagram (where it is needed that ϵ and δ are even in order to ensure commutativity):

$$\begin{tikzcd} 0 \arrow[r] & \underline{\mathrm{Hom}}_B(M', N') \arrow[r] & \underline{\mathrm{Hom}}_B(G', N') \arrow[r] & \underline{\mathrm{Hom}}_B(F', N') \\ 0 \arrow[r] & \underline{\mathrm{Hom}}_A(M, N) \otimes B \arrow[r] \arrow[u, "\alpha_M"] & \underline{\mathrm{Hom}}_A(G, N) \otimes B \arrow[r] \arrow[u, "\alpha_G"] & \underline{\mathrm{Hom}}_A(F, N) \otimes B \arrow[u, "\alpha_F"] \end{tikzcd} \quad . \tag{6.52}$$

From the first part of the proof we know that α_F and α_G are isomorphisms. By a simple diagram-chasing argument it follows that α_M is also an isomorphism. $\square$

Corollary 6.5.10. *Let A be a superalgebra over k and let V_1 and V_2 be super vector spaces over k of finite dimensions $p_1|q_1$ and $p_2|q_2$ respectively. Then*

$$\underline{\mathrm{Hom}}_{\mathbf{sVec}}(V_1, V_2) \otimes_k A \cong \underline{\mathrm{Hom}}_A(A^{p_1|q_1}, A^{p_2|q_2})\,. \tag{6.53}$$

Proof. Since k is a field, A and V_2 are free k-modules and hence A is flat and V_2 is finitely presented. It is easy to see that $A^{r|s} \cong k^{r|s} \otimes_k A$ for any r and s. $\square$

We remark that the situation in corollary 6.5.10 is very similar to the situation of lemma 3.7.7, but there is a difference. Corollary 6.5.10 just states that there is an isomorphism of A-modules. For the case of super vector spaces, it is not too hard this result directly, which we in fact did in section 3.7. Lemma 3.7.7 additionally gives an isomorphism of superalgebras.

Corollary 6.5.11. *Let A be a superring and $\mathfrak{p}$ a prime ideal in A. Then we have $\underline{\mathrm{Hom}}_A(M,N)_{\mathfrak{p}} \cong \underline{\mathrm{Hom}}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$.*

Proof. Using the fact that $A_{\mathfrak{p}}$ is a flat A-module, and that $M_{\mathfrak{p}} \cong M \otimes_A A_{\mathfrak{p}}$ the result follows immediately from lemma 6.2.7. $\square$

Corollary 6.5.12. *Let A be a Noetherian superring and M a finitely generated module. Then M is projective if and only if all localizations $M_{\mathfrak{p}}$ at prime ideals $\mathfrak{p}$ are free $A_{\mathfrak{p}}$-modules.*

Proof. We claim that if M is projective as an A-module, then $M_{\mathfrak{p}}$ is a projective $A_{\mathfrak{p}}$-module. Suppose $p : K \to L$ is a surjective morphism of $A_{\mathfrak{p}}$-modules and $f : M_{\mathfrak{p}} \to L$ any morphism of $A_{\mathfrak{p}}$-modules. Denote $i : M \to M_{\mathfrak{p}}$ the canonical morphism associated to the localization. Then K and L can also be viewed as A-modules, where the action of A goes via i. We thus find a morphism $\varphi : M \to K$ such that $p \circ \varphi = f \circ i$. For all elements $a \in A - \mathfrak{p}$ the linear homothety $l_a : K \to K$ along a is invertible and hence by the universal property of localization (see 5.1.14) there is a unique morphism $\psi : M_{\mathfrak{p}} \to K$ such that the following diagram commutes:

$$\begin{tikzcd}
K \arrow[r, "h"] & L \arrow[r] & 0 \\
M \arrow[u, "\varphi"] \arrow[r, "i"'] & M_{\mathfrak{p}} \arrow[u, "f"'] \arrow[ul, "\psi"] &
\end{tikzcd}\quad . \tag{6.54}$$

This proves that $M_{\mathfrak{p}}$ is projective. Since $A_{\mathfrak{p}}$ is local we know by lemma 6.4.6 that $M_{\mathfrak{p}}$ is a free module. Conversely, suppose that for all prime ideals $\mathfrak{p}$ the localized module $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module. If $f : K \to L$ is a surjective morphism of A-modules, then using the maps $\alpha_K : \underline{\mathrm{Hom}}_A(M,K) \otimes_A A_{\mathfrak{p}} \to \underline{\mathrm{Hom}}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, K_{\mathfrak{p}})$ and $\alpha_L : \underline{\mathrm{Hom}}_A(M,L) \otimes_A A_{\mathfrak{p}} \to \underline{\mathrm{Hom}}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, L_{\mathfrak{p}})$ defined in theorem 6.5.9 give rise to the following commutative diagram:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, K_{\mathfrak{p}}) \arrow[r, "(f_{\mathfrak{p}})^*"] & \underline{\mathrm{Hom}}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, L_{\mathfrak{p}}) \arrow[r] & 0 \\
\underline{\mathrm{Hom}}_A(M,K) \otimes_A A_{\mathfrak{p}} \arrow[u, "\alpha_K"] \arrow[r, "\check{f}_{\mathfrak{p}}"] & \underline{\mathrm{Hom}}_A(M,L) \otimes_A A_{\mathfrak{p}} \arrow[u, "\alpha_L"'] &
\end{tikzcd}\quad , \tag{6.55}$$

where $f_{\mathfrak{p}}$ is the induced morphism $K_{\mathfrak{p}} \to L_{\mathfrak{p}}$, $\check{f}_{\mathfrak{p}}$ sends $\frac{a}{s} \otimes \varphi \in \underline{\mathrm{Hom}}_A(M,K) \otimes_A A_{\mathfrak{p}}$ to $\frac{a}{s} \otimes f \circ \varphi$ and $(f_{\mathfrak{p}})^*$ is given by $(f_{\mathfrak{p}})^*(u) = f_{\mathfrak{p}} \circ u$ for $u \in \underline{\mathrm{Hom}}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, K_{\mathfrak{p}})$. The upper row of diagram (6.55) is exact since free modules are projective. The vertical arrows are isomorphisms, as $A_{\mathfrak{p}}$ is a flat A-module and M is finitely presented by proposition 6.5.6. Hence the morphism $\check{f}_{\mathfrak{p}}$ is surjective for all prime ideals. But then by lemma 5.1.21 the sequence

$$\underline{\mathrm{Hom}}_A(M,K) \longrightarrow \underline{\mathrm{Hom}}_A(M,L) \longrightarrow 0 \tag{6.56}$$

is exact. Hence M is projective. $\square$

Chapter 7

Dimension theory of superrings

In this chapter we discuss dimension theory of superrings. First we investigate the dimension of the Zariski tangent space $\mathfrak{m}/\mathfrak{m}^2$ for a local superring with maximal ideal $\mathfrak{m}$. Then we use Hilbert functions to give some more general results.

7.1 Dimension parameters

Definition 7.1.1. *Let A be a local Noetherian superring with maximal ideal $\mathfrak{m}$ and with canonical ideal $J = (A_{\bar{1}})$. We call the total dimension of A the minimal numbers of generators for $\mathfrak{m}$ and we write $T(A)$ for the total dimension. We call the bare dimension of A the minimal number of generators of $\bar{\mathfrak{m}}$ in $\bar{A}$ and we denote the bare dimension of A by $B(A)$. We call the odd dimension of A the minimal number of generators for J and we denote it by $O(A)$. We call the Krull dimension of A the maximal chain length of prime ideals in A, which is thus equivalent to the Krull dimension of $\bar{A}$ and we denote it by $K(A)$.*

An immediate consequence of the definition is:

Lemma 7.1.2. *We have $K(A) \leq B(A)$ and equality if and only if $\bar{A}$ is regular.*

In the present section we try to relate the dimension parameters defined in definition 7.1.1. For the remainder of the section we write

$$p|q = \dim_{A/\mathfrak{m}} \left(\mathfrak{m}/\mathfrak{m}^2\right). \tag{7.1}$$

Furthermore we fix a set of even elements $\{e_1, \ldots, e_p\}$ and a set of odd elements $\{\eta_1, \ldots, \eta_q\}$ such that the images of these elements in $\mathfrak{m}/\mathfrak{m}^2$ are a basis of $\mathfrak{m}/\mathfrak{m}^2$ over $A/\mathfrak{m}$. Using the lemma of Nakayama we know that the set $\{e_i, \eta_\alpha\}$ generate $\mathfrak{m}$ (also see the proof of lemma 6.4.6). We conclude that $p + q = T(A)$. We now claim that $e_i \bmod J \neq 0$; indeed, if $e_i \in J$ then as the e_i are even, we must have $e_i \in J^2 \subset \mathfrak{m}^2$. But that is impossible. Furthermore, the images of e_i in $\bar{A}$ span $\bar{\mathfrak{m}}$; indeed, if $x \in \bar{\mathfrak{m}}$, then $x = y \bmod J$ for some $y \in \mathfrak{m}_0$. We can write $y = \sum \lambda_i e_i$, where no η_α-terms appear as the η_α are in J. Hence $x = \sum \bar{\lambda}_i \bar{e}_i$. In other words, we have $p \geq B(A)$ and we have proved first half of the lemma:

Lemma 7.1.3. *We have $B(A) = p$.*

Proof. Write $r = B(A)$ and assume $f_1, \ldots, f_r$ are elements such that the images in $\bar{A}$ generate $\bar{\mathfrak{m}}$ minimally. We may assume $f_i \in \mathfrak{m}_{\bar{0}}$. Let $v \in \left(\mathfrak{m}/\mathfrak{m}^2\right)_{\bar{0}}$ and write $v = w \bmod \mathfrak{m}^2$ for some

$w \in \mathfrak{m}_{\bar{0}}$. Then there are $\lambda_i \in A$ such that $\bar{w} = \sum \bar{\lambda}_i \bar{f}_i$, and we may assume the λ_i to be even. Then $w - \sum \lambda_i f_i$ lies in $J \cap \mathfrak{m}_{\bar{0}}$. But $J \cap A_{\bar{0}} \subset (A_{\bar{1}})^2 \subset J^2 \subset \mathfrak{m}^2$ and hence $(w - \sum \lambda_i f_i) \bmod \mathfrak{m}^2 = 0$. Hence the f_i span $\left(\mathfrak{m}/\mathfrak{m}^2\right)_{\bar{0}}$ and thus $r \geq p$. □

As in the proof of lemma 7.1.3, let $f_1, \ldots, f_r$ be even elements such that the images in $\bar{A}$ generate $\bar{\mathfrak{m}}$ minimally. Further, let $\xi_1, \ldots, \xi_t$ be a set of homogeneous elements that generate J minimally; then we have $O(A) = t$. By proposition 3.3.5 we know that the ξ_α are odd. Now let $x \in \mathfrak{m}$, then we now that there are $\lambda_i \in A$ such that $x \equiv \sum \lambda_i f_i \bmod J$ and hence the set $\{f_i, \xi_\alpha\}$ generates $\mathfrak{m}$. It thus follows that $r + t \leq p + q = T(A)$, from which we conclude that $O(A) \geq q$. However, we even have:

Lemma 7.1.4. *We have $O(A) = q$.*

Proof. Write $s = O(A)$ and assume that $\theta_1, \ldots, \theta_s$ generate J minimally. We will prove that the $\theta_i \bmod \mathfrak{m}^2$ are linearly independent over $A/\mathfrak{m}$ and hence $O(A) = s \leq q = \dim(\mathfrak{m}/\mathfrak{m}^2)_{\bar{1}}$.

Suppose $\sum \alpha_i \theta_i \bmod \mathfrak{m}^2 = 0$ for some $\alpha_i \in R/\mathfrak{m}$, which are not all zero. Then there are $a_i \in A_{\bar{0}}$ such that $\sum a_i \theta_i \in \mathfrak{m}^2$ and not all a_i are in $\mathfrak{m}$. We may assume $a_1 \notin \mathfrak{m}$ and thus a_1 is invertible. It follows that $\theta_1 + \sum_{i\geq 2} b_i \theta_i \in \mathfrak{m}^2 \cap A_{\bar{1}} = \mathfrak{m}_{\bar{0}} A_{\bar{1}}$. This implies there are $\lambda_i \in \mathfrak{m}_{\bar{0}}$ such that

$$\theta_1 + \sum_{i\geq 2} b_i \theta_i = \sum_{i=1}^{s} \lambda_i \theta_i \,, \tag{7.2}$$

and hence we can write

$$(1 - \lambda_1)\theta_1 = \sum_{i=2}^{s} c_i \theta_i \,, \tag{7.3}$$

for some numbers c_i. But since $\lambda_1 \in \mathfrak{m}$, the element $1 - \lambda_1$ is invertible and we can express θ_1 in terms of the other θ_i, which is contradicting the assumption that the θ_i are a minimal generating set. □

Summarizing we have

Theorem 7.1.5. *Let A be a local superring with maximal ideal $\mathfrak{m}$ and denote $p|q = \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$, then $O(A) = q$, $B(A) = p$ and $T(A) = p + q$.*

As an aside, we propose the following definition of smoothness for superrings:

Definition 7.1.6. *Let A be a local superring with maximal ideal $\mathfrak{m}$ such that A contains a copy of $k = A/\mathfrak{m}$. Let $\dim_k \mathfrak{m}/\mathfrak{m}^2 = p|q$. Then we call A a regular local superring if the completion $\hat{A}$ of A with respect to the $\mathfrak{m}$-adic grading is isomorphic to the superring $k[[x_1, \ldots, x_p | \eta_1, \ldots, \eta_q]]$.*

An immediate consequence is that A is regular if and only if $\hat{A}$ is regular. Some other consequences are that any regular local superring is a super domain and that the body of a regular local superring is a regular local ring. For a general superring A we say that A is regular if all localizations at prime ideals are regular local superrings.

The above definition of smoothness was also used by Fioresi [59] to show that affine algebraic groups are smooth. In other words, any affine algebraic supergroup is a Lie supergroup, see for example [5, 8].

7.2 Hilbert functions

In this section we define Hilbert functions for superrings. We let A be a Noetherian $\mathbb{Z}$-graded superring $A = \bigoplus_{i\geq 0} A_i$. Call $A_+ = \bigoplus_{i\geq 0}$, then A_+ is a finitely generated ideal. Therefore, A is finitely generated as an A_0-superalgebra. Since $A_0 = A/A_+$ is a quotient of A, the superring A_0 is also Noetherian. We assume that A is generated as an A_0-superalgebra by the elements of A_1.

We consider the category $\mathcal{C}$ of finitely generated $\mathbb{Z}$-graded modules. Any such module M is bounded in degree below, that is, there is an integer d such that $M = \bigoplus_{k\geq d} M_k$. We claim that any summand M_d of M in $\mathcal{C}$ is a finitely generated A_0-module: If A is generated as an A_0-module by homogeneous generators $x_1, \dots, x_r$ and M is generated by homogeneous elements $m_1, \dots, m_s$, then M_d is generated as an A_0-module by all elements $m = \sum a_i m_i$, where a_i are monomials in the x_i of $\mathbb{Z}$-degree $d - \deg(m_i)$. But there are only finitely many such monomials, hence M_d is a finitely generated A_0-module. The morphisms in the category $\mathcal{C}$ are those A-module morphisms that preserve the $\mathbb{Z}$- and the $\mathbb{Z}_2$-grading. There is a natural functor T in this category that shifts the $\mathbb{Z}$-degree of each module M in $\mathcal{C}$: $(TM)_k = M_{k+1}$. We use the notation $M[i] = T^i M$.

Definition 7.2.1. *We call additive function, any function on the class of finitely generated of A_0-modules with values in $\mathbb{Z}$, such that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of finitely generated A_0-modules, then $\lambda(M') - \lambda(M) + \lambda(M'') = 0$.*

Some immediate consequences are: $\lambda(0) = 0$ and if $M \cong M'$ then $\lambda(M) = \lambda(M')$. To see the first claim: we note that $0 \to 0 \to 0 \to 0 \to 0$ is exact. Applying λ then gives $\lambda(0) = 0$. If $M \cong M'$, then $0 \to M \to M' \to 0 \to 0$ is a short exact sequence. Applying λ and using $\lambda(0) = 0$ gives $\lambda(M) = \lambda(M')$.

Given an additive function λ we introduce a formal Laurent series $H^\lambda_M(t)$ for each A-module M in $\mathcal{C}$ by the formula

$$H^\lambda_M(t) = \sum_{\mu\in\mathbb{Z}} \lambda(M_\mu) t^\mu \,. \tag{7.4}$$

The power series $H^\lambda_M(t)$ is bounded below by the assumptions on M. We call $H^\lambda_M(t)$ the Hilbert function of M with respect to the additive function λ. We write $K^\lambda_M(t)$ for the function $H^\lambda_M(t) + H^\lambda_{\Pi M}(t)$. The following lemma gives the most elementary properties of Hilbert functions.

Lemma 7.2.2.

(i) For any M in $\mathcal{C}$ we have $H^\lambda_{M[d]}(t) = t^{-d} H^\lambda_M(t)$.

(ii) Suppose we have a finite exact sequence of finitely generated A_0-modules

$$0 \longrightarrow M_1 \xrightarrow{f_1} \dots \longrightarrow M_{k-1} \xrightarrow{f_{k-1}} M_k \longrightarrow 0 \quad , \tag{7.5}$$

then we have

$$\sum_{i=1}^{k} (-1)^i \lambda(M_i) = 0 \,. \tag{7.6}$$

(iii) Let M be in $\mathcal{C}$ and let $\varphi : M[-d] \to M$ be a morphism, then we have

$$H^\lambda_M(t) - t^d H^\lambda_M(t) = H^\lambda_{\mathrm{Coker}\,(\varphi)}(t) - t^d H^\lambda_{\mathrm{Ker}\,(\varphi)}(t) \,. \tag{7.7}$$

Proof. The proof of (i) is standard and follows directly from the definitions. For (ii) we remark that for $1 \leq i \leq k$ all sequences

$$0 \longrightarrow \mathrm{Ker}\, f_i \longrightarrow M_i \longrightarrow \mathrm{Im}\, f_i \longrightarrow 0 \tag{7.8}$$

are exact. But since the sequence (7.5) is exact we have $\mathrm{Ker}\, f_i = \mathrm{Im}\, f_{i-1}$. Applying λ to all short exact sequences (7.8) and adding up the results we obtain equation (7.6). Then (*iii*) readily follows. □

We would like to know what the possible Hilbert functions are for a given additive function λ and a given A-module M in $\mathcal{C}$. The following proposition only uses the formal properties of the functors T and Π to reduce the possible Hilbert function to a rational function with at most three different poles.

Proposition 7.2.3. *Let A be a Noetherian $\mathbb{Z}$-graded superring that is graded as an A_0-superalgebra by A_1 and let M be a finitely generated $\mathbb{Z}$-graded A-module. Then the Hilbert function $H^\lambda_M(t)$ is of the form:*

$$H^\lambda_M(t) = t^d \frac{Q(t)}{(1-t)^m(1+t)^n}\,, \tag{7.9}$$

for some polynomial $Q(t) \in \mathbb{Z}[t]$ and some integers d, m, n, with m, n nonnegative.

Proof. We use induction on the number of generators of A as an A_0-superalgebra. If there are zero generators, the module M is just an A_0-module, and since M is finitely generated, we may assume that there is an integer e such that M_μ contains no generators for any $\mu \geq e$. Hence $M_\mu = 0$ for all $\mu \geq e$ and thus $H^\lambda_M(t)$ is polynomial times a power of t.

Assume that A is generated by r even elements in A_1 and s odd elements of A_1, and $r+s \geq 0$. Then we pick one generator x and consider the maps $\varphi : M \to M[1]$ given by multiplication with x; $\varphi(m) = xm$. We write N for the A-module M/xM whose dth graded component is $N_d = M_d/xM_{d-1}$. There are two cases to distinguish; x even or x odd. If x is even we have an exact sequence

$$0 \longrightarrow \mathrm{Ker}\,\varphi \longrightarrow M \longrightarrow M[1] \longrightarrow N[1] \longrightarrow 0 \quad . \tag{7.10}$$

Both the $\mathrm{Ker}\,\varphi$ and the cokernel N are finitely generated $A/(x)$-modules and we may apply the induction hypothesis to get:

$$(1-t)H^\lambda_M(t) = H^\lambda_N(t) - tH^\lambda_{\mathrm{Ker}\,\varphi}(t) = t^d \frac{Q(t)}{(1-t)^m(1-t)^n}\,. \tag{7.11}$$

In the case that x is odd, we get two exact sequences

$$\begin{array}{c} 0 \longrightarrow \Pi\mathrm{Ker}\,\varphi \longrightarrow \Pi M \longrightarrow M[1] \longrightarrow N[1] \longrightarrow 0 \\[1em] 0 \longrightarrow \mathrm{Ker}\,\varphi \longrightarrow M \longrightarrow \Pi M[1] \longrightarrow \Pi N[1] \longrightarrow 0 \end{array} \quad . \tag{7.12}$$

Consequently, we get two equations upon applying λ:

$$0 = \lambda((\Pi\mathrm{Ker}\,\varphi)_\mu) - \lambda((\Pi M)_\mu) + \lambda(M_{\mu+1}) - \lambda(N_{\mu+1}) \tag{7.13}$$

$$0 = \lambda((\mathrm{Ker}\,\varphi)_\mu) - \lambda(M_\mu) + \lambda((\Pi M)_{\mu+1}) - \lambda((\Pi N)_{\mu+1})\,. \tag{7.14}$$

After multiplying eqns.(7.13,7.14) with $t^{\mu+1}$ and adding up, we can use the induction hypothesis to obtain:

$$t^c \frac{Q_1(t)}{(1-t)^a(1+t)^b} = tH^\lambda_{\Pi M}(t) - H^\lambda_M(t) \qquad a, b, c \in \mathbb{Z}\,, \quad a, b \geq 0\,, \tag{7.15}$$

$$t^k \frac{Q_2(t)}{(1-t)^m(1+t)^n} = tH^\lambda_M(t) - H^\lambda_{\Pi M}(t) \qquad m, n, k \in \mathbb{Z}\,, \quad m, n \geq 0\,. \tag{7.16}$$

Solving the above system of equations finishes the proof. □

We have not yet used that a superring is super commutative, and it is this property that ensures that the Hilbert function is regular at $t = -1$.

Theorem 7.2.4. *Let A be a Noetherian $\mathbb{Z}$-graded superring that is generated as an A_0-superalgebra by A_1 and let M be a finitely generated $\mathbb{Z}$-graded A-module. Then the Hilbert function is a rational function $f(t)/g(t)$ with $f(t) \in \mathbb{Z}[t]$ and $g(t) = t^m(1-t)^n$ for some nonnegative integers m, n.*

Proof. Assume that A is generated as an A_0-superalgebra by even elements $x_1, \ldots, x_p \in A_{1,\bar{0}}$ and odd elements $\eta_1, \ldots, \eta_q \in A_{1,\bar{1}}$. We will do induction on the number p.

Assume first that $p = 0$, then $A = A_0[\eta_1, \ldots, \eta_q]$ and we have a finite decomposition $A = \bigoplus_{i=0}^q A_i$. Then M is also finitely generated as an A_0-module. Indeed, if M is generated by elements $m_1, \ldots, m_k$ with $\mathbb{Z}$-degrees $d_1, \ldots, d_k$ respectively, then M is generated over A_0 by all products $m_i u$ where u is a monomial in the η_α. Since there are only finitely many of such monomials u - their number being 2^q - we conclude that $M_e = 0$ for all $e \geq \max_i(d_i + q)$. In this case, the Hilbert function is just a finite sum of powers of t with coefficients in $\mathbb{Z}$.

If $p > 0$ we proceed as in the proof of proposition 7.2.3. If x is an even generator of A, consider the morphism $l_x : M \to M$ given by $l_x(m) = xm$. Then we have an exact sequence $0 \to K \to M \to M[1] \to N[1] \to 0$ of finitely generated A-modules, where $K = \mathrm{Ker}(l_x)$ and $N = M/xM$. We have the following relation between the Hilbert functions:

$$(1-t)H_M^\lambda(t) = H_N^\lambda(t) - tH_K^\lambda(t)\,. \tag{7.17}$$

Since K and N are finitely generated $A/(x)$-modules, we can apply the induction and we are done. ◻

Corollary 7.2.5. *Let $A = \bigoplus_d A_d$ be a Noetherian $\mathbb{Z}$-graded superring that is generated as an A_0-superalgebra by elements $x_1, \ldots, x_p \in A_{1,\bar{0}}$ and $\eta_1, \ldots, \eta_q \in A_{1,\bar{1}}$. Let M be a finitely generated $\mathbb{Z}$-graded A-module. Then for any additive function λ on the class of finitely generated A_0-modules, there is an integer N such that the $\lambda(M_\mu)$ are given by a polynomial of a degree less than or equal to p for $\mu \geq N$.*

Proof. By theorem 7.2.4 the Hilbert function of M with respect to λ is given by $H_M^\lambda(t) = \frac{f(t)}{t^m(1-t)^n}$ with $n \leq p$ and with $f(t)$ a polynomial with integer coefficients. We have the expansions

$$H_M^\lambda(t) = \sum_{\mu\in\mathbb{Z}} \lambda(M_\mu)t^\mu\,, \quad \frac{1}{(1-t)^n} = \sum_{k\geq 0}\binom{n-1+k}{n-1}t^k\,. \tag{7.18}$$

Let N be the degree of f and write $f(t) = \sum_0^N f_l t^l$, then for $\mu \geq N$ we have

$$\lambda(M_{\mu-m}) = \sum_{k=0}^N f_k\binom{n-1+N-k}{N-k}\,, \tag{7.19}$$

which is obviously polynomial in μ and of degree less than or equal to n. ◻

Remark 7.2.6. The reason we first proved proposition 7.2.3 and not theorem 7.2.4 right away is to clarify the role of the super-commutativity. If we change the setting to commutative rings with a $\mathbb{Z}\times\mathbb{Z}_2$-grading the Hilbert function can get poles at $t = -1$. Consider for example the commutative ring $R = k[x_1, \ldots, x_p, y_1, \ldots, y_q]$ over a field k generated by elements x_i and y_j with $1 \leq i \leq p$ and $1 \leq j \leq q$. We give the generators x_i the $\mathbb{Z}\times\mathbb{Z}_2$-grading $(0, \bar{0})$ and the generators y_j the $\mathbb{Z}\times\mathbb{Z}_2$-grading $(0, \bar{1})$. Thus $R_0 = k$, so that the category of $\mathbb{Z}_2$-graded R_0-modules is the category of super vector spaces and we consider the additive function $\lambda(V) = \dim_k(V_{\bar{0}})$. The dimension of $R_{n,\bar{0}}$ is given by the number of different monomials XY with $X = x_1^{a_1}\cdots x_p^{a_p}$ and $Y = y_1^{b_1}\cdots y_q^{b_q}$ such that $a = \sum_i a_i$ and $b = \sum_j b_j$ add up to n and such that 2 divides b. Hence we have

$$\dim_k(R_{n,\bar{0}}) = \sum_{a+b=n,2|b} \binom{p-1+a}{p-1}\binom{q-1+b}{b}. \tag{7.20}$$

It follows that

$$\sum_{n\geq 0} t^n \dim_k(R_{n,\bar{0}}) = \frac{1}{(1-t)^p}\Big(\frac{1}{2(1-t)^q} + \frac{1}{2(1+t)^q}\Big). \tag{7.21}$$

Another interesting case is obtained when we replace the $\mathbb{Z}_2$-grading by a $\mathbb{Z}_3$-grading. Consider the commutative polynomial ring $S = k[x_i, y_j, z_k]$ where $1 \leq i \leq p$, $1 \leq j \leq q$ and $1 \leq k \leq r$. We give all generators $\mathbb{Z}$-degree 1 and the x_i we give $\mathbb{Z}_3$-degree $0 \bmod 3$, the y_j we give $\mathbb{Z}_3$-degree $1 \bmod 3$ and the z_j we give $\mathbb{Z}_3$-degree $2 \bmod 3$. Now let $\omega \neq 1$ be a third root of unity, then

$$\begin{aligned}\sum_{n\geq 0} t^n \dim_k(S_{n,0 \bmod 3}) &= \frac{1}{3(1-t)^{p+q+r}} + \frac{1}{3(1-t)^p(1-\omega t)^q(1-\omega^2 t)^r}\\ &+ \frac{1}{3(1-t)^p(1-\omega^2 t)^q(1-\omega t)^r}.\end{aligned} \tag{7.22}$$

Example 7.2.7. Let A be $k[x_1,\ldots,x_p|\eta_1,\ldots,\eta_q]$. Consider the additive function $\lambda(M_\mu) = \mathrm{dim_k}(\mathrm{M}_{\mu,\bar{0}})$. There are $\binom{\mu+p-1}{p-1}$ monomials of the form $x_1^{m_1}\cdots x_p^{m_p}$ where $\sum_i m_i = \mu$, there are $\binom{\mu+p-2}{p-1}\binom{q}{1}$ binomials of the form $x_1^{m_1}\cdots x_p^{m_p}\eta_\alpha$ where $\sum_i m_i = \mu - 1$, and $\binom{\mu+p-3}{p-1}\binom{q}{2}$ binomials of the form $x_1^{m_1}\cdots x_p^{m_p}\eta_\alpha\eta_\beta$ where $\sum_i m_i = \mu-2$. Continuing in this way we see that

$$\lambda(A_\mu) + \lambda(\Pi A_\mu) = \sum_{\alpha+\beta=\mu} \binom{\alpha+p-1}{p-1}\binom{q}{\beta}. \tag{7.23}$$

The function $K_A^\lambda(t) = H_A^\lambda(t) + H_{\Pi A}^\lambda(t)$ is thus given by

$$K_A^\lambda(t) = \frac{(1+t)^q}{(1-t)^p}. \tag{7.24}$$

To calculate the dimension of the even part of A_μ we only should take those monomials with an even number of η_α's, hence

$$\lambda(A_{\mu,\bar{0}}) = \sum_{j\geq 0} \binom{\mu+p-1-2j}{p-1}\binom{q}{2j}. \tag{7.25}$$

which is a finite sum as $\binom{q}{2j}$ is zero if $2j > q$. We then obtain the following formula for the Hilbert function of A:

$$H_A^\lambda(t) = \frac{(1+t)^q + (1-t)^q}{2(1+t)^p}. \qquad \triangle \tag{7.26}$$

An even element $x \in A_{\bar{0}}$ is a nonzerodivisor of M if $xm = 0$ implies $m = 0$ for all $m \in M$. An l-tuple $(x_1,\ldots,x_l)$ in $(A_{\bar{0}})^l$ is an even M-regular sequence if x_1 is a nonzerodivisor of M and x_i is a nonzerodivisor of $M/(x_1M + \ldots + x_{i-1}M)$ for $2 \leq i \leq l$. We generalize these notions for odd elements as follows:

Definition 7.2.8. *We call an odd element $\eta \in A_{\bar{1}}$ an odd M-regular element if $\eta m = 0$ implies $m \in \eta M$. We call an n-tuple $(\eta_1, \dots, \eta_n) \in (A_{\bar{1}})^n$ an odd M-regular sequence if η_1 is an odd M-regular element is and η_i an odd $M^{(i)}$-regular element is for $2 \le i \le n$, where $M^{(i)} = M/(\eta_1 M + \dots + \eta_{i-1} M)$.*

Proposition 7.2.9. *Let A be a Noetherian $\mathbb{Z}$-graded superring generated as A_0-superalgebra by A_1 and let M be a finitely generated A-module.*

(i) If $x \in A_{1,\bar{0}}$ is a nonzerodivisor on M, then $(1-t)H^{\lambda}_M(t) = H^{\lambda}_{M/xM}(t)$.

(ii) If $\eta \in A_{1,\bar{1}}$ is an odd M-regular element, then $K^{\lambda}_M(t) = (1+t)K^{\lambda}_{M/\eta M}(t)$.

Proof. (i): Under the assumptions the kernel of the morphism $\varphi : M \to M$, $\varphi(m) = xm$, is zero and the exact sequence of eqn.(7.10) becomes

$$0 \longrightarrow M \longrightarrow M[1] \longrightarrow M/xM[1] \longrightarrow 0 \quad . \tag{7.27}$$

Hence $\lambda(M_n) - \lambda(M_{n+1}) + \lambda((M/xM)_{n+1}) = 0$ which gives rise to $(1-t)H^{\lambda}_M(t) = H^{\lambda}_{M/xM}(t)$.

(ii): Under the assumptions the following sequences are exact

$$\begin{array}{l}
0 \longrightarrow (\eta M)_n \longrightarrow M_n \xrightarrow{m \mapsto \eta m} \Pi M_{n+1} \longrightarrow \Pi(M/\eta M)_{n+1} \longrightarrow 0 \\[1em]
0 \longrightarrow (\Pi\eta M)_n \longrightarrow \Pi M_n \xrightarrow{m \mapsto \eta m} M_{n+1} \longrightarrow (M/\eta M)_{n+1} \longrightarrow 0 \\[1em]
0 \longrightarrow (\eta M)_n \longrightarrow M_n \xrightarrow{m \mapsto \eta m} (\eta \Pi M)_{n+1} \longrightarrow 0 \\[1em]
0 \longrightarrow (\Pi\eta M)_n \longrightarrow \Pi M_n \xrightarrow{m \mapsto \eta m} (\eta M)_{n+1} \longrightarrow 0
\end{array} \quad , \tag{7.28}$$

where in all cases the first map is the canonical injection. Applying λ, multiplying with t^{n+1} and adding up one obtains

$$\begin{aligned}
0 &= tH^{\lambda}_{\eta M}(t) - tH^{\lambda}_M(t) + H^{\lambda}_{\Pi M}(t) - H^{\lambda}_{\Pi(M/\eta M)}(t)\,, \\
0 &= tH^{\lambda}_{\eta\Pi M}(t) - tH^{\lambda}_{\Pi M}(t) + H^{\lambda}_M(t) - H^{\lambda}_{M/\eta M}(t)\,, \\
0 &= tH^{\lambda}_{\eta M}(t) - tH^{\lambda}_M(t) + H^{\lambda}_{\eta\Pi M}(t)\,, \\
0 &= tH^{\lambda}_{\eta\Pi M}(t) - tH^{\lambda}_{\Pi M}(t) + H^{\lambda}_{\eta M}(t)\,.
\end{aligned} \tag{7.29}$$

From the last two equations of eqn.(7.29) we obtain

$$\begin{aligned}
(1-t^2)H^{\lambda}_{\eta M}(t) &= tH^{\lambda}_{\Pi M}(t) - t^2 H^{\lambda}_M(t)\,, \\
(1-t^2)H^{\lambda}_{\eta\Pi M}(t) &= tH^{\lambda}_M(t) - t^2 H^{\lambda}_{\Pi M}(t)\,,
\end{aligned} \tag{7.30}$$

from which we get

$$(1+t)K^{\lambda}_{\eta M}(t) = tK^{\lambda}_M(t)\,. \tag{7.31}$$

From the first two equations of eqn.(7.29) we obtain

$$\begin{aligned}
(1-t^2)H^{\lambda}_{\Pi M}(t) &= tH^{\lambda}_{M/\eta M}(t) + H^{\lambda}_{\Pi(M/\eta M)}(t) - tH^{\lambda}_{\eta M}(t) - t^2 H^{\lambda}_{\eta\Pi M}(t)\,, \\
(1-t^2)H^{\lambda}_M(t) &= tH^{\lambda}_{\Pi(M/\eta M)}(t) + H^{\lambda}_{M/\eta M}(t) - tH^{\lambda}_{\eta\Pi M}(t) - t^2 H^{\lambda}_{\eta M}(t)\,,
\end{aligned} \tag{7.32}$$

and thus we get

$$(1-t)K^{\lambda}_M(t) = K^{\lambda}_{M/\eta M}(t) - tK^{\lambda}_{\eta M}(t)\,. \tag{7.33}$$

Combining the eqns.(7.31,7.33) we get $K^{\lambda}_M(t) = (1+t)K^{\lambda}_{M/\eta M}(t)$. $\square$

Corollary 7.2.10. *Let M, A be as before and suppose that there exists an odd M-regular sequence $(\eta_1, \ldots, \eta_s)$ with all $\eta_i \in A_{1,\bar{1}}$, then $K_M^\lambda(t)$ has a zero at $t = -1$ of order at least s. If $(x_1, \ldots, x_l)$ is an even M-regular sequence, then $(1-t)^l H_M^\lambda(t) = H_{M/N}^\lambda(t)$ for $N = x_1M + \ldots + x_lM$.*

Proof. We use inductively $K_M^\lambda(t) = (1+t)^i K_{M^{(i)}}^\lambda(t)$ where $M^{(0)} = M$ and $M^{(i)} = M^{(i-1)}/\eta_i M^{(i-1)}$ so that for $i > 1$ we have $M^{(i)} = M/(\eta_1 M + \ldots \eta_{i-1}M)$. The second statement is trivial. ☐

7.3 Application to local superrings

Let $(A, \mathfrak{m})$ be a Noetherian local superring, let $\mathfrak{q}$ be $\mathfrak{m}$-primary and write $k = A/\mathfrak{m}$ for the residue field of A. Let M be a finitely generated A-module with a $\mathfrak{q}$-stable filtration $M = M_0 \supset M_1 \supset M_2 \supset \ldots$ (also see section 5.7). The superring A has a natural $\mathfrak{q}$-stable filtration $A = \mathfrak{q}^0 \supset \mathfrak{q}^1 \supset \mathfrak{q}^2 \supset \ldots$ Let $\mathrm{gr}(A) = \bigoplus_{k\geq 0} \mathfrak{q}^l/\mathfrak{q}^{l+1}$ and $\mathrm{gr}(M) = \bigoplus_{k\geq 0} M_l/M_{l+1}$ be the associated graded superring and associated graded module respectively.

Lemma 7.3.1. *Let $(A, \mathfrak{m})$ be a Noetherian local superring, and let $\mathfrak{q}$ be $\mathfrak{m}$-primary. Then $A/\mathfrak{q}$ is an Artinian superring.*

Proof. Since $\mathfrak{m}$ is finitely generated and $\mathfrak{q}$ is $\mathfrak{m}$-primary, there is an integer l such that $\mathfrak{m}^l \subset \mathfrak{q}$. Clearly then $\mathfrak{m}^l \subset \mathfrak{q} \subset \mathfrak{m}$. Since

$$\mathrm{tdim}_k(\mathfrak{m}/\mathfrak{m}^{l+1}) = \sum_{i=1}^{l-1} \mathrm{tdim}_k \mathfrak{m}^i/\mathfrak{m}^{i+1}\,,$$

where tdim is the total dimension (sum of dimension of even part and of odd part), and each of the total dimensions on the right-hand side is finite, we see that the vector space $\mathfrak{m}/\mathfrak{m}^l$ is finite-dimensional over $k = A/\mathfrak{m}$. Therefore $\mathfrak{m}/\mathfrak{q}$ is finite-dimensional over k and $\mathrm{tdim}_k(A/\mathfrak{q}) = \mathrm{tdim}_k(A/\mathfrak{m}) + \mathrm{tdim}_k \mathfrak{m}/\mathfrak{q}$ is finite. Thus $A/\mathfrak{q}$ is a finite-dimensional superring over k, hence Artinian. ☐

For a fixed A-module M with a $\mathfrak{q}$-stable filtration $\{M_i\}$ we write $l_n(M)$ for the length of $(M/M_n)_{\bar{0}}$ when viewed as an $(A/\mathfrak{q})_{\bar{0}}$-module. For convenience we write $B = (A/\mathfrak{q})_{\bar{0}}$. We write $l_B(M)$ for the length of a B-module. Thus $l_n(M) = l_B(M/M_n)$.

Lemma 7.3.2. *Let A be a local superring with maximal ideal $\mathfrak{m}$ and let $\mathfrak{q}$ be $\mathfrak{m}$-primary. If M is a finitely generated A-module with a $\mathfrak{q}$-stable filtration, then $l_n(M) < \infty$.*

Proof. Let $B = (A/\mathfrak{q})_{\bar{0}}$. Since each M_n is a finitely generated A-module and $\mathfrak{q}M_n \subset M_{n+1}$, each M_n/M_{n+1} is a finitely generated $A/\mathfrak{q}$-module. Therefore $(M_n/M_{n+1})_{\bar{0}}$ is finitely generated B-module. Since A is Noetherian, J_A is finitely generated and thus M_n/M_{n+1} is a finitely generated B-module. Since B is Artinian by lemma 7.3.1, the B-module M_n/M_{n+1} is an Artinian module.

Since $A/\mathfrak{q}$ is Noetherian, so is its even part B by proposition 3.3.6. Hence M_n/M_{n+1} has finite length as an B-module by theorem 3.4.9. In [16] it is shown that the length is additive on the class of all finite length modules of an Artinian commutative ring. Hence the length $l_B(M/M_n)$ is given by the sum $\sum_{i=1}^n l_B(M_i/M_{i+1})$. This shows that $l_B(M/N_n)$ is finite. But $(M/M_n)_{\bar{0}}$ is a submodule of M/M_n and thus also has finite length. Thus $l_n(M) = l_B((M/M_n)_{\bar{0}})$ is finite. ☐

The following lemma assures that assigning a finitely generated $A/\mathfrak{q}$-module M the value of the length of $M_{\bar{0}}$ as a B-module gives rise to an additive function on the class of finitely generated $A/\mathfrak{q}$-modules.

Lemma 7.3.3. *Let A be a Noetherian and Artinian superring, the function that assigns to each finitely generated A-module M the length of $M_{\bar{0}}$ as an $A_{\bar{0}}$-module, is additive on the class of finitely generated A-modules.*

Proof. Clearly, as the canonical ideal of A is finitely generated, A is a finite $A_{\bar{0}}$-module. Hence any finitely generated A-module is finitely generated as an $A_{\bar{0}}$-module. And thus any finitely generated A-module is an Artinian $A_{\bar{0}}$-module.

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-module maps, with all morphisms preserving the $\mathbb{Z}$-grading. Then we can restrict to the even parts and thus $0 \to M'_{\bar{0}} \to M_{\bar{0}} \to M''_{\bar{0}} \to 0$ is an exact sequence of $A_{\bar{0}}$-modules. Applying proposition 6.9 of [16] gives the required result. □

Lemma 7.3.4. *Let M be a finitely generated A-module with a $\mathfrak{q}$-stable filtration $M = M_0 \supset M_1 \supset M_2 \supset \dots$. Then there is a polynomial $f(t) \in \mathbb{Z}[t]$ such that $l_B(M/M_n) = f(n)$ for n large enough.*

Proof. The $\mathbb{Z}$-graded superring $\mathrm{gr}(A)$ is generated by the elements of $\mathbb{Z}$-degree 1. By corollary 7.2.5 there is a polynomial $g(t) \in \mathbb{Z}[t]$ such that for large enough n we have

$$g(n) = l_B(M_n/M_{n+1})\,. \tag{7.34}$$

The following sequence is exact

$$0 \longrightarrow M_n/M_{n+1} \longrightarrow M/M_{n+1} \longrightarrow M/M_n \longrightarrow 0 \quad , \tag{7.35}$$

as a sequence of B-modules. Hence

$$l_B(M/M_{n+1}) = l_B(M/M_n) + l_B(M/M_n)\,. \tag{7.36}$$

By induction we then see that there is a polynomial $f(t)$ such that $f(n) = l_n(M)$ for large enough n. □

Lemma 7.3.5. *Let M be a finitely generated A-module with a $\mathfrak{q}$-stable filtration $M = M_0 \supset M_1 \supset M_2 \supset \dots$. Let $f(t) \in \mathbb{Z}$ be such that for large enough n we have $f(n) = l_n(M)$. The degree and the leading coefficient of f are independent of the $\mathfrak{q}$-stable filtration on M.*

Proof. Let $M = \tilde{M}_0 \supset \tilde{M}_1 \supset \tilde{M}_2 \supset \dots$ be another $\mathfrak{q}$-stable filtration on M and let $\tilde{f}(t) \in \mathbb{Z}[t]$ be such that $\tilde{f}(n) = l_B(M/M_n)$ for large enough n. As both filtrations are $\mathfrak{q}$-stable, there is an integer N such that $M_{k+1} = \mathfrak{q}M_k$ and $\tilde{M}_{k+1} = \mathfrak{q}\tilde{M}_k$ for all $k \geq N$. It follows that $M_{N+k} = \mathfrak{q}^k M_N \subset \mathfrak{q}^k M_0 \subset \tilde{M}_k$, and similarly $\tilde{M}_{N+k} \subset M_k$ for all $k \geq 0$. Hence $l_B(M/M_{N+k}) \geq l_B(M/\tilde{M}_k)$ and $l_B(M/\tilde{M}_{N+k}) \geq l_B(M/M_k)$ for all $k \geq 0$. It follows that

$$1 \leq \frac{l_B(M/\tilde{M}_{N+k})}{l_B(M/M_k)} \leq \frac{l_B(M/M_{2N+k})}{l_B(M/M_k)}\,. \tag{7.37}$$

Taking the limit $k \to \infty$ the right-hand side of eqn.(7.37) goes tends to 1 as $l_B(M/M_k)$ becomes polynomial in k. Hence also the middle term of eqn.(7.37) tends to 1, which can only happen of $\tilde{f}(t)$ and $f(t)$ have the same leading coefficient and the same degree. □

For any finitely generated A-module M with a $\mathfrak{q}$-stable filtration $M = M_0 \supset M_1 \supset \dots$ we define the characteristic function $\chi_{\mathfrak{q}}^M(t) \in \mathbb{Z}[t]$ to be that polynomial for which $\chi_{\mathfrak{q}}^M(n) = l_n(M)$ for n large enough. We write $\chi_{\mathfrak{q}}^A(t) = \chi_{\mathfrak{q}}(t)$ and $\chi_{\mathfrak{q}}^{\Pi A}(t) = \check{\chi}_{\mathfrak{q}}(t)$.

Lemma 7.3.6. *Let A be a local superring with maximal ideal $\mathfrak{m}$ and $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal. Then the degrees of the characteristic functions $\chi_{\mathfrak{m}}(t)$ and $\chi_{\mathfrak{q}}(t)$ are the same. Similarly, the degrees of $\check{\chi}_{\mathfrak{q}}(t)$ and $\check{\chi}_{\mathfrak{m}}(t)$ are the same.*

Proof. We have $\mathfrak{m}^r \subset \mathfrak{q} \subset \mathfrak{m}$ for some r. Hence $\mathfrak{m}^{rn} \subset \mathfrak{q}^n \subset \mathfrak{m}^n$ for all n. Then $\chi_m(n) \leq \chi_{\mathfrak{q}}(n) \leq \chi_{\mathfrak{m}}(nr)$ and similarly $\check{\chi}_m(n) \leq \check{\chi}_{\mathfrak{q}}(n) \leq \check{\chi}_{\mathfrak{m}}(nr)$; taking $n \to \infty$ proves the lemma. □

Proposition 7.3.7. *Let $(A, \mathfrak{m})$ be a local regular superring with residue field k. Suppose* $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = p|q$ *and that A contains a field. Then $\chi_{\mathfrak{m}}(t)$ has degree p.*

Proof. Let $\hat{A}$ be the completion with respect to the $\mathfrak{m}$-adic filtration, and let $\hat{m}$ be the maximal ideal of $\hat{A}$. Then by proposition 5.5.1 we have $\hat{A}/\hat{\mathfrak{m}}^k \cong A/\mathfrak{m}^k$ for all k. Hence we may replace A by $\hat{A}$. Since A is regular and contains a field, we have by theorem 5.12.4

$$\mathrm{gr}(A) = k[x_1, \ldots, x_p|\eta_1, \ldots, \eta_q]\,, \tag{7.38}$$

where $\mathrm{gr}(A)$ is the associated $\mathbb{Z}$-graded superring to the filtration $A = \mathfrak{m}^0 \supset \mathfrak{m}^1 \supset \mathfrak{m}^2 \supset \ldots$. By example 7.2.7 we have

$$\dim_k(\mathfrak{m}^l/\mathfrak{m}^{l+1})_{\bar{0}} = \sum_{a+b=l, 2|b} \binom{p-1+a}{p-1}\binom{q}{b}\,. \tag{7.39}$$

Any summand of eqn.(7.39) is of the form

$$\frac{((l-2j)+1)((l-2j)+2)\cdots((l-2j)+p-1)}{(p-1)!}\binom{q}{2j} = \frac{l^{p-1}}{(p-1)!}\binom{q}{2j} + O_{p-1}(l)\,, \tag{7.40}$$

for some j with $0 \leq 2j \leq l$ and where $O_r(l)$ stands for a polynomial in l of degree less than r. Noting that $\sum_{2|j}\binom{q}{2j} = 2^{q-1}$ we obtain the following expression:

$$\dim_k(\mathfrak{m}^l/\mathfrak{m}^{l+1})_{\bar{0}} = \frac{2^{q-1}l^{p-1}}{(p-1)!} + O_{p-1}(l)\,. \tag{7.41}$$

But then

$$\dim_k(A/\mathfrak{m}^l)_{\bar{0}} = \frac{2^{q-1}l^p}{p!} + O_p(l)\,. \tag{7.42}$$

□

The proof of proposition 7.3.7 shows that the degree of $\check{\chi}_{\mathfrak{m}}(t)$ also equals p if A is a regular local ring with $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = p|q$ and A contains a field. The odd dimension q can be read off from the leading term. This is related to the fact that $\hat{A}$ is as a module over the commutative ring $R = k[x_1, \ldots, x_p]$ isomorphic to a free module with 2^q copies of R.

Chapter 8

Algebraic supergroups and super Hopf algebras

The goal of this chapter is to introduce algebraic supergroups and discuss their relation with super Hopf algebras. The first part of this section is devoted to introduce the notion of a super Hopf algebra. The discussion is parallel to the presentation in standard textbooks such as [20–22], with perhaps the main difference that here care is taken that all notions respect the $\mathbb{Z}_2$-grading. We first recall some basics of linear algebra for infinite dimensional super vector spaces over a fixed base field k. We note that in this chapter not all algebras are commutative, and hence we need to distinguish on occasion between left and right ideals.

8.1 Linear algebra

With $\mathrm{Sgn}(a)$ we mean $(-1)^a$. We fix a ground field k. For two super vector spaces A, B, the set of k-linear maps $A \to B$ forms a super vector space $\underline{\mathrm{Hom}}_{\mathbf{sVec}}(A, B)$. If A, B, C, D are super vector spaces and $f : A \to C$ and $g : B \to D$ are linear maps, then $f \otimes g : A \otimes B \to C \otimes D$ is given by $f \otimes g(x \otimes y) = \mathrm{Sgn}(|x||g|)f(x) \otimes g(y)$.

Given a super vector space V we denote $V^* = \underline{\mathrm{Hom}}_{\mathbf{sVec}}(V, k)$ the dual space, consisting of all k-linear maps $V \to k$, where k is interpreted as the super vector space $k = k^{1|0}$. V^* is in a natural way $\mathbb{Z}_2$-graded. Since all super vector spaces are free modules, any super vector space is projective. Any super vector space admits a homogeneous basis, since both the even part and the odd part admit a (Hamel) basis. If V, W are super vector spaces and $V \subset W$, then since W/V and V admit a homogeneous basis, there exists a homogeneous basis of W containing a homogeneous basis of V. This shows that any super vector space is injective, and for any exact sequence of super vector spaces $0 \to U \to V \to W \to 0$ the sequence $0 \to W^* \to V^* \to U^* \to 0$ is exact. Often we will use an asterisk and write $v^*, w^*, \ldots$ for elements of V^*.

For a sub super vector space $X \subset V$ we denote $X^\perp$ the set of elements $w \in V^*$ such that $w(x) = 0$ for all $x \in X$. If X is $\mathbb{Z}_2$-graded,then $X^\perp$ is $\mathbb{Z}_2$-graded. For a sub super vector space $Y \subset V^*$ we denote $Y^\perp$ the set of all vectors v in V such that $y(v) = 0$ for all $y \in Y$. When Y is $\mathbb{Z}_2$-graded, then $Y^\perp$ is $\mathbb{Z}_2$-graded. If $X \subset V$ is a sub super vector space, then one easily shows that $(V/X)^* \cong X^\perp$. This observation gives the following lemma:

Lemma 8.1.1. *Let V be a super vector space and let X be a sub super vector space. Then $X^{\perp\perp} = X$.*

Proof. Clearly $X \subset X^{\perp\perp}$. The lemma follows if we can show that if $v \notin X$, then $v \notin X^{\perp\perp}$. Suppose $v \notin X$, then $v \bmod X \neq 0$. Hence there exists $w \in (V/X)^*$ such that $w(v \bmod X) \neq 0$ and hence composing w with the projection $V \to V/X$ defines a nonzero element w' in $X^\perp$ such that $w'(v) \neq 0$. $\square$

We say a subspace $Y \subset V^*$ is dense if $Y^\perp = 0$.

Lemma 8.1.2. *Let V be a super vector space, then the canonical morphism $d : V \to V^{**}$ given by $d(v)(v^*) = (-1)^{|v||v^*|}v^*(v)$, for all $v \in V$ and $v^* \in V^*$, is injective and $d(V)$ is dense in V^{**}, that is, $d(V)^\perp = 0$.*

Proof. Suppose that $d(v) = 0$ for some homogeneous v, then $v^*(v) = 0$ for all $v^* \in V^*$. Let U be a complement in V to $k \cdot v$ such that $V = k \cdot v \oplus U$ (this can always be achieved, since the super vector space $V/k \cdot v$ admits a basis). Take any nonzero $\lambda \in k^* \cong (k \cdot v)^*$ and $p : V \to V/U$ the canonical projection. Then we can define $v^* = \lambda \circ p \in V^*$. As $\lambda(v \bmod U) = 0$, we need $v \in U$, which forces $v \in k \cdot v \cap U = 0$. Hence d is injective.

For the second claim we compute

$$\begin{aligned} d(V)^\perp &= \{v^* \in V^* \mid d(v)(v^*) = 0\,, \quad \forall v \in V\} \\ &= \{v^* \in V^* \mid v^*(v) = 0\,, \quad \forall v \in V\} \\ &= 0\,. \end{aligned}$$

$\square$

Lemma 8.1.3. *Let V be a super vector space. Let X be a finite-dimensional sub super vector space of V^*. Then the morphism $V \to X^*$ induced by the inclusion $X \to V^*$, is an epimorphism.*

Proof. Let $d : V \to V^{**}$ be the inclusion of lemma 8.1.2, $i : X \to V^*$ be the inclusion of X in V^* and let $p : V^{**} \to X^*$ be the projection given by $p(w)(x) = w(i(x))$ for all $w \in V^{**}, x \in X$. Consider the composite map $\varphi = p \circ d : V \to V^{**} \to X^*$. Then $\varphi(v)(x) = \mathrm{Sgn}(|x||v|)x(v)$ for all $x \in X$ and $v \in V$. Decompose X^* as $X^* = \varphi(V) \oplus Z$. Consider the subspace $\varphi(V)^\perp = \{x \in X \mid x(v) = 0\,, \ \forall v \in V\} \subset X$. Then $i(\varphi(V)^\perp)$ lies in $d(V)^\perp$. Hence $\varphi(V)^\perp = 0$, but $\dim_k \varphi(V)^\perp = \dim_k Z$. Hence $\varphi(V) = X^*$. $\square$

Lemma 8.1.4. *Let V and W be super vector spaces, then the morphism $e : V^* \otimes W^* \to (V \otimes W)^*$ given by $e(v^* \otimes w^*)(v \otimes w) = (-1)^{|v||w^*|}v^*(v)w^*(w)$ is injective and the image is dense.*

Proof. Suppose $u \in V^* \otimes W^*$. Then by definition of the tensor product, u is a finite sum $\sum_i v_i^* \otimes w_i^*$ with $v_i^* \in V^*$ and $w_i^* \in W^*$ and we may assume that the v_i^* and w_i^* are homogeneous.

Take $X \subset V^*$ to be the span of the v_i^* and $Y \subset W^*$ to be the span of the w_i^*. The inclusion $X \to V^*$ induces an epimorphism $V \to X^*$ by lemma 8.1.3. Similarly, we have an epimorphism $W \to Y^*$. Take $v \in V$ such that $d(v)(v_1^*) = 1$ and $d(v)(v_i^*) = 0$ for $i > 1$, where $d : V \to V^{**}$ is the inclusion of lemma 8.1.2. Similarly, find $w \in W$ such that $d'(w)(w_1^*) = 1$ and $d(w)(w_i^*) = 0$ for $i > 1$, where $d' : W \to W^{**}$ is the inclusion of lemma 8.1.2. Then $u(v \otimes w) = 1$, hence $e(u) \neq 0$.

For the second claim, suppose $x \in (e(V^* \otimes W^*))^\perp$. Write $x = \sum_i v_i \otimes w_i$, which is a finite sum, and suppose the w_i are linearly independent. Call Y the span of the w_i, then Y is finite-dimensional and the inclusion $Y \to W$ induces an epimorphism $W^* \to Y^*$. Thus we can choose $w^* \in W^*$, such that $w^*(w_1) = 1$ and $w^*(w_i) = 0$ for $i \geq 2$. But then for all $v^* \in V^*$ we have $v^* \otimes w^*(x) = v^*(v_1) = 0$. Hence $v_1 = 0$ and choosing different w we see that $x = 0$. $\square$

In the proof of lemma 8.1.4 we have seen some important techniques how to deal with infinite-dimensional super vector spaces. A corollary to the proof is that if X is a finite-dimensional sub super vector space in V^*, then $X^* \cong V/X^\perp$.

Lemma 8.1.5. *Let V_1, V_2, W_1, W_2 be super vector spaces and $f_1 : V_1 \to W_1$, $f_2 : V_2 \to W_2$ two homogeneous linear maps, then $f_1 \otimes f_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ given by $f_1 \otimes f_2(v \otimes v') = (-1)^{|f_2||v|} f_1(v) \otimes f_2(v')$ has $\mathbb{Z}_2$-grading $|f_1 \otimes f_2| = |f_1| + |f_2|$ and* $\mathrm{Ker}\,(f_1 \otimes f_2) = \mathrm{Ker}\,(f_1) \otimes V_2 + V_1 \otimes \mathrm{Ker}\,(f_2)$.

Proof. The inclusion $\mathrm{Ker}\,(f_1) \otimes V_2 + V_1 \otimes \mathrm{Ker}\,(f_2) \subset \mathrm{Ker}\,(f_1 \otimes f_2)$ is obvious. For the converse, suppose $\sum v_{1,i} \otimes v_{2,i}$ is in the kernel of $\mathrm{Ker}\,(f_1 \otimes f_2)$. We may assume that the images of $v_{1,i}$ in $V_1/\mathrm{Ker}\,(f_1)$ are linearly independent over k (this perhaps needs some substraction of an element in $\mathrm{Ker}\,(f_1) \otimes V_2$). Then the elements $f(v_{1,i})$ are linearly independent in W_1 and then from $\sum_i f_1(v_{1,i}) \otimes f_2(v_{2,i}) = 0$ it follows that $v_{2,i} \in \mathrm{Ker}\,(f_2)$ for all i. □

Lemma 8.1.6. *Let V, W be super vector spaces and $X \subset V^*$ and $Y \subset W^*$ sub super vector spaces in the dual spaces. Then $(X \otimes Y)^\perp = X^\perp \otimes W + V \otimes Y^\perp$.*

Proof. Define $r : V \to X^*$ and $s : W \to Y^*$ by $r(v)(x) = (-1)^{|x||v|} x(v)$ and $s(w)(y) = (-1)^{|y||w|} y(w)$. Then $X^\perp = \mathrm{Ker}\,(r)$ and $Y^\perp = \mathrm{Ker}\,(s)$ and since r, s are even maps, the kernels are $\mathbb{Z}_2$-graded. The morphism $f : V \otimes W \to (X \otimes Y)^*$ factors over $r \otimes s : V \otimes W \to X^* \otimes Y^*$ as $f = i \circ r \otimes s$, where i is the injective map $i : X^* \otimes Y^* \to (X \otimes Y)^*$. We have $(X \otimes Y)^\perp = \mathrm{Ker}\,(f) = \mathrm{Ker}\,(r \otimes s)$ and thus by lemma 8.1.5 we have $(X \otimes Y)^\perp = X^\perp \otimes W + V \otimes Y^\perp$. □

8.2 Super coalgebras

A super coalgebra is a super vector space C over k together with morphisms of super vector spaces $\Delta : C \to C \otimes C$ and $\epsilon : C \to k$ that satisfy:

$$\mathrm{id} \otimes \epsilon \circ \Delta = \epsilon \otimes \mathrm{id} \circ \Delta = \mathrm{id}\,, \tag{8.1}$$

where we identify $k \otimes C \cong C \otimes k \cong C$. The map Δ is called the coproduct, or comultiplication, and ϵ is called the counit. We always assume that a super coalgebra is co-associative, which means: $\Delta \otimes \mathrm{id} \circ \Delta = \mathrm{id} \otimes \Delta \circ \Delta$. The properties of ϵ and Δ required by eqn.(8.1) and the co-associativity, can be summarized by saying that the following diagrams commute

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta\otimes\mathrm{id}"] \\ C \otimes C \arrow[r, "\mathrm{id}\otimes\Delta"'] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} C \otimes C \arrow[r, "\mathrm{id}\otimes\epsilon"] & C & C \arrow[l, "\epsilon\otimes\mathrm{id}"'] \\ & C \arrow[ul, "\Delta"] \arrow[u, "\mathrm{id}"] \arrow[ur, "\Delta"'] & \end{tikzcd} \quad . \tag{8.2}$$

We will often use Sweedler notation, where one writes $\Delta(c) = \sum c' \otimes c''$, and we will often omit the summation sign when we use Sweedler notation. It is important to note that the c' and c'' that appear in $\Delta(c) = c' \otimes c''$ are not unique. In Sweedler notation equation (8.1) reads as: $c = c'\epsilon(c'') = \epsilon(c')c''$, for all $c \in C$. For more on Sweedler notation we refer to for example [20, 22].

Example 8.2.1. Consider the superalgebra A of endomorphisms of $k^{p|q}$. We can identify A with the superalgebra of $(p+q) \times (p+q)$-matrices with entries in k, where $A_{\bar{0}}$ consists of all matrices of the form

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \tag{8.3}$$

where X is a $p \times p$-matrix and Y a $q \times q$-matrix. The odd part $A_{\bar{1}}$ consists of all matrices of the form

$$\begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix}, \tag{8.4}$$

where U is a $p \times q$-matrix and V a $q \times p$-matrix. Consider now the super vector space $C = A^* = \underline{\mathrm{Hom}}_{\mathbf{sVec}}(A, k)$. Then C has a basis of elements X_{ij}, where for any $x \in A$ we define $X_{ij}(x)$ to be the (i,j)-entry of the matrix of x. The basis element X_{ij} is even when $1 \leq i, j \leq p$ or when $p+1 \leq i, j \leq p+q$ and odd otherwise. The matrix multiplication $A \otimes A \to A$ induces a map $\Delta : C \to C \otimes C$ defined by $\Delta(\alpha)(x \otimes y) = \alpha(xy)$ for all $\alpha \in C$ and $x, y \in A$. For the basis elements X_{ij} one finds

$$\Delta : X_{ij} \mapsto \sum_{k=1}^{p+q} X_{ik} \otimes X_{kj}\,. \tag{8.5}$$

Furthermore, we have a map $\epsilon : C \to k$ defined $\epsilon(\alpha) = \alpha(1)$, which for the basis elements is given by $\epsilon(X_{ij}) = 0$ if $i \neq j$ and $\epsilon(X_{ij}) = 1$ when $i = j$. One easily verifies that C is a super coalgebra with comultiplication Δ and counit ϵ. $\triangle$

Definition 8.2.2. *Let C be a super coalgebra. We say that a sub super vector space V of C is a if $\Delta V \subset V \otimes C + C \otimes V$ and $\epsilon(V) = 0$. We call a sub super vector space of C a sub super coalgebra if $\Delta V \subset V \otimes V$ (the counit works automatically).*

Remark 8.2.3. For superalgebras the even part is a subalgebra. For super coalgebras this need not be the case since $\Delta(C_{\bar{0}}) \subset C_{\bar{0}} \otimes C_{\bar{0}} + C_{\bar{1}} \otimes C_{\bar{1}}$. On the other hand, every coalgebra can be seen as a super coalgebra with trivial odd part.

Definition 8.2.4. *Let C, D be super coalgebras with comultiplications Δ_C and Δ_D respectively and counits ϵ_C and ϵ_D respectively. A morphism of super vector spaces $f : C \to D$ is a morphism of super coalgebras if $\Delta_D \circ f = f \otimes f \circ \Delta_C$ and $\epsilon_D \circ f = \epsilon_C$.*

We will often omit the subscripts on the symbols for comultiplication and counit and simply write Δ for Δ_C, ϵ for ϵ_C, etcetera, when a clear reading is not at risk.

The sum of two sub super coalgebras is again a sub super coalgebra. One easily checks that with the given definitions, if C is a sub super coalgebra of D, then the inclusion $C \to D$ is a morphism of super coalgebras. Note that lemma 8.1.5 implies that $(C \bmod V) \otimes (C \bmod V) \cong C \otimes C \bmod (C \otimes V + V \otimes C)$. This observation proves the following lemma:

Lemma 8.2.5. *If C is a super coalgebra and V a coideal, then C/V becomes in a natural way a super coalgebra with super coalgebra structure maps $\bar{\Delta}$ and $\bar{\epsilon}$ defined by: $\bar{\Delta}(c \bmod V) = (\Delta c) \bmod (C \otimes V + V \otimes C) = \sum c' \bmod V \otimes c'' \bmod V$ and $\bar{\epsilon}(c \bmod V) = \epsilon(c)$. In particular, the projection $C \to C/V$ is a morphism of super coalgebras.*

Proposition 8.2.6. *Let C be a super coalgebra. The odd part $C_{\bar{1}}$ is a coideal and we can make $C_{\bar{0}}$ into a coalgebra.*

Proof. Note that as super vector spaces $C_{\bar{0}} \cong C/C_{\bar{1}}$. Hence the second statement follows from the first statement and lemma 8.2.5. The first statement follows from the fact that Δ is a morphism of super vector spaces (also see remark 8.2.3). $\square$

The morphism $C \to C_{\bar{0}}$ is the super coalgebra equivalent of the projection to the body $A \to \bar{A}$ of superrings. We therefore write $\bar{C} = C/C_{\bar{1}}$. If $f : C \to D$ is a morphism of super coalgebras, then

$f(C_{\bar{1}}) \subset D_{\bar{1}}$ and thus there is an induced morphism $\bar{f} : C/C_{\bar{1}} \to D/D_{\bar{1}}$ such that the following diagram commutes

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & D \arrow[d] \\ \bar{C} \arrow[r, "\bar{f}"'] & \bar{D} \end{tikzcd} . \tag{8.6}$$

As in the case of superrings we have the following adjointness theorem:

Theorem 8.2.7. *Let $\mathcal{C}$ denote the category of coalgebras and $\mathcal{D}$ the category of super coalgebras. Let $\mathcal{S}$ be the functor $\mathcal{S} : \mathcal{C} \to \mathcal{D}$ that assigns to any coalgebra C the super coalgebra C, but then viewed as a super coalgebra with trivial odd part and that is the identity on morphisms of coalgebras. Let $\mathcal{B}$ be the functor $\mathcal{B} : \mathcal{D} \to \mathcal{C}$ that assigns to each super coalgebra C the coalgebra $\bar{C}$ and that assigns to a morphism of super coalgebras $f : C \to D$ the morphism of coalgebras $\bar{f} : \bar{C} \to \bar{D}$ defined by diagram (8.6). Then $\mathcal{B}$ is left-adjoint to $\mathcal{S}$.*

Proof. Let C be a coalgebra and D a super coalgebra. Any morphism of super coalgebras $f : D \to \mathcal{S}(C)$ factors uniquely over $f' : \bar{D} \to \mathcal{C}(S)$ as $f(D_{\bar{1}}) \in \mathrm{Ker}(f)$. But $f' : \bar{D} \to \mathcal{S}(C)$ can be viewed as a morphism in the category of coalgebras. Conversely, any morphism $f' : \bar{D} \to C$ of coalgebras gives rise to a morphism $f : D \to \mathcal{S}(C)$ of super coalgebras by composing f' with the projection $D \to \bar{D}$. This establishes $\underline{\mathrm{Hom}}_{\mathcal{D}}(D, \mathcal{S}(C)) \cong \mathrm{Hom}_{\mathcal{C}}(\bar{D}, C)$. Using the commutativity of diagram (8.6), naturality is straightforwardly verified. □

Proposition 8.2.8. *If $f : C \to D$ is a morphism of super coalgebra, then the image of f is a sub super coalgebra of D.*

Proof. Let $d \in f(C)$, we have to show that there are d_i and e_i in $f(C)$ such that $\Delta(d) = \sum_i d_i \otimes e_i$. This is obvious since $d = f(c)$ for some $c \in C$ and thus, using Sweedler notation for c, $\Delta d = f \otimes f \circ \Delta(c) = f \otimes f(c' \otimes c'') = f(c)' \otimes f(c)''$. □

Proposition 8.2.9. *Let $f : C \to D$ be a morphism of super coalgebras. Then* $\mathrm{Ker}(f)$ *is a coideal in C.*

Proof. Let $c \in \mathrm{Ker}(f)$, then $0 = \Delta(f(c)) = f \otimes f \circ \Delta(c)$ and hence $\Delta(c) \in \mathrm{Ker}(f \otimes f) = \mathrm{Ker}(f) \otimes C + C \otimes \mathrm{Ker}(f)$, where we used lemma 8.1.5. Clearly, $\mathrm{Ker}(f)$ is $\mathbb{Z}_2$-graded. □

Definition 8.2.10. *We say a super coalgebra C is cocommutative if $T \circ \Delta = \Delta$, where T is the braiding map $T : C \otimes C \to C \otimes C$ given by $T(c \otimes d) = Sgn(|c||d|)d \otimes c$.*

Lemma 8.2.11. *Let C be a super coalgebra. Then C^* is in natural way an associative unital superalgebra. The product is defined by $v \cdot w(c) = Sgn(|w||c'|)v(c')w(c'') = m \circ v \otimes w \circ \Delta(c)$ where m is the multiplication $k \otimes k \to k$. The unit element of C^* is the counit. If C is cocommutative, then C^* is commutative.*

Proof. Let $\mu : C^* \otimes C^* \to C^*$ be the map defined by $\mu(c^* \otimes d^*)(c) = c^* \otimes d^* \circ \Delta(c)$, then μ is the multiplication in C^*. We have

$$\mu(c^*, \epsilon)(c) = c^*(c')\epsilon(c'') = c^*(c'\epsilon(c'')) = c^*(c)\,, \tag{8.7}$$

showing that $\mu(c^*,\epsilon) = c^*$. Similarly one finds $\mu(\epsilon, c^*) = c^*$. Hence C^* is a unital superalgebra. Writing id_X for identity map on a super vector space X, we have for any $c^*, d^*, e^* \in C^*$

$$\begin{aligned}\mu(c^*, \mu(d^*, e^*)) &= m \circ c^* \otimes \mu(d^*, e^*) \circ \Delta \\ &= m \circ c^* \otimes (m \circ d^* \otimes e^* \circ \Delta) \circ \Delta \\ &= m \circ \mathrm{id}_k \otimes m \circ c^* \otimes d^* \otimes e^* \circ \mathrm{id}_C \otimes \Delta \circ \Delta\,.\end{aligned} \tag{8.8}$$

Using associativity of m and coassociativity of Δ one recognizes that the expression on the final line of eqn.(8.8) equals $\mu(\mu(c^*, d^*), e^*)$. Hence C^* is associative. Distributivity is obvious.

The last claim follows from the identity

$$c^* \otimes d^* \circ T(c \otimes d) = T^*(c^* \otimes d^*)(c \otimes d)\,, \tag{8.9}$$

where T^* is the braiding map $T^* : C^* \otimes C^* \to C^* \otimes C^*$ sending $c^* \otimes d^*$ to $\mathrm{Sgn}(|c^*||d^*|) d^* \otimes c^*$. $\square$

The converse of lemma 8.2.11 is not true: If A is any associative unital superalgebra, then the dual need not be a super coalgebra. This problem already shows up for non-$\mathbb{Z}_2$-graded coalgebras. The problem lies in the fact that $A^* \otimes A^*$ is a proper subspace of $(A \otimes A)^*$ in the infinite-dimensional case. For finite-dimensional algebras however $A^* \otimes A^* \cong (A \otimes A)^*$ and the dual of a finite-dimensional superalgebra is a super coalgebra.

Lemma 8.2.12. *Let C be a super coalgebra and $D \subset C$ a sub super coalgebra, then $D^\perp$ is a $\mathbb{Z}_2$-graded two-sided ideal in C^*.*

Proof. We write $c^* \cdot d^*$ for the product in C^*. Suppose $x \in C^*$ and $y, z \in D^\perp$. Then we have for any $d \in D$:

$$x \cdot (y + z)(d) = m \circ (x \otimes y + x \otimes z) \circ \Delta(c) = 0\,, \tag{8.10}$$

where m is the multiplication map $k \otimes k \to k$. Hence $D^\perp$ is a left ideal. A similar calculation shows that $I^\perp$ is a right ideal. $\square$

Lemma 8.2.13. *Let C be a super coalgebra and let I be a two-sided $\mathbb{Z}_2$-graded ideal in C^*, then $I^\perp$ is a sub super coalgebra of C.*

Proof. It is obvious that $I^\perp$ is closed under taking k-linear sums. We have to show that $\Delta(I^\perp) \subset I^\perp \otimes I^\perp$. Take $x \in I^\perp$ and write $\Delta(x) = \sum x_i \otimes y_i$ and choose the y_i homogeneous and linearly independent. Choose $y^* \in C^*$ with $y^*(y_i) = 0$ for $i \neq 1$ and $y^*(y_1) = 1$, then for all $c^* \in I$ we have $c^* \cdot y^*(x) = 0$ and thus $c^*(x_1) = 0$. Hence all x_i lie in $I^\perp$ and we have shown that $\Delta(x) \in I^\perp \otimes C^*$. In a similar fashion one shows $\Delta(x) \in C^* \otimes I^\perp$ and hence $\Delta(x) \in I^\perp \otimes C^* \cap C^* \otimes I^\perp$. Using the fact that one can find a basis of $I^\perp$ that can be extended to a basis of C, one shows that $(I^\perp \otimes C^*) \cap (C^* \otimes I^\perp) = I^\perp \otimes I^\perp$. $\square$

Corollary 8.2.14. *A $\mathbb{Z}_2$-graded subspace $D \subset C$ of a super coalgebra C is a sub super coalgebra if and only if $D^\perp$ is a $\mathbb{Z}_2$-graded two-sided ideal in C^*.*

Proof. The proof is found by combining lemma 8.2.12, lemma 8.2.13 and lemma 8.1.1. $\square$

The following proposition is the dual statement to corollary 8.2.14:

Proposition 8.2.15. *Let C be a super coalgebra and V a sub super vector space, then V is a $\mathbb{Z}_2$-graded coideal in C if and only if $V^\perp$ is a sub superalgebra of C^*.*

Proof. If V is a $\mathbb{Z}_2$-graded coideal, then $\Delta(V) \subset V \otimes C + C \otimes V$ and thus if c^* and d^* are in $V^\perp$, then so is $c^* \cdot d^*$. Hence $V^\perp$ is a sub superalgebra of C^*.

Now suppose $A \subset C^*$ is a sub superalgebra. We have to show that $\Delta(A^\perp) \subset A^\perp \otimes C + C \otimes A^\perp$, since then lemma 8.1.1 proves the proposition. If $c \in A^\perp$ then $\Delta(c) \in (A \otimes A)^\perp$. But by lemma 8.1.6 this equals $A^\perp \otimes C + C \otimes A^\perp$. Now apply lemma 8.1.1 to $A = V^\perp$. □

Proposition 8.2.16. *The intersection of sub super coalgebras is again a sub super coalgebra.*

Proof. Let $\{C_i\}_{i\in I}$ be a set of sub super coalgebras in C. Then $C_i^\perp$ is a $\mathbb{Z}_2$-graded two-sided ideal of C^*. We have $(\cap_i C_i)^\perp = \sum_i C_i^\perp$, which is a $\mathbb{Z}_2$-graded two-sided ideal of C^*. □

Definition 8.2.17. *Let C be a super coalgebra and S a set of homogeneous elements of C. Then we call the intersection of all sub super coalgebras of C that contain S the sub super coalgebra generated by S.*

A notion that we will only use on occasion is that of a left , which is defined as follows: Let C be a super coalgebra, then a sub super vector space L is a left coideal if $\Delta L \subset L \otimes C$. Using the same techniques as to prove corollary 8.2.14 and proposition 8.2.16, one shows

Proposition 8.2.18. *Let C be a super coalgebra, then a sub super vector space $L \subset C$ is a left coideal if and only if $L^\perp$ is a right $\mathbb{Z}_2$-graded ideal, that is $L^\perp C^* \subset L^\perp$.*

Proposition 8.2.19. *Let C be a super coalgebra and let $\{L_\alpha\}$ be a collection of left coideals. Then the intersection $\cap_\alpha L_\alpha$ is also a left .*

Proof. Each $L_\alpha^\perp$ is a $\mathbb{Z}_2$-graded right ideal in C^* and $(\cap_\alpha C_\alpha)^\perp = \sum_\alpha L_\alpha^\perp$ is a $\mathbb{Z}_2$-graded right ideal. □

Let $\{C_i\}_{i\in I}$ be a collection of super coalgebras with comultiplications Δ_i and counits ϵ_i. The direct sum of super coalgebras $\{C_i\}_{i\in I}$ is as a super vector space the direct sum $\bigoplus_{i\in I} C_i$. The comultiplication and counit of the direct sum are given by the componentwise action: $\Delta((x_i)_i) = (\Delta_i(x_i))_i \subset \bigoplus_{i\in I} C_i \otimes C_i \subset \bigoplus_{i\in I} C_i \otimes \bigoplus_{i\in I} C_i$ and $\epsilon((x_i)_i) = \sum_i \epsilon(x_i)$, where the last summation makes sense as any element $(x_i)_i \in \bigoplus_{i\in I} C_i$ only has finitely many nonzero components. The injections $C_j \to \bigoplus_{i\in I} C_i$ are easily seen to be morphisms of super coalgebras. However, the projections $\bigoplus_{i\in I} C_i \to C_j$ are not morphisms of super coalgebras.

Let C and D be super coalgebra with structure maps Δ_C, ϵ_C and Δ_D, ϵ_D respectively. Then define a linear map $\Delta : C \otimes D \to C \otimes D \otimes C \otimes D$ and $\epsilon : C \otimes D$ by

$$\Delta : c \otimes d \mapsto \sum \mathrm{Sgn}(|c''||d'|) c' \otimes d' \otimes c'' \otimes d'' , \quad \epsilon(c \otimes d) = \epsilon_C(c)\epsilon_D(d) , \tag{8.11}$$

for all homogeneous $c \in C$ and $d \in D$ and extend Δ and ϵ by linearity. We can write $\Delta = T_{23} \circ \Delta_C \otimes \Delta_D$, where T_{23} exchanges the second and third factor in the tensor product $C \otimes C \otimes D \otimes D$ with the appropriate sign. The following lemma is then easily obtained by using the definitions.

Lemma 8.2.20. *Let C, D be super coalgebras, then $C \otimes D$ becomes a super coalgebra with the structure maps Δ and ϵ as defined in eqn.(8.11).*

We call the super vector space $C \otimes D$ of two super coalgebras, equipped with the comultiplication and counit from eqn.(8.11), the tensor product super coalgebras of C and D. The tensor product of C and D has the following universal property: Define the projections $\pi_1 : C \otimes D \to C$ and $\pi_2 : C \otimes D \to D$ given by $\pi_1 : c \otimes d \mapsto c\epsilon_D(d)$ and $\pi_2(c \otimes d) = \epsilon_C(c)d$. Then for any two morphisms of super coalgebras $x : E \to C$ and $y : E \to D$, there is a unique morphism $f : E \to C \otimes D$ such that $x = \pi_1 \circ f$ and $y = \pi_2 \circ f$; that is, the following diagram commutes:

$$\begin{tikzcd} C & C \otimes D \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & D \\ & E \arrow[ul, "x"] \arrow[u, "f"] \arrow[ur, "y"'] & \end{tikzcd} . \tag{8.12}$$

The morphism f is given explicitly by $f(e) = x \otimes y \circ \Delta(e)$.

We close this introduction to super coalgebras by introducing two special kinds of elements in a super coalgebra, grouplike elements and primitive elements. In chapter 9 we will proceed with the discussion of the structure of super coalgebras. Then the role of primitive and grouplike elements will be essential in discussing properties of representations of algebraic supergroups.

Definition 8.2.21. *Let C be a super coalgebra. We say an element g of a super coalgebra C is grouplike if $\Delta c = c \otimes c$.*

It is not too hard to think of an example of an inhomogeneous grouplike element. However, no odd grouplike elements exist, as is obvious. The even grouplike elements correspond to one-dimensional sub super coalgebras, which have no odd part. If g is an even grouplike element in C, we have $\epsilon(g)g = g$ and hence either $g = 0$, or $\epsilon(g) = 1$.

Definition 8.2.22. *Let C be a super coalgebra and let g be a nonzero even grouplike element of C. We call an element h of a super coalgebra C primitive over g if $\Delta h = h \otimes g + g \otimes h$. We write $P_g(C)$ for the set of all primitive elements over g.*

The sum of two primitive elements over an even grouplike element g is again primitive over g. Hence $P_g(C)$ is a sub super vector space of C. If h is primitive over g, then $\epsilon(h)g + \epsilon(g)h = h$ and thus we have $\epsilon(h) = 0$.

8.3 Super bialgebras

When one combines the notion of a superalgebra and that of a super coalgebra, one obtains a super bialgebra. To give the definition of a super bialgebra, we need the notion of the tensor product algebra of two superalgebras. For any two superalgebras A, B over a field k, the tensor product $A \otimes B = A \otimes_k B$ becomes a superalgebra with the multiplication $a \otimes b \cdot a' \otimes b' = \mathrm{Sgn}(|b||a'|)aa' \otimes bb'$.

Definition 8.3.1. *A super bialgebra is an associative superalgebra B over a field k that is at the same time a super coalgebra, such that the comultiplication $\Delta : B \to B \otimes B$ and the counit $\epsilon : B \to k$ are superalgebra morphisms: $\Delta(xy) = \Delta(x)\Delta(y)$ and $\epsilon(xy) = \epsilon(x)\epsilon(y)$.*

From the requirement that in a super bialgebra B the comultiplication is a morphism of superalgebras it follows that $\Delta(1) = 1 \otimes 1$. The zero super vector space $\{0\}$ cannot be given the structure of a super bialgebra. As 0 is the unit element of the multiplication in $\{0\}$, we need that $\epsilon(0) = 1$, but then $\epsilon(0) = \epsilon(0 + 0) = \epsilon(0) + \epsilon(0) = 2$. In contrary, $\{0\}$ can be given the structure of a superalgebra, but in $\{0\}$ we then have $1 = 0$ and the map $k \to \{0\}$ is not injective. Often this construction is excluded by hand; on the other hand, it is not possible to make $\{0\}$ into a super bialgebra.

Example 8.3.2. Consider the superalgebra A from example 8.2.1 and the dual super coalgebra $C = A^*$. Now define B as the superalgebra generated by the elements X_{ij} of C, that is, $B = k[C]$. Then any element $b \in B$ is a polynomial in the X_{ij}. There is only one way to extend the comultiplication Δ and ϵ to maps $B \to B \otimes B$ and $B \to k$ respectively in such a way that B becomes a super bialgebra. Namely we require for any positive integer s

$$\Delta\Big(\prod_{l=1}^{s} X_{i_l j_l}\Big) = \prod_{l=1}^{s} \Delta X_{i_l j_l}\,, \qquad \epsilon\Big(\prod_{l=1}^{s} X_{i_l j_l}\Big) = \prod_{l=1}^{s} \epsilon(X_{i_l j_l})\,. \tag{8.13}$$

One can view the obtained super bialgebra as the super bialgebra of regular functions on the superalgebra A. △

Definition 8.3.3. *Let B be a super bialgebra. A bi-ideal in a B is a super sub vector space I that is a two-sided $\mathbb{Z}_2$-graded ideal of B as a superalgebra and a coideal of B as a super coalgebra.*

If B is a super bialgebra and I is a bi-ideal, then one easily verifies that the quotient B/I becomes a super bialgebra with the obvious multiplication, comultiplication and counit.

Proposition 8.3.4. *Let B be a super bialgebra and let g be an even grouplike element of B. Then $P_g(B)$ becomes a Lie superalgebra with the bracket $[x, y] = xy - (-1)^{|x||y|}yx$.*

Proof. The proof is a matter of writing out: Let x, y be in $P_g(B)$, then

$$\Delta(xy) = (x \otimes g + g \otimes x)(y \otimes g + g \otimes y)\,. \tag{8.14}$$

One then easily verifies that

$$\Delta([x, y]) = \Delta(x)\Delta(y) - (-1)^{|x||y|}\Delta(y)\Delta(x) = \Delta([x, y]) \otimes g + g \otimes \Delta([x, y])\,. \tag{8.15}$$

□

We already noted that in a super bialgebra the element 1 is grouplike and even. One then calls an element of a super bialgebra primitive if it is primitive over 1.

Lemma 8.3.5. *Let C be a superalgebra and let $T : C \otimes C \to C \otimes C$ the swapping map sending $c \otimes d$ to $Sgn(|c||d|)d \otimes c$, then T is an algebra map: $T(c \otimes c' \cdot d \otimes d') = T(c \otimes c')T(d \otimes d')$ for all $c, c', d, d' \in C$.*

Proof. This is a matter of writing out both sides of $T(c \otimes c' \cdot d \otimes d') = T(c \otimes c')T(d \otimes d')$. □

Proposition 8.3.6. *Suppose B is a super bialgebra and let C be a sub super coalgebra of B, which generates B as a superalgebra. Then if C is cocommutative, B is cocommutative.*

Proof. Any element in B is a sum of monomials $c_1 \cdot c_2 \cdots c_k$. It suffices to show that for all monomials $T \circ \Delta = \Delta$, where $T : B \otimes B \to B \otimes B$ is the map $T(b \otimes b') = \mathrm{Sgn}(|b||b'|)b' \otimes b$. One calculates

$$T \circ \Delta(c_1 \cdots c_k) = T(\Delta(c_1) \cdot \Delta(c_2) \cdots \Delta(c_k))\,, \tag{8.16}$$

which by lemma 8.3.5 and the assumption on C equals $\Delta(c_1) \cdot \Delta(c_2) \cdots \Delta(c_k)$. □

Let B and B' be two super bialgebras. We can turn $B \otimes B'$ into a superalgebra with product $b \otimes b' \cdot h \otimes h' = \mathrm{Sgn}(|b'||h|)bh \otimes b'h'$. But on the other hand we can turn $B \otimes B'$ into a super coalgebra by defining $\Delta^{\otimes}(b \otimes b') = T^{(4)}_{23}\Delta(b) \otimes \Delta(b')$, where $T^{(4)}_{23} : B \otimes B \otimes B' \otimes B' \to B \otimes B' \otimes B \otimes B'$ is the morphism that sends $a \otimes b \otimes a' \otimes b'$ to $\mathrm{Sgn}(|b||a'|)a \otimes a' \otimes b \otimes b'$, and $\epsilon(b \otimes b') = \epsilon(b)\epsilon(b')$. In fact these two structures are compatible and $B \otimes B$ is a super bialgebra. That the counit is a morphism of superalgebras, is trivial. Hence we need to check that $\Delta^{\otimes}$ is a super-algebra morphism. We calculate $\Delta^{\otimes}(a \otimes b \cdot c \otimes d)$:

$$\begin{aligned}
\Delta^{\otimes}(a \otimes b \cdot c \otimes d) &= \Delta^{\otimes}(\mathrm{Sgn}(|b||c|) ac \otimes bd) \\
&= \mathrm{Sgn}(|b||c|) T_{23}^{(4)} \circ \Delta \otimes \Delta(ac \otimes bd) \\
&= \mathrm{Sgn}(|b||c|) T_{23}^{(4)} \Delta(ac) \otimes \Delta(bd) \\
&= \mathrm{Sgn}(|b||c|) T_{23}^{(4)} \Delta a \Delta c \otimes \Delta b \Delta d \\
&= \mathrm{Sgn}(|b||c|) T_{23}^{(4)} ((a' \otimes a'' \cdot c' \otimes c'') \otimes (b' \otimes b'' \cdot d' \otimes d'')) \\
&= L\, a'c' \otimes b'd' \otimes a''c'' \otimes b''d'' \,,
\end{aligned} \tag{8.17}$$

where we defined

$$L = \mathrm{Sgn}(|b||c| + |a''||c'| + |b''||d'| + (|a''| + |c''|)(|b'| + |d'|)) \,. \tag{8.18}$$

On the other hand we have:

$$\begin{aligned}
\Delta^{\otimes}(a \otimes b) \Delta^{\otimes}(c \otimes d) &= \mathrm{Sgn}(|a''||b'| + |c''||d'|) a' \otimes b' \otimes a'' \otimes b'' \cdot c' \otimes d' \otimes c'' \otimes d'' \\
&= R\, a'c' \otimes b'd' \otimes a''c'' \otimes b''d''
\end{aligned} \tag{8.19}$$

where we defined

$$R = \mathrm{Sgn}(|a''||b'| + |c''||d'| + |c'||b''| + |c'||a''| + |c'||b'| + |d'||b''| + |d'||a''| + |c''||b''|) \,. \tag{8.20}$$

After comparing the terms in exponents one concludes $R = L$ and hence $\Delta^{\otimes}$ is a superalgebra morphism.

8.3.1 The algebra of linear maps

Let B_1 and B_2 be two super bialgebras over a field k. We denote the multiplication map of B_2 by μ, given by $\mu : a \otimes b \mapsto ab$ for all $a, b \in B_2$, and the comultiplication of B_1 by Δ. We consider the k-linear maps from B_1 to B_2 and we provide this super vector space with a product structure:

$$f * g = \mu \circ f \otimes g \circ \Delta \,, \quad f * g(x) = \mathrm{Sgn}(|g||x'|) f(x') g(x'') \tag{8.21}$$

We view the map $\epsilon : B_1 \to k$ as a map to B_2 by considering the image of ϵ within B_2. We have

$$f * \epsilon(x) = f(x')\epsilon(x'') = f(x'\epsilon(x'')) = f(x) \,, \tag{8.22}$$

and similarly $\epsilon * f = f$, hence ϵ is an identity element with respect to the product $*$. We denote by $\mathcal{A}(B_1, B_2)$ the algebra of linear maps from B_1 to B_2 with the product $*$ and identity element ϵ.

The following calculation shows that $\mathcal{A}(B_1, B_2)$ is an associative algebra:

$$\begin{aligned}(f*g)*h &= \mu\circ(f*g)\otimes h\circ\Delta \\ &= \mu\circ(\mu\circ f\otimes g\circ\Delta)\otimes h\circ\Delta \\ &= \mu\circ\mu\otimes\mathrm{id}\circ f\otimes g\otimes\mathrm{id}\circ\Delta\otimes\mathrm{id}\circ\mathrm{id}\otimes h\circ\Delta \\ &= \mu\circ\mu\otimes\mathrm{id}\circ f\otimes g\otimes\mathrm{id}\circ\mathrm{id}\otimes h\circ\Delta\otimes\mathrm{id}\circ\Delta \\ &= \mu\circ\mu\otimes\mathrm{id}\circ f\otimes g\otimes h\circ\mathrm{id}\otimes\Delta\circ\Delta \\ &= \mu\circ\mathrm{id}\otimes\mu\circ f\otimes g\otimes h\circ\mathrm{id}\otimes\Delta\circ\Delta \\ &= \mu\circ\mathrm{id}\otimes\mu\circ\mathrm{id}\otimes g\otimes h\circ f\otimes\mathrm{id}\circ\mathrm{id}\otimes\Delta\circ\Delta \\ &= \mu\circ\mathrm{id}\otimes\mu\circ\mathrm{id}\otimes g\otimes h\circ\mathrm{id}\otimes\Delta\circ f\otimes\mathrm{id}\circ\Delta \\ &= \mu\circ\mathrm{id}\otimes(g*h)\circ f\otimes\mathrm{id}\circ\Delta \\ &= \mu\circ f\otimes(g*h)\circ\Delta \\ &= f*(g*h)\,.\end{aligned} \tag{8.23}$$

We note that:

$$|f*g(x)| = |f(x')| + |g(x'')| = |f| + |g| + |x'| + |x''| = |f| + |g| + |x|\,, \tag{8.24}$$

from which it follows that $|f*g| = |f| + |g|$ and hence $\mathcal{A}(B_1, B_2)$ is an associative superalgebra with identity. We call $\mathcal{A}(B_1, B_2)$ the superalgebra of linear maps from B_1 to B_2.

Remark 8.3.7. The calculation 8.23 is typical for calculations involving super Hopf algebras and might not be familiar to a broad audience. We therefore have chosen to display all steps. In the following chapters we will omit some steps in such calculations, but try to be very explicit in this chapter, which should serve as an introduction. The advantage of manipulating expressions with maps, such as the comultiplication and the multiplications, instead of dealing with elements is twofold. Working with tensor products that involve multiple factors becomes readily clumsy. For super bialgebras the bookkeeping of the signs becomes rather involved; encapsulating the sign-changes in the linear operators bypasses this difficulty.

8.4 Super Hopf algebras

Definition 8.4.1. *A super Hopf algebra is a super bialgebra H together with an even linear map $S : H \to H$, called the antipode, such that for all $x \in H$ we have $x'S(x'') = S(x')x'' = \epsilon(x)$.*

Let H and H' be two super Hopf algebras with antipodes S and S', comultiplications Δ and Δ' and counits ϵ and ϵ' respectively. A morphism of super Hopf algebras is a morphism of superalgebras $f : H \to H'$ that is also a morphism of super coalgebras and that satisfies $S' \circ f = f \circ S$.

A Hopf ideal is a bi-ideal that is stable under the action of S. If H is super Hopf algebra and I a Hopf ideal, then $\Delta I \subset I \otimes H + H \otimes I$, $\epsilon(I) = 0$ and $S(I) \subset (I)$. The quotient H/I is again a super Hopf algebra with the structure maps: $\bar\Delta(\bar x) = \Delta x \bmod (I \otimes H + H \otimes I)$, $\bar\epsilon(\bar x) = \epsilon(x)$ and $\bar S(\bar x) = \overline{S(x)}$, where we wrote $\bar x = x \bmod I$. Below, in lemma 8.4.2, we will see that the antipode is unique and hence $\bar S$ is the only choice to make H/I into a super Hopf algebra. If $f : H \to H'$ is a morphism of super Hopf algebras, then the kernel of f is a Hopf ideal.

We now discuss some elementary properties of a super Hopf algebra, which are similar, if not identical, to the corresponding properties of ordinary Hopf algebras. From subsection 8.3.1 we conclude that a super bialgebra H is an super Hopf algebra if there is an inverse to the identity map $\mathrm{id}_H : h \mapsto h$ on H, when viewed as an element in the algebra of linear maps $\mathcal{A}(H,H)$;

$$S * \mathrm{id}_H(x) = S(x')x'' = \epsilon(x)\,, \quad \mathrm{id}_H * S(x) = x'S(x'') = \epsilon(x)\,. \tag{8.25}$$

From this observation the following lemma is an easy result, which is often paraphrased by saying that the antipode is unique:

Lemma 8.4.2. *Let H be a super bialgebra. There exists at most one way to make H into a super Hopf algebra.*

Proof. Since $\mathcal{A}(H,H)$ is associative, any inverse to id_H is uniquely determined. That is, there is at most one inverse to id_H in $\mathcal{A}(H,H)$. □

Lemma 8.4.3. *The antipode S of a super Hopf algebra H satisfies: $S(xy) = Sgn(|x||y|)S(y)S(x)$, for all $x, y \in H$.*

Proof. We consider the algebra $\mathcal{A}(H\otimes H, H)$ and claim that $\rho : x\otimes y \mapsto S(xy)$ is a left inverse of $\mu : x \otimes y \mapsto xy$ and that $\nu : x\otimes y \mapsto \mathrm{Sgn}(|x||y|)S(y)S(x)$ is a right inverse of μ. The claim then follows from the uniqueness of inverses and equality of left and right inverses.

We have

$$\begin{aligned}
\rho * \mu(x\otimes y) &= \mathrm{Sgn}(|x''||y'|)\rho(x'\otimes y')x''y'' \\
&= \mathrm{Sgn}(|x''||y'|)S(x'y')x''y'' \\
&= S\otimes \mathrm{id}\,(x'\otimes x''\cdot y'\otimes y'') \\
&= S\otimes \mathrm{id}\,(\Delta(xy)) \\
&= \epsilon(xy)\,,
\end{aligned} \tag{8.26}$$

and, writing $\Delta^{\otimes}$ for the comultiplication in the super bialgebra $H\otimes H$, we have

$$\begin{aligned}
\mu * \nu(x\otimes y) &= \mu\circ\mu\otimes\nu\circ\Delta^{\otimes}(x\otimes y) \\
&= \mathrm{Sgn}(|x''||y'|)\mu\circ\mu\otimes\nu(x'\otimes y'\otimes x''\otimes y'') \\
&= \mathrm{Sgn}(|x''||y'| + |x''||y''|)\mu(x'y'\otimes S(y'')S(x'')) \\
&= \mathrm{Sgn}(|x''||y|)x'\epsilon(y)S(x'') \\
&= x'S(x'')\epsilon(y) \\
&= \epsilon(xy)\,,
\end{aligned} \tag{8.27}$$

where we used that $|\epsilon(y)| = |y|$. □

If A is a superalgebra with multiplication map $\mu : A\otimes A \to A$, then we define the opposite superalgebra A^{opp} to be the same super vector space as A, but with multiplication map $\mu^{\mathrm{opp}} : A^{\mathrm{opp}} \otimes A^{\mathrm{opp}} \to A^{\mathrm{opp}}$ given by $\mu^{\mathrm{opp}} = \mu\circ T$, where $T : A\otimes A \to A\otimes A$ is the swapping map $T : a\otimes b \mapsto \mathrm{Sgn}(|a||b|)b\otimes a$.

Theorem 8.4.4. *Let H be a super Hopf algebra. Then the antipode is a morphism of superalgebras $H \to H^{\mathrm{opp}}$.*

Proof. Lemma 8.4.3 shows that S, interpreted as a morphism of superalgebras $H \to H^{\mathrm{opp}}$ preserves the product. The final step is achieved by proving $S(1) = 1$, which follows from: $\mathrm{id}_H * S(1) = 1\cdot S(1) = \epsilon(1) = 1$. □

An immediate consequence from theorem 8.4.4 is that in a commutative super Hopf algebra H, the antipode is a superalgebra morphism. It is well-known that there is an intimate relation between groups and commutative Hopf algebras, see for example [21,22,60]. In the next section we establish this relation for commutative super Hopf algebras, where commutative means commutative as superalgebras in the sense of eqn.(3.1). The following results are intimately related to more familiar properties of groups: proposition 8.4.5 is the super Hopf algebra equivalent of the statement that the inverse of the unit element of a group is the unit element, proposition 8.4.6 is related to the fact that in any group G the inverse of the inverse of $g \in G$ is g, and theorem 8.4.8 is the super Hopf algebra version of the statement that $(gh)^{-1} = h^{-1}g^{-1}$ for all g, h in a group G.

Proposition 8.4.5. *In a super Hopf algebra the antipode satisfies:* $\epsilon \circ S = \epsilon$.

Proof. $\epsilon(S(x)) = \epsilon(S(x'\epsilon(x''))) = \epsilon(S(x'))\epsilon(x'') = \epsilon(S(x')x'') = \epsilon(\epsilon(x)) = \epsilon(x)$. □

Proposition 8.4.6. *If H is a commutative super Hopf algebra - meaning that it is commutative as a superalgebra -, then* $S^2 = \mathrm{id}_H$.

Proof. We show that in $\mathcal{A}(H, H)$ the map S^2 is also an inverse to S, implying it must be id_H. Using commutativity, we have

$$S * S^2(x) = S(x')S^2(x'') = S(x'S(x'')) = S(\epsilon(x)) = \epsilon(x)S(1) = \epsilon(x)\,, \tag{8.28}$$

and similarly $S^2 * S = \epsilon$. □

Lemma 8.4.7. *Let x be an element of a super bialgebra, then we have the identity:*

$$(x')' \otimes (x')'' \otimes (x'')' \otimes (x'')'' = x' \otimes ((x'')')' \otimes ((x'')')'' \otimes (x'')''\,. \tag{8.29}$$

Proof. The equality follows from applying the identity

$$\Delta \otimes \Delta \circ \Delta = \Delta \otimes \mathrm{id} \otimes \mathrm{id}\circ = \mathrm{id} \otimes \Delta \otimes \mathrm{id} \circ \mathrm{id} \otimes \Delta \circ \Delta\,, \tag{8.30}$$

to $x \in H$. □

Theorem 8.4.8. *Let H be a super Hopf algebra, then for all $x \in H$ we have:*

$$\Delta(S(x)) = \sum (-1)^{|x'||x''|} S(x'') \otimes S(x')\,. \tag{8.31}$$

Proof. We consider $\mathcal{A}(H, H \otimes H)$ and claim that $\rho = \Delta \circ S$ is a left inverse to Δ and that $\nu = S \otimes S \circ T \circ \Delta$ is a left inverse to Δ, where $T : x \otimes y \mapsto \mathrm{Sgn}(|x||y|) y \otimes x$. The theorem then follows. Multiplication and counit in $H \otimes H$ will also be denoted μ and ϵ respectively.

We have:

$$\rho * \Delta(x) = \rho(x')\Delta(x'') = \Delta(S(x'))\Delta(x'') = \Delta(S(x')x'') = \Delta(\epsilon(x)) = \epsilon(x)\,. \tag{8.32}$$

On the other hand, using lemma 8.4.7 we have

$$\begin{aligned}
\Delta * \nu(x) &= \Delta(x')\nu(x'') \\
&= \mathrm{Sgn}(|(x'')'||(x'')''|)\Delta(x')S((x'')'') \otimes S((x'')') \\
&= \mathrm{Sgn}(|(x'')'||(x'')''|)\mu \circ \mathrm{id}_2 \otimes S \otimes S\,((x')' \otimes (x')'') \otimes (x'')'' \otimes (x'')' \\
&= \mu \circ (\mathrm{id}_2 \otimes S \otimes S) \circ T^{(4)}_{34}\,((x')' \otimes (x')'' \otimes (x'')' \otimes (x'')'') \\
&= \mu \circ (\mathrm{id}_2 \otimes S \otimes S) \circ T^{(4)}_{34}\,(x' \otimes ((x'')')' \otimes ((x'')')'' \otimes (x'')'') \\
&= \mathrm{Sgn}(|(x'')''||(x'')'|)x'S((x'')'') \otimes ((x'')')'S(((x'')')'') \\
&= \mathrm{Sgn}(|(x'')''||(x'')'|)x'S((x'')'') \otimes \epsilon((x'')') \\
&= x'S(\epsilon((x'')')(x'')'') \\
&= x'S(x'') = \epsilon(x)\,,
\end{aligned} \tag{8.33}$$

where we used id_2 to denote the identity map on $H \otimes H$ and the map $T_{34}^{(4)}$ is the swapping map $T_{34}^{(4)} : x \otimes y \otimes z \otimes w \mapsto \mathrm{Sgn}(|z||w|)x \otimes y \otimes w \otimes z$. $\square$

8.5 Affine algebraic supergroups

This section gives the basic definitions of affine algebraic supergroups and gives some of their elementary properties. In chapter 9, when we discuss comodules of super coalgebras, we can say more on representations of affine algebraic supergroups.

In this section the following notational convention for natural transformations is in use: Let $F, G : \mathcal{A} \to \mathcal{B}$ be two functors from the category $\mathcal{A}$ to the category $\mathcal{B}$ and suppose $\phi : F \to G$ is a natural transformation. Then ϕ is a collection of $\mathcal{B}$-morphisms $\phi^A : F(A) \to G(A)$ for all objects A in $\mathcal{A}$. In this section all superalgebras will be commutative, unless otherwise stated.

Let **sAlg** denote the category of commutative (as in the sense of eqn.(3.1) superalgebras over a fixed ground field k and **Sets** the category of sets. We call a functor $G : \mathbf{sAlg} \to \mathbf{Sets}$ a group functor if G factors over the category of groups, by which we mean that $G(A)$ is a group for all superalgebras A and any morphism of superalgebras $A \to B$ induces a group morphism $G(A) \to G(B)$. Another way of saying this, is that G is a group functor if there are natural transformations $\mu : G \times G \to G$, $\iota : G \to G$ and $u : E \to G$ with $E : \mathbf{sAlg} \to \mathbf{Grp}$ given by $E(A) = \{1_A\}$. Thus E is the functor that assigns to each superalgebra A the identity element of the group $G(A)$. The natural transformations have to satisfy the following commutative diagrams:

$$
\begin{tikzcd}
G \times G \times G \arrow[r, "\mu\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\mu"] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"] & G
\end{tikzcd}
$$

$$
\begin{tikzcd}
G \arrow[r, "(\mathrm{id},\iota)"] \arrow[d] & G \times G \arrow[d, "\mu"] & G \arrow[l, "(\iota,\mathrm{id})"'] \arrow[d] \\
E \arrow[r, "u"] & G & E \arrow[l, "u"']
\end{tikzcd}
\tag{8.34}
$$

$$
\begin{tikzcd}
G \times E \arrow[r, "\mathrm{id}\times u"] \arrow[dr, "\mathrm{proj}_1"'] & G \times G \arrow[d, "\mu"] & E \times G \arrow[l, "u\times\mathrm{id}"'] \arrow[dl, "\mathrm{proj}_2"] \\
& G &
\end{tikzcd}
$$

We call u the identity transformation, μ the multiplication transformation and ι the inverse transformation. If G is a group functor, then a subfunctor H is a subgroup functor is a subfunctor such that for all commutative superalgebras $H(A)$ is a subgroup of $G(A)$. Recall that H a subfunctor of G if for all commutative superalgebras we have $H(A) \subset G(A)$ and if $f : A \to B$ is a morphism of superalgebras, then $H(f) : H(A) \to H(B)$ is the restriction of $G(f)$ to $H(A)$.

For any superalgebra A we have a functor $F_A : \mathbf{sAlg} \to \mathbf{Sets}$, given by $F_A(B) = \mathrm{Hom}_{\mathbf{sAlg}}(A, B)$. We say a functor is F representable if there is an object A such that there is a natural isomorphism

$F(B) \cong F_A(B)$ for all B. One then says that A represents the functor F. The object that represents a representable functor is unique up to isomorphism by the lemma of Yoneda. We call a group functor G an affine algebraic supergroup if G is a representable group functor and the representing superalgebra is finitely generated. The following theorem relates affine algebraic supergroups to super Hopf algebras.

Theorem 8.5.1. *Let $G : \mathbf{sAlg} \to \mathbf{Sets}$ be a representable group functor and suppose $k[G]$ represents G. Then $k[G]$ is a commutative super Hopf algebra. Conversely, if A is a super Hopf algebra, then the functor $B \mapsto \mathrm{Hom}_{\mathbf{sAlg}}(A, B)$ defines a representable group functor.*

Proof. Let G be a representable group functor, represented by $k[G]$. The functor $E : A \mapsto \{1_A\}$ is representable by k. The functor $G \times G$ is representable by $k[G] \otimes k[G]$. By the lemma of Yoneda, the natural transformation $u : E \to G$ corresponds to a superalgebra morphism $\epsilon : k[G] \to k$ and the natural transformation $\mu : G \times G \to G$ corresponds to a superalgebra morphism $\Delta : k[G] \to k[G] \otimes k[G]$. The natural transformation ι corresponds likewise to a morphism of superalgebras $S : k[G] \to k[G]$. The diagrams (8.34) commute if and only if Δ, ϵ and S satisfy the conditions that make them a comultiplication, a counit and an antipode respectively. □

The isomorphism between the functors defined by $A \mapsto \mathrm{Hom}_{\mathbf{sAlg}}(k[G], A) \times \mathrm{Hom}_{\mathbf{sAlg}}(k[G], A)$ and $A \mapsto \mathrm{Hom}_{\mathbf{sAlg}}(k[G] \otimes k[G], A)$ is given by

$$\begin{aligned} &i : \mathrm{Hom}(k[G], A) \times \mathrm{Hom}(k[G], A) \to \mathrm{Hom}(k[G] \otimes k[G], A)\,, \\ &i(x, y) : p \otimes q \mapsto x(p)y(q)\,, \quad \text{for all } x, y \in \mathrm{Hom}(k[G], A)\,, \end{aligned} \tag{8.35}$$

with the inverse

$$\begin{aligned} j \,&:\mathrm{Hom}(k[G] \otimes k[G], A) \to \mathrm{Hom}(k[G], A) \times \mathrm{Hom}(k[G], A)\,, \\ j \,&:X \mapsto (j_1(X), j_2(X))\,, \quad j_1(X)(p) = X(p \otimes 1)\,, \quad j_2(X)(q) = X(1 \otimes q)\,. \end{aligned} \tag{8.36}$$

We now briefly make the group structure alluded in theorem 8.5.1 explicit. Let H be a super Hopf algebra and suppose A is a superalgebra. Write $\mu_A : A \otimes A \to A$ for the multiplication in A. Then the multiplication in $\mathrm{Hom}_{\mathbf{sAlg}}(H, A)$ is given by

$$x \cdot y(h) = \mu_A \cdot x \otimes y \circ \Delta(h)\,, \qquad x, y \in \mathrm{Hom}_{\mathbf{sAlg}}(H, A)\,,\ h \in H\,. \tag{8.37}$$

The inverse of $x \in \mathrm{Hom}_{\mathbf{sAlg}}(H, A)$ is given by

$$x^{-1}(h) = x \circ S(h)\,, \qquad h \in H\,. \tag{8.38}$$

The unit element in $\mathrm{Hom}_{\mathbf{sAlg}}(H, A)$ is given by composing ϵ with the inclusion $k \to A$.

Let G and H be two group functors from **sAlg** to **Sets**. We define a morphism of group functors $\phi : G \to H$ to be a natural transformation from G to H that respects the group structure. To respect the group structure means that for any superalgebra A, the map $\phi^A : G(A) \to H(A)$ is a group morphism. The following lemma then shows that for affine algebraic supergroups the morphisms of group functors are in one-to-one correspondence with super Hopf algebra morphisms:

Lemma 8.5.2. *Let G and H be affine algebraic supergroups represented by $k[G]$ and $k[H]$ respectively. Suppose that $\phi : G \to H$ is a morphism of group functors, then there is a morphism of super Hopf algebras $\psi : k[H] \to k[G]$ such that $\phi^A : G(A) \to H(A)$ is given by $\phi^A(x) = x \circ \psi$. Conversely, any super Hopf algebra morphism $\psi : k[H] \to k[G]$ induces a morphism of group functors $G \to H$, by composing with ψ.*

Proof. By the lemma of Yoneda, there exists a morphism of superalgebras $\psi : k[H] \to k[G]$ such that $\phi^A : G(A) \to H(A)$ is given by $\phi^A(x) = x \circ \psi$. We need to check that ψ is a morphism of super Hopf algebras. Let $\mu^G : G \times G \to G$ and $\mu^H : H \times H \to H$ be the group multiplication transformations of G and H. Let Δ^G and Δ^H be the comultiplication of G and H respectively. From $\phi^A \circ \mu^G = \mu^H \circ (\phi^A \times \phi^A)$ we see that the following diagram has to commute

$$\begin{tikzcd} \mathrm{Hom}(k[G] \otimes k[G], A) \arrow[r, "\psi\otimes\psi"] \arrow[d, "\circ\Delta^G"] & \mathrm{Hom}(k[H] \otimes k[H], A) \arrow[d, "\circ\Delta^H"] \\ \mathrm{Hom}(k[G], A) \arrow[r, "\phi^A"] & \mathrm{Hom}(k[H], A) \end{tikzcd} \quad . \tag{8.39}$$

Applying the diagram to $A = k[G] \otimes k[G]$ and on the element $\mathrm{id}_{k[G]\otimes k[G]}$, we immediately obtain

$$\psi \otimes \psi \circ \Delta^H = \Delta^G \circ \psi \,. \tag{8.40}$$

Let $u^G : E \to G$ and $u^H : E \to H$ denote the identity transformations of G and H respectively. Preserving the identity elements requires that the diagram

$$\begin{tikzcd} \mathrm{Hom}(k, A) \arrow[r] \arrow[dr] & \mathrm{Hom}(k[G], A) \arrow[d] \\ & \mathrm{Hom}(k[H], A) \end{tikzcd} \tag{8.41}$$

commutes. Applying to $A = k$ and the element $1 \mapsto 1$ of $\mathrm{Hom}(k, k)$ (as superalgebra morphisms) gives

$$\varepsilon^G \circ \psi = \varepsilon^H \,. \tag{8.42}$$

Let ι^G and ι^H be the inverse transformations of G and H respectively. Preservation of the inverse requires that $\iota^H \circ \phi = \phi \circ \iota^G$, or in terms of diagrams: the following diagram has to commute for all superalgebras A

$$\begin{tikzcd} \mathrm{Hom}(k[G], A) \arrow[r, "\circ\psi"] \arrow[d, "\circ S^G"] & \mathrm{Hom}(k[H], A) \arrow[d, "\circ S^H"] \\ \mathrm{Hom}(k[G], A) \arrow[r, "\circ\psi"] & \mathrm{Hom}(k[H], A) \end{tikzcd} \quad . \tag{8.43}$$

Putting $A = k[G]$ and applying to the identity map on $k[G]$, we get

$$S^G \circ \psi = \psi \circ S^H \,. \tag{8.44}$$

Conversely, using the explicit description in eqns.(8.37,8.38) of the group structure on $G(A)$ and $H(A)$ for any superalgebra A, it is not hard to show that any morphism of super Hopf algebras induces a morphism $\varphi : G \to H$ of group functors by composition $\varphi^A(x) \mapsto x \circ \psi$. $\square$

Corollary 8.5.3. *Let H, H' be commutative super Hopf algebras, with comultiplications Δ and Δ' respectively, and let $f : H \to H'$ be a morphism of superalgebras satisfying $f \otimes f \circ \Delta = \Delta' \circ f$. Then f is a morphism of super Hopf algebras.*

Proof. Let ϵ and ϵ' be the counits of H and H' respectively and let S and S' be the antipodes of H and H' respectively. We have to show that $\epsilon' \circ f = \epsilon$ and that $S' \circ f = f \circ S$. This follows from the fact that $\varphi : G \to G'$ is a morphism of groups if and only $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G$. Indeed, if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G$, then it follows that φ preserves the identity and $\varphi(a)^{-1} = \varphi(a^{-1})$ for all $a \in G$.

Let G be the group functor represented by H and G' be the group functor represented by H' and let $\varphi : G' \to G$ be the natural transformation induced by $f : H \to H'$. Then from $f \otimes f \circ \Delta = \Delta' \circ f$ it follows that $\varphi^A : G'(A) \to G(A)$ is a morphism of groups for all superalgebras A. Indeed, write μ_A for the multiplication in A and let $x, y \in G'(A)$ and $h \in H$, then

$$\varphi^A(x \cdot y)(h) = \mu_A \circ x \otimes y \circ \Delta' \circ f(h) = \mu_A \circ (x \circ f) \otimes (y \circ f) \circ \Delta(h) = \varphi^A(x) \cdot \varphi^A(y)(h)\,. \quad (8.45)$$

But then φ^A preserves the identity element for all A. Take $A = k$, then $\epsilon' \in G'(k)$ and $\epsilon \in G(k)$ are the identity elements. Hence we have $\varphi^k(\epsilon') = \epsilon' \circ f = \epsilon$. Now take $A = H'$, then $S' \in G'(H')$ by 8.4.4 as H' is commutative. But S' is the inverse to $\mathrm{id}_{H'}$ and is mapped to $S' \circ f$, where $\mathrm{id}_{H'}$ is mapped to f. We conclude that in the group $G(H')$, the element $S \circ f$ is the inverse of f. Thus we have $\mu_{H'} \circ (S' \circ f) \otimes f \circ \Delta = \epsilon$. On the other hand, since f is k-linear and preserves products, we have for all $h \in H$

$$\epsilon(h) = f(\mu_H \circ S \otimes \mathrm{id}_H \circ \Delta(h)) = \mu_{H'} \circ (f \circ S) \otimes f \circ \Delta(h)\,, \quad (8.46)$$

which shows that $f \circ S$ is an inverse to f. Hence we conclude that $S' \circ f = f \circ S$. $\square$

Let G be an affine algebraic supergroup represented by $k[G]$. Write $X = \mathrm{Spec}(k[G])$ and let $\mathcal{O}_X$ be the structure sheaf on X. In the category of affine superschemes over k, the product $X \times_k X$ is represented by $\mathrm{Spec}(k[G] \otimes k[G])$. Since for commutative super Hopf algebras, the antipode $S : k[G] \to k[G]$, the comultiplication $\Delta : k[G] \to k[G] \otimes k[G]$ and the counit $\epsilon : k[G] \to 1$ are all morphism of superalgebras, we can conclude that X is a group superscheme. In other words, X is a group object in the category of superschemes over k. We therefore will use the name affine group superscheme for affine algebraic supergroups.

Let G be an affine algebraic supergroup with representing super Hopf algebra $k[G]$. Suppose $\mathfrak{a}$ is a Hopf ideal. Then $k[G]/\mathfrak{a}$ is again a super Hopf algebra and thus defines an affine group superscheme. Let H be the affine algebraic supergroup defined by $k[G]/\mathfrak{a}$. We have for all superalgebras A

$$H(A) \cong \{g \in G(A) \mid g(\mathfrak{a}) = 0\}\,, \quad (8.47)$$

and thus H can be viewed as a subgroup of G. This leads us to define the following:

Definition 8.5.4. *Let G and H be two affine algebraic supergroups with representing super Hopf algebras $k[G]$ and $k[H]$ respectively. We call a morphism $i : H \to G$ a closed embedding if $k[H] \cong k[G]/\mathfrak{a}$ for some Hopf ideal $\mathfrak{a}$ and i is induced by the projection $k[G] \to k[G]/\mathfrak{a}$. In this case we call H a closed subgroup of G.*

The group defined by $E : A \mapsto \{1_A\}$ is a closed subgroup of any affine algebraic supergroup G. The morphism $E \to G$ is defined by the counit $\epsilon : k[G] \to k$ of G. We have $k[E] = k \cong \mathrm{Ker}(\epsilon)$. The ideal $\mathrm{Ker}(\epsilon)$ is called the augmentation ideal of G and one easily checks that it is a Hopf ideal.

Proposition 8.5.5. *Let $f : G \to H$ be a morphism of affine algebraic supergroups. Let $k[G]$ and $k[H]$ be the super Hopf algebras representing G and H respectively. The kernel of f is a closed subgroup of G and $\mathrm{Ker}(f)$ is represented by the super Hopf algebra $k[G]/\mathfrak{a}$, where $\mathfrak{a}$ is the $\mathbb{Z}_2$-graded ideal in $k[G]$ generated by the image of the augmentation ideal of $k[H]$.*

Proof. We need the fibred product in this proof; also see the discussion after definition 5.4.11: The kernel of f is the fibred product of G and E over H. Thus we have a commutative diagram

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r] \arrow[d] & E \arrow[d] \\ G \arrow[r, "f"'] & H \end{tikzcd} , \tag{8.48}$$

and $\mathrm{Ker}(f)$ has the appropriate universal property. Let $k[H]$ and $k[G]$ be the super Hopf algebras for H and G respectively and suppose that f is induced by a morphism of super Hopf algebras $\phi : k[H] \to k[G]$. Then the dual picture in terms of super Hopf algebras is given by

$$\begin{tikzcd} k[H] \arrow[r, "\epsilon"] \arrow[d, "\phi"] & k \\ k[G] & \end{tikzcd} . \tag{8.49}$$

The tensor product gives the required universal property and hence the kernel is represented by the super Hopf algebra $k[G] \otimes_{k[H]} k$. The map $p : k[G] \to k[G] \otimes_{k[H]} k$, given by $a \mapsto a \otimes 1$, defines the embedding of the kernel as a subgroup in G. As p is surjective and the kernel of p is an ideal, we have shown that the kernel of f is a closed subgroup of G.

Define the ideal $\mathfrak{a} = \phi(I_H) \cdot k[G]$, where I_H is the augmentation ideal of H, and let $q : k[G] \to k[G]/\mathfrak{a}$ be the canonical projection. One easily checks that $k[G]/\mathfrak{a}$ makes the diagram

$$\begin{tikzcd} k[H] \arrow[r, "\epsilon"] \arrow[d, "\phi"] & k \arrow[d, "i"] \\ G \arrow[r, "q"'] & k[G]/\mathfrak{a} \end{tikzcd} \tag{8.50}$$

commute, where $i : k \to k[G]/\mathfrak{a}$ is the inclusion $x \mapsto x \cdot 1$. Suppose A is any superalgebra over k and that $g : k[G] \to A$ is a morphism of superalgebras such that $g \circ f = i \circ \epsilon$. Then g maps $\mathfrak{a}$ to zero and g factors over $k[G]/\mathfrak{a}$. Hence $k[G]/\mathfrak{a}$ has the same universal property as $k[G] \otimes_{k[H]} k$ and thus $\mathfrak{a} \cong \mathrm{Ker}(p)$. Furthermore, using that I_H is a Hopf ideal in $k[H]$, one easily checks that $\mathfrak{a}$ is a Hopf ideal. $\square$

It can be quite tedious to check whether a given $\mathbb{Z}_2$-graded ideal in a super Hopf algebra is a Hopf ideal. In order to facilitate the recognition of closed subgroups, we state the following lemma:

Lemma 8.5.6. *Let G be an affine algebraic supergroup represented by the super Hopf algebra $k[G]$ and let H be a group subfunctor. If $\mathfrak{a}$ is a $\mathbb{Z}_2$-graded ideal in $k[G]$ such that*

$$H(A) \cong \{g \in G(A) \mid g(\mathfrak{a}) = 0\} , \tag{8.51}$$

for all superalgebras A, then H is a closed subgroup and $\mathfrak{a}$ is a Hopf ideal.

Proof. Since H is a subgroup functor, the identity of G is the identity of H. Since ϵ is the unit of $G(k)$, it follows that $\mathfrak{a} \in \operatorname{Ker} \epsilon$. Now take $A = k[G]/\mathfrak{a}$, then the canonical projection $\pi : k[G] \to A$ is an element of $H(A)$, hence also its inverse, as H is a subgroup functor. Therefore $\pi^{-1} = \pi \circ S$ annihilates $\mathfrak{a}$. Consider now the map $g : k[G] \to A \otimes A$ given by $g : x \mapsto \pi(x) \otimes 1$ and $h : k[G] \to A \otimes A$ given by $h : x \mapsto 1 \otimes \pi(x)$. Then $g, h \in H(A \otimes A)$ and thus also their product, which is the map $\pi \otimes \pi \circ \Delta$. Hence $\Delta\mathfrak{a} \subset \operatorname{Ker}(\pi \otimes \pi) = \mathfrak{a} \otimes k[G] + k[G] \otimes \mathfrak{a}$. Thus $\mathfrak{a}$ is indeed a Hopf ideal and $H(B) \cong \operatorname{Hom}_{\mathbf{sAlg}}(k[G]/\mathfrak{a}, B)$ for any superalgebra B. □

Remark 8.5.7. There is a slight generalization to lemma 8.5.6: $H(A)$ needs only be a subgroup for finitely generated commutative superalgebras A. And in fact, $H(A)$ only needs to be a group for $A = k$, $A = k[G]\mathfrak{a}$ and $A = k[G]/\mathfrak{a} \otimes k[G]/\mathfrak{a}$, as they were the only ones we needed to conclude that $\mathfrak{a}$ is Hopf ideal.

Definition 8.5.8. *Let G be an affine algebraic supergroup represented by the super Hopf algebra $k[G]$. A closed subgroup $H \subset G$ is called a normal subgroup if for all superalgebras A the subgroup $H(A)$ is a normal subgroup of $G(A)$.*

In section 9.1.1 we will say more on normal subgroups and give a characterization on the level of super Hopf algebras. Notice that if $f : G \to H$ is a morphism of affine algebraic supergroups, the kernel of f is a normal subgroup. We say a morphism of group functors $f : G \to H$ is surjective if the morphism $f^A : G(A) \to H(A)$ is surjective for all superalgebras A.

Proposition 8.5.9. *Let G and H be affine algebraic supergroups represented by super Hopf algebras $k[G]$ and $k[H]$ respectively. Then a morphism $f : G \to H$ is surjective if and only if the morphism of super Hopf algebras $\varphi : k[H] \to k[G]$ that induces f, has a left inverse as a morphism of superalgebras. In particular, if f is surjective, then φ is injective.*

Proof. If f is surjective, which by definition means that $f^A : G(A) \to H(A)$ is surjective, there is a morphism of superalgebras $\chi : k[G] \to k[H]$ such that $f(\chi) = \chi \circ \varphi = \mathrm{id}_{k[H]}$. Then χ is a left inverse to φ. This implies that φ is injective.

Conversely, suppose that φ has a left inverse $\chi \circ \varphi = \mathrm{id}_{k[H]}$. Let A be any superalgebra and suppose $h \in H(A) = \operatorname{Hom}_{\mathbf{sAlg}}(k[H], A)$ is given. Then $h' = h \circ \chi \in \operatorname{Hom}_{\mathbf{sAlg}}(k[G], A)$ and $f(h') = h' \circ \varphi = f$. □

Definition 8.5.10. *Let G be an affine algebraic group. We call G an abelian group if for each superalgebra A, the group $G(A)$ is abelian.*

Lemma 8.5.11. *Let G be an affine algebraic group represented by the super Hopf algebra $k[G]$. Then G is an abelian group superscheme if and only if $k[G]$ is cocommutative.*

Proof. Let $\tau : G \times G \to G \times G$ be the morphism of groups that interchanges the two factors. Then τ is induced by the morphism $T : k[G] \otimes k[G] \to k[G] \otimes k[G]$ given by $a \otimes b \mapsto \operatorname{Sgn}(|a||b|) b \otimes a$. Let $\mu : G \otimes G \mapsto G$ be the multiplication transformation, then μ is induced by the comultiplication $\Delta : k[G] \to k[G] \otimes k[G]$ of $k[G]$. The group G is commutative if and only if the diagram

$$\begin{tikzcd} G \times G \arrow[rr, "\mu"] \arrow[dr, "\tau"'] & & G \\ & G \times G \arrow[ur, "\mu"'] & \end{tikzcd} \tag{8.52}$$

commutes, which means that the diagram obtained by application to a superalgebra A must commute for all A. If the corresponding diagram

$$\begin{tikzcd} k[G] \times k[G] & & k[G] \arrow[ll, "\Delta"'] \arrow[dl, "\Delta"] \\ & k[G] \times k[G] \arrow[ul, "T"] & \end{tikzcd} \tag{8.53}$$

commutes, then surely G is commutative. Conversely, if diagram (8.52) commutes, then in particular, the diagram applied to $k[G] \otimes k[G]$ commutes. That is, for all morphisms $g : k[G] \otimes k[G] \to k[G] \otimes k[G]$ we have $g \circ T \circ \Delta = g \circ \Delta$. Taking $g = \mathrm{id}_{k[G]\otimes k[G]}$ shows that diagram (8.53) commutes. □

An example of an abelian algebraic supergroup is given by the affine algebraic supergroup T^1, which is the affine group superscheme defined by the super Hopf algebra $A^1 = k[x, x^{-1}]$ with comultiplication $\Delta(x) = x \otimes x$, antipode $S(x) = x^{-1}$ and counit $\epsilon(x) = 1$. We give A^1 the $\mathbb{Z}_2$-grading where $(A^1)_{\bar{1}} = 0$. We call the affine algebraic supergroup defined by A^1 the one-dimensional torus and denote it by T^1. Let G be an affine algebraic supergroup. A torus in G is an abelian subgroup in G isomorphic to a direct product of several copies of T^1. We write $(T^1)^n = T^n$ for the n-fold fibred product of T^1 over k. We call $T^n \subset G$ an n-torus in G.

Lemma 8.5.12. *Let G be an affine algebraic supergroup with representing super Hopf algebra $k[G]$. Then the body $A = \overline{k[G]}$ is a commutative Hopf algebra over k.*

Proof. Let J be the ideal in $k[G]$ generated by the odd elements. Then clearly we have $\epsilon(J) = 0$ and $S(J) \subset J$. Since $\Delta(k[G]_{\bar{1}}) \subset k[G]_{\bar{1}} \otimes k[G]_{\bar{0}} + k[G]_{\bar{0}} \otimes k[G]_{\bar{1}}$ we conclude that $\Delta(J) \subset J \otimes k[G] + k[G] \otimes J$. Hence J is a Hopf ideal and $A = \overline{k[G]} = k[G]/J$ is a Hopf algebra. □

Definition 8.5.13. *Let G be an affine algebraic supergroup represented by the super Hopf algebra $k[G]$. The affine algebraic group defined by the body of $k[G]$ is called the underlying algebraic group of G.*

The morphism $k[G] \to \overline{k[G]}$ defines a subgroup of G. Hence the underlying algebraic group of G is a closed subgroup of G. In general, it is not a normal subgroup. The commutative diagram (3.2) from section 3.1 applies to give the following statement:

Lemma 8.5.14. *Let G, H be two affine algebraic supergroups and $f : G \to H$ be a morphism of group functors. If G' and H' are the underlying algebraic groups of G and H respectively, then there is a morphism $f' : G' \to H'$ such that the following diagram commutes*

$$\begin{tikzcd} G' \arrow[r, "f'"] \arrow[d] & H' \arrow[d] \\ G \arrow[r, "f"'] & H \end{tikzcd} , \tag{8.54}$$

where the vertical arrows are the injections of the underlying algebraic groups into the algebraic supergroups.

If T^n is a torus inside G for some number n, then there is also a T^n inside the underlying algebraic group of G. This can be seen as follows. Suppose $\mathfrak{a}$ is a Hopf ideal in $k[G]$ such that $k[G]/\mathfrak{a} \cong k[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$. Then, by definition of a torus, all the x_i are even in $k[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ and are not nilpotent. Thus the ideal $\mathfrak{a}$ has to contain J and there is a projection $p : k[G]/J \to k[G]/\mathfrak{a}$, where J is the ideal in $k[G]$ generated by the odd elements. But then p realizes the torus T^n as a closed subgroup in the underlying algebraic group of G. Conversely, let T^n be a torus in the underlying algebraic group G' of G. Then there is an ideal $\mathfrak{b} \subset \overline{k[G]}$ such that $k[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] \cong \overline{k[G]}$. Consider the projection $\pi : k[G] \to \overline{k[G]}$ and take the preimage $\mathfrak{a} = \pi^{-1}(\mathfrak{b})$. Then $\mathfrak{a}$ contains J and $k[G]/\mathfrak{a} \cong \overline{k[G]}/\mathfrak{b}$ and hence the torus T^n is also a closed subgroup of G. All in all we have proved the following lemma:

Lemma 8.5.15. *Let $k[G]$ be a super Hopf algebra representing an affine algebraic supergroup G. There is a one-to-one correspondence between the n-tori in G and the n-tori in the underlying affine algebraic group represented by $\overline{k[G]}$.*

Example 8.5.16. Consider the group functor $\mathrm{GL}_{p|q} : A \mapsto \mathrm{GL}_{p|q}(A)$, which was already defined in section 3.7. For the moment, fix a commutative superalgebra A over k. Any element $g \in \mathrm{GL}_{p|q}(A)$ can be written in matrix form as

$$g = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}, \tag{8.55}$$

where a_{00} is a $p \times p$-matrix with entries in $A_{\bar{0}}$, a_{10} and a_{01} are $q \times p$-, respectively $p \times q$-matrices with entries in $A_{\bar{1}}$ and where a_{11} is a $q \times q$-matrix with entries in $A_{\bar{0}}$. However, not all matrices with this prescribed form are elements of $\mathrm{GL}_{p|q}(A)$; g has to be invertible as well. From lemma 3.7.3 it follows that g is invertible if and only if $\det(a_{00})$ and $\det(a_{11})$ are invertible elements in $A_{\bar{0}}$. We will now show that any $g \in \mathrm{GL}_{p|q}(A)$ determines a morphism of superalgebras from a superalgebra $k[\mathrm{GL}_{p|q}]$ to A.

Let $\mathcal{A}$ be the free superalgebra over k generated by the even elements x_{ij}, $y_{\alpha\beta}$, λ and μ and by the odd elements $\xi_{i\alpha}$ and $\eta_{\alpha i}$ where $1 \le i, j \le p$ and $1 \le \alpha, \beta \le q$. Let X be the $p \times p$-matrix with entries $X_{ij} = x_{ij}$ and let Y be the $q \times q$-matrix with entries $Y_{\alpha\beta} = y_{\alpha\beta}$. For any superalgebra A, any morphism $\phi : \mathcal{A} \to A$ is completely determined by choosing for each generator of $\mathcal{A}$ an element in A with the same $\mathbb{Z}_2$-grading. All morphisms $\phi : \mathcal{A} \to A$ such that $\det(X)$ and $\det(Y)$ are invertible elements in $A_{\bar{0}}$ can be described as those morphisms that satisfy $\phi(\det(X))\phi(\lambda) = 1$ and $\phi(\det(Y))\phi(\mu) = 1$. Let us call I the $\mathbb{Z}_2$-graded ideal in $\mathcal{A}$ generated by $\det(X)\lambda - 1$ and $\det(Y)\mu - 1$. Then any morphism $g : \mathcal{A}/I \to A$ maps $\det(X) \bmod I$ and $\det(Y) \bmod I$ to invertible elements in A. But that means that the matrix

$$\begin{pmatrix} g(x_{ij}) & g(\xi_{ia}) \\ g(\eta_{ai}) & g(y_{ab}) \end{pmatrix}, \tag{8.56}$$

is an element of $\mathrm{GL}_{p|q}(A)$. Hence there is a one-to-one correspondence between the morphisms $\mathcal{A}/I \to A$ and the elements of the group $\mathrm{GL}_{p|q}(A)$. We have thus shown

$$\mathrm{GL}_{p|q}(A) \cong \mathrm{Hom}_{\mathbf{sAlg}}(\mathcal{A}/I, A), \tag{8.57}$$

for all superalgebras A. If $f : A \to B$ is a morphism of superalgebras, then there is a natural morphism $\mathrm{GL}_{p|q}(A) \to \mathrm{GL}_{p|q}(B)$ by applying f componentwise to each invertible matrix $g \in \mathrm{GL}_{p|q}(A)$. But if ϕ is the morphism $\phi : \mathcal{A}/I \to A$ corresponding to g, then the element of $\mathrm{GL}_{p|q}(B)$ obtained by applying f componentwise to g is precisely that element of $\mathrm{GL}_{p|q}(B)$ representing the morphism $f \circ \phi : \mathcal{A}/I \to A$. Hence, the functor $\mathrm{GL}_{p|q}$ is isomorphic to the functor $\mathrm{Hom}_{\mathbf{sAlg}}(\mathcal{A}/I, -)$, and $\mathrm{GL}_{p|q}$ is representable by the superalgebra $k[\mathrm{GL}_{p|q}] = \mathcal{A}/I$. Since $\mathrm{GL}_{p|q}(A)$ is a group for each

A, it is guaranteed that $k[\mathrm{GL}_{p|q}]$ is a super Hopf algebra. For a calculational verification and explicit expressions of the antipode we refer to [61]. It is clear by the group structure, which is matrix multiplication, and by example 8.2.1 that the comultiplication on the generators x_{ij} is given by

$$\Delta x_{ij} = \sum_k x_{ik} \otimes x_{kj} + \sum_\alpha \xi_{i\alpha} \otimes \eta_{\alpha j}\,. \tag{8.58}$$

The tori inside $\mathrm{GL}_{p|q}$ are well-studied, see for instance [62, 63]. △

Example 8.5.17. Let Ω be the $(p+2q)\times(p+2q)$-matrix defined by

$$\Omega = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & J_q \end{pmatrix}\,, \qquad J_q = \begin{pmatrix} 0 & -\mathbb{1}_q \\ \mathbb{1}_q & 0 \end{pmatrix}\,, \tag{8.59}$$

where for any natural number m, $\mathbb{1}_m$ denotes the $m\times m$ identity matrix. Define the group functor $\mathrm{Osp}_{p|2q}$ by

$$\mathrm{Osp}_{p|2q}(A) = \left\{X \in \mathrm{GL}_{p|2q}(A) \mid X^{ST}\Omega X = \Omega\right\}\,, \tag{8.60}$$

which is a group subfunctor of $\mathrm{GL}_{p|2q}$. If $f : A \to B$ is a morphism of superalgebras over k, then by applying f to each matrix entry we obtain a morphism of groups $\mathrm{Osp}_{p|2q}(A) \to \mathrm{Osp}_{p|2q}(B)$. To show that $\mathrm{Osp}_{p|2q}$ is representable is straightforward: The equations in eqn.(8.60) define a $\mathbb{Z}_2$-graded ideal in $k[\mathrm{GL}_{p|2q}]$, which one can check (but there is no need to) to be a Hopf ideal. Hence $\mathrm{Osp}_{p|2q}$ is a closed subgroup of $\mathrm{GL}_{p|2q}$. △

8.6 Lie algebras to algebraic supergroups

In this section we associate to any affine algebraic supergroup G a super vector space, called the Lie superalgebra of G. Later we will define a functor for an affine algebraic supergroup that associates a Lie algebra to each superalgebra. We first consider tangent spaces to affine superschemes, derivations and differentials.

8.6.1 Differentials

Let R be a superring and let A be an R-superalgebra. We may assume without loss of generality that R is a sub superring of A. For any A-module M we define a derivation of A into M to be an R-linear map $D : A \to M$ such that for all $a, b \in A$ and all $r \in R$ we have

$$D(ab) = D(a)b + (-1)^{|a||b|}D(b)a\,, \qquad D(ar+b) = D(a)r + D(b)\,. \tag{8.61}$$

The sum of two derivations of A into M is again a derivation of A into M and if D is a derivation of A into M and $a \in A$, then aD is also an derivation of A into M. We can thus also define Da to be the derivation $(Da)(a') = (-1)^{|a||a'|}(Da')a$. We call a derivation $D : A \to M$ of A into M even (resp. odd) if $D(A_{\bar{x}}) \subset M_{\bar{x}}$ (resp. $D(A_{\bar{x}}) \subset M_{\bar{x}+1}$) for $\bar{x} \in \mathbb{Z}_2$. Hence, in a natural way, the derivations of A into M make up an A-module. We denote the module of derivations of A into M by $\mathrm{Der}_R(A, M)$.

We define the A-module $U_{A/R}$ to be the free right A-module generated by the elements $\mathrm{d}a$ where a runs over all elements of A and where d is just an abstract symbol. Consider the submodule N in $U_{A/R}$ generated by all elements of the form $\mathrm{d}(ab) - (\mathrm{d}a)b - (-1)^{|a||b|}(\mathrm{d}b)a$ and all elements of the form $\mathrm{d}(ar+b) - (\mathrm{d}a)r - \mathrm{d}b$, where a, b run over all elements of A and r runs over all elements of R. We define the A-module of Kähler differentials relative to R to be the A-module $\Omega_{A/R} = U_{A/R}/N$. We have a canonical map $\mathrm{d} : A \to \Omega_{A/R}$ given by $a \mapsto \mathrm{d}a$, which we call the canonical derivation.

Lemma 8.6.1. *Let A be an R-superalgebra and $\Omega_{A/R}$ be the module of Kähler differentials relative to R. Then $\Omega_{A/R}$ has the following universal property: The map $\mathrm{d} : A \to \Omega_{A/R}$ is a derivation of A into $\Omega_{A/R}$ and if $D \in \mathrm{Der}_R(A, M)$ is any derivation of A into M, then there is a unique homomorphism of A-modules $f : \Omega_{A/R} \to M$ such that the following diagram commutes*

$$\begin{tikzcd} A \arrow[rr, "D"] \arrow[dr, "\mathrm{d}"'] & & M \\ & \Omega_{A/R} \arrow[ur, "f"'] & \end{tikzcd} \quad . \tag{8.62}$$

In other words, $\underline{\mathrm{Hom}}_A(\Omega_{A/R}, M) \cong \mathrm{Der}_R(A, M)$ holds as an isomorphism of right A-modules.

Proof. The first assertion is obvious: by construction d is a derivation of A into $\Omega_{A/R}$. Suppose $D \in \mathrm{Der}_R(A, M)$ is an even derivation of A into M, then we define $f(\mathrm{d}a) = Da$. Then because D is a derivation of A into M, f is well-defined and defines a homomorphism of A-modules. Uniqueness is clear as $f(\mathrm{d}a) = Da$ is the only possibility.

Now suppose that D is an odd derivation of A into M, we replace M by ΠM and then D is an even derivation of A into ΠM. For general D we decompose D into its homogeneous parts $D = D_{\bar{0}} + D_{\bar{1}}$. We can use the isomorphism of right A-modules $\Pi\underline{\mathrm{Hom}}_A(M, N) \cong \underline{\mathrm{Hom}}_A(M, \Pi N)$ from section 6.1.2 to conclude $\mathrm{Der}_R(A, M) \cong \underline{\mathrm{Hom}}_A(\Omega_{A/R}, M)$. ☐

Example 8.6.2. Let $R = k$ and $A = k[x_1, \ldots, x_p|\eta_1, \ldots, \eta_q]$, then $\Omega_{A/k}$ is the free A-module on the generators $\mathrm{d}x_i$ and $\mathrm{d}\eta_\alpha$. The generators $\mathrm{d}x_i$ are even and the generators $\mathrm{d}\eta_\alpha$ are odd. △

Now we specialize to superschemes over a field k. For a superalgebra A over k we write Ω_A for the module of Kähler differentials $\Omega_{A/k}$. If A and B are two superalgebras over k and $f : A \to B$ is a morphism of superalgebras over k, we obtain an induced morphism $Tf : \Omega_A \to \Omega_B$ as follows: Let $\mathrm{d}_A : A \to \Omega_A$ and $\mathrm{d}_B : B \to \Omega_B$ be the canonical derivations, then $\mathrm{d}_B \circ f$ is easily seen to be a derivation of A into B. Hence by lemma 8.6.1 we obtain a morphism of A-modules $Tf : \Omega_A \to \Omega_B$ such that $Tf \circ \mathrm{d}_A = f \circ \mathrm{d}_B$.

Let $f : A \to B$ be a morphism of superalgebras over k. We define the super vector space of derivations of A into B over f as the super vector space $\mathrm{Der}_k^f(A, B)$, where we view B as an A-module via f. When the morphism f is clear, we sometimes simply write $\mathrm{Der}_k(A, B)$. Of particular interest to us will be those morphisms f that factor over k. More specifically, let X be a superscheme over k. Then a k-point on X is a closed point such that the residue field of the structure sheaf on X is k. If k is algebraically closed and there is a covering of X by open affine superschemes $\mathrm{Spec}(A)$, with each such A being a finitely generated k-superalgebra, then all closed points are in fact k-points. A k-point $x \in X$ defines a morphism $\pi : \mathcal{O}_{X,x} \to k$ given by $\pi(a) = a \bmod \mathfrak{m}_x$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$.

Definition 8.6.3. *If $x \in X$ is a k-point and $\pi : \mathcal{O}_{X,x} \to k$ the projection to the residue field at x, then we define the tangent space of X at x to be the super vector space $\mathrm{Der}_k^\pi(\mathcal{O}_{X,x}, k)$. We denote the tangent space of X at x by T_xX.*

Let $x \in X$ be a k-point of X and $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$, then by lemma 8.6.1 we have $T_xX \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{X,x}}(\Omega_{\mathcal{O}_{X,x}}, k)$, where k becomes an $\mathcal{O}_{X,x}$-module via the map $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_x \cong k$. Since $\mathcal{O}_{X,x}$ is a k-superalgebra, we identify the residue field at x with k. Since the action of $\mathcal{O}_{X,x}$ factors over k, describing T_xX as an $\mathcal{O}_{X,x}$-module or as a super vector space makes no difference. The next lemma formalizes this:

Lemma 8.6.4. *Let X be a superscheme over k and let x be a k-point of X. If $\mathfrak{m}_x$ is the maximal ideal in the local superring $\mathcal{O}_{X,x}$ then there is an isomorphism of super vector spaces*

$$T_xX \cong \underline{\mathrm{Hom}}_{\mathbf{sVec}}(\mathfrak{m}_x/\mathfrak{m}_x^2, k)\,. \tag{8.63}$$

Proof. Write $A = \mathcal{O}_{X,x}$, $\mathfrak{m} = \mathfrak{m}_x$ and write π for the canonical projection $A \to A/\mathfrak{m} = k$, where we use that we identify the residue field at x with k. The superring A is a superalgebra over k and writing $a \in A$ as $a - \pi(a) + \pi(a)$ gives a decomposition $A = k \oplus \mathfrak{m}$. Let a derivation $D \in T_xX = \mathrm{Der}_k^{\pi}(A, k)$ over π be given, then we can associate to D the super vector space homomorphism $\phi_D : a \bmod \mathfrak{m}^2 \mapsto D(a)$, which is well-defined as $D(\mathfrak{m}^2) = 0$. Conversely, if $\phi : \mathfrak{m}/\mathfrak{m}^2 \to k$ is a super vector space homomorphism, then we can assign to ϕ the derivation $D_\phi : a \mapsto \phi(a - \pi(a)) \bmod \mathfrak{m}^2$. Then D_ϕ is indeed a derivation since

$$\begin{aligned} D_\phi(ab) =&\phi((ab - \pi(ab)) \bmod \mathfrak{m}^2) = \phi(((a - \pi(a))\pi(b) + \pi(a)(b - \pi(b))) \bmod \mathfrak{m}^2) \\ =&D_\phi(a)\pi(b) + \pi(a)D_\phi(b)\,. \end{aligned} \tag{8.64}$$

Clearly, the assignments $D \mapsto \phi_D$ and $\phi \mapsto D_\phi$ are k-linear, preserve the $\mathbb{Z}_2$-grading and are inverse to each other. □

We can thus identify the tangent space of X at x with the dual of the super vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. For this reason, we call $\mathfrak{m}_x/\mathfrak{m}_x^2$ the cotangent space of X at x.

For any superalgebra A write A^+ for the augmented superalgebra $A^+ = A[x]/(x^2)$, where we define x to be an even element of A^+. If $f : A \to B$ is a morphism of superalgebras, there is a natural map $f^+ : A^+ \to B^+$, namely, $f^+(a + a'x) = f(a) + f(a')x$. If $p_A : A^+ \to A$ is the projection $p_A : a + a'x \mapsto a$ and $i_A : A \to A^+$ is the inclusion $i : a \mapsto a$, then for any morphism $f : A \to B$ we have $f^+ \circ i_A = i_B \circ f$ and $p_B \circ f^+ = f \circ p_A$.

Definition 8.6.5. *We call the functor $T_A : B \mapsto \mathrm{Hom}_{\mathbf{sAlg}}(A, B^+)$ the total tangent bundle functor associated to the representable functor $B \mapsto \mathrm{Hom}_{\mathbf{sAlg}}(A, B)$. For the action of the total tangent bundle functor on morphisms $f : B \to C$, we define $T_A(f) : \mathrm{Hom}_{\mathbf{sAlg}}(A, B^+) \to \mathrm{Hom}_{\mathbf{sAlg}}(A, C^+)$ by $T_A(f)(g) = f^+ \circ g$.*

Lemma 8.6.6. *Let A and B be superalgebras over k and let the augmented superalgebra of B be defined by $B^+ = B[x]/(x^2)$, where x is given the $\mathbb{Z}_2$-grading $|x| = \bar{0}$. Then $\mathrm{Hom}_{\mathbf{sAlg}}(A, B^+)$ consists of all tuples (φ, D), where $\varphi \in \mathrm{Hom}_{\mathbf{sAlg}}(A, B)$ and D is an even derivation of A into B over φ.*

Proof. Let $f : A \to B^+$ be a morphism of superalgebras, then we write f as a sum $f(a) = f'(a) + f''(a)x$. Then $f(ab) = f(a)f(b) = f'(a)f'(b) + f'(a)f''(b)x + f''(a)f'(b)x$, which shows that $f' \in \mathrm{Hom}_{\mathbf{sAlg}}(A, B)$ and that $f'' \in \mathrm{Der}_k^{f'}(A, B)_{\bar{0}}$, where B carries the A-module structure defined by f'. Conversely, if $f \in \mathrm{Hom}_{\mathbf{sAlg}}(A, B)$ and D is an even derivation of A into B over f, then $a \mapsto f(a) + D(a)x$ defines an element of $\mathrm{Hom}_{\mathbf{sAlg}}(A, B^+)$. □

Remark 8.6.7. In the previous lemma 8.6.6 one can also incorporate the odd derivations as follows: One defines for any superalgebra B the superalgebra $B^\dagger = B[x, \eta]/(x^2, x\eta)$, where x is $\mathbb{Z}_2$-even and η is $\mathbb{Z}_2$-odd. It is easily verified that $\mathrm{Hom}_{\mathbf{sAlg}}(A, B^\dagger)$ consists off all triples (f, D_+, D_-) where $f : A \to B$ is a morphism of superalgebras, $D_+ : A \to B$ is an even derivation of A into B over f and where D_- is an odd derivation of A into B over f.

8.6.2 Derivations on super Hopf algebras

An affine algebraic supergroup G over k has a distinguished k-point, the identity element, which is defined by the counit on the super Hopf algebra $k[G]$ representing G. The tangent space at this point will play the role of the Lie superalgebra. All affine algebraic supergroups are superschemes over a fixed ground field k.

Definition 8.6.8. *Let A be a super Hopf algebra over k, and let M be an A-module. Let ϵ be the counit of A. We call a derivation of A into M over ϵ any k-linear map $D : A \to M$ such that $D(ab) = D(a)\epsilon(b) + (-1)^{|a||b|}D(b)\epsilon(a)$.*

If M is a module over a super Hopf algebra A, then M admits a second A-module structure, namely $m \cdot a = m\epsilon(a)$. The derivations of A into M over ϵ are the derivations of A into M with respect to this alternative A-module structure. We write $\mathrm{Der}_k^\epsilon(A, M)$ for the A-module of derivations of A into M over ϵ. The proof of lemma 8.6.4 can be repeated to show the following lemma:

Lemma 8.6.9. *Suppose A is a super Hopf algebra and M is an A-module. Let $\mathfrak{m}_E$ be the augmentation ideal of A and write $\pi : A \to \mathfrak{m}_E/\mathfrak{m}_E^2$ for the projection that sends $a \in A$ to $(a - \epsilon(a)) \bmod \mathfrak{m}_E^2$. Then the map $\underline{\mathrm{Hom}}_{\mathbf{sVec}}(\mathfrak{m}_E/\mathfrak{m}_E^2, M) \to \mathrm{Der}_k^\epsilon(A, M)$ that sends ϕ to $\phi \circ \pi$, is an isomorphism of super vector spaces.*

The following proposition shows that the new object $\mathrm{Der}_k^\epsilon(A, M)$ is actually not that new.

Proposition 8.6.10. *Let A be a commutative super Hopf algebra and let M be an A-module. Then we have an isomorphism of super vector spaces $\mathrm{Der}_k(A, M) \cong \mathrm{Der}_k^\epsilon(A, M)$.*

Proof. Let $m : M \otimes A \to M$ be the multiplication from the right of A on M, $\mu : A \otimes A \to A$ be the multiplication of A, $\Delta : A \to A \otimes A$ be the comultiplication and $S : A \to A$ be the comultiplication of A. We have the identity

$$m \circ m \otimes \mathrm{id}_A = m \circ \mathrm{id}_M \otimes \mu \,. \tag{8.65}$$

We define a map $f : \mathrm{Der}_k(A, M) \to \mathrm{Der}_k^\epsilon(A, M)$ as follows

$$f(D) = m \circ D \otimes S \circ \Delta \,, \qquad f(D)(a) = D(a') \cdot S(a'') \,. \tag{8.66}$$

Clearly, $f(D)$ is k-linear and for $a, b \in A$ we have

$$\begin{aligned} f(D)(ab) &= \mathrm{Sgn}(|a''||b'|)D(a'b') \cdot S(a''b'') \\ &= D(a') \cdot (S(a')b'S(b'')) + \mathrm{Sgn}(|a||b'|)D(b') \cdot (a'S(a'')S(b'')) \\ &= f(D)(a)\epsilon(b) + \mathrm{Sgn}(|a||b|)f(D)(b)\epsilon(a) \,, \end{aligned} \tag{8.67}$$

which proves that $f(D) \in \mathrm{Der}_k^\epsilon(A, M)$. We define a map $g : \mathrm{Der}_k^\epsilon(A, M) \to \mathrm{Der}_k(A, M)$ as follows

$$g(\delta) = m \circ \delta \otimes \mathrm{id}_A \circ \Delta \,, \qquad g(\delta)(a) = \delta(a') \cdot (a'') \,. \tag{8.68}$$

To show that $g(\delta)$ is an element of $\mathrm{Der}_k(A, M)$ we calculate

$$\begin{aligned} m \circ \delta \otimes \mathrm{id}_A \circ \Delta(ab) &= (-1)^{|a''||b'|}\delta(a'b') \cdot (a''b'') \\ &= \delta(a') \cdot (a''\epsilon(b')b'') + (-1)^{|a||b|}\delta(b') \cdot (b''\epsilon(a')a'') \\ &= g(\delta)(a) \cdot b + (-1)^{|a||b|}g(\delta) \cdot a \,. \end{aligned} \tag{8.69}$$

Furthermore, if $\delta \in \mathrm{Der}_k^\epsilon(A, M)$ then we have

$$\begin{aligned} f(g(\delta)) &= m \circ g(\delta) \otimes S \circ \Delta \\ &= m \circ m \otimes \mathrm{id}_A \circ \delta \otimes \mathrm{id}_A \otimes \mathrm{id}_A \circ \Delta \otimes \mathrm{id}_A \circ \mathrm{id}_A \otimes S \circ \Delta \\ &= m \circ \mathrm{id}_M \otimes \mu \circ \delta \otimes \mathrm{id}_A \otimes \mathrm{id}_A \circ \mathrm{id}_A \otimes \mathrm{id}_A \otimes S \circ \mathrm{id}_A \otimes \Delta \circ \Delta \\ &= m \circ \delta \otimes \mathrm{id}_A \circ \mathrm{id}_A \otimes (\mu \circ \mathrm{id}_A \otimes S \circ \Delta) \circ \Delta \\ &= m \circ \delta \otimes \epsilon \circ \Delta \\ &= \delta\,, \end{aligned} \tag{8.70}$$

where we used eqn.(8.66) and elementary properties of super Hopf algebras. The proof that for any $D \in \mathrm{Der}_k(A, M)$ we have $g \circ f(D) = D$ is similar. $\square$

Lemma 8.6.11. *Let A be a super Hopf algebra over a field k. Write $\mathfrak{m}_E$ for the augmentation ideal of A. The module of Kähler differentials relative to k is given by $\Omega_A = \mathfrak{m}_E/\mathfrak{m}_E^2 \otimes A$, where the tensor product is over k. The canonical derivation $\mathrm{d} : A \to \Omega_A$ is given by $\pi \otimes \mathrm{id}_A \circ \Delta$, where $\pi : A \to \mathfrak{m}_E/\mathfrak{m}_E^2$ is the projection $\pi : a \mapsto (a - \epsilon(a)) \bmod \mathfrak{m}_E^2$.*

Proof. We show that the A-module $\mathfrak{m}_E/\mathfrak{m}_E^2 \otimes A$ has the required universal property.

Suppose $D : A \to M$ is any k-linear derivation of A into M. By proposition 8.6.10 there is a unique $\delta \in \mathrm{Der}_k^\epsilon(A, M)$ such that $D = m \circ \delta \otimes \mathrm{id}_A \circ \Delta$, where $m : M \otimes A \to M$ is the right action of A on M. By lemma 8.6.9 δ is given by $\delta = f \circ \pi$ for a unique $f \in \underline{\mathrm{Hom}}_{\mathbf{sVec}}(\mathfrak{m}_E/\mathfrak{m}_E^2, M)$. Hence we find that $D = m \circ f \otimes \mathrm{id}_A \circ \mathrm{d}$. Define $f_D = m \circ f \otimes \mathrm{id}_A$, then f_D makes the following diagram commute

$$\begin{tikzcd} A \arrow[rr, "D"] \arrow[dr, "\mathrm{d}"'] & & M \\ & \mathfrak{m}_E/\mathfrak{m}_E^2 \otimes A \arrow[ur, "f_D"'] & \end{tikzcd} \quad . \tag{8.71}$$

For uniqueness of f_D, we note that we have an isomorphism of super vector spaces $\underline{\mathrm{Hom}}_A(\mathfrak{m}_E/\mathfrak{m}_E^2 \otimes A, M) \cong \underline{\mathrm{Hom}}_{\mathbf{sVec}}(\mathfrak{m}_E/\mathfrak{m}_E^2, M)$, as any homomorphism $\phi : \mathfrak{m}_E/\mathfrak{m}_E^2 \otimes A \to M$ is completely determined by its action on the elements of the form $x \otimes 1$, with $x \in \mathfrak{m}_E/\mathfrak{m}_E^2$. The isomorphism thus sends $g \in \underline{\mathrm{Hom}}_A(\mathfrak{m}_E/\mathfrak{m}_E^2 \otimes A, M)$ to the map $m \circ g \otimes \mathrm{id}_A \in \underline{\mathrm{Hom}}_A(\mathfrak{m}_E/\mathfrak{m}_E^2 \otimes A, M)$. Hence f_D is uniquely determined by f, which was uniquely determined by D. $\square$

Definition 8.6.12. *Let A be a super Hopf algebra over k with comultiplication Δ. We call a k-linear map $L : A \to A$ left-invariant if $\mathrm{id}_A \otimes L \circ \Delta = \Delta \circ L$.*

We write $\mathfrak{X}_A^L$ for the super vector space of left-invariant derivations from A into A. As to be expected from the experience of left-invariant vector fields on Lie groups, we expect that a left-invariant derivation of A into A is completely determined by its "value at the origin". For super Hopf algebras, to be determined by the value at the origin, means that a left-invariant derivation $D : A \to A$ is completely determined by $\epsilon \circ D$.

Lemma 8.6.13. *Let A be a super Hopf algebra over k and write $\mathfrak{X}_A^L$ for the super vector space of left-invariant derivations of A into A. Then the map $\sigma : \mathfrak{X}_A^L \to \mathrm{Der}_k^\epsilon(A, k)$ that sends $D \in \mathfrak{X}_A^L$ to $\epsilon \circ D$ is an isomorphism of super vector spaces.*

Proof. The proof is similar to the proof of proposition 8.6.10. The inverse of σ is given by

$$\sigma^{-1} : \mathrm{Der}_k^{\epsilon}(A,k) \to \mathfrak{X}_A^L\,, \qquad \sigma^{-1}(\delta) = \mathrm{id}_A \otimes \delta \circ \Delta\,, \tag{8.72}$$

where we identify $A \otimes k \cong A$. Indeed, we have for $D \in \mathfrak{X}_A^L$

$$\begin{aligned} \mathrm{id}_A \otimes (\epsilon \circ D) \circ \Delta &= \mathrm{id}_A \otimes \epsilon \circ \mathrm{id}_A \otimes D \circ \Delta \\ &= \mathrm{id}_A \otimes \epsilon \circ \Delta \circ D \\ &= \mathrm{id}_A \circ D = D\,. \end{aligned} \tag{8.73}$$

For the converse, again identifying $A \otimes k \cong A$ we have for any $\delta \in \mathrm{Der}_k^{\epsilon}(A,k)$

$$\epsilon \circ \mathrm{id}_A \otimes \delta \circ \Delta = \delta \circ \epsilon \otimes \mathrm{id}_A \circ \Delta = \delta\,. \tag{8.74}$$

□

Theorem 8.6.14. *Let A be a Noetherian super Hopf algebra over a field k, $\mathfrak{X}_A^L$ the super vector space of left-invariant derivations of A into A, $\beta : A \to k$ any superalgebra morphism and let $\mathfrak{m}_E$ be the augmentation ideal of A. Then we have isomorphisms*

$$(\mathfrak{m}_E/\mathfrak{m}_E^2)^* \cong \mathfrak{X}_A^L \cong \mathrm{Der}_k^{\epsilon}(A,k) \cong \mathrm{Der}_k^{\beta}(A,k) \tag{8.75}$$

Proof. Since A is Noetherian, the super vector space $\mathfrak{m}_E/\mathfrak{m}_E^2$ is finite-dimensional and for finite-dimensional super vector spaces $\underline{\mathrm{Hom}}_{\mathbf{sVec}}(V,W) \cong W \otimes V^*$. Taking this into account, the proof follows from lemmas 8.6.11, 8.6.13 and proposition 8.6.10. □

Another variation of theorem 8.6.14 is the following theorem:

Theorem 8.6.15. *Let A be a Noetherian super Hopf algebra over k, $\mathfrak{m}_E$ the augmentation ideal of A, ϵ the counit of A and Ω_A the module of Kähler differentials of A relative to k. Suppose B is a superalgebra over k and that we are given a morphism $A \to B$, by which we can view B as an A-module. Then we have isomorphisms of super vector spaces*

$$\underline{\mathrm{Hom}}_A(\Omega_A, B) \cong \mathrm{Der}_k(A,B) \cong \mathrm{Der}_k^{\epsilon}(A,B) \cong \underline{\mathrm{Hom}}_{\mathbf{sVec}}(\mathfrak{m}_E/\mathfrak{m}_E^2, B) \cong B \otimes (\mathfrak{m}_E/\mathfrak{m}_E^2)^* \tag{8.76}$$

Proof. The proof follows immediately from lemmas 8.6.11, 8.6.13 and proposition 8.6.10 together from the observation that $\mathfrak{m}_E/\mathfrak{m}_E^2$ is a finite-dimensional super vector space over k. □

8.6.3 Lie superalgebras of supergroups

In this section we fix a ground field k and all superalgebras are over this ground field k.

Definition 8.6.16. *Let G be an affine algebraic supergroup with representing super Hopf algebra $k[G]$. We define the Lie superalgebra of G to be the super vector space $\mathrm{Der}_k^{\epsilon}(k[G],k)$. The Lie bracket is given as follows: for homogeneous $x, y \in \mathrm{Der}_k^{\epsilon}(k[G],k)$ we define $[x,y] = m \circ (x \otimes y - (-1)^{|x||y|} y \otimes x) \circ \Delta$, where m is the multiplication $k \otimes k \to k$.*

One easily sees that the Lie bracket on $\mathrm{Der}_k^{\epsilon}(k[G],k)$ makes $\mathrm{Der}_k^{\epsilon}(k[G],k)$ into a Lie superalgebra. The Lie bracket on the left-invariant derivations is given by a similar formula: If $D, D' \in \mathfrak{X}_A^L$ are left-invariant derivations, then $[D,D'] = \mu \circ (D \otimes D' - (1)^{|D||D'|} D' \otimes D) \circ \Delta$, where μ is the multiplication map $\mu : A \otimes A \to A$. If $\beta : k[G] \to k$ is a morphism of superalgebras, then β defines the k-point $x_\beta = \mathrm{Ker}(\beta)$. The tangent space at x_β is then by theorem 8.6.14 isomorphic to the tangent space at the counit and carries an isomorphic Lie superalgebra structure. An easy consequence of the definition of the Lie bracket is the following lemma:

Lemma 8.6.17. *Let G be an abelian affine algebraic supergroup, then the Lie superalgebra of G is abelian.*

Proposition 8.6.18. *Let G and H be affine algebraic supergroups with Lie superalgebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. If $\phi : G \to H$ is a morphism of supergroups, then we have an induced natural morphism of Lie superalgebras $\mathrm{d}\phi : \mathfrak{g} \to \mathfrak{h}$. In particular, if G is a closed subgroup of H, then $\mathfrak{g}$ is a Lie sub superalgebra of $\mathfrak{h}$.*

Proof. Suppose that $\varphi : k[H] \to k[G]$ is the morphism of super Hopf algebras that induces ϕ, then we define the map $\mathrm{d}\phi : \mathrm{Der}_k^\epsilon(k[G], k) \to \mathrm{Der}_k^\epsilon(k[H], k)$ by

$$\mathrm{d}\phi(D) = D \circ \varphi\,, \qquad D \in \mathrm{Der}_k^\epsilon(k[H], k)\,. \tag{8.77}$$

It follows from the properties of φ that $\mathrm{d}\phi$ is linear and preserves the $\mathbb{Z}_2$-grading. Writing $m : k \otimes k \to k$ for the multiplication map of k we have for $x, y \in \mathrm{Der}_k^\epsilon(k[H], k)$

$$\begin{aligned} \mathrm{d}\phi[x, y] &= m \circ (x \otimes y - (-1)^{|x||y|} y \otimes x) \circ \Delta \circ \varphi \\ &= m \circ (x \otimes y - (-1)^{|x||y|} y \otimes x) \circ \varphi \otimes \varphi \circ \Delta \\ &= [\mathrm{d}\phi(x), \mathrm{d}\phi(y)]\,. \end{aligned} \tag{8.78}$$

Hence $\mathrm{d}\phi$ is a morphism of Lie superalgebras. For the last claim we note that, if G is a closed subgroup of H, then φ is a surjective map and for that reason the map $\mathrm{d}\phi$ is injective. ☐

Combining the last results, one sees that the Lie superalgebra of a torus in an affine algebraic supergroup G is an abelian Lie sub superalgebra of the Lie superalgebra of G. For an affine algebraic supergroup G, the underlying affine algebraic group $\bar{G}$ is a closed subgroup. Therefore from proposition 8.6.18 we conclude that the Lie algebra of $\bar{G}$ is a Lie subalgebra of the Lie superalgebra of G. The following proposition singles out which Lie algebra.

Proposition 8.6.19. *Let G be an affine algebraic supergroup with representing super Hopf algebra $k[G]$ and with Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$. Then the Lie algebra of the underlying affine algebraic group $\bar{G}$, represented by the Hopf algebra $\overline{k[G]}$, has Lie algebra $\bar{\mathfrak{g}} = \mathfrak{g}_{\bar{0}}$.*

Proof. It suffices to show that $\mathrm{Der}_k^\epsilon(k[G], k)_{\bar{0}} \cong \mathrm{Der}_k^\epsilon(\overline{k[G]}, k)$ as super vector spaces. Since $\overline{k[G]} \cong k[G]_{\bar{0}}/(k[G]_{\bar{1}})^2$ this is obvious: any even derivation $D : k[G] \to k$ is zero on $(k[G]_{\bar{1}})^2$ and hence descends to a derivation $\bar{D} : \overline{k[G]} \to k$. Conversely, any derivation $\bar{D} : \overline{k[G]} \to k$ can be lifted for the same reason to a derivation $D : k[G] \to k$ that is zero on the odd part, hence is even. ☐

Suppose G, H are affine algebraic supergroups with underlying affine algebraic groups $\bar{G}, \bar{H}$ and with Lie superalgebras $\mathfrak{g}, \mathfrak{h}$ respectively. If $f : G \to H$ is a morphism of groups, then we have an induced morphism of underlying groups $\bar{f} : \bar{G} \to \bar{H}$ and by commutativity of the diagram (3.2) we have a commutative diagram

$$\begin{array}{ccc} \bar{G} & \longrightarrow & G \\ \downarrow & & \downarrow \\ \bar{H} & \longrightarrow & H \end{array} \,. \tag{8.79}$$

If $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie superalgebras of G and H respectively, the morphism $\mathrm{d}f : \mathfrak{g} \to \mathfrak{h}$ of Lie superalgebras maps $\mathfrak{g}_{\bar{0}}$ into $\mathfrak{h}_{\bar{0}}$.

Remark 8.6.20. Let A be a superalgebra and let $\beta : A \to k$ be an morphism of superalgebras. It is tempting to think that $\mathrm{Der}_k^{\beta}(A,k)_{\bar{0}} = \mathrm{Der}_k^{\beta}(A_{\bar{0}},k)$. Clearly, we have the inclusion $\mathrm{Der}_k^{\beta}(A,k)_{\bar{0}} \subset \mathrm{Der}_k^{\beta}(A_{\bar{0}},k)$. But in general the inclusion is proper: Consider $A = k[\eta_1,\eta_2]$ and $\beta : A \to k$ given by $\beta(\eta_1) = \beta(\eta_2) = 0$. Then $A_{\bar{0}} \cong k[x]/(x^2)$ and the k-linear map $D : ax+b \bmod x^2 \mapsto a$ is a derivation of $A_{\bar{0}}$ that cannot be lifted to an even derivation of A, as $D(\eta_1\eta_2) = \beta(\eta_1)D(\eta_2) + D(\eta_1)\beta(\eta_2) = 0$. Intuitively, the derivations of $A_{\bar{0}}$ don't see that $(A_{\bar{1}})^2 \subset A_{\bar{0}}$.

Proposition 8.6.21. *Let G_1, G_2 and G_3 be affine algebraic supergroups and suppose $f : G_1 \to G_2$ and $g : G_2 \to G_3$ are morphisms of affine super groupschemes. Then $\mathrm{d}(g \circ f) = \mathrm{d}g \circ \mathrm{d}f$.*

Proof. Let A_1, A_2 and A_3 be the super Hopf algebras representing G_1, G_2 and G_3 respectively and write $\varphi : A_2 \to A_1$ and $\chi : A_3 \to A_2$ for the super Hopf algebra morphisms that induces f and g. Then if $D \in \mathrm{Der}_k^{\epsilon}(A_1,k)$, we have $\mathrm{d}(g\circ f)(D) = D \circ \varphi \circ \chi = (D\circ\varphi)\circ\chi = \mathrm{d}g(\mathrm{d}f(D))$. □

Lemma 8.6.22. *Let $f : G \to H$ be a morphism of affine algebraic supergroups and let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie superalgebras of G and H respectively. Then the Lie superalgebra of the kernel of f is the kernel of $\mathrm{d}f$.*

Proof. Let A, B be the super Hopf algebras representing H and G respectively and let $\mathfrak{m}_A$ and $\mathfrak{m}_B$ denote the augmentation ideal of A and B respectively. Furthermore, let $\varphi : A \to B$ be the morphism of super Hopf algebras inducing f. The kernel of f is by proposition 8.5.5 represented by the super Hopf algebra $C = B/\mathfrak{m}_A \cdot B$. Since $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$ the augmentation ideal of C, which we denote $\mathfrak{m}_C$, is given by the image of $\mathfrak{m}_B$ under the projection $\pi : B \to B/\mathfrak{m}_A \cdot B$. We conclude that

$$\mathfrak{m}_C/\mathfrak{m}_C^2 \cong \mathfrak{m}_B/(\mathfrak{m}_B^2 + \mathfrak{m}_A \cdot B)\,. \tag{8.80}$$

The morphisms φ and π induce morphisms of super vector spaces $\varphi^* : \mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ and $\pi^* : \mathfrak{m}_B/\mathfrak{m}_B^2 \to \mathfrak{m}_C/\mathfrak{m}_C^2$. The map π^* is clearly surjective and $\pi^* \circ \varphi^* = 0$. We claim that the sequence

$$\mathfrak{m}_A/\mathfrak{m}_A^2 \xrightarrow{\varphi^*} \mathfrak{m}_B/\mathfrak{m}_B^2 \xrightarrow{\pi^*} \mathfrak{m}_C/\mathfrak{m}_C^2 \longrightarrow 0 \tag{8.81}$$

is exact. To prove the claim, we only need to verify that the kernel of π^* is contained in the image of φ^*. Let $b \in \mathfrak{m}_B$ be such that $\pi^*(b \bmod \mathfrak{m}_B^2) = 0$, then $b \in \mathfrak{m}_B^2 + \mathfrak{m}_A \cdot B$, and thus there are $a_i \in \mathfrak{m}_A$ and $b_i \in B$ such that $b \equiv \sum \varphi(a_i)b_i \bmod \mathfrak{m}_B^2$. Using that $\varphi(a_i) \in \mathfrak{m}_B$ we see that $\varphi(a_i)b_i \equiv \varphi(a_i)\epsilon(b_i) \bmod \mathfrak{m}_B^2$, where ϵ is the counit of B, and we conclude that $b \in \mathfrak{m}_B^2 + \varphi(\mathfrak{m}_A)$. This proves the claim. The lemma then follows by dualizing the exact sequence (8.81) and noting that the kernel of a morphism of Lie superalgebras is a Lie sub superalgebra. □

Proposition 8.6.23. *Let $E \to G' \to G \to G'' \to E$ be an exact sequence of affine algebraic supergroups with E the trivial group and where exact means that for all superalgebras A the sequence $1 \to G'(A) \to G(A) \to G''(A) \to 1$ is exact. Let $\mathfrak{g}'$, $\mathfrak{g}$, and $\mathfrak{g}''$ denote the Lie superalgebras of G', G and G'' respectively. Then we have an exact sequence of super vector spaces $0 \to \mathfrak{g}' \to \mathfrak{g} \to \mathfrak{g}'' \to 0$.*

Proof. Since G' is the kernel of the morphism $G \to G''$, G' is a closed subgroup of G and by lemma 8.6.22 the sequence $0 \to \mathfrak{g}' \to \mathfrak{g} \to \mathfrak{g}''$ is exact. We thus only need to prove that $\mathrm{d}f : \mathfrak{g} \to \mathfrak{g}''$ is surjective if $f : G \to G''$ is surjective. Let A and B be the super Hopf algebras representing G'' and G respectively and let $\varphi : A \to B$ be the morphism of super Hopf algebras inducing f. From proposition 8.5.9 it follows that φ has a left inverse $\chi : B \to A$. Let D' be a derivation of A into k over the counit, then $D = D' \circ \chi : B \to k$ is a derivation of B into k over $\epsilon_A \circ \chi$, where ϵ_A is the counit of A. Denote $m : k \otimes B \to k$ the map that sends $1 \otimes b$ to $\epsilon_A(\chi(b))$ and write S for the antipode of B and Δ for the comultiplication of B. Then from proposition 8.6.10 we know that $\tilde{D} = m \circ D \otimes S \circ \Delta$ is a derivation of B to k over the counit of B. A short calculation shows that $\mathrm{d}f(\tilde{D}) = \tilde{D} \circ \varphi = D'$ and thus $\mathrm{d}f$ is surjective. □

Example 8.6.24. Let $G = \mathrm{GL}_{p|q}$ be the general linear supergroup introduced in example 8.5.16. We write $k[G] = k[X_{ij}, \lambda, \mu]/I$, where I is the ideal generated by $\det(X_{00})\lambda - 1$ and $\det(X_{11})\mu - 1$ and where $X_{00} = (X_{ij})_{1\le i,j\le p}$ and $X_{11} = (X_{ij})_{p+1\le i,j\le p+q}$; also see example 8.5.16. The comultiplication in $k[G]$ is determined by $\Delta(X_{ij}) = \sum_k X_{ik} \otimes X_{kj}$.

Any derivation on $k[G]$ is uniquely determined by its values on the X_{ij}. Denote $\frac{\partial}{\partial X_{ij}}$ the derivation from $k[G]$ to k sending X_{kl} to 1 if $k = i, j = l$ and zero otherwise. Thus the parity of $\frac{\partial}{\partial X_{ij}}$ is $|i| + |j|$.

Any element of $\mathfrak{gl}_{p|q} = \mathrm{Der}_k^\epsilon(k[G], k)$ can be expanded in terms of the $\frac{\partial}{\partial X_{ij}}$. We consider the map $\varphi : \mathrm{Mat}_{p|q}(k) \to \mathfrak{gl}_{p|q}$ that sends any matrix $M = (M_{ij})$ to the derivation $\varphi(M)$ given by

$$\varphi(M) = \sum_{ij} (-1)^{|i|(|i|+|j|)} M_{ij} \frac{\partial}{\partial X_{ij}} . \tag{8.82}$$

For all $M, N \in \mathrm{Mat}_{p|q}(k)$ the map φ satisfies $\varphi(M) * \varphi(N) = \varphi(MN)$, where $*$ is the product introduced in section 8.3.1 and MN is the matrix product of M and N. Hence the map φ sets up an isomorphism of Lie superalgebras $\mathfrak{gl}_{p|q} \cong \mathrm{Mat}_{p|q}(k)$. (We introduced the Lie superalgebra structure on $\mathrm{Mat}_{p|q}(k)$ in section 2.2.)

Now we consider the Lie algebra functor to G. It is easy to see that any derivation $D \in \mathfrak{gl}_{p|q}(A) = \mathrm{Der}_k^\epsilon(k[G], A)_{\bar 0}$ can be written as

$$D = \sum_{i,j} D_{ij} \frac{\partial}{\partial X_{ij}} , \tag{8.83}$$

where $D_{ij} \in A_{\bar 0}$ if $1 \le i, j \le p$ or when $p+1 \le i, j \le p+q$ and $D_{ij} \in A_{\bar 1}$ in the other cases. When we consider the map that sends the derivation D to the matrix $D = (D_{ij}) \in \mathrm{Mat}_{p|q}(A)_{\bar 0}$, we obtain an isomorphism of $A_{\bar 0}$-modules $\mathfrak{gl}_{p|q}(A) \cong \mathrm{Mat}_{p|q}(A)_{\bar 0}$. In fact, this is an isomorphism of Lie algebras over $A_{\bar 0}$ since the Lie bracket in $\mathfrak{gl}_{p|q}(A)$ is given by

$$[D_1, D_2] = \sum_{ij} [D_1, D_2]_{ij} \frac{\partial}{\partial X_{ij}} = \sum_{ij} [D_1, D_2](X_{ij}) \frac{\partial}{\partial X_{ij}} , \tag{8.84}$$

and

$$[D_1, D_2](X_{ij}) = \sum_k D_1(X_{ik}) D_2(X_{kj}) - D_2(X_{ik}) D_2(X_{kj}) . \tag{8.85}$$

Thus, upon the identification of $\mathfrak{gl}_{p|q}(A)$ with $\mathrm{Mat}_{p|q}(A)_{\bar 0}$, the Lie bracket of $\mathfrak{gl}_{p|q}(A)$ becomes the Lie bracket of $\mathrm{Mat}_{p|q}(A)_{\bar 0}$. △

8.6.4 Lie algebra functors

As we defined affine algebraic supergroups as representable functors, it turns out to be convenient to have a functorial definition of Lie algebras of affine algebraic supergroups as well. In order to do so, we need to broaden the concept of a Lie algebra to a Lie algebra over some ring.

Let R be a commutative ring. A Lie algebra over a ring R is an A-module L together with an R-bilinear map $[,] : L \times L \to L$ satisfying $[x, y] = -[y, x]$ and $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$. The only thing different from ordinary Lie algebras is that the vector space structure relative to some field is replaced by the module structure with respect to some ring and that we do not require $[x, x] = 0$ but the more stringent condition $[x, y] = -[y, x]$. If L is a Lie algebra over the ring R and L' is a Lie algebra over the ring R', then we define a morphism of Lie algebras

$f : L \to L'$ to be a pair of (f, ϕ) where $\phi : R \to R'$ is a morphism of rings and where $f : L \to L'$ is a morphism of abelian groups satisfying $f(rx) = \phi(r)f(x)$ for all $r \in R$ and $x \in L$ and satisfying $f([x, y]) = [f(x), f(y)]$ for all $x, y \in L$.

Definition 8.6.25. *Let G be an affine algebraic supergroup represented by a super Hopf algebra $k[G]$. We define the Lie algebra functor of G to be the functor $L_G : \mathbf{sAlg} \to \mathbf{sVec}$ that associates to a superalgebra B over k the super vector space $\mathrm{Der}_k^\epsilon(k[G], B)_{\bar{0}}$ and that associates to a morphism $f : B \to C$ the morphism $L_G(f) : \mathrm{Der}_k^\epsilon(k[G], B)_{\bar{0}} \to \mathrm{Der}_k^\epsilon(k[G], C)_{\bar{0}}$ given by $L_G(f) : D \mapsto f \circ D$.*

Let A be a super Hopf algebra with comultiplication Δ and B a superalgebra with multiplication map $\mu : B \otimes B \to B$. The Lie algebra structure on $\mathrm{Der}_k^\epsilon(A, B)_{\bar{0}}$ is the following: For $D_1, D_2 \in \mathrm{Der}_k^\epsilon(A, B)$ we put $[D_1, D_2] = \mu \circ (D_1 \otimes D_2 - D_2 \otimes D_1) \circ \Delta$. Then $\mathrm{Der}_k^\epsilon(A, B)_{\bar{0}}$ is a Lie algebra over $B_{\bar{0}}$. We now briefly explore the relation with the total tangent bundle functor and the definition of Fioresi and Lledó [64] of the Lie algebra to an affine algebraic supergroup.

Lemma 8.6.26. *Let G be an affine algebraic supergroup represented by a super Hopf algebra $k[G]$. Write TG for the total tangent bundle functor $TG(A) = \mathrm{Hom}_{\mathbf{sAlg}}(k[G], A^+)$. Then there is a natural isomorphism $TG(A) \cong G(A) \times L_G(A)$.*

Proof. The proof is essentially the same as the proof of proposition 8.6.10. By lemma 8.6.6 an element of $\mathrm{Hom}_{\mathbf{sAlg}}(k[G], B^+)$ is a pair (f, D) with $f \in \mathrm{Hom}_{\mathbf{sAlg}}(k[G], B)$ and $D \in \mathrm{Der}_k^f(k[G], B)$. Let μ_B be the multiplication in B, Δ the comultiplication of $k[G]$ and S the antipode of $k[G]$. Then define a map $\sigma_B : \mathrm{Hom}_{\mathbf{sAlg}}(k[G], B^+) \to G(B) \times L_G(B)$ as follows

$$\sigma_B : (f, D) \mapsto (f, \mu_B \circ D \otimes (f \circ S) \circ \Delta)\,. \tag{8.86}$$

To show that $\mu_B \circ D \otimes (f \circ S) \circ \Delta$ is an element of $\mathrm{Der}_k^\epsilon(A, B)$ is practically the same calculation as is done in proposition 8.6.10. The inverse to σ_B is given by

$$\sigma_B^{-1} : (f, \delta) \mapsto (f, \mu_B \circ \delta \otimes f \circ \Delta)\,. \tag{8.87}$$

Suppose we have a morphism of superalgebras $\chi : B \to C$, then there is an induced morphism $\chi^* : \mathrm{Hom}_{\mathbf{sAlg}}(k[G], B^+) \to \mathrm{Hom}_{\mathbf{sAlg}}(k[G], C^+)$ sending $(f, D) \in \mathrm{Hom}_{\mathbf{sAlg}}(k[G], B^+)$ to $(\chi \circ f, \chi \circ D)$. Similarly, we have an induced morphism $\chi^\# : G(B) \times L_G(B) \to G(C) \times L_G(C)$ sending $(f, \delta) \in G(B) \times L_G(B)$ to $(\chi \circ f, \chi \circ \delta)$. One easily sees that $\sigma_C \circ \chi^* = \chi^\# \circ \sigma_B$, proving naturality. □

Geometrically, the last lemma says that the tangent bundle of an affine algebraic supergroup is trivial. Let G be an affine algebraic supergroup represented by a Noetherian super Hopf algebra $k[G]$ and let $\mathfrak{g}$ be the Lie superalgebra of G. By theorem 8.6.15 we have $\mathrm{Der}_k^\epsilon(k[G], B) \cong (\mathfrak{g} \otimes B)_{\bar{0}}$. In fact, it is not hard to see that the isomorphism is natural in the second variable, so that any morphism of superalgebras $\varphi : B \to C$ induces the map $\varphi^* : (\mathfrak{g} \otimes B)_{\bar{0}} \to (\mathfrak{g} \otimes C)_{\bar{0}}$ given by $\varphi^* : x \otimes b \mapsto x \otimes \varphi(b)$. We thus have

Lemma 8.6.27. *Let G be an affine algebraic supergroup represented by a Noetherian super Hopf algebra $k[G]$ and with Lie superalgebra $\mathfrak{g}$. Then the Lie algebra functor L_G is isomorphic to the functor that assigns to each superalgebra B the Lie algebra $(\mathfrak{g} \otimes B)_{\bar{0}}$.*

The Lie algebra structure on $(\mathfrak{g} \otimes B)_{\bar{0}}$ is in fact more transparent. If $x \otimes b$ and $y \otimes b'$ are elements of $(\mathfrak{g} \otimes B)_{\bar{0}}$, then we have $[x \otimes b, y \otimes b'] = (-1)^{|y||b|}[x, y] \otimes bb'$. Thus $L_G(B)$ and $(\mathfrak{g} \otimes B)_{\bar{0}}$ are isomorphic as Lie algebras over the ring $B_{\bar{0}}$. We will therefore often write $\mathfrak{g}$ for the functor L_G if no confusion is possible.

Let G and H be affine algebraic supergroups with Lie superalgebras $\mathfrak{g}$ and $\mathfrak{h}$. We have seen above that a morphism $f : G \to H$ of group functors induces a morphism of Lie superalgebras

$\mathrm{d}f : \mathfrak{g} \to \mathfrak{h}$. However, there is also an induced natural transformation $L_G \to L_H$, given by composition with f. One easily checks that for any superalgebra B this natural transformation is given by $\mathrm{d}f \otimes \mathrm{id}_B : (\mathfrak{g} \otimes B)_{\bar{0}} \to (\mathfrak{h} \otimes B)_{\bar{0}}$. By the theorem of Deligne and Morgan 3.6.1 these are the only natural transformations $L_B \to L_C$ that respect the Lie algebra structure. In fact, the theorem provides an alternative proof that shows that the natural transformation $L_G \to L_H$ induced by f is in fact of the form $h \otimes \mathrm{id}_B : (\mathfrak{g} \otimes B)_{\bar{0}} \to (\mathfrak{h} \otimes B)_{\bar{0}}$ for all superalgebras B and for some morphism of Lie superalgebras $h : \mathfrak{g} \to \mathfrak{h}$.

The morphism $f : G \to H$ induces a natural transformation $Tf : TG \to TH$ between the total tangent bundle functors. If A is any superalgebra, then $\alpha \in TG(A) = \mathrm{Hom}_{\mathbf{sAlg}}(k[G], A^+)$ we define $Tf(\alpha) = \alpha \circ \varphi : \mathrm{Hom}_{\mathbf{sAlg}}(k[G], A^+) \to \mathrm{Hom}_{\mathbf{sAlg}}(k[H], A^+)$, where $\varphi : k[H] \to k[G]$ is the morphism of super Hopf algebras that induces f. Evaluating Tf at the counit of G we obtain a morphism $L_G \to L_H$, which is the derived transformation $\mathrm{d}f$:

Lemma 8.6.28. *Let G and H be affine algebraic supergroups represented by super Hopf algebras $k[G]$ and $k[H]$ respectively and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebra functors. Suppose we have a morphism $f : G \to H$, then if $Tf : TG \to TH$ is the induced morphism between total tangent bundle functors, then the differential $\mathrm{d}f : \mathfrak{g} \to \mathfrak{h}$ is given by*

$$(\epsilon, \mathrm{d}f(D)) = Tf(\epsilon, D)\,. \tag{8.88}$$

Proof. The proof follows immediately from the definition of the total tangent bundle functor and of the way we constructed Tf. □

When B is a superalgebra over k, then we have maps $p_B : B^+ \to B$ and $i_B : B \to B^+$, see subsection 8.6.1. Then if A is a super Hopf algebra, we have a morphism of groups $\rho_B : \mathrm{Hom}_{\mathbf{sAlg}}(A, B^+) \to \mathrm{Hom}_{\mathbf{sAlg}}(A, B)$. In [64] the Lie algebra of an affine algebraic supergroup is defined as the functor that assigns to B the kernel of ρ_B. Remembering that $\mathrm{Hom}_{\mathbf{sAlg}}(A, B^+)$ consisted of all pairs (f, D) with $f \in \mathrm{Hom}_{\mathbf{sAlg}}(A, B)$ and D an even derivation of A into B over f, we easily see that $\mathrm{Ker}\,(\rho_B) = \mathrm{Der}_k^{\epsilon}(A, B)_{\bar{0}}$. Therefore, our definition of a Lie algebra of an affine algebraic supergroup coincides with the definition of Fioresi and Lledó. For the case of an algebraic group scheme the use of the total tangent bundle functor was used and developed extensively in for example [65, 66].

Example 8.6.29. Consider the group functor $SL_{p|q}$ that is defined as the functor that assigns to any superalgebra A the set of all elements M of $\mathrm{GL}_{p|q}(A)$ with $\mathrm{Ber}(M) = 1$. One calls $SL_{p|q}$ the special linear supergroup. Writing $M \in \mathrm{GL}_{p|q}(A)$ in block-matrices as

$$M = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix}\,, \tag{8.89}$$

we can rewrite $\mathrm{Ber}(M) = 1$ as

$$\det(m_{00} - m_{01} m_{11}^{-1} m_{10}) = \det(m_{11})\,. \tag{8.90}$$

We conclude that $SL_{p|q}$ is a closed subgroup of $\mathrm{GL}_{p|q}$. We use the notation of example 8.5.16 and introduce the matrices $X_{00} = (x_{ij})$, $X_{01} = (x_{ia})$, $X_{10} = (x_{ai})$ and $X_{11} = (x_{ab})$. To obtain the Lie superalgebra $\mathrm{Der}_k^{\epsilon}(k[SL_{p|q}], k)$ we note that $\mathrm{Der}_k^{\epsilon}(k[SL_{p|q}], k)$ consists of all derivations of $k[\mathrm{GL}_{p|q}]$ to k over the counit ϵ that vanish on the $\mathbb{Z}_2$-graded ideal in $k[\mathrm{GL}_{p|q}]$ generated by $\det(X_{00} - X_{01} X_{11}^{-1} X_{10}) - \det(X_{11}) \in k[\mathrm{GL}_{p|q}]$. Noting that any matrix element of $X_{01} X_{11}^{-1} X_{10}$ lies in $(\mathrm{Ker}\,\epsilon)^2$, and using example 8.6.24 we see that the Lie superalgebra $\mathfrak{sl}_{p|q}$ consists of all elements in $\mathrm{Mat}_{p|q}(k)$ with zero supertrace:

$$\mathfrak{sl}_{p|q} = \left\{ N \in \mathrm{Mat}_{p|q}(k) \mid \mathrm{str}(N) = 0 \right\}\,. \tag{8.91}$$

For the Lie algebra functor $\mathfrak{sl}_{p|q} : A \mapsto \mathfrak{sl}_{p|q} = \mathrm{Der}_k^{\epsilon}(k[SL_{p|q}], A)_{\bar{0}}$ we then get

$$\mathfrak{sl}_{p|q}(A) = \left\{ Y \in \mathrm{Mat}_{p|q}(A)_{\bar{0}} \mid \mathrm{str}(Y) = 0 \right\} . \tag{8.92}$$

The Lie bracket is the same as in $\mathfrak{gl}_{p|q}$, see for example 8.6.24. △

Example 8.6.30. Let Ω be the $(p+2q) \times (p+2q)$-matrix defined in eqn.(8.59). Then for any superalgebra A, the group $\mathrm{Osp}_{p|2q}(A)$ is given by all $(p+2q) \times (p+2q)$-matrices M that satisfy

$$M^{ST} \Omega M = \Omega \, . \tag{8.93}$$

It follows that each $M \in \mathrm{Osp}_{p|2q}(A)$ is invertible and hence is a subgroup of $\mathrm{GL}_{p|2q}(A)$. Writing $M \in \mathrm{GL}_{p|q}(A)$ in block-matrices as

$$M = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix} , \tag{8.94}$$

we see that $M \in \mathrm{Osp}_{p|2q}(A)$ if and only if we have

$$m_{00}^T m_{00} - m_{10}^T J m_{10} = 1 \, , \quad m_{00}^T m_{01} - m_{10}^T J m_{11} = 0 \, , \quad m_{01}^T m_{01} + m_{11}^T J m_{11} = J \, . \tag{8.95}$$

The equations $M^{ST} \Omega M - \Omega = 0$ clearly define a $\mathbb{Z}_2$-graded ideal in $k[\mathrm{GL}_{p|2q}]$ and thus $\mathrm{Osp}_{p|2q}$ is a closed subgroup of $\mathrm{GL}_{p|2q}$. We can now proceed as in the case of $SL_{p|q}$ to calculate the Lie superalgebra $\mathfrak{osp}_{p|2q}$ and the Lie algebra functor. Using the explicit isomorphism $\mathfrak{gl}_{p|2q} \cong \mathrm{Mat}_{p|2q}(k)$ constructed in example 8.6.24 one finds

$$\mathfrak{osp}_{p|2q} \cong \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{p|2q}(k) \mid a^T + a = b + c^T J = d^T J + J d = 0 \right\} , \tag{8.96}$$

which is the super vector space of all matrices $M \in \mathrm{Mat}_{p|2q}(k)$ satisfying $M^{ST}\Omega + \Omega M = 0$. Hence we see that $\mathfrak{osp}_{p|2q}$ is indeed isomorphic to the orthosymplectic Lie superalgebra $\mathfrak{osp}_{p|2q}(k)$ defined in section 2.2. Note that when we write an element $D \in \mathfrak{osp}_{p|2q}$ as $\sum D_{ij} \frac{\partial}{\partial X_{ij}}$ the matrix (D_{ij}) satisfies $D^+ \Omega + \Omega D = 0$ as a super vector space, where D^+ stands for $((D^{ST})^{ST})^{ST}$. The appearance of D^+ is due to the asymmetry between the supertranspose for odd and even elements.

Taking A any superalgebra, tensoring $\mathfrak{osp}_{p|2q}$ with A, and using lemma 3.7.7 and example 8.6.24 one finds that we have a further isomorphism

$$\mathfrak{osp}_{p|2q}(A) \cong \left\{ Y \in \mathrm{Mat}_{p|2q}(A)_{\bar{0}} \mid Y^{ST} \Omega + \Omega Y = 0 \right\} . \tag{8.97}$$

Computing $\mathrm{Der}_k^{\epsilon}(k[\mathrm{Osp}_{p|2q}], A)$ directly also gives the result of eqn.(8.97). The Lie bracket is the same as for $\mathfrak{gl}_{p|2q}$. We remark that $\mathfrak{osp}_{p|2q} \subset \mathfrak{sl}_{p|2q}$. △

Remark 8.6.31. In the previous example we have seen there is an asymmetry between the signs in the definition of the Lie superalgebra and in the definition of the Lie algebra functor. Most authors seem to notice this asymmetry, but solve the problem by simply redefining the supertranspose (see for example [5]).

Chapter 9

Representations and comodules

In this chapter we will study some aspects of representations of affine algebraic supergroups. This will give insight into the structure of affine algebraic supergroups. To discuss representations of supergroups, we will need to develop a little bit of machinery to deal with comodules of super coalgebras and we will need to know more about the structure of super coalgebras. Because of the duality between superalgebras and super coalgebras, we are forced to consider not only commutative superalgebras. Therefore, in this chapter, a superalgebra will not be commutative unless otherwise mentioned.

The first section 9.1 will be devoted to define representations of affine algebraic supergroups and to establish the link between comodules of super coalgebras and representations of affine algebraic supergroups. In section 9.2 we study the structure of super coalgebras and comodules. In section 9.3 this will be used to derive some properties of representations and to show that all affine algebraic supergroups are closed subgroups of the general linear supergroup $\mathrm{GL}_{p|q}$ for some p and q. Then we end this chapter with a section on the representations of the Lie algebra of an affine algebraic supergroup and on the adjoint representation of the supergroup in its Lie algebra. We work over a fixed field k of characteristic zero.

9.1 Representations versus comodules

Let V be a super vector space over k. The super vector space can be viewed as the functor that assigns to any commutative superalgebra A the Grassmann envelop $V(A) = (V \otimes A)_{\bar{0}}$. We define the group functor $\mathrm{GL}_V : \mathbf{sAlg} \to \mathbf{Sets}$ to be the functor that assigns to each commutative superalgebra A the even invertible elements of $\mathrm{End}_A(V \otimes A)$. If V is finite-dimensional and if we fix a basis of V, we have an isomorphism $V \cong k^{p|q}$ for some p and q, which induces an isomorphism of group functors $\mathrm{GL}_V \cong \mathrm{GL}_{p|q}$. Therefore, GL_V is an affine algebraic supergroup for finite-dimensional V. The action of $\mathrm{GL}_V(A)$ on $(V \otimes A)_{\bar{0}}$ is easily seen to be a natural transformation $\mathrm{GL}_V \times V \to V$.

Definition 9.1.1. *Let G be a group functor and V a super vector space, viewed as a functor. A linear representation of G in V is a morphism of group functors $G \to \mathrm{GL}_V$. Equivalently, a linear representation of G in V is a natural transformation $G \times V \to V$ that factors over $\mathrm{GL}_V \times V \to V$.*

Often we will omit the adjective linear and just write representation. If G is an affine algebraic supergroup represented by $k[G]$, then for finite-dimensional V a representation is equivalently described by a morphism of super Hopf algebras $\phi : k[\mathrm{GL}_V] \to k[G]$. If G has a linear representation in V, we call V a G-module. If $W \subset V$ is a sub super vector space, we call W a submodule of

V if the action of G on V restricts to an action on W. In other words, for each commutative superalgebra A, the image of $W(A)$ under the map $G(A) \times V(A) \to V(A)$ lies in $W(A)$. If W is a submodule of V, then there is a natural representation of G in V/W.

Let V and V' be super vector spaces. By the theorem of Deligne and Morgan 3.6.1 there is a one-to-one correspondence between the natural transformation from V to V', when we view the super vector spaces as functors, and the super vector space morphisms from V to V'. Suppose that G has a representation in V and in V'. We call a super vector space morphism $f : V \to V'$ a morphism of G-modules if for all commutative superalgebras the diagram

$$\begin{tikzcd} G(A) \times V(A) \arrow[r] \arrow[d, "\mathrm{id}\times f_A"] & V(A) \arrow[d, "f_A"] \\ G(A) \times V'(A) \arrow[r] & V'(A) \end{tikzcd} \tag{9.1}$$

commutes, where the horizontal rows are the actions of G on V and V', and where f_A is the induced morphism $f \otimes \mathrm{id}_A : (V \otimes A)_{\bar{0}} \to (V' \otimes A)_{\bar{0}}$. Note that an action of G on V can easily be extended to an representation on $V \oplus \Pi V$ by using the embedding $(V \otimes A)_{\bar{0}} \to V \otimes A$ for each commutative superalgebra A. Indeed, $\mathrm{GL}_V(A)$ naturally acts on $V \otimes A$.

We want to relate representations of the affine algebraic supergroup G to comodules of $k[G]$, which we will now define.

Definition 9.1.2. *Let C be a super coalgebra with comultiplication map Δ and counit ϵ. A left comodule over C is a super vector space V together with a morphism of super vector spaces $\psi : V \to V \otimes C$ such that $\psi \otimes \mathrm{id}_C \circ \psi = \mathrm{id}_V \otimes \Delta \circ \psi$ and $\mathrm{id}_V \otimes \epsilon \circ \psi = \mathrm{id}_V$, where we identified $V \otimes k \cong V$.*

The conditions on the morphisms ψ, Δ and ϵ can be compactly formulated by requiring the commutativity of the following diagrams:

$$\begin{tikzcd} V \arrow[r, "\cong"] \arrow[d, "\psi"'] & V \otimes k \\ V \otimes C \arrow[ur, "\mathrm{id}\otimes\epsilon"'] & \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\psi"] \arrow[d, "\psi"'] & V \otimes C \arrow[d, "\mathrm{id}\otimes\Delta"'] \\ V \otimes C \arrow[r, "\psi\otimes\mathrm{id}"'] & V \otimes C \otimes C \end{tikzcd} . \tag{9.2}$$

A right comodule is defined in a similar way as a left comodule, replacing $\psi : V \to V \otimes C$ by $\psi : V \to C \otimes V$ and so on.

Let V be a left comodule over a super coalgebra C with the structure map $\psi : V \to V \otimes C$. If W is a sub super vector space in V such that $\psi(W) \subset W \otimes C$, then we call W a sub comodule of V. One easily verifies that W is indeed a left comodule over C with structure map the restriction of ψ to W. If V' is a second comodule over C with structure map $\psi' : V' \to V' \otimes C$, then we call a super vector space morphism $f : V \to V'$ a morphism of comodules if $\psi' \circ f = f \otimes \mathrm{id}_C \circ \psi$. The image $f(V)$ is easily seen to be a sub comodule of V', and the kernel of f is a sub comodule of V. For a submodule $W \subset V$, there is an induced comodule structure on the quotient V/W and one easily checks that the projection $\pi : V \to V/W$ is a morphism of comodules.

Theorem 9.1.3. *Let G be an affine algebraic supergroup represented by the super Hopf algebra $k[G]$. Then a linear representation $\Phi : G \to \mathrm{GL}_V$ on V corresponds to a unique k-linear map*

$\rho : V \to V \otimes k[G]$ such that the following diagrams commute:

$$
\begin{tikzcd}
V \arrow[r, "\rho"] \arrow[dr, "\cong"] & V \otimes k[G] \arrow[d, "id_V \otimes \epsilon"] \\
& V \otimes k
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[rr, "\rho"] \arrow[d, "\rho"] & & V \otimes k[G] \arrow[d, "\rho \otimes id_{k[G]}"] \\
V \otimes k[G] \arrow[rr, "id_V \otimes \Delta"] & & V \otimes k[G] \otimes k[G]
\end{tikzcd}
. \tag{9.3}
$$

Conversely, given a map $\rho : V \to V \otimes k[G]$ such that the above diagrams commute, then ρ defines a representation. In other words, there is a one-to-one correspondence between G-modules and left $k[G]$-comodules.

Proof. The proof is actually no more than Yoneda's lemma. We follow the approach of [60]. We extend for each commutative superalgebra A the action of $\mathrm{GL}_V(A)$ on $V(A)$ to an action on $V \otimes A$.

We write $X = k[G]$ and define for each $v \in V$ a k-linear map $\rho(v) = \Phi^X(\mathrm{id}_X)(v \otimes 1)$. Let A be any commutative superalgebra and $g \in G(A) = \mathrm{Hom}_{\mathbf{sAlg}}(X, A)$, then the action of g is determined by ρ as the commutative diagram shows:

$$
\begin{tikzcd}
V(X) \arrow[r, "\Phi^X(\mathrm{id}_X)"] \arrow[d, "\mathrm{id}_V \otimes g"'] & V(X) \arrow[d, "\mathrm{id}_V \otimes g"'] \\
V(A) \arrow[r, "\Phi^A(g)"] & V(A)
\end{tikzcd}
. \tag{9.4}
$$

We indeed have for any $v \otimes a \in V \otimes A$

$$
\begin{aligned}
\Phi^A(g)(v \otimes a) &= \Phi^A(g)(v \otimes 1)a \\
&= \Phi^A(g)(\mathrm{id}_V \otimes g)(v \otimes 1)a \\
&= \mathrm{id}_V \otimes g \circ \Phi^X(\mathrm{id}_X)(v \otimes 1)a \\
&= (\mathrm{id}_V \otimes g \circ \rho(v))a\,.
\end{aligned} \tag{9.5}
$$

Requiring $\Phi^A(e) = \mathrm{id}_{V(A)}$, where e is the identity element of the group $G(A)$ and is the composition of the counit with the inclusion $k \to A$, we obtain immediately the first commutative diagram. To see what happens when we require $\Phi^A(gh) = \Phi^A(g)\Phi^A(h)$ we first write out the action of $\Phi^A(gh)$:

$$\Phi^A(gh)(v \otimes 1) = \mathrm{id}_V \otimes \mu_A \circ \mathrm{id}_V \otimes g \otimes h \circ \mathrm{id}_V \otimes \Delta \circ \rho(v) \tag{9.6}$$

where $\mu_A : A \otimes A \to A$ is the multiplication map of A. The action of $\Phi^A(g)\Phi^A(h)$ is given by

$$\Phi^A(g)\Phi^A(h)(v \otimes 1) = \mathrm{id}_V \otimes \mu_A \circ \mathrm{id}_V \otimes g \otimes h \circ \rho \otimes \mathrm{id}_X \circ \rho(v)\,. \tag{9.7}$$

If the second diagram in (9.3) commutes the actions of $\Phi^A(g)\Phi^A(h)$ and $\Phi^A(gh)$ are the same. For the converse, we take $A = X \otimes X$ and g and h the superalgebra morphisms $g : a \mapsto a \otimes 1$ and $h : a \mapsto 1 \otimes a$. Then $\mu_A \circ g \otimes h : X \otimes X \otimes X \otimes X$ is the identity map on $X \otimes X$. Therefore the second diagram in (9.3) commutes if and only if $\Phi^A(g)\Phi^A(h) = \Phi^A(gh)$ for all A and for all $g, h \in G(A)$.

On the other hand, if we have an even k-linear map $\rho : V \to V \otimes X$ such that the diagrams (9.3) commute, we get a natural transformation $G \to \mathrm{End}(V)$ by putting $\Phi^A(g)(v \otimes a) = \mathrm{id}_V \otimes g \circ \rho(v)a$. We only need to check that the image lies in GL_V, which is obvious since $\Phi^A(g^{-1})\Phi^A(g) = \mathrm{id}_{V(A)}$. This completes the proof. □

Proposition 9.1.4. *Let G be an affine algebraic supergroup represented by the super Hopf algebra $k[G]$ and let V and V' be two G-modules. A morphism $f : V \to V'$ is a morphism of G-modules if and only if f is a morphism of left $k[G]$-comodules.*

Proof. Let $\psi : V \to V \otimes k[G]$ and $\psi' : V' \to V' \otimes k[G]$ be the structure maps that induce the representations of G in V and V' respectively. Clearly, if f is a morphism of left $k[G]$-comodules, then the diagram (9.1) commutes. Conversely, suppose that diagram (9.1) commutes for all commutative superalgebras A. Then extending the action of $G(A)$ to $V \otimes A$, the diagram still commutes. Now take $A = k[G]$, $\mathrm{id}_{k[G]} \in G(k[G])$ and consider $v \otimes 1 \in V \otimes k[G]$, then the commutativity of diagram (9.1) implies that

$$\mathrm{id}_V \otimes \mathrm{id}_{k[G]} \circ \psi'(f(v)) = f \otimes \mathrm{id}_{k[G]} \circ \psi(v)\,. \tag{9.8}$$

As v was arbitrary, we conclude that f is a morphism of left $k[G]$-comodules. ☐

An immediate consequence is that if $W \subset V$ is a submodule of V, then W is a sub comodule as well. Even more, by proposition 9.1.4 there is an equivalence of categories of G-modules and $k[G]$-comodules. Due to its importance, we state this as a corollary:

Corollary 9.1.5. *Let G be an affine algebraic supergroup represented by a super Hopf algebra $k[G]$. There is an equivalence of categories between G-modules and $k[G]$-comodules. In particular, if V is a G module, and $W \subset V$ is a sub super vector space, then W is a G-submodule if and only if W is a $k[G]$-subcomodule.*

Consider a finite-dimensional super vector space V of dimension $p|q$ and equip V with a standard homogeneous basis $\{e_i\}_{1 \le i \le p+q}$. Then we may identify $k[\mathrm{GL}_V]$ with $k[X_{ij}, \lambda, \mu]/I$, where I is the ideal generated by $\det(X_{00})\lambda - 1$ and $\det(X_{11})\mu - 1$ and where $X_{00} = (X_{ij})_{1 \le i,j \le p}$ and $X_{11} = (X_{ij})_{p+1 \le i,j \le p+q}$; also see example 8.5.16. We write $\Phi : \mathrm{GL}_V \times V \to V$ for the natural action of GL_V on V. For the comodule structure map $\psi : V \to V \otimes k[\mathrm{GL}_V]$ we then find

$$\psi(e_i) = \Phi^{k[\mathrm{GL}_V]}(\mathrm{id}_{k[\mathrm{GL}_V]})(e_i \otimes 1) = e_j \otimes X_{ji}\,, \tag{9.9}$$

since the identity map on $k[\mathrm{GL}_V]$ is given by $X_{kl} \mapsto X_{kl}$ and is therefore represented by the matrix (X_{kl}). Now we consider any affine algebraic supergroup G. If we have a linear representation of G on V, then that corresponds to a morphism of super Hopf algebras $\varphi : k[\mathrm{GL}_V] \to k[G]$. Let us denote by a_{ij} the image of the X_{ij} under the morphism φ. If Δ is the comultiplication of $k[G]$, then since φ commutes with comultiplication, we have $\Delta a_{ij} = \sum_k a_{ik} \otimes a_{kj}$. If $g \in G(A)$ for some commutative superalgebra A, then the image of g in $\mathrm{GL}_V(A)$ is represented by the matrix with entries $g(a_{ij})$.

An important representation of an affine group superscheme is given by the regular representation, where the vector space is the super Hopf algebra $k[G]$ and the comodule map ψ is the comultiplication Δ. It is easy to see that the map $\Delta : k[G] \to k[G] \otimes k[G]$ indeed makes the diagrams (9.3) commute. Although $k[G]$ is in general not finite-dimensional, we will see later that each element in $k[G]$ is contained in a finite-dimensional submodule inside $k[G]$. In other words, $k[G]$ is as a comodule the sum of its finite-dimensional sub comodules.

If V and W are G-modules, then $V \oplus W$ can also be equipped with the structure of a G-module. If $\phi_V : G \to \mathrm{GL}_V$ and $\phi_W : G \to \mathrm{GL}_W$ are the natural transformations defining the representations, we define $\phi : G \to \mathrm{GL}_{V \oplus W}$ by mapping $g \in G(A)$ to the linear transformation that maps $(v, w) \in V \oplus W$ to $(\phi_V^A(g)(v), \phi_W^A(g)(w))$ for all commutative superalgebras A. Let $\psi_V : V \to V \otimes k[G]$ and $\psi_W : W \to W \otimes k[G]$ be the comodule structure maps corresponding to ϕ_V and ϕ_W respectively. One easily checks that the comodule structure map corresponding to ϕ is given by $\psi : (v, w) \mapsto (\psi_V(v), \psi_W(w))$.

Given super vector space V and W, then to $V \otimes W$ corresponds the functor $A \mapsto (V \otimes W \otimes A)_{\bar{0}}$. We have isomorphisms

$$\begin{aligned} &r : (V \otimes A) \otimes_A (V \otimes A) \to V \otimes W \otimes A\,, \\ &r((v \otimes a) \otimes_A (w \otimes b)) = v \otimes w \otimes (-1)^{|a||w|} ab\,, \quad \text{and} \\ &s : V \otimes W \otimes A \to (V \otimes A) \otimes_A (V \otimes A)\,, \\ &s(v \otimes w \otimes a) = (v \otimes 1) \otimes_A (w \otimes a)\,. \end{aligned} \tag{9.10}$$

Given representations ϕ_1 and ϕ_2 of G on V and W respectively, then for a representation $\phi_\otimes$ on $V \otimes W$ we require

$$\phi_\otimes^A(g)(v \otimes w \otimes 1) = r\Big(\phi_1^A(g)(v \otimes 1) \otimes_A \phi_2^A(g)(w \otimes 1)\Big)\,, \tag{9.11}$$

for any commutative superalgebra A and $g \in G(A)$. The corresponding comodule map $\psi^\otimes : V \otimes W \to V \otimes W \otimes k[G]$ is calculated to be

$$\psi^\otimes(v \otimes w) = r\Big(\psi_1 \otimes \psi_2(v \otimes w)\Big)\,, \tag{9.12}$$

where ψ_1 and ψ_2 are the comodule maps corresponding to ϕ_1 and ϕ_2 respectively. Using the isomorphisms r and s it is an easy matter to verify that $\psi^\otimes$ makes the diagrams (9.3) commute, and therefore defines a representation.

Proposition 9.1.6. *Let G be an affine algebraic supergroup and let V be a G-module. If U is a sub super vector space of V, the subfunctors $\mathrm{Stab}_G^U : \mathbf{sAlg} \to \mathbf{Sets}$ and $G^U : \mathbf{sAlg} \to \mathbf{Sets}$ defined by*

$$\begin{aligned} \mathrm{Stab}_G^U(A) &= \{g \in G(A) \mid g(U(A)) \subset U(A)\}\ , \\ G^U(A) &= \{g \in G(A) \mid g \cdot u \otimes 1 = u \otimes 1\,, \quad \forall u \in U\}\ , \end{aligned} \tag{9.13}$$

for all commutative superalgebras A, are closed subgroups of G.

Proof. Clearly, for all commutative superalgebras A, $G^U(A)$ is a subgroup of $G(A)$. To show that $\mathrm{Stab}_G^U(A)$ is also a subgroup, we first remark that clearly, the identity of $G(A)$ is in $\mathrm{Stab}_G^U(A)$ and for any pair $g, g' \in \mathrm{Stab}_G^U(A)$ also $g \cdot g' \in \mathrm{Stab}_G^U(A)$. From $g \in \mathrm{Stab}_G^U(A)$ it follows that $U \subset g^{-1}(U)$ and we obtain a chain of $A_{\bar{0}}$ modules $U \subset g^{-1}U(A) \subset g^{-2}U(A) \cdots$. If $A_{\bar{0}}$ is a finitely generated superalgebra and V is finite-dimensional, then $V(A)$ is a Noetherian $A_{\bar{0}}$-module and thus the chain has to become stable. Let N be such that $g^{-N}U(A) = g^{-N-1}U(A)$. Then for any $u \in U$ we have $g^{-1}(u \otimes 1) = g^N(g^{-N-1}(u \otimes 1) = g^N g^{-N} u' = u'$ for some $u' \in U(A)$. Hence $g^{-1} \in \mathrm{Stab}_G^U(A)$. Thus for all finitely generated A, $\mathrm{Stab}_G^U(A)$ is a subgroup of $G(A)$.

Let $\{u_i\}_i$ be a homogeneous basis for U in V and choose any complement $W \subset V$ such that $W \oplus U = V$. Choose a homogeneous basis $\{w_a\}_a$ of W and write $k[G]$ for the super Hopf algebra representing G. There are $\sigma_{ia} \in k[G]$ and $\tau_{ij} \in k[G]$ such that the comodule morphism $\psi : V \to V \otimes k[G]$ satisfies $\psi(u_i) = \sum_a w_a \otimes \sigma_{ia} + \sum_j u_j \otimes \tau_{ji}$. Then $g \in \mathrm{Stab}_G^U(A)$ if and only if g vanishes on the $\mathbb{Z}_2$-graded ideal generated by the σ_{ia}. Further, $g \in G^U(A)$ if and only if g vanishes on the $\mathbb{Z}_2$-graded ideal generated by the elements σ_{ia}, $\tau_{ij} - \delta_{ij}$, where δ_{ij} is 1 if $i = j$ and zero otherwise. Hence by lemma 8.5.6 and remark 8.5.7 these ideals are Hopf ideals and the proposition is proved. □

9.1.1 Application to normal subgroups

Let G be an affine algebraic supergroup with representing super Hopf algebra $k[G]$. Write $\Delta : k[G] \to k[G] \otimes k[G]$ for the comultiplication, $\mu : k[G] \otimes k[G] \to k[G]$ for the multiplication and $S : k[G] \to k[G]$ for the antipode. We have a natural transformation $C : G \times G \to G$ given by $C^A : (g,h) \mapsto ghg^{-1}$ for any superalgebra A and $g, h \in G(A)$. For any superalgebra A and $g, h \in G(A)$, the morphism $C(g,h) : k[G] \to A$ is given by

$$
\begin{aligned}
C(g,h) &= \mu_A \circ \mu_A \otimes \mathrm{id}_A \circ g \otimes h \otimes (g \circ S) \circ \Delta \otimes \mathrm{id}_{k[G]} \circ \Delta \\
&= \mu_A \circ g \otimes h \circ \mathrm{id}_{k[G]} \otimes T \circ \mathrm{id}_{k[G]} \otimes \mathrm{id}_{k[G]} \otimes S \circ \Delta \otimes \mathrm{id}_{k[G]} \circ \Delta \,,
\end{aligned} \tag{9.14}
$$

where $\mu_A : A \otimes A \to A$ is the multiplication in A and $T : k[G] \otimes k[G] \to k[G] \otimes k[G]$ is the map given by $T : x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$. Hence we find that the corresponding morphism of superalgebras $c : k[G] \to k[G] \otimes k[G]$ is given by

$$
c = \mu \otimes \mathrm{id}_{k[G]} \circ \mathrm{id}_{k[G]} \otimes T \circ \mathrm{id}_{k[G]} \otimes \mathrm{id}_{k[G]} \otimes S \circ \Delta \otimes \mathrm{id}_{k[G]} \circ \Delta \,. \tag{9.15}
$$

Now let H be a closed subgroup of G with representing super Hopf algebra $k[H] = k[G]/\mathfrak{a}$ for some Hopf ideal $\mathfrak{a}$. We say H is a normal subgroup if C restricts to a map $C : G \times H \to H$. That is, if $i : H \to G$ is the closed embedding induced by the projection $\pi : k[G] \to k[H]$, we require the existence of a map $C' : G \times H \to H$ such that the following diagram commutes

$$
\begin{tikzcd}
G \times H \arrow[rr, "C\circ(\mathrm{id}_G, i)"] \arrow[dr, "C'"'] & & G \\
& H \arrow[ur, "i"'] &
\end{tikzcd} \quad . \tag{9.16}
$$

On the level of super Hopf algebras we thus need that there is a map $c' : k[H] \to k[G] \otimes k[H]$ such that the following diagram commutes

$$
\begin{tikzcd}
k[G] \otimes k[H] & & G \arrow[ll, "\mathrm{id}_{k[G]}\otimes\pi\circ c"'] \arrow[dl, "\pi"] \\
& k[H] \arrow[ul, "c'"] &
\end{tikzcd} \quad . \tag{9.17}
$$

In other words, we require that $c' : k[H] \to k[G] \otimes k[H]$ defined $c'(x \bmod \mathfrak{a}) = \mathrm{id}_{k[G]} \otimes \pi \circ c(x)$ is well-defined and turns $k[H]$ into a left $k[G]$-comodule. In particular, we need that $\mathfrak{a} \subset \mathrm{Ker}\,(\mathrm{id}_{k[G]} \otimes \pi \circ c)$. Using lemma 8.1.5 we conclude that c has to satisfy $c(\mathfrak{a}) \subset \mathrm{Ker}\,(\mathrm{id}_{k[G]} \otimes \pi) = k[G] \otimes \mathfrak{a}$. Conversely, if $c(\mathfrak{a}) \subset k[G] \otimes \mathfrak{a}$, then c' is well-defined. One calls a $\mathbb{Z}_2$-graded Hopf ideal $\mathfrak{a}$ of $k[G]$ with the property $c(\mathfrak{a}) \subset k[G] \otimes \mathfrak{a}$ a normal Hopf ideal. The preceding discussion thus establishes a one-to-one correspondence between normal Hopf ideals and closed normal subgroups.

Let $f : G \to H$ be a morphism of affine algebraic supergroups. Then the kernel of f is a normal closed subgroup of G. Indeed, for each superalgebra A, the kernel of $f^A : G(A) \to H(A)$ is a normal subgroup. In proposition 8.5.5 we showed that the kernel of f is a closed subgroup. If follows that the Hopf ideal defining the kernel must be a normal Hopf ideal, but we will show this now using (super) Hopf techniques and following [21].

Let $k[G]$ and $k[H]$ be the super Hopf algebras representing G and H respectively. Suppose $\phi : k[H] \to k[G]$ is the morphism of super Hopf algebras that induces the group morphism f. By proposition 8.5.5 the kernel is a closed subgroup of G defined by the Hopf ideal $\phi(I_H)k[G]$, where I_H is the augmentation ideal of $k[H]$.

Let $\mathfrak{a}$ be the kernel of ϕ and let furthermore $\Delta_G, S_G, \epsilon_G$ and $\Delta_H, S_H, \epsilon_H$ be the comultiplication, antipode and counit of $k[G]$ respectively $k[H]$. Since $\epsilon_H = \epsilon_G \circ \phi$, $S_G \circ \phi = \phi \circ S_H$ and $\Delta_G \circ \phi = \phi \otimes \phi \circ \Delta_H$, we conclude that $\epsilon_H(\mathfrak{a}) = 0$, $S_H(\mathfrak{a}) \subset \mathfrak{a}$ and $\Delta(\mathfrak{a}) \subset \mathrm{Ker}\,(\phi \otimes \phi) = \mathfrak{a} \otimes k[H] + k[H] \otimes \mathfrak{a}$. Hence $\mathfrak{a}$ is a Hopf ideal and defines a closed subgroup of H. Since the kernel of f is defined by the ideal generated by the image under ϕ of the augmentation ideal of $k[H]$ and since the augmentation ideal of $k[H]$ contains $\mathfrak{a}$, we may replace $k[H]$ by $k[H]/\mathfrak{a}$. (One can think of the affine algebraic supergroup defined by $k[H]/\mathfrak{a}$ as the image of G.) We thus think of $k[H]$ as being a sub super Hopf algebra of $k[G]$.

Under the assumption that $k[H]$ is a sub super Hopf algebra of $k[G]$ with augmentation ideal I_H, we now show that $I_H \cdot k[G]$ is a normal Hopf ideal. Let $c : k[G] \to k[G] \otimes k[G]$ be the morphism defined in eqn.(9.15) and let $a \in I_H$, then writing $\Delta \otimes \mathrm{id}_{k[G]} \circ \Delta = \sum a_i \otimes b_i \otimes c_i$ gives

$$\mathrm{id}_{k[G]} \otimes \epsilon \circ c(a) = \sum (-1)^{|b_i||c_i|} a_i S(c_i) \epsilon(b_i)\,. \tag{9.18}$$

Using coassociativity, commutativity and the identity $\mathrm{id}_{k[G]} = \mathrm{id}_{k[G]} \otimes \epsilon \circ \Delta$, one obtains

$$\sum (-1)^{|b_i||c_i|} a_i S(c_i) \epsilon(b_i) = \mu \circ \mathrm{id}_{k[G]} \otimes S \circ \Delta(a) = \epsilon(a) = 0\,. \tag{9.19}$$

Therefore $c(I_H) \subset \mathrm{Ker}\,(\mathrm{id}_{k[G]} \otimes \epsilon) = k[G] \otimes I_G$, where I_G is the augmentation ideal of $k[G]$. However, since $k[H]$ is a sub super Hopf algebra, we have $c(k[H]) \subset k[H] \otimes k[H]$ and thus $c(I_H) \subset k[H] \otimes I_H$. But then

$$c(I_H \cdot k[G]) \subset (k[H] \otimes I_H)(k[G] \otimes k[G]) \subset k[G] \otimes (I_H \cdot k[G])\,, \tag{9.20}$$

which shows that $I_H \cdot k[G]$ is a normal Hopf algebra.

9.2 Structure of comodules and super coalgebras

In this section we will introduce some notions to study the structure of super coalgebras. As guidelines for this section served the references [21, 22]. Having studied the structure of super coalgebras, we can say more on the form of the representations of affine algebraic supergroups in section 9.3.

9.2.1 Rational modules

Let C be a super coalgebra and suppose M is a left comodule over C with structure map $\psi : M \to M \otimes C$. We can make M into a left C^*-module by the following action:

$$\rho_\psi(c^* \otimes m) = \sum (-1)^{|m_i||c_i|} c^*(c_i) m_i\,, \quad \text{where} \quad \psi(m) = \sum m_i \otimes c_i\,. \tag{9.21}$$

Our first task is to consider which left C^*-modules arise in this way.

Proposition 9.2.1. *Let ψ be any even linear map $M \to M \otimes C$ and define ρ_ω as above. Then (M, ψ) is a right C-comodule if and only if (M, ρ_ψ) is a left C^*-module.*

Proof. Suppose $\psi : M \to M \otimes C$ makes M into a left C-comodule. Recall that the 1 of C^* is the counit map. For all $m \in M$ we have $\rho_\psi(\epsilon \otimes m) = \mathrm{id}_M \otimes \epsilon(m) = m$. Let $a, b \in C^*$, then

$$\rho_\psi(ab \otimes m) = \mathrm{id}_M \otimes ab \circ \psi(m) = \mathrm{id}_M \otimes a \otimes b \circ \mathrm{id}_M \otimes \Delta \circ \psi(m)\,, \tag{9.22}$$

where we identify $M \otimes k \otimes k \cong M$. On the other hand, we have

$$\rho_\psi(a \otimes \rho_\psi(b \otimes m)) = \mathrm{id}_M \otimes a \otimes b \circ \psi \otimes \mathrm{id}_C \circ \psi\,, \tag{9.23}$$

using the same identification $M \otimes k \otimes k \cong M$. Using the definition of a comodule, we see that $\rho_\psi(a \otimes \rho_\psi(b \otimes m))$. Thus M is a C^*-module with structure map $\rho_\psi : C^* \otimes M \to M$.

Conversely, if ρ_ψ makes M into a left C^*-module, then acting on M with ϵ we see that $\mathrm{id}_M \otimes \epsilon \circ \psi(m) = m$ for all $m \in M$. If for all $a, b \in C^*$ we have $\rho_\psi(a \otimes \rho_\psi(b \otimes m))$, then for all $a, b \in C^*$ we have

$$\mathrm{id}_M \otimes a \otimes b \circ (\mathrm{id}_M \otimes \Delta \circ \psi - \psi \otimes \mathrm{id}_C \circ \psi) = 0\,, \tag{9.24}$$

as linear map $M \to M \otimes k \otimes k \cong M$. Take $m \in M$ and expand $(\mathrm{id}_M \otimes \Delta \circ \psi - \psi \otimes \mathrm{id}_C \circ \psi)(m) = \sum m_i \otimes c_i \otimes d_i$ with $m_i \in M$ and $c_i, d_i \in C$. Then the sum is in fact finite and we may assume the m_i are linearly independent. This implies that for each i and for all $a, b \in C^*$ we have $a \otimes b(c_i \otimes d_i) = 0$. Thus for all i, $c_i \otimes d_i$ is in $(C^* \otimes C^*)^\perp$. But by lemma 8.1.4 the image of $C^* \otimes C^*$ in $(C \otimes C)^*$ is dense. Hence for all i, we have $c_i \otimes d_i = 0$. □

Consider the natural inclusions $M \otimes C \to M \otimes C^{**} \to \underline{\mathrm{Hom}}_{\mathbf{sVec}}(C^*, M)$. We consider $M \otimes C$ as a subspace in $\underline{\mathrm{Hom}}(C^*, M)$ and the action of $m \otimes c$ on c^* is given by $m \otimes c(c^*) = (-1)^{|c||c^*|} m c^*(c)$. Suppose that the map $\rho : C^* \otimes M \to M$ makes M into a left C^*-module, then we have a map $\psi_\rho : M \to \underline{\mathrm{Hom}}_{\mathbf{sVec}}(C^*, M)$ defined by

$$\psi_\rho(m)(c^*) = (-1)^{|m||c^*|} \rho(c^* \otimes m)\,. \tag{9.25}$$

Definition 9.2.2. *Let C be a super coalgebra and M a left C^*-module with structure map $\rho : C^* \otimes M \to M$. We say that M is a rational C^*-module if the associated map $\psi_\rho : M \to \underline{\mathrm{Hom}}_{\mathbf{sVec}}(C^*, M)$ has image in $M \otimes C$, where we view $M \otimes C$ as a subspace of $\underline{\mathrm{Hom}}_{\mathbf{sVec}}(C^*, M)$.*

Proposition 9.2.3. *If M is a rational C^*-module with structure map $\rho : C^* \otimes M \to M$, then the associated map $\psi_\rho : M \to M \otimes C$ determines a comodule structure on C. Furthermore, the C^*-module structure derived from ψ_ρ is again ρ.*

Proof. By proposition 9.2.1 the associated map $\psi_\rho : M \to M \otimes C$ makes M into a left C-comodule. By construction, the C^*-action on M derived from ψ_ρ is given by:

$$c^* \otimes m \mapsto \mathrm{id}_M \otimes c^* \circ \psi_\rho(m)\,, \tag{9.26}$$

but by definition, the right-hand side coincides with $\rho(c^* \otimes m)$. □

Proposition 9.2.4. *Let M be a rational C^*-module, then any submodule or quotient of M is also a rational C^*-module.*

Proof. Let $\rho : C^* \otimes M \to M$ be the multiplication map and let $\sigma : M \to \underline{\mathrm{Hom}}_{\mathbf{sVec}}(C^*, M)$ be the map $\sigma(m)(c^*) = (-1)^{|c^*||m|} \rho(c^* \otimes m)$. By assumption, $\sigma(M)$ lies in the image of $M \otimes C$ in $\underline{\mathrm{Hom}}_{\mathbf{sVec}}(C^*, M)$. Therefore, if N is a submodule, then so lies $\sigma(N)$ in the image of $M \otimes C$ in $\underline{\mathrm{Hom}}_{\mathbf{sVec}}(C^*, M)$. Since in addition $\rho(C^* \otimes N) \subset N$, it follows that $\sigma(N)$ lies in the image of $N \otimes C$ in $\underline{\mathrm{Hom}}_{\mathbf{sVec}}(C^*, M)$. Writing $\sigma' : M/N \mapsto \underline{\mathrm{Hom}}_{\mathbf{sVec}}(C^*, M/N)$ for the morphism determined by $\sigma'(m \bmod N)(c^*) = \sigma(m)(c^*) \bmod N$, then σ' is well-defined and can be written as $\sigma' = \pi \otimes \mathrm{id}_C \circ \sigma$ and thus has image in $M/N \otimes C$. □

Proposition 9.2.5. *Let M, N be rational C^*-modules and $f : M \to N$ a linear even map, then f is a morphism of modules if and only if f is a morphism of comodules.*

Proof. Let $\psi : M \to M \otimes C$ and $\psi' : N \to N \otimes C$ be the associated comodule structure maps. We simply write $c^* \cdot m$ for the action of c^* on m. Let $m \in M$ and write $\psi(m) = \sum m_i \otimes c_i$ and $\psi'(f(m)) = \sum n_j \otimes d_j$. Then

$$f(c^* \cdot m) = \sum (-1)^{|c_i||m_i|} c^*(c_i) f(m_i)\,, \tag{9.27}$$

and on the other hand

$$c^* \cdot f(m) = \sum (-1)^{|d_j||n_j|} c^*(d_i) n_i\,. \tag{9.28}$$

If f is a comodule map, then clearly eqns.(9.27,9.28) coincide. For the converse, we observe that if f is a module map, then the element $x = \sum f(m_i) \otimes c_i - \sum n_j \otimes d_i$ is annihilated by all elements $\mathrm{id}_N \otimes c^*$. A basis argument then shows that $x = 0$. □

Proposition 9.2.6. *Let M be a rational C^*-module, then any submodule that is generated by a finite number of elements is finite-dimensional.*

Proof. It suffices to show that a single element $m \in M$ lies in a finite submodule. Let $\psi : M \to M \otimes C$ be the associated comodule morphism, and write $\psi(m) = \sum_i m_i \otimes c_i$. Then the sum is finite and the super vector space V, defined as the span of the m_i, is finite-dimensional. Since $\mathrm{id}_M \otimes \epsilon \circ \psi(m) = m$, we have $m \in V$ and therefore $C^* \cdot m \subset V$. □

From the correspondence between the rational modules and the comodules (propositions 9.2.3 and 9.2.1) it follows that each comodule is the sum of its finite-dimensional sub comodules. Applying proposition 9.2.6 to the regular representation of an affine algebraic supergroup on its representing super Hopf algebra, shows that the super Hopf algebra is the sum of its finite-dimensional sub comodules.

Corollary 9.2.7. *Let G be an affine algebraic supergroup with representing super Hopf algebra $k[G]$. Then the regular representation of G on $k[G]$ is locally finite, that is, each element $a \in k[G]$ is contained in a finite-dimensional submodule.*

We now move one step further to show that not only comodules are locally finite, but super coalgebras as well. Recall that we defined the sub super coalgebra generated by a set S as the intersection of all sub super coalgebras containing S, see for example proposition 8.2.16.

Theorem 9.2.8. *Let C be a super coalgebra and S a finite set of homogeneous elements of C, then the sub super coalgebra of C generated by S is finite-dimensional.*

Proof. We proof that any element $c \in C$ generates a finite-dimensional sub super coalgebra. The theorem then follows, since if $D_1, D_2 \subset C$ are finite-dimensional sub super coalgebras, then so is $D_1 + D_2$.

With respect to the comultiplication $\Delta : C \to C \otimes C$, C is a left C-comodule. We write $c^* \cdot c$ for the associated action of $c^* \in C^*$ on $c \in C$. Thus $c^* \cdot c = \mathrm{id}_C \otimes c^* \circ \Delta(c)$. Then C is a rational left C^*-module. Hence for any $c \in C$, the submodule $X = C^* \cdot c$ is finite-dimensional. The action of C^* on X induces a map $f : C^* \to \mathrm{End}_k(X)$ and since $\mathrm{End}_k(X)$ is a finite-dimensional superalgebra over k, the kernel of f is a two-sided $\mathbb{Z}_2$-graded ideal such that $C^*/\mathrm{Ker}(f)$ is finite-dimensional. Therefore $\mathrm{Ker}(f)^\perp$ is a finite-dimensional sub super coalgebra of C. For any $y \in \mathrm{Ker}(f)$ we have $\epsilon(y \cdot c) = 0$, which equals $\epsilon \otimes y \circ \Delta(c) = y(c)$. Hence $c \in \mathrm{Ker}(f)^\perp$ and thus c is contained in a finite-dimensional sub super coalgebra. □

9.2.2 Simplicity and irreducibility

One defines an algebra to be simple if it has no nontrivial two-sided ideals. We say a superalgebra is simple if it has no nontrivial two-sided $\mathbb{Z}_2$-graded ideals. A commutative simple superalgebra is easily seen to be a commutative simple algebra, as the ideal generated by the odd part must be zero.

Definition 9.2.9. *We say a super coalgebra is irreducible if any two nonzero sub super coalgebras have a nonzero intersection. We call a super coalgebra C simple, if it contains no sub super coalgebras except* 0 *and C itself.*

By Zorn's lemma, any super coalgebra contains a simple sub super coalgebra. Let now C be an irreducible super coalgebra. Then the intersection of all sub super coalgebras is nonempty and hence C contains a unique sub super coalgebra, which necessarily is simple. Conversely, suppose C contains a unique simple sub super coalgebra D. Then the intersection of two arbitrary sub super coalgebras $C_1, C_2 \subset C$ cannot be empty, since C_1 and C_2 both contain a simple sub super coalgebra. Hence we have shown that a super coalgebra is irreducible if and only if there is a unique simple sub super coalgebra.

Lemma 9.2.10. *Let C be a super coalgebra.*

(i) C is irreducible if and only if all its sub super coalgebras are irreducible.

(ii) If D, E are nonzero simple sub super coalgebras then either $E \cap D = 0$ or $E = D$.

(iii) Any simple super coalgebra is finite-dimensional.

Proof. The third claim follows from theorem 9.2.8. The second claim is obvious as the intersection of two sub super coalgebras is a sub super coalgebra. For the first claim, if all sub super coalgebras are irreducible, then so C. For the converse, if D is a sub super coalgebra and E_1, E_2 are two nonzero sub super coalgebras of D, then they are two nonzero sub super coalgebras of C and thus have nonempty intersection. □

Proposition 9.2.11. *Let C be a super coalgebra. The map that sends any sub super vector space $D \subset C$ to $D^\perp \subset C^*$ sets up a one-to-one correspondence between the simple sub super coalgebras of C and the non-dense maximal $\mathbb{Z}_2$-graded two-sided ideals of C^*.*

Proof. Let D be a simple sub super coalgebra of C. Then D is finite-dimensional and D^* is a finite-dimensional algebra. Suppose that I is a two-sided $\mathbb{Z}_2$-graded ideal in D^*, then $I^\perp \subset D$ is by corollary 8.2.14 a sub super coalgebra of D and hence $I^\perp = 0$ or $I^\perp = D$. Since D is finite-dimensional, $I^\perp = 0$ implies $I = D^*$ and $I^\perp = D$ implies $I = 0$. Therefore D^* is a simple superalgebra.

If I is a two-sided $\mathbb{Z}_2$-graded ideal of C^* containing $D^\perp$, then $I \bmod D^\perp$ is an ideal in $C^*/D^\perp$, which is isomorphic to D^*. Hence $I \bmod D^\perp = 0$ or $I \bmod D^\perp = C^* \bmod D^\perp$; in the first case, $D^\perp = I$ and in the second case $C^*/I \cong (C^* \bmod D^\perp)/(I \bmod D^\perp) = 0$. Hence $D^\perp$ is a maximal $\mathbb{Z}_2$-graded two-sided ideal in C^*. As $(D^\perp)^\perp = D$ by lemma 8.1.1 is nonzero, $D^\perp$ is not dense.

Is $\mathfrak{m}$ is a non-dense maximal $\mathbb{Z}_2$-graded two-sided ideal of C^*, then $\mathfrak{m}^\perp$ is a sub super coalgebra of C. If D is a nonzero simple sub super coalgebra of $\mathfrak{m}^\perp$, then $D^\perp$ is a non-dense maximal $\mathbb{Z}_2$-graded two-sided ideal of C^* and contains $\mathfrak{m}^{\perp\perp}$. Since $\mathfrak{m}^{\perp\perp} \supset \mathfrak{m}$ we must have $\mathfrak{m}^{\perp\perp} = \mathfrak{m} = D^\perp$ and thus $D = D^{\perp\perp} = \mathfrak{m}^\perp$. Hence $\mathfrak{m}^\perp$ is a simple sub super coalgebra.

If D is any sub super coalgebra, we have $D^{\perp\perp} = D$ by lemma 8.1.1. If $\mathfrak{m}$ is a two-sided non-dense $\mathbb{Z}_2$-graded maximal ideal of C^*, then $\mathfrak{m}^\perp$ is a nonzero sub super coalgebra, and thus $\mathfrak{m}^{\perp\perp}$ is not dense, as $\mathfrak{m}^{\perp\perp\perp}$ equals $\mathfrak{m}^\perp$ by lemma 8.1.1. But $\mathfrak{m}^{\perp\perp}$ is $\mathbb{Z}_2$-graded and two-sided. Therefore $\mathfrak{m}^{\perp\perp} = \mathfrak{m}$. Hence we indeed have a bijection. □

An easy consequence of proposition 9.2.11 is that if C is an irreducible super coalgebra, then C^* has only one non-dense two-sided $\mathbb{Z}_2$-graded maximal ideal. In the finite-dimensional case, this implies that C^* is an Artinian local superalgebra.

Lemma 9.2.12. *If C is a super coalgebra and $\bar{C}$ is irreducible as a coalgebra, then C is irreducible.*

Proof. Let D, E be sub super coalgebras of C. Since it is not possible that $D \cap C_{\bar{0}} = 0$ or that $E \cap C_{\bar{0}}$, we conclude that $\bar{D}$ and $\bar{E}$ are both nonzero. Hence they have a nonzero intersection. Since as super vector spaces $\bar{C} \cong C_{\bar{0}}$, E and D contain an element $e = e_{\bar{0}} + e_{\bar{1}}$ and $d = d_{\bar{0}} + d_{\bar{1}}$ respectively with $e_{\bar{0}} = d_{\bar{0}}$. But since D and E are sub super coalgebras, we have $e_{\bar{0}} \in E$ and $d_{\bar{0}} \in D$. □

Remark 9.2.13. Consider the finite-dimensional superalgebra A of $(1+1) \times (1+1)$-supermatrices with entries in k. Then A is a simple superalgebra. As in the proof of proposition 9.2.11 we conclude that A^* is a simple super coalgebra. However, $\bar{A}^*$ is not simple, as it consists of two copies of the unique one-dimensional super coalgebra. Also, as these two copies are disjoint, the converse of lemma 9.2.12 is false in general.

On the other hand, consider the super vector space C spanned by elements g, θ, η, where we define g to be even and θ and η to be odd. Define a comultiplication and counit by $\Delta(g) = g \otimes g$, $\Delta(\theta) = \theta \otimes g + g \otimes \theta$, $\Delta(\eta) = g \otimes \eta + \eta \otimes g$, $\epsilon(g) = 1$ and $\epsilon(\eta) = \epsilon(\theta) = 0$. Then C is a super coalgebra and is not simple, since g and η span a sub super coalgebra. On the other hand, $\bar{C}$ is one-dimensional and thus simple.

We conclude that C being simple does not imply that $\bar{C}$ is simple, and conversely, that simplicity of $\bar{C}$ does not guarantee that C is simple.

Definition 9.2.14. *We call a super coalgebra C pointed if all simple sub super coalgebras of C are one-dimensional.*

Any even grouplike element defines a simple one-dimensional sub super coalgebra, and thus if a super coalgebra is pointed and irreducible, there is only one grouplike element.

Proposition 9.2.15. *Let C be a super coalgebra and suppose we can write C as a sum $C = \sum_\alpha C_\alpha$, where C_α are sub super coalgebra.*

(i) Any simple sub super coalgebra of C lies in one of the C_α.

(ii) C is irreducible if and only if each C_α is irreducible and $\cap_\alpha C_\alpha \neq 0$.

(iii) C is pointed if and only if each C_α is pointed.

(iv) C is pointed irreducible if and only if all C_α are pointed irreducible and $\cap_\alpha C_\alpha \neq 0$.

Proof. (i): If D is simple, then D is finite-dimensional and is contained in finitely many C_α. Therefore it suffices to consider $D \subset C_1 + C_2$. If $d \in D$, we may assume d to be homogeneous, and we write $d = d_1 + d_2$, with $d_i \in C_i$. As D is simple, we have $D \cap C_1 = D \cap C_2 = 0$. Hence the image of D under the projection $C \to C/C_2$ is nonzero and we can view D as a sub super vector space in C/C_2. Hence the induced morphism $(C/C_2)^* \to D^*$ is surjective. It follows that we can find c^* in C^* such that $c^*|_D$ is the counit of D and such that $c^*|_{C_2} = 0$. But then $d = \mathrm{id}_C \otimes c^* \circ \Delta(d)$ lies in D_1, where Δ is the comultiplication of C. (ii): Lemma 9.2.10(i) shows that all C_α are irreducible and that $\cap_\alpha C_\alpha$ is not empty if C is irreducible. Conversely, if all C_α are irreducible and $R = \cap_\alpha C_\alpha \neq 0$, then any simple sub super coalgebra lies in R and hence is unique. (iii): If C is pointed, then clearly, all C_α must be pointed. Conversely, if all C_α are pointed and $D \subset C$ is simple, then (i) shows that $D \subset C_\beta$ for some β and hence D is one-dimensional. (iv) follows from (ii) and (iii). □

If C is a super coalgebra and $\{D_\alpha\}_\alpha$ is a set of irreducible sub super coalgebras with nonempty intersection, then the sum $\sum_\alpha D_\alpha$ is again irreducible by the previous proposition 9.2.15. Hence by Zorn's lemma, if $D \subset C$ is an irreducible sub super coalgebra, their exists a maximal irreducible sub super coalgebra of C containing C.

Definition 9.2.16. *A sub super coalgebra D of C is an irreducible component of C if it is a maximal irreducible sub super coalgebra. D is a pointed irreducible component if in addition the unique simple sub super coalgebra of D is one-dimensional.*

Theorem 9.2.17. *Let C be any super coalgebra.*

(i) Any irreducible sub super coalgebra lies in an irreducible component.

(ii) The sum of irreducible components of C is direct.

(iii) If C is cocommutative, then C is the direct sum of its irreducible components.

Proof. (i): Is proved in the paragraph above definition 9.2.16. (ii): If C_1, C_2 are two irreducible components having a nonzero intersection, then $C_1 + C_2$ is irreducible and contains C_1 and C_2. (iii): As the sum of irreducible components is direct, it suffices to show that C is the sum of its irreducible components and we have to show that any element is contained in an a sum of irreducible sub super coalgebra. If c is in C, then by theorem 9.2.8 there is a finite-dimensional sub super coalgebra D containing c. It will be sufficient to show that D contains some irreducible sub super coalgebras that contain c. Since D is finite-dimensional and commutative, D^* is an Artinian commutative superalgebra and by corollary 5.2.3 D^* is a direct sum of local Artinian superalgebras. Thus we write $D^* = \bigoplus_{i=1}^n A_i$, where the A_i are local, from which it follows that $D \cong \bigoplus_{i=1}^n A_i^*$ as super vector spaces. But in fact, one easily verifies that the A_i are $\mathbb{Z}_2$-graded ideals in D^* and thus the A_i^* are sub super coalgebras and since the A_i are local, the A_i^* are irreducible. □

Corollary 9.2.18. *Let C be any super coalgebra.*

(i) The sum of distinct simple sub super coalgebras is direct.

(ii) C is irreducible of and only if any element of C lies in an irreducible sub super coalgebra.

(iii) C is pointed irreducible if and only if any element of C lies in a pointed irreducible sub super coalgebra.

(iv) A pointed cocommutative super coalgebra is the direct sum of its pointed irreducible components.

Proof. (i): Since distinct simple sub super coalgebras lie in distinct irreducible components, this follows from theorem 9.2.17(ii). (ii): This follows immediately from proposition 9.2.15(ii). (iii): Follows from (ii) and proposition 9.2.15(iv). (iv): This follows from theorem 9.2.17(iii) and proposition 9.2.15(iii). □

If C is a super bialgebra, then C contains the simple sub super coalgebra 1. Hence if a super bialgebra is irreducible, it is automatically pointed. Even more, using the same techniques as in [22] one shows that an irreducible super bialgebra always has an antipode and is thus a super Hopf algebra. We close this exhibition on irreducibility and simplicity by the following theorem:

Theorem 9.2.19. *Let C be an irreducible super coalgebra, R its unique simple sub super coalgebra and $f : C \to E$ a super coalgebra epimorphism.*

(i) If F is a nonzero sub super coalgebra of E, then $F \cap f(R)$ is nonzero.

(ii) The image $f(R)$ contains all the simple sub super coalgebras of E.

(iii) E is irreducible if and only if $f(R)$ is irreducible.

(iv) If R is cocommutative, then E is irreducible with unique simple sub super coalgebra $f(R)$.

(v) The homomorphic image of a pointed irreducible super coalgebra is pointed irreducible.

Proof. (i): For each $x \in F$, there is a $y \in C$ with $f(y) = x$. We can replace C by a finite-dimensional sub super coalgebra containing y, since this sub super coalgebra automatically contains R and $f(C) \cap F$ is a sub super coalgebra of E containing x. Hence we assume that C and E are finite-dimensional. It follows that C^* is a local ring with $\mathbb{Z}_2$-graded two-sided maximal ideal $R^\perp$ - it is two-sided as it is the kernel of the surjective map $C^* \mapsto R^*$. We note that we can use the proofs of lemmas 4.1.15 and 6.4.5 for $\mathbb{Z}_2$-graded anticommutative superalgebras as well and thus we conclude that if M is a finitely generated module over C^* with $R^\perp M = M$, then $M = 0$. Since $R^\perp$ is finite-dimensional, there must be a positive integer such that $(R^\perp)^n = 0$. But then $\mathfrak{m} = E^* \cap R^\perp$ is an ideal in E^*, which we can view as a subalgebra of C as $C \to E$ is surjective, so that $E^* \to C^*$ is injective. Thus $\mathfrak{m}$ is a nilpotent ideal in E^* and is thus contained in any $\mathbb{Z}_2$-graded maximal ideal in E^*. We conclude that $\mathfrak{m}^\perp$ is a sub super coalgebra of E containing all simple sub super coalgebras of E. Viewing E^* as a sub superalgebra of C^* and thus E as a quotient of C shows that $\mathfrak{m}^\perp = f(R)$, and since any simple sub super coalgebra of F is contained in $\mathfrak{m}^\perp$, this proves (i). (ii): This we proved on the way in proving (i). (iii): Follows from (i). (iv): As R is finite-dimensional, cocommutative and simple, R^* is a simple, finite-dimensional commutative k-superalgebra, and thus a simple finite-dimensional commutative k-algebra, and hence a finite field extension of k. Thus all subalgebras of R^* are also simple, and thus all quotients of R are simple. Hence $f(R)$ is simple and in particular irreducible, hence E is irreducible. (v): Follows from (iii). □

9.3 Properties of group representations

Proposition 9.3.1. *Let G be an affine algebraic supergroup. Then every finite-dimensional representation of G is isomorphic to a submodule of a finite number of copies of the regular representation.*

Proof. The proof follows the presentation of [60]. Let V be a finite-dimensional G-module and let $\psi : V \to V \otimes k[G]$ be the corresponding $k[G]$-module structure map, where $k[G]$ is the super Hopf algebra representing G. Consider the super vector space $W = V \otimes k[G]$ and equip W with the comodule structure defined by the map $\psi' = \mathrm{id}_V \otimes \Delta : W \to W \otimes k[G]$, where Δ is the comultiplication of $k[G]$. As a G-module, W consists of a finite number of copies of $k[G]$. Since V is a $k[G]$-comodule, we have $\mathrm{id}_V \otimes \Delta \circ \psi = \psi \otimes \mathrm{id}_{k[G]} \circ \psi$, which now reads $\psi' \circ \psi = \psi \otimes \mathrm{id}_{k[G]} \circ \psi$. But the last equality says that $\psi : V \to W$ is a morphism of $k[G]$-comodules, and thus a morphism of G-modules. As $\mathrm{id}_V \otimes \epsilon \circ \psi = \mathrm{id}_V$, where ϵ is the counit of $k[G]$, it follows that ψ is an injective map. Hence, we can identify V with a sub G-module of W. □

The previous proposition 9.3.1 says that in order to study representations of G, it suffices to study $k[G]$-subcomodules of the regular representation. For ordinary affine algebraic groups, one can proceed to construct all representations by taking submodules, direct sums and tensor products of the regular representation. It would be interesting to extend this technique for affine algebraic supergroups, but here a difficulty arises. Consider the super Hopf algebra of $\mathrm{GL}_{p|q}$: we can write it as $k[X_{ij}, 1/\det(X_{00})], 1/\det(X_{11})$, where X_{00} is the matrix (X_{ij}) where $1 \leq i, j \leq p$ and where X_{11} is the matrix (X_{ij}) for $p+1 \leq i, j \leq p+q$. In contrast to the case for affine algebraic groups, the elements $1/\det(X_{00})$ and $1/\det(X_{11})$ do not define one-dimensional comodules. However, $\mathrm{Ber}(X)$, where X is the matrix (X_{ij}) with $1 \leq i, j \leq p+q$, does define a one-dimensional comodule.

The following theorem says that as for the ordinary non-super case, any affine algebraic supergroup is a closed subgroup of $\mathrm{GL}_{p|q}$ for some positive integers p and q. The key ingredient for its proof lies again in the regular representation.

Theorem 9.3.2. *Let G be an affine algebraic supergroup. Then there are positive integers p and q such that G is a closed subgroup of $\mathrm{GL}_{p|q}$.*

Proof. Since the super Hopf algebra representing G is finitely generated there exists by corollary 9.2.7 a finite-dimensional $k[G]$-subcomodule V in $k[G]$ containing all generators. Thus V is a finite-dimensional G-module inside the regular representation, say $\dim V = p|q$. This means that we have a morphism of super Hopf algebras $f : k[\mathrm{GL}_{p|q}] \to k[G]$. Let $\{e_i\}_{1\leq i\leq p+q}$ be a homogeneous basis of V. The comodule structure map $\psi : V \to V \otimes k[G]$ is the comultiplication $\Delta : k[G] \to k[G] \otimes k[G]$ restricted to V. We can write $\psi(e_i) = \sum_j e_j \otimes a_{ji}$ and the a_{ij} are the images of the elements X_{ij} of $k[\mathrm{GL}_{p|q}]$ under the morphism f (see example 8.6.24 and the discussion around eqn.(9.9) for the conventions on $\mathrm{GL}_{p|q}$). In particular, the a_{ij} are contained in $f(k[\mathrm{GL}_{p|q}])$. Now consider the identity

$$e_i = \mathrm{id}_{k[G]} \otimes \epsilon \circ \Delta(e_i) = \epsilon \otimes \mathrm{id}_{k[G]} \circ \Delta(e_i) = \sum_j \epsilon(e_j) a_{ji}\,, \tag{9.29}$$

from which we conclude that all e_i are in the image of f. But then $f : k[\mathrm{GL}_{p|q}] \to k[G]$ is a surjective morphism and thus $k[G] \cong k[\mathrm{GL}_{p|q}]/\mathfrak{a}$ for some Hopf ideal $\mathfrak{a}$, which shows that G is a closed subgroup of $\mathrm{GL}_{p|q}$. □

Let G be an affine algebraic supergroup and let V be a G-module. We call the representation of G in V irreducible if there is no nontrivial submodule. If the representation is not irreducible, we call it reducible. If V is a representation, such that for all submodules W, there exists a submodule $W' \subset V$ such that $W \oplus W' = V$, we call V completely reducible. We call a representation in V diagonalizable if V splits as a sum of one-dimensional submodules. For a comodule over a super coalgebra we use the same nomenclature, so for example, a comodule is irreducible if it contains no nontrivial sub comodules. And in fact, we use the same nomenclature for modules of superalgebras. The justification for this overall use of the same names is due to corollary 9.1.5, and propositions 9.2.5 and 9.2.1, the last two of which state that the category of rational C^*-modules is equivalent to the category of C-comodules.

Remark 9.3.3. For superalgebras there are two equivalent definitions of complete reducibility: A module is completely reducible if (1) it is a sum of irreducible submodules, or (2) it is a direct sum of irreducible submodules. We omit the proof, which can be found in for example [50, XVII§2] and requiring that all submodules are $\mathbb{Z}_2$-graded and all elements homogeneous. Similarly, for the proof of the claim that any submodule of a completely irreducible module is again completely irreducible, we refer to [50, XVII§2].

Let C be a super coalgebra and let V be a left C-comodule, with structure map $\psi : V \to V \otimes B$. Then V is a rational left C^*-module. If C is even a super bialgebra, so that $1 \in C$, then we define a vector $v \in V$ to be invariant if $\psi(v) = v \otimes 1$. It follows, that the sub super vector space V^{inv} spanned by the invariant vectors is $\mathbb{Z}_2$-graded.

Theorem 9.3.4. *Let C be a super bialgebra with its dual superalgebra C^* and assume V is a rational left C^*-algebra. Denote $\psi : V \to V \otimes C$ the corresponding comodule structure map. Then:*

$$\{v \in V | f \cdot v = f(1)v,\ \forall f \in C^*\} = \{v \in V | \psi(v) = v \otimes 1\}\ . \tag{9.30}$$

Proof. By the definition of the action of C^* on V, it is clear that the right-hand side is contained in the left-hand side. Suppose that v is homogeneous and that $f \cdot v = f(1)v$ for all $f \in C^*$ and write $\psi(v) = \sum v_i \otimes c_i$. We may assume the c_i are linearly independent. Then we can choose $f \in C^*$ such that $f(c_1) = 1$ and $f(c_i) = 0$ for $i \neq 1$. It follows that v_1 is a multiple of v, and similarly, each v_i is a multiple of v, so that we can write $\psi(v) = v \otimes c$ for some $c \in C$. By the assumption on v it follows that $f(c-1) = 0$ for all $f \in C^*$, hence $c = 1$. □

We call a comodule V of a super bialgebra trivial if for all $v \in V$, the comodule structure map is given by $\psi : v \mapsto v \otimes 1$. If G is an affine algebraic supergroup, we say that a representation in V is trivial if V is a trivial comodule of the super Hopf algebra representing G.

Definition 9.3.5. *Let C be a super coalgebra and let V be a comodule over C with structure map $\psi : V \to V \otimes C$. We call the support of V the smallest sub super coalgebra D of C such that $\psi(V) \subset V \otimes D$. We write $C(V)$ for the support of V.*

Lemma 9.3.6. *Let C be a super coalgebra and let V be a left comodule over C with structure map $\psi : V \to V \otimes C$. Let $\{v_i\}_i$ be a homogeneous basis of V, then the support of V is given by*

$$C(V) = \mathrm{Span}\left\{c_i \in C | \psi(v) = \sum v_i \otimes c_i,\ \forall v \in V\right\}. \tag{9.31}$$

Proof. Clearly, the right-hand side of eqn.(9.31) is contained in the left-hand side of eqn.(9.31), so it suffices to show that the right-hand side of eqn.(9.31) is a sub super coalgebra. But the identity $\psi \otimes \mathrm{id}_C \circ \psi = \mathrm{id}_V \otimes \Delta \circ \psi$, where Δ is the comultiplication of C, just tells us that the right-hand side of eqn.(9.31) is indeed a sub super coalgebra. In particular, the right-hand side of eqn.(9.31) is independent of the choice of the basis. □

The properties of the support that are important for us, are summarized in the following lemma:

Lemma 9.3.7. *Let C be a super coalgebra and let V be a left comodule over C. Then:*

(i) If V is one-dimensional, then so is $C(V)$.

(ii) If V is a sub comodule of W, then $C(V) \subset C(W)$.

(iii) If $V = V_1 + V_2$, then $C(V) = C(V_1) + C(V_2)$.

(iv) If V is an irreducible comodule, then $C(V)$ is a simple sub super coalgebra of C.

(v) If D is a simple sub super coalgebra of C, then there exists an irreducible left C-comodule W, such that $C(W) = D$.

Proof. (*i*): This is obvious. (*ii*): Choose a basis of V and then extend it to a basis of W, then use lemma 9.3.6. (*iii*): Clearly, $C(V_1) + C(V_2) \subset C(V)$. Choose a basis for V_1 and a basis for V_2, then the union of the basis spans V, and we can delete some elements to obtain a basis for V. Since any $v \in V$ can be written as $v = v_1 + v_2$ with $v_i \in V_i$, lemma 9.3.6 proves the claim. (*iv*): We fix a homogeneous basis $\{v_i\}$ of V, and write $\psi(v_i) = \sum_j v_j \otimes c_{ji}$. Then $C(V)$ is spanned by the c_{ji}. We write $c^* \cdot v$ for the action of C^* on V defined by $c^* \cdot v = \mathrm{id}_V \otimes c^* \circ \psi(v)$. Then V is a rational left C^*-module and the two-sided $\mathbb{Z}_2$-graded ideal I of all elements $c^* \in C^*$ that annihilate V consists precisely of those elements $c^* \in C^*$ such that $c^*(c_{kl}) = 0$ for all k, l. Hence $C(V) \subset I^\perp$. Since V is irreducible and $V \cong C^*/I$, we conclude that I is a maximal two-sided $\mathbb{Z}_2$-graded ideal, and since $C(V) \subset I^\perp$, I is not dense. Hence proposition 9.2.11 says that $I^\perp$ is a simple sub super coalgebra of C. Since $C(V) \subset I^\perp$ and $C(V) \neq 0$, we conclude that $C(V) = I^\perp$ and $C(V)$ is simple. (*v*): The claim follows if we can show that D has a minimal left coideal (see proposition 8.2.18 and the preceding paragraph). By proposition 8.2.19 the arbitrary intersection of left coideals is again a left coideal. But then a standard application of Zorn's lemma, shows that minimal left coideals exist. □

We now come to study the relations between certain properties of super coalgebras, like simplicity and irreducibility, and properties of comodules, such as irreducibility and complete reducibility.

Corollary 9.3.8. *Let C be a super coalgebra. Then the following are equivalent:*

(i) All rational left C^*-modules that are irreducible, are one-dimensional.

(ii) All minimal left coideals of C are one-dimensional.

(iii) C is a pointed super coalgebra.

Proof. $(i) \Rightarrow (ii)$: Let V be a minimal left coideal, then V is a rational left C^*-module and irreducible. Therefore, V is one-dimensional. $(ii) \Rightarrow (iii)$: If D is a simple sub super coalgebra of C, then D is in particular a left coideal, and hence one-dimensional. $(iii) \Rightarrow (i)$: Let V be an irreducible rational left C^*-module. Then V is an irreducible left C-comodule. By lemma 9.3.7 the support of V is a simple sub super coalgebra of C. Hence $C(V)$ is one-dimensional. Using the identity $\mathrm{id}_V \otimes \epsilon \circ \psi = \mathrm{id}_V$, we see that this implies that for any $v \in V$, the comodule structure map $\psi : V \to V \otimes C$ is given by $\psi(v) = \lambda v \otimes g$, where g spans $C(V)$ and $\lambda \in k$. Thus each homogeneous vector in V defines a sub comodule, and thus, as V is supposed to be irreducible, V has to be one-dimensional. □

Let C be a super coalgebra. We define the coradical of C to be the sum of all simple sub super coalgebras. By corollary 9.2.18 the sum is direct. The coradical plays a role, similar to the nilradical of a superalgebra. In fact, proposition 9.2.11 shows that if R is the coradical of C, then $R^\perp$ is the intersection of all non-dense $\mathbb{Z}_2$-graded two-sided maximal ideals of C^*. For noncommutative superalgebras, we define the nilradical to be the intersection of all $\mathbb{Z}_2$-graded maximal left ideals. Thus $R^\perp$ is contained in the nilradical of C^*. We call a superalgebra semi-simple if the nilradical is zero. Note that a noncommutative superalgebra is simply a noncommutative algebra with the additional structure of a $\mathbb{Z}_2$-grading. Therefore, if the intersection of all maximal $\mathbb{Z}_2$-graded left ideals is zero, then so is the intersection of all maximal left ideals. Hence if a superalgebra is semi-simple, then it is also semi-simple as an algebra.

Proposition 9.3.9. *Let C be a super coalgebra and let R be its coradical.*

(i) C is the sum of all supports $C(V)$ where V ranges over all finite-dimensional left C-comodules.

(ii) R is the sum of all supports $C(V)$, where V ranges over all irreducible sub super coalgebras.

Proof. (i): By theorem 9.2.8 every element is contained in a finite-dimensional sub super coalgebra. Thus for every homogeneous element $c \in C$, there is a finite-dimensional sub super coalgebra D, with $c \in D$. The comultiplication makes D into a left C-comodule, and clearly $C(D) = D$. (ii) By lemma 9.3.7(v) each simple sub super coalgebra is the support of some irreducible comodule. By lemma 9.3.7(iv) each $C(V)$ is simple whenever V is an irreducible left comodule. □

Theorem 9.3.10. *Let C be a super coalgebra, then the following are equivalent:*

(i) All left C-comodules are completely reducible.

(ii) C equals its coradical.

Proof. $(i) \Rightarrow (ii)$: As a C^*-module, C itself has to be a direct sum of irreducible submodules. Hence C is the direct sum of its simple sub super coalgebras, hence equals its coradical. $(ii) \Rightarrow (i)$: Any C-comodule is the sum of its finite-dimensional sub comodules by proposition 9.2.6. Hence by remark 9.3.3 it suffices to prove that any finite-dimensional comodule V is completely reducible. Let V be a finite-dimensional comodule over C. One easily sees that then the support of V is a finite-dimensional sub super coalgebra $D' \subset C$. Hence D' is contained in a finite direct sum of simple sub super coalgebras, and we write D for this direct sum. Then D^* is a semi-simple finite-dimensional superalgebra. Thus V is a finite-dimensional D^*-module and is thus by lemma 9.3.11 below, which is a slightly adapted version of a theorem of Wedderburn, completely reducible. But the action of C^* on V' factors over the action of D^* on V', hence V' is completely reducible as a C^*-module. □

Lemma 9.3.11. *Let A be a finite-dimensional semisimple superalgebra. Then any left module over A is completely reducible.*

Proof. As the intersection of all $\mathbb{Z}_2$-graded maximal left ideals is zero and as A is finite-dimensional, we can choose finitely many $\mathbb{Z}_2$-graded maximal left ideals $\mathfrak{m}_i$ such that $\cap_i \mathfrak{m}_i = 0$. Say, we choose n of them. Then the canonical morphism of left A-modules $A \to \bigoplus_{i=1}^n A/\mathfrak{m}_i$ is injective. Each $A/\mathfrak{m}_i$ is irreducible and hence $\bigoplus_{i=1}^n A/\mathfrak{m}_i$ is completely reducible. Thus A is a completely reducible left A-module. For a general left A-module M, let $f : F \to M$ be a surjective left A-module morphism. Then F is completely reducible as it is a sum of copies of A. Hence $F = \ker(f) \oplus N$ for some submodule N. But $N \cong M$, and since N is a submodule of F, N is also completely reducible. □

Theorem 9.3.12. *Let G be an affine algebraic group, represented by a Hopf algebra H. Then we have:*

(i) Any representation of G is completely reducible if and only if $k[G]$ equals its coradical.

(ii) Any irreducible representation of G is one-dimensional, if and only if $k[G]$ is pointed.

(iii) Any representation of G is diagonalizable if and only if $k[G]$ is pointed and equals its coradical.

(iv) Any irreducible representation of G is trivial if and only if $k[G]$ is irreducible.

Proof. Due to corollary 9.1.5 all four claims can be reduced to properties of the comodules of $k[G]$. (i): By theorem 9.3.10 any representation is completely irreducible if and only if C equals its nilradical. (ii): Corollary 9.3.8 claims that all irreducible comodules are one-dimensional if and only if $k[G]$ is pointed. (iii): In order that a representation be diagonalizable, it suffices and is necessary that all representations are completely reducible and all irreducible representations are one-dimensional. Hence this follows from (i) and (ii). (iv): If $k[G]$ is irreducible, it contains only one simple sub super coalgebra, which is 1. Hence by (ii) all irreducible representations are one-dimensional. Thus if V is a one-dimensional representation, then also the support $C(V)$ is automatically one-dimensional, hence $C(V) = k \cdot 1$. Hence V is a trivial representation. If all irreducible representations are trivial, they are all one-dimensional and hence $k[G]$ is pointed. Thus the only simple sub super coalgebras are those of the grouplike elements. If g is a grouplike element, then $\Delta g = g \otimes g$, which, by triviality, must equal $g \otimes 1$. Hence $g = 1$ and $k[G]$ is irreducible. □

For ordinary algebraic groups, one calls a group all of whose representations are completely reducible a linearly reductive group. Nagata classified the linearly reductive algebraic groups for arbitrary characteristic [67]. We are not aware of any such classification for affine algebraic supergroups.

Remark 9.3.13. We have restricted to finitely generated super Hopf algebras. This is in fact not a severe restriction for most results. The techniques as for example displayed in [60] apply equally well to conclude that a super Hopf algebra is a directed union of its finitely generated sub super Hopf algebras. Thus, any representable group functor is an inverse limit of affine algebraic supergroups.

Remark 9.3.14. We have focussed on linear actions of affine algebraic supergroups. Most of the arguments apply as well to make statements on actions of affine algebraic supergroups on affine superschemes. We briefly sketch the situation.

Let $G = \mathrm{Spec}(k[G])$ be an affine algebraic supergroup and let $X = \mathrm{Spec}(k[X])$ be an affine superscheme. Then an action of G on X is a natural transformation $\rho : G \times X \to X$ satisfying $\rho(g, \rho(h, x) = \rho(g \cdot h, x)$ and $\rho(e, x) = x$, where e is the identity element.

Since in the affine case, any natural transformation $\mathrm{Spec}(k[G]) \times \mathrm{Spec}(k[X]) \to \mathrm{Spec}(k[X])$ is equivalent to a morphism of superalgebras $\rho^* : k[X] \to k[X] \otimes k[G]$, the same conclusion of theorem 9.1.3 applies and thus $k[X]$ is a comodule over $k[G]$. We call the invariant sub superalgebra of $k[X]$ under G the sub superalgebra consisting of the elements $a \in k[X]$ such that $\rho^*(a) = a \otimes 1$ and we denote it $k[X]^G$. We define the natural quotient to be the morphism $p : \mathrm{Spec}(k[X]) \to \mathrm{Spec}(k[X]^G)$ induced by the inclusion $k[X]^G \to k[X]$.

Let $k[X]$ be finitely generated, say by homogeneous elements $a_1, \ldots, a_N$. Then the sub comodule generated by the a_i is finite-dimensional. If we write V for the sub comodule generated by the a_i, we have linearized the action of G: the inclusion of V in $k[X]$ is a morphism of comodules and induces a G-equivariant morphism of superschemes $\mathrm{Spec}(k[X]) \to \mathrm{Spec}(k[V])$. Hence the restriction to linear representations is not a severe restriction when one only considers affine superschemes.

9.4 Lie superalgebras and representations

9.4.1 The adjoint representation

Let G be an affine algebraic supergroup with representing super Hopf algebra $k[G]$ and with Lie superalgebra $\mathfrak{g} = \mathrm{Der}_k^\epsilon(k[G], k)$. Then as $\mathfrak{g}$ is a super vector space, we can view it as a functor $\mathfrak{g} : A \mapsto (\mathfrak{g} \otimes A)_{\bar{0}} \cong \mathrm{Der}_k^\epsilon(k[G], A)_{\bar{0}}$ for any commutative superalgebra A. To define the adjoint representation of the group G with module $\mathfrak{g}$, we first note that the usual approach does not work: We cannot fix an element $g \in G(A)$, and then differentiate ghg^{-1} with respect to h, since we have to be able to change A in a functorial way. In [64] they then used a trick to get something that is close to differentiating functorially. We will present an equivalent approach that gives a simple and equivalent formula in the end. Recall the total tangent bundle functor, $TG : A \mapsto \mathrm{Hom}_{\mathbf{sAlg}}(k[G], A^+)$, where A^+ was the augmented superalgebra obtained by adding one even variable to A and requiring that it squares to zero: $A^+ = A[x]/(x^2)$. By lemma 8.6.6, $TG(A)$ consists of all pairs (g, D) with $g \in G(A)$ and $D : k[G] \to A$ is a derivation over g. Hence we have a natural inclusion $G \to TG$, sending g to $(g, 0)$. Similarly, we can embed the Lie algebra in TG, as follows: to any derivation $D : k[G] \to A$ over the counit ϵ, we assign $(\epsilon, D) \in TG(A)$. Now let $C : G \times G \to G$ be the natural transformation defined by $C(g, h) = ghg^{-1}$ for any $g, h \in G(A)$ for any commutative superalgebra A. The natural transformation C is already defined in eqn.(9.14) and the associated morphism $c : k[G] \to k[G] \otimes k[G]$ was given in eqn.(9.15). We can now define the adjoint action of G on $\mathfrak{g}$.

Definition 9.4.1. *Let G be an affine algebraic supergroup G with representing super Hopf algebra $k[G]$ and with Lie superalgebra $\mathfrak{g} = \mathrm{Der}_k^\epsilon(k[G], k)$, which we view as a Lie algebra functor. The adjoint action $\mathrm{Ad}^G : G \times \mathfrak{g} \to \mathfrak{g}$ is defined as follows: Fix a commutative superalgebra A for each $g \in G(A)$ and $D \in \mathrm{Der}_k^\epsilon(k[G], A)$ and consider their images $(g, 0)$ and (ϵ, D) in $TG(A)$. One then* *easily shows using the explicit formulae that*

$$C((g,0),(\epsilon,D)) = (g,0)\cdot(\epsilon,D)\cdot(g,0) \tag{9.32}$$

is again of the form $(\epsilon, D') \in TG(A)$ *for some* $D \in \mathrm{Der}_k^\epsilon(k[G],A)$. *We then define* $\mathrm{Ad}^G_g(D) = \mathrm{Ad}^G(g,D) = D'$ *so that we have*

$$C((g,0),(\epsilon,D)) = (\epsilon, \mathrm{Ad}^G_g(D))\,. \tag{9.33}$$

We remark that if $*$ denotes the product in the algebra of linear map $\underline{\mathrm{Hom}}_{\mathbf{sVec}}(k[G],A)$ given by $x*y = \mu_A \circ x \otimes y \circ \Delta$ (also see section 8.3.1), where μ_A is the multiplication in A, then in $TG(A)$ we have

$$(g,D)\cdot(g',D') = (g\cdot g', g*D' + D*g)\,. \tag{9.34}$$

Using the explicit expressions for the conjugation morphism C, we immediately obtain the following lemma that provides an explicit formula for the adjoint representation:

Lemma 9.4.2. *Let G be an affine algebraic supergroup with representing super Hopf algebra $k[G]$ and Lie superalgebra $\mathfrak{g} = \mathrm{Der}_k^\epsilon(k[G],k)$. Let $c : k[G] \to k[G]\otimes k[G]$ be the morphism defined by eqn.(9.15). Then for any commutative superalgebra A, $g \in G(A)$ and $D \in \mathrm{Der}_k^\epsilon(k[G],A)$ we have*

$$\mathrm{Ad}^G_g(D) = \mu_A \circ g \otimes D \circ c\,, \tag{9.35}$$

where $\mu_A : A\otimes A \to A$ is the multiplication in A. In particular, Ad *is a natural transformation.*

Proof. Using the explicit multiplication in $TG(A)$ for any commutative superalgebra A, one finds that

$$\mathrm{Ad}^G_g(D) = \mu_A \circ \mu_A \otimes \mathrm{id}_A \circ g \otimes D \otimes (g\circ S) \circ \Delta \otimes \mathrm{id}_{k[G]} \circ \Delta\,, \tag{9.36}$$

where $\Delta : k[G] \to k[G]\otimes k[G]$ is the comultiplication of $k[G]$ and $S : k[G] \to k[G]$ the antipode of $k[G]$. Using eqn.(9.15) one finds eqn.(9.35). For any morphism of superalgebras $f : A \to B$ we have $\mu_B \circ f \otimes f = f \circ \mu_A$, where μ_A and μ_B are the multiplication maps of A and B respectively, that $\mathrm{Ad}^G_{f\circ g}(f\circ D) = f \circ \mathrm{Ad}^G_g(D)$. This proves that Ad^G is a natural transformation. □

Lemma 9.4.3. *Let G be an affine algebraic supergroup. For any commutative superalgebra A and for any $g,h \in G(A)$ we have $\mathrm{Ad}^G_{gh}(D) = \mathrm{Ad}^G_g \circ \mathrm{Ad}^G_h$.*

Proof. Let $\mathfrak{g}$ be the Lie algebra functor of G and fix a commutative superalgebra A. Write μ_A for the multiplication in A and let $k[G]$ be the super Hopf algebra representing G. We first remark that it follows from theorem 8.4.8 that for $g,h \in G(A)$ we have $(g\cdot h)\circ S = (h\circ S)\cdot(g\circ S)$, where S is the antipode and where we wrote a dot for multiplication in the group $G(A)$. Using this and the explicit formula eqn.(9.36) one finds for $g,h\in G(A)$ and $D \in \mathfrak{g}(A)$ that

$$\mathrm{Ad}^G_{gh}(D) = \mu^{(4)} \circ g \otimes h \otimes D \otimes (h\circ S)\otimes (g\circ S) \circ \Delta^{(4)}\,, \tag{9.37}$$

where

$$\mu^{(4)} = \mu_A \circ \mu \otimes \mathrm{id}_A \circ \mu_A \otimes \mathrm{id}_A \otimes \mathrm{id}_A \circ \mu_A \otimes \mathrm{id}_A \otimes \mathrm{id}_A \otimes \mathrm{id}_A \otimes \mathrm{id}_A\,, \tag{9.38}$$

and

$$\Delta^{(4)} = \Delta \otimes \mathrm{id}_{k[G]} \otimes \mathrm{id}_{k[G]} \otimes \mathrm{id}_{k[G]} \circ \Delta \otimes \mathrm{id}_{k[G]} \otimes \mathrm{id}_{k[G]} \circ \Delta \otimes \mathrm{id}_{k[G]} \circ \Delta\,. \tag{9.39}$$

But we also have

$$\mathrm{Ad}^G_g(\mathrm{Ad}^G_h(D)) = \mu_A \circ \mu_A \otimes \mathrm{id}_A \circ g \otimes \mathrm{Ad}^G_h \otimes (g\circ S)\circ \Delta \otimes \mathrm{id}_{k[G]} \circ \Delta\,, \tag{9.40}$$

and expanding $\mathrm{Ad}^G_h(D)$ in this equation gives again eqn.(9.37). □

Lemma 9.4.4. *Let $\phi : G \to H$ be a morphism of supergroups. Suppose $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebra functors of G and H respectively. The induced map $\mathrm{d}\phi : \mathfrak{g} \to \mathfrak{h}$ satisfies*

$$\mathrm{d}\phi(\mathrm{Ad}^G_g(D)) = \mathrm{Ad}^H_{\phi(g)}(\mathrm{d}\phi(D))\,. \tag{9.41}$$

Proof. This follows from the fact that if $\varphi : k[H] \to k[G]$ is the morphism of super Hopf algebras that induces ϕ, then the we have $\varphi \otimes \varphi \circ c = c \circ \varphi$. $\square$

The result of lemma 9.4.4 is of great importance. With the notation as in the lemma, it says that the following diagram commutes:

$$\begin{tikzcd} G \times \mathfrak{g} \arrow[r, "\mathrm{Ad}^G"] \arrow[d, "(\phi,\mathrm{d}\phi)"'] & \mathfrak{g} \arrow[d, "\mathrm{d}\phi"] \\ H \times \mathfrak{h} \arrow[r, "\mathrm{Ad}^H"'] & \mathfrak{h} \end{tikzcd} \quad . \tag{9.42}$$

When no confusion is clear, we will write Ad instead of Ad^G. One should however not forget that Ad is always defined with respect to a certain group structure.

Example 9.4.5. Let $G = \mathrm{GL}_{p|q}$ and consider its Lie algebra functor $\mathfrak{gl}_{p|q}$. We use the notation introduced in example 8.6.24.

If $g \in \mathrm{GL}_{p|q}(A)$, we can write g in a matrix form $g = (g_{ij})$, where $g_{ij} = g(X_{ij})$. When we write g^{-1} for the inverse of g, then g^{-1} has the matrix representation $g^{-1}_{ij} = g(S(X_{ij}))$, where S is the antipode of $k[G]$. Using the explicit expressions from lemma 9.4.2 one obtains:

$$\mathrm{Ad}_g(D) = \mathrm{Ad}_g(\sum D_{ij}\frac{\partial}{\partial X_{ij}}) = \sum_{ijmn} g_{im}D_{mn}g^{-1}_{nj}\frac{\partial}{\partial X_{ij}}\,. \tag{9.43}$$

We can thus identify Ad_g with a matrix $(Z_{ij,mn}(g))$ given by

$$\mathrm{Ad}_g(D) = \sum D_{ij}Z_{ij,mn}(g)\frac{\partial}{\partial X_{mn}}\,, \tag{9.44}$$

where

$$Z_{ij,mn}(g) = (-1)^{|g_{ij}||g_{mi}|}g_{mi}g^{-1}_{jn}\,. \tag{9.45}$$

Let us write W for the super vector space $\mathrm{Der}^{\epsilon}_k(k[G], k)$, then W is spanned by the $\frac{\partial}{\partial X_{ij}}$ and any element Ω in $\mathrm{GL}_W(A)$ can be written as a matrix $(\Omega_{ij,mn})$ according to

$$\Omega(\frac{\partial}{\partial X_{ij}}) = \sum_{mn}\Omega_{ij,mn}\frac{\partial}{\partial X_{mn}}\,. \tag{9.46}$$

Then we can proceed as in example 8.5.16 to write $k[\mathrm{GL}_W]$ as a quotient of $k[Z_{ij,mn}, s, t]$, where s and t are even variables, and are used to impose the invertibility constraint (we do not write out the details here, which are cumbersome, since the procedure is clear; the interested reader can for example figure out when $Z_{ij,mn}$ is even or odd). Tracing back the steps, which amounts to taking $A = k[\mathrm{GL}_{p|q}]$ and $g = \mathrm{id}_{k[\mathrm{GL}_{p|q}]}$ in eqn.(9.45), we find that the adjoint representation $\mathrm{Ad} : \mathrm{GL}_{p|q} \to \mathrm{GL}_W$ is given by the morphism of super Hopf algebras $\phi : k[\mathrm{GL}_W] \to k[\mathrm{GL}_{p|q}]$ that is uniquely determined by

$$\phi(Z_{ij,mn}) = (-1)^{|X_{ij}||X_{mi}|}X_{mi}S(X_{jn})\,. \tag{9.47}$$

$\triangle$

The importance of the previous example 9.4.5 lies in the fact that every affine algebraic supergroup is isomorphic to a closed subgroup of $\mathrm{GL}_{p|q}$ for some p and q. Therefore also the Lie algebra is isomorphic to a Lie subalgebra of $\mathfrak{gl}_{p|q}$ by proposition 8.6.18. Let G be a closed subgroup of $\mathrm{GL}_{p|q}$, and suppose the embedding $i : G \to \mathrm{GL}_{p|q}$ is induced by the morphism of super Hopf algebras $\pi : k[\mathrm{GL}_{p|q}] \to k[G]$. Then, using the notation of example 9.4.5, the adjoint action of G is determined by the morphism of super Hopf algebras $\psi : k[\mathrm{GL}_W] \to k[G]$ given by $\psi : Z_{ij,mn} \mapsto (-1)^{|X_{ij}||X_{mi}|}\pi(X_{mi})\pi(S(X)_{jn})$. By lemma 9.4.4 we have $\mathrm{d}i \circ \mathrm{Ad}_g = \mathrm{Ad}_{i(g)} \circ \mathrm{d}i$ and thus G acts on its Lie algebra by restriction of the adjoint action of $\mathrm{GL}_{p|q}$.

9.4.2 Derived representations

Consider a finite-dimensional Lie superalgebra $\mathfrak{g}$ and view $\mathfrak{g}$ as a functor $\mathfrak{g} : \mathbf{sAlg} \to \mathbf{Sets}$ defined by $\mathfrak{g}(A) = (\mathfrak{g} \otimes A)_{\bar{0}}$. Then $\mathfrak{g}(A)$ is a Lie algebra for each commutative superalgebra A. For a super vector space V the Lie superalgebra $\mathfrak{gl}_V$ can in a natural way be seen as a functor: $\mathfrak{gl}_V(A) = \mathrm{End}((V \otimes A)_{\bar{0}}) \cong \mathrm{Hom}_{\mathbf{A\text{-}mod}}(V \otimes A, V \otimes A)$. Note that $\mathfrak{gl}_V(A)$ is a Lie algebra (also see section 2.2, section 3.7 and the discussion around lemma 3.7.7) with Lie bracket $[X, Y] = X \circ Y - Y \circ X$. Note that the form of the Lie bracket is very similar to the Lie bracket of a Lie algebra related to an affine algebraic group. By the Deligne–Morgan theorem 3.6.1, there is a one-to-one correspondence between the representations of Lie superalgebra in a super vector space V and the natural transformations $\mathfrak{g} \to \mathfrak{gl}_V$. We will therefore not distinguish the two. The representations of Lie superalgebras are quite well-known, see for example [29, 31, 33, 68–74], which is by no means a complete list of references but can be used to trace more references.

Given a representation $\phi : G \to \mathrm{GL}_V$ of the supergroup G in V, induced by a morphism of super Hopf algebras $\varphi : k[\mathrm{GL}_V] \to k[G]$ we have a comodule map $\psi : V \to V \otimes k[G]$. Let $\mathrm{d}\phi : \mathfrak{g} \to \mathfrak{gl}_V$ be the morphism of Lie superalgebras $\mathrm{d}\phi : D \mapsto D \circ \varphi$ for any $D \in \mathrm{Der}^{\epsilon}_k(k[G], k)$. In proposition 8.6.18 it was proved that $\mathrm{d}\phi$ is a morphism of Lie superalgebras. We will call this representation the derived representation associated to ϕ. If we fix a basis homogeneous v_i of V, then as in eqn.(9.9), we know that the comodule structure map is given by $\psi(v_i) = \sum_k v_k \otimes \varphi(X_{ki})$, where $X_{ij} \in k[\mathrm{GL}_V]$ are as in examples 8.6.24 and 9.4.5. If $g \in G(A)$, then the action of g on $V(A)$ is determined by $g(v_i \otimes 1) = \sum_k v_k \otimes g(\varphi(X_{ki}))$. Thus, if D is an element of the Lie algebra $\mathrm{Der}^{\epsilon}_k(k[G], A)$, then $\mathrm{d}\phi(D)$ is the derivation that acts on $v_i \otimes 1$ by $\mathrm{d}\phi(D)(v_i \otimes 1) = \sum_k v_k \otimes D(\varphi(X_{ki}))$. This follows since, as in example 9.4.5, $\mathrm{d}\phi(D)$ is identified with the matrix $(D(\varphi(X_{ij})))$. We summarize:

Proposition 9.4.6. *Let G be an affine algebraic supergroup with representing super Hopf algebra $k[G]$ and with Lie algebra functor $\mathfrak{g}$. If V is a G-module, with comodule morphism $\psi : V \to V \otimes k[G]$ and with natural transformation $\phi : G \to \mathrm{GL}_V$, then the associated derived representation is given by the formula:*

$$\mathrm{d}\phi(D) : v \otimes 1 \mapsto 1 \otimes D \circ \psi(v)\,, \qquad v \in V\,,\ D \in \mathfrak{g}(A) = \mathrm{Der}^{\epsilon}_k(k[G], A)\,, \tag{9.48}$$

for any commutative superalgebra A.

We get an immediate corollary:

Corollary 9.4.7. *Let G, G' be affine algebraic supergroups with Lie algebra functors $\mathfrak{g}$ and $\mathfrak{g}'$ respectively. Suppose we have natural transformations $\Phi : G \times V \to V$ and $\Phi' : G' \times W \to W$ that define linear representations, a super vector space morphism $f : V \to W$ and a morphism of groups*

$\phi : G \to H$ such that the following diagram commutes:

$$\begin{tikzcd} G \times V \arrow[r, "\Phi"] \arrow[d, "(\phi,f)"'] & V \arrow[d, "f"] \\ G' \times W \arrow[r, "\Phi'"'] & W \end{tikzcd} \quad . \tag{9.49}$$

Then also the following diagram commutes

$$\begin{tikzcd} \mathfrak{g} \times V \arrow[r, "\mathrm{d}\Phi"] \arrow[d, "(\phi,f)"'] & V \arrow[d, "f"] \\ \mathfrak{g}' \times W \arrow[r, "\mathrm{d}\Phi'"'] & W \end{tikzcd} \quad , \tag{9.50}$$

where $\mathrm{d}\Phi : \mathfrak{g} \times V \to V$ and $\mathrm{d}\Phi : \mathfrak{g}' \times W \to W$ are the derived representations.

Proof. Let $\psi : V \to V \otimes k[G]$ and $\psi' : W \to k[G']$ be the associated comodule morphisms and $\varphi : k[G'] \to k[G]$ the morphism of super Hopf algebras that induces ϕ. The commutativity of the first diagram (9.49) is equivalent to the identity $f \otimes \mathrm{id}_{k[G]} \circ \psi = \mathrm{id}_W \otimes \varphi \circ \psi' \circ f$; the identity is clearly sufficient, that it is necessary follows from applying the diagram to the commutative superalgebra $k[G]$ and the group element $g = \mathrm{id}_{k[G]}$. Using this identity and proposition 9.4.6 one easily verifies that the second diagram (9.50) commutes for all commutative superalgebras A. □

Remark 9.4.8. Let G, G' be affine algebraic supergroups and $\phi : G \to G'$ a morphism of groups. The adjoint representation of G and G' on their Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ respectively, and the morphism $f = \mathrm{d}\phi : \mathfrak{g} \to \mathfrak{g}'$ satisfy the premises of corollary 9.4.7.

Proposition 9.4.9. *Let G be an affine algebraic group, represented by the super Hopf algebra $k[G]$, and let $\mathfrak{g}$ be the Lie algebra functor $\mathfrak{g} : A \mapsto \mathrm{Der}_k^\epsilon(k[G], A)_{\bar{0}}$. Write $C : G \times G \to G$ for the conjugation morphism $C : (g,h) \mapsto ghg^{-1}$ and $TC : TG \times TG \to TG$ for the associated morphism between total tangent bundle functors. Then for any fixed commutative superalgebra A and $D_1, D_2 \in \mathrm{Der}_k^\epsilon(k[G], A)_{\bar{0}}$*

$$TC^A((\epsilon, D_1), (\epsilon, D_2)) = [D_1, D_2] = D_1 * D_2 - D_2 * D_1 \,. \tag{9.51}$$

Proof. If S is the antipode of $k[G]$, then for any derivation $D : k[G] \to A$ over the counit, we have $D \circ S = -D$. Then inserting the definitions in the formulae proves the proposition. □

Let G be an affine algebraic supergroup with Lie algebra functor $\mathfrak{g}$. Then $\mathfrak{g}$ admits a natural Lie algebra representation on itself $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}_{\mathfrak{g}}$ defined by $\mathrm{ad}^A(D_1)(D_2) = [D_1, D_2]$ for any commutative superalgebra A and $D_1, D_2 \in \mathrm{Der}_k^\epsilon(k[G], A) = \mathfrak{g}(A)$. The previous proposition 9.4.9 and lemma 8.6.28 suggest that we have the identity $\mathrm{dAd} = \mathrm{ad}$. We first check this for $G = \mathrm{GL}_{p|q}$.

Example 9.4.10. We use the same notation as in example 9.4.5. The adjoint representation $\mathrm{Ad} : \mathrm{GL}_{p|q} \to \mathrm{GL}_W$ is induced by the morphism of super Hopf algebras $\phi : k[\mathrm{GL}_W] \to k[\mathrm{GL}_{p|q}]$

$$\phi : Z_{ij,mn} \mapsto (-1)^{|X_{ij}||X_{mi}|} X_{mi} S(X_{jn}) \,. \tag{9.52}$$

See example 9.4.5 for a description of the $Z_{ij,mn}$ and W. Let then $D = \sum_{ij} D_{ij} \frac{\partial}{\partial X_{ij}}$ be any even derivation from $k[\mathrm{GL}_{p|q}]$ to A, for some commutative superalgebra A. Then $\mathrm{d}\phi(D) \in \mathfrak{gl}_W(A)$ is identified with the matrix $(D(\phi(Z_{ij,mn})))$. Using $D \circ S = -D$ we find that

$$\mathrm{d}\phi(D)(Z_{ij,mn}) = D((-1)^{|X_{ij}||X_{mi}|} X_{mi} S(X_{jn})) = \Big((-1)^{|X_{ij}||X_{mi}|} D_{mi}\delta_{jn} - D_{jn}\delta_{mi}\Big) . \tag{9.53}$$

This means that if $E = \sum_{mn} E_{mn} \frac{\partial}{\partial X_{mn}} \in \mathfrak{gl}_{p|q}(A)$ is a second even derivation, we have

$$\mathrm{d}\phi(D)(E) = \sum_{i,m,n} \Big(D_{mi}E_{in} - E_{mi}D_{in}\Big) \frac{\partial}{\partial X_{mn}} , \tag{9.54}$$

which equals $[D, E]$. △

Corollary 9.4.11. *Let G be an affine algebraic supergroup with Lie superalgebra $\mathfrak{g}$. For any commutative superalgebra A and $D_1, D_2 \in \mathfrak{g}(A) = \mathrm{Der}_k^{\epsilon}(k[G], A)_{\bar{0}}$ we have*

$$\mathrm{dAd}_{D_1}(D_2) = [D_1, D_2] = \mathrm{ad}(D_1)(D_2) . \tag{9.55}$$

Proof. Since any affine algebraic supergroup is isomorphic to a closed subgroup of $\mathrm{GL}_{p|q}$ for some p and q, we can reduce the problem to showing the claim for closed subgroups of $\mathrm{GL}_{p|q}$. Let $i : G \to \mathrm{GL}_{p|q}$ be a closed embedding and write $\mathrm{d}i : \mathfrak{g} \to \mathfrak{gl}_{p|q}$ for the induced embedding of Lie algebra functors. By lemma 9.4.4 the adjoint representation satisfies the premises of corollary 9.4.7. Hence we have

$$\mathrm{d}i(\mathrm{dAd}^G(D_1)(D_2)) = \mathrm{dAd}^{\mathrm{GL}_{p|q}}(\mathrm{d}i(D_1))(\mathrm{d}i(D_2)) = [\mathrm{d}i(D_1), \mathrm{d}(D_2)] = \mathrm{d}i([D_1, D_2]) . \tag{9.56}$$

As $\mathrm{d}i$ is injective, the claim follows. □

Chapter 10

Rational supergeometry

In this section we scratch the surface of the theory of rational geometry of superschemes. Most notions of rational geometry carry over without problems to superschemes. This task will be initiated in the second section, after having said something on the different pictures of superschemes. In the third section we apply the results from the first section to give a sensible definition of Cayley supergroups and give an example of a family of Cayley supergroups.

10.1 A note on pictures

In the previous two chapters 8 and 9 we have treated superrings as certain representable functors. In section 5.4 we have treated superrings as locally ringed spaces. By proposition 5.4.5 these are the same things: Any morphism of superrings $f : A \to B$ corresponds uniquely with a morphism of superschemes $\phi : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ and, by the Yoneda lemma, corresponds uniquely with a natural transformation $\varphi : \mathrm{Hom}_{\mathbf{sRng}}(B, -) \to \mathrm{Hom}_{\mathbf{sRng}}(A, -)$. Furthermore, any superscheme X gives rise to a functor $h_X : \mathbf{sRng} \to \mathbf{Sets}$ by putting $h_X(A) = \mathrm{Hom}_{\mathbf{Ssch}}(\mathrm{Spec}(A), X)$. Thus superrings can be viewed as different objects, as a covariant and a contravariant representable functor from **sRng** to **Sets**, as topological spaces and as superrings of course. These different pictures are in a certain, not to be specified, sense equivalent. It is not always clear how to go from one picture to the other and what the relations between the different pictures are. For example: Let X, Y be two superschemes, not necessarily affine ones, and consider the functors $f_X, f_Y : \mathbf{sRng} \to \mathbf{Sets}$ defined by $f_X(A) = \mathrm{Hom}_{\mathbf{Ssch}}(\mathrm{Spec}(A), X)$ and $f_Y(A) = \mathrm{Hom}_{\mathbf{Ssch}}(\mathrm{Spec}(A), Y)$. Now suppose we have a surjective morphism $\phi : X \to Y$, in the sense that the continuous map of topological spaces $X \to Y$ is surjective. Is it then also true that for all superrings A, the induced morphism $\phi_A : f_X(A) \to f_Y(A)$ is surjective? We are not aware of conclusive answers in this direction.

In this chapter we think of superrings as the space of prime ideals together with a sheaf, that is, as affine superschemes. When possible we try to relate to the other pictures. The functor picture is commonly used for algebraic groups, see for example [60, 66, 75] or the more recent [76].

10.2 Rational functions and rational maps

We call a superscheme X integral, if X is connected and if there exists an affine open covering $X = \cup X_i$, with $X_i = \mathrm{Spec}(A_i)$ such that all A_i are super domains. Recall that a superring B is a super domain if the body $\bar{B}$ is an integral domain. Thus equivalently, a superscheme X is integral if the underlying scheme $\bar{X}$ is integral. But $\bar{X}$ is integral if and only if any open affine subscheme is the spectrum of an integral domain if and only if $\bar{X}$ is irreducible and the open affine sub superschemes are the spectra of reduced rings.

Recall that a superring A is a super domain if and only if the $\mathbb{Z}_2$-graded ideal J_A generated by the odd elements, is a prime ideal. In that case we can build the superring of fractions $\mathrm{Frac}(A) = A_{J_A}$, see section 5.1. It follows from proposition 5.1.3 that if A is a super domain and S is a multiplicative set in $A_{\bar{0}}$, then $\mathrm{Frac}(S^{-1}A) \cong \mathrm{Frac}(A)$. For an integral superscheme X we can define a sheaf of rational functions by putting for all affine $U \cong \mathrm{Spec}(A)$ $\mathcal{K} : U \mapsto \mathrm{Frac}(A)$. However, as the next lemma shows, this is not really an exciting sheaf:

Lemma 10.2.1. *Let X be an integral superscheme. Suppose $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$ are affine open sub superschemes of X. Then $\mathcal{K}(U) \cong \mathcal{K}(V)$.*

Proof. We note that by proposition 5.1.12 we can write the superring of fractions of A as an inductive limit over localizations A_a with a even and $\bar{a} \neq 0$. Furthermore, any two open affine sub superschemes intersect in an open dense subset. We write $U_a = \mathrm{Spec}(A_a) \subset U$ for any even $a \in A$ with $\bar{a} \neq 0$ and similarly $V_b = \mathrm{Spec}(B_b) \subset V$ whenever $b \in B$ is even and $\bar{b} \neq 0$. Then for any such U_a, there is a V_b contained in U_a and there is a $U_{a'}$ contained in V_b. The inclusions $U_{a'} \to U_a$, $U_{a'} \to V_b$ and $V_b \to U_a$ induce a commutative triangle:

$$\begin{tikzcd} A_{a'} \arrow[rr] \arrow[dr] & & A_a \\ & B_b \arrow[ur] & \end{tikzcd} \quad . \tag{10.1}$$

Hence we conclude as in lemma 5.5.2 that the category of cones over the direct system $(A_a \to A_{a'} : a' \in \sqrt{(a)})$ is equivalent to the category of cones over the direct system $(B_b \to B_{b'} : b' \in \sqrt{(b)})$. Hence the limits are isomorphic. □

Remark 10.2.2. On occasion it is useful to know the isomorphism $\mathcal{K}(U) \to \mathcal{K}(V)$ from lemma 10.2.1. We use the same notation as in lemma 10.2.1 and its proof. For $a/s \in \mathcal{K}(U)$ there is $t \in B$ with $V_t \subset U_s$, and hence there is a morphism $\chi : A_s \to B_t$ induced by the inclusion $V_t \subset U_s$. Thus we can map $\chi(a/s)$ into $\mathcal{K}(V)$ and one easily checks that this is the sought for isomorphism.

The sheaf $\mathcal{K}$ is thus rather uninteresting on an integral superscheme. In particular, when X is affine, the sheaf $\mathcal{K}$ is a constant sheaf. For non-affine but integral X, we define the superring of rational functions on X to be not the sheaf $\mathcal{K}$, but to be the superring $\mathcal{K}(X) = \mathcal{K}(U)$ for any affine sub superscheme U.

Definition 10.2.3. *Let X be an integral superscheme. We say a point $\xi \in X$ is a generic point if the closure of ξ is the whole of X.*

Lemma 10.2.4. *If X is an integral superscheme, there is precisely one generic point.*

Proof. First we proof existence. Let $U = \mathrm{Spec}(A)$ be an open affine sub superscheme of X. Then the $\mathbb{Z}_2$-graded ideal $J_A \subset A$ is a prime ideal and is contained in all prime ideals of A. Hence $\overline{\{J_A\}} = U$ in U. But $\overline{\{J_A\}}$ is a closed subset in X containing an open dense subset, hence $\overline{\{J_A\}} = X$.

Now for uniqueness, let ξ_1 and ξ_2 be two generic points inside $U = \mathrm{Spec}(A)$, then $\xi_1 \subset \overline{\{\xi_2\}} = V(\xi_2)$ and thus $\xi_2 \subset \xi_1$. Exchanging ξ_1 and ξ_2 we get $\xi_1 = \xi_2$. Now suppose ξ_1 and ξ_2 are generic points in X, then ξ_1 is contained in an open affine sub superscheme U_1 and ξ_2 is contained in an open affine sub superscheme. The intersection $U_1 \cap U_2$ also contains an open affine sub superscheme and by the existence part contains a generic point ξ_3. But then ξ_1 and ξ_3 are generic points in U_1 and hence $\xi_1 = \xi_3$. Similarly, $\xi_2 = \xi_3$ and thus $\xi_1 = \xi_2$. □

The notion of a generic point provides an alternative definition of the superring of rational functions. If X is an integral superscheme with generic point ξ, then we define $\mathcal{K}(X)$ to be the local superring $\mathcal{O}_{X,\xi}$. This is indeed equivalent to the previous definition: the generic point ξ can be seen in any chart $U = \mathrm{Spec}(A)$ where it equals the prime ideal J_A.

Suppose $f : X \to Y$ is a morphism of superschemes and X and Y are integral. Then take any open affine sub superscheme $U = \mathrm{Spec}(A)$ of Y and consider an affine open sub superscheme $V = \mathrm{Spec}(B) \subset f^{-1}(U)$. We have an induced morphism $\tilde{f} : V \to U$, which induces a morphism $\phi : A \to B$. We want to extend ϕ to a morphism $\psi : \mathrm{Frac}(A) \to \mathrm{Frac}(B)$, which is only possible if $a \notin J_A$, where J_A is the $\mathbb{Z}_2$-graded ideal in A generated by the odd elements, implies $\phi(a) \notin J_B$. This is equivalent to requiring that the induced morphism $\bar{\phi} : \bar{A} \to \bar{B}$ be injective, which is equivalent to $\mathrm{Ker}\,(\phi) \subset J_B$. This in turn means that the image of V in U is dense in U, and hence dense in Y.

Definition 10.2.5. *Let X and Y be integral superschemes. We call a morphism $f : X \to Y$ dominant if the image of X is dense in Y.*

Now suppose X and Y are integral superschemes and $f : X \to Y$ is a dominant morphism. Then we can choose an open dense affine sub superscheme $U \cong \mathrm{Spec}(A) \subset Y$ and an affine open sub superscheme $V \cong \mathrm{Spec}(B) \subset f^{-1}(U)$ and obtain an extension

$$\phi_{\mathcal{K}} : \mathcal{K}(Y) \cong \mathrm{Frac}(A) \to \mathrm{Frac}(B) \cong \mathcal{K}(X)\,. \tag{10.2}$$

Using lemma 10.2.1 and remark 10.2.2 one easily checks that this is independent of the choices made. Alternatively, one uses the next lemma:

Lemma 10.2.6. *Let X and Y be integral superschemes with generic points ξ and η respectively. If $f : X \to Y$ is a dominant morphism, then $f(\xi) = \eta$.*

Proof. Let $V = \mathrm{Spec}(A)$ be an open affine sub superscheme of Y, then Y contains ξ and takes the form J_A. Now take any open affine sub superscheme $U = \mathrm{Spec}(B)$ in $f^{-1}(V)$. Then the restriction of f to U induces a morphism $\phi : A \to B$. As f is dominant, the morphism $\bar{\phi} : \bar{A} \to \bar{B}$ is injective. In any way, we have $\phi(J_A) \subset J_B$, so that $\phi^{-1}(J_B)$ contains J_A, but if $a \in A$ does not lie in a, then $\bar{\phi}(\bar{a}) \neq 0$ and thus $\phi(a) \notin J_B$. Hence $\phi^{-1}(J_B) = J_A$, which means $f(\xi) = \eta$. □

It follows from lemma 10.2.6 that if $f : X \to Y$ is a dominant morphism between integral superschemes with generic points $\xi \in X$ and $\eta \in Y$ we have an induced morphism $\phi_\xi : \mathcal{O}_{Y,\eta} \to \mathcal{O}_{X,\xi}$. The induced morphism is precisely the morphism $\phi_{\mathcal{K}} : \mathcal{K}(Y) \to \mathcal{K}(X)$ of equation (10.2).

Definition 10.2.7. *We call two integral superschemes X and Y birational if $\mathcal{K}(X) \cong \mathcal{K}(U)$.*

We have seen that a dominant morphism $f : X \to Y$ induces a morphism $\mathcal{K}(X) \to \mathcal{K}(Y)$, but there are more ways to get such morphisms:

Definition 10.2.8. *Let X and Y be integral superschemes. We define a rational map to be a dominant morphism from an open dense sub superscheme $U \subset X$ to Y.*

By the same procedure as above, a rational map induces a morphism $\mathcal{K}(X) \cong \mathcal{K}(U) \to \mathcal{K}(Y)$. If X is an integral superscheme, the body of the superring of rational functions is a field. If furthermore X is birational to Y, then $\overline{\mathcal{K}(X)} \cong \overline{\mathcal{K}(Y)}$, and thus:

Lemma 10.2.9. *Let X and Y be two integral superschemes. If X and Y are birational, then $\bar{X}$ and $\bar{Y}$ are birational in the sense of ordinary schemes.*

Remark 10.2.10. Now that we have introduced rational maps, we can in fact go further and define Cartier divisors on superschemes. We will however not do so in this thesis, but leave this for future research.

We close this section by introducing some terminology that can be useful when discussing rational maps between affine superschemes. Let $f : X = \mathrm{Spec}(A) \to Y = \mathrm{Spec}(B)$ be a rational map between X and Y. We say f is a principal rational map if f is defined on a principal open subset of X. Thus, there is an even element $a \in A$ such that f is a dominant morphism $\mathrm{Spec}(A_a) \to \mathrm{Spec}(B)$. Let X and Y be two representable functors from the category of superalgebras over a field k represented by $k[X]$ and $k[Y]$ respectively. We call a rational transformation between X and Y any natural transformation $\mathrm{Hom}_{\mathbf{sAlg}}(U, -) \to \mathrm{Hom}_{\mathbf{sAlg}}(k[Y], -) \cong Y$, where U is an open set of $\mathrm{Spec}(k[X])$. We call such a rational transformation fine, if there is a multiplicative set $S \in k[X]_{\bar{0}}$ such that $U = \mathrm{Spec}(S^{-1}k[X])$. We call a fine rational transformation a principal rational transformation if $S = \langle s \rangle$ is the multiplicative set of powers of a single element of $k[X]_{\bar{0}}$.

10.3 Cayley maps

Let G be an affine algebraic supergroup with representing super Hopf algebra $k[G]$ and Lie algebra functor $\mathfrak{g}$. We can view G as an affine superscheme by writing $\mathcal{G} = \mathrm{Spec}(k[G])$. When the underlying group $\mathrm{Spec}(\overline{k[G]})$ is irreducible, then $\mathcal{G}$ is an integral superscheme. It is natural to ask, whether we can also view the Lie algebra functor as a superscheme. In other words, is there a superscheme X, such that for all superalgebras A over k we have $\mathfrak{g}(A) \cong \mathrm{Hom}_{\mathbf{Ssch}}(\mathrm{Spec}(A), X)$? If X would be affine with representing superalgebra $k[X]$, then we would have $\mathfrak{g}(A) \cong \mathrm{Hom}_{\mathbf{sAlg}}(k[X], A)$. But then it is obvious what X should be, as the functor $\mathfrak{g}$ is of the form $A \mapsto (V \otimes A)_{\bar{0}}$ for a finite-dimensional super vector space V. And for any such super vector space and any superalgebra A we have

$$(V \otimes A)_{\bar{0}} \cong \mathrm{Hom}_{\mathbf{sVec}}(V*, A) \cong \mathrm{Hom}_{\mathbf{sAlg}}(k[V], A)\,. \tag{10.3}$$

Thus the superscheme representing the Lie algebra functor is a rather simple object; if $\mathrm{Der}_k^{\epsilon}(k[G], k)$ has dimension $p|q$, then $\mathfrak{g}$ is represented by the superscheme $\mathrm{Spec}(k[p|q])$, where

$$k[p|q] = k[X_1, \dots, X_p|\eta_1, \dots, \eta_q]\,. \tag{10.4}$$

We will on occasion write $k[\mathfrak{g}]$ for the affine superscheme representing $\mathfrak{g}$. When viewing $\mathfrak{g}$ as a superscheme, one loses the Lie algebra structure: The Lie algebra structure can be enforced in the linear part of $k[\mathfrak{g}]$, but there seems to be no canonical way to enlarge the Lie algebra structure to the whole of $k[\mathfrak{g}]$. However, viewing $\mathfrak{g}$ as a superscheme makes it possible to study rational maps of a certain kind.

Definition 10.3.1. *Let G be an integral affine algebraic supergroup with representing super Hopf algebra $k[G]$ and with Lie algebra functor $\mathfrak{g}$. A Cayley map is a rational map transformation $\Phi : G \to \mathfrak{g}$ such that for all superalgebras A and all $g, h \in G(A)$ we have $\Phi^A(ghg^{-1}) = \mathrm{Ad}_g(\Phi^A(h))$.*

If an integral affine algebraic supergroup admits a Cayley map, then its superring of rational functions is of a very simple kind: it is isomorphic to $\mathrm{Frac}(k[p|q])$ for some positive integers p and q. We call an integral affine algebraic supergroup that admits a Cayley map a Cayley supergroup. The following theorem shows that they exist:

Theorem 10.3.2. *For any positive integers the integral affine algebraic supergroup $\mathrm{Osp}_{p|2q} \cap SL_{p|2q}$ is a Cayley supergroup.*

Proof. Let us recall the definitions and properties of the affine algebraic supergroup $\mathrm{Osp}_{p|2q}$ from example 8.6.30. For any fixed superalgebra the group $\mathrm{Osp}_{p|2q}(A)$ corresponds to all matrices $X \in \mathrm{Mat}_{p|2q}(A)_{\bar{0}}$ satisfying

$$X^{ST}\Omega X = \Omega\,, \tag{10.5}$$

where Ω is defined in eqn.(8.59). The Lie algebra functor assigns to A all even elements $Y \in \mathrm{Mat}_{p|2q}(A)$ such that

$$Y^{ST}\Omega + \Omega Y = 0\,. \tag{10.6}$$

It is clear that the underlying algebraic group of $\mathrm{Osp}_{p|2q}$ is the direct product of $\mathrm{O}_p \times \mathrm{Sp}_{2q}$. This is not an irreducible variety, and hence $\mathrm{Osp}_{\mathrm{p|2q}}$ is not integral as a superscheme. However, the intersection $\mathrm{Osp}_{p|2q} \cap \mathrm{SL}_{p|2q}$ has underlying variety $\mathrm{SO}_p \times \mathrm{Sp}_{2q}$, which is an irreducible variety and hence $\mathrm{Osp}_{p|2q} \cap \mathrm{SL}_{p|2q}$ is an integral superscheme. For a fixed superalgebra A, we now define the map $\Phi^A : \mathrm{Osp}_{p|2q}(A) \to \mathrm{Der}_k^{\epsilon}(k[\mathrm{Osp}_{p|2q}], A)_{\bar{0}}$ by

$$\Phi^A : X \mapsto (1 - X)(1 + X)^{-1}\,, \tag{10.7}$$

where 1 denotes the $(p + 2q) \times (p + 2q)$ identity matrix. Clearly, when Φ^A is defined, then $\Phi^A \circ \Phi^A(M) = M$ for any $M \in \mathrm{Mat}_{p|2q}(A)_{\bar{0}}$. Furthermore, when X satisfies eqn.(10.5) then $\Phi^A(X)$ satisfies eqn.(10.6) and vice versa. The whole construction is functorial in the superalgebra A and thus Φ defines a rational transformation. As $\Phi^A(X)$ is only not defined when $(1 + X)$ is not invertible, Φ is a principal rational transformation. We conclude that Φ sets up a birational equivalence between a principal open subset of $\mathrm{Spec}(k[\mathrm{Osp}_{p|2q}])$ and a principal open subset of $\mathrm{Spec}(k[\frac{1}{2}p(p-1) + q(2q+1)|2pq])$.

It remains to check the Ad-equivariance. This is straightforward as the adjoint action from the group $\mathrm{Osp}_{p|2q}(A)$ on the Lie algebra is given by matrix conjugation and we easily verify that

$$\begin{aligned}\Phi^A(MNM^{-1}) &= M(1-N)M^{-1}(1 + MNM^{-1})^{-1}\\ &= M(1-N)(1+N)^{-1}M^{-1} = \mathrm{Ad}_M(\Phi^A(N))\,,\end{aligned} \tag{10.8}$$

for any $M, N \in \mathrm{Osp}_{p|2q}(A)$. □

From the defining equation of $\mathrm{Osp}_{p|2q}$ one easily sees that if $X \in \mathrm{Osp}_{p|2q}(A)$ for some superalgebra A, then $\mathrm{Ber}(X) = \pm 1$; indeed, $\mathrm{Ber}(X^{ST}\Omega X) = (\mathrm{Ber}(X))^2\mathrm{Ber}(\Omega)$, which should equal $\mathrm{Ber}(\Omega) = \pm 1$, hence dividing by $\mathrm{Ber}(\Omega)$ gives the statement. We now apply the Cayley map to prove the following property of $\mathrm{Osp}_{p|2q}$:

Lemma 10.3.3. *If* $X \in \mathrm{Osp}_{p|2q}(A)$ *for some superalgebra* A *and* $\mathrm{Ber}(X) = -1$*, then* $1 + X$ *is not invertible.*

Proof. If X were invertible, then $X = \Phi^A(Y)$ for some $Y \in \mathfrak{osp}_{p|2q}(A)$ and thus

$$\begin{aligned}\mathrm{Ber}(X) &= \frac{\mathrm{Ber}(1-Y)}{\mathrm{Ber}(1+Y)} = \frac{\mathrm{Ber}\Omega^{ST}(1-Y)\Omega}{\mathrm{Ber}(1+Y)}\\ &= \frac{\mathrm{Ber}(1-Y)}{\mathrm{Ber}(1+Y)} = 1\,,\end{aligned} \tag{10.9}$$

where we used that $\Omega^{ST}\Omega = 1$, $(\Omega^{ST})^{ST} = \Omega$ and that Y satisfies $Y^{ST}\Omega + \Omega Y = 0$, whence $\Omega^{ST}Y\Omega = -Y$. Thus if $1 + X$ is invertible, then the Berezinian is positive. □

Remark 10.3.4. The Cayley map is named after Arthur Cayley who was the first to introduce such a map for the orthogonal groups [77]. In [78] the Cayley map was defined slightly differently, however, one can also extend their definition and show that the natural transformation from theorem 10.3.2 makes $\mathrm{Osp}_{p|2q}$ into a Cayley group in their sense. However, the definition of [78] is not functorial and hence not well-suited for affine algebraic supergroups. In [79] all simple reductive algebraic groups that are Cayley were classified.

We have not succeeded in classifying all affine algebraic supergroups that admit a Cayley map. But, the following result at least gives a first test:

Proposition 10.3.5. *Let G be an irreducible affine algebraic supergroup that admits a Cayley map. Then the underlying affine algebraic group is a Cayley group.*

Proof. We apply the rational transformation $\Phi : G \to \mathfrak{g}$ to the field k, which we then view as a superalgebra. Then we get a rational map between varieties, that satisfies the definition of a Cayley map in the sense of [79]. $\square$

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Abstract

This thesis is concerned with extending the theory of commutative rings to a theory of superrings. The goal is to develop enough machinery to do algebraic geometry with superrings. Special attention is given to a suitable definition of algebraic supergroups and their basic properties.

A superring A is an associative ring with unit that admits a $\mathbb{Z}_2$-decomposition $A = A_{\bar{0}} \oplus A_{\bar{1}}$ such that for two homogeneous elements $a \in A_\alpha$ and $b \in A_\beta$, with $\alpha, \beta \in \mathbb{Z}_2$ we have $ab = (-1)^{\alpha\beta} ba$. In particular, a general superring is not commutative and any odd element is nilpotent. That the odd elements are nilpotent, guarantees that many theorems of commutative algebra can be extended to superrings. The goal of this thesis is to carry out the programm of extending as much commutative algebra theorems as possible to superrings, to find a suitable setting to do algebraic geometry based on the set of prime ideals of a superring and to define and discuss algebraic supergroups.

The motivation to study superrings and to develop a framework to do supergeometry mainly comes from physics. Mirror symmetry and supersymmetry play an important role in modern theoretical physics and are promising research areas when it comes to fundamental deep results in string theory, or more general particle physics, and in mathematics. Both concepts use algebraic varieties with additional 'anticommuting coordinates', that is, the local coordinates are elements of superrings. Another motivation comes from the trend in mathematics to try to construct a framework for noncommutative geometry, a quest which is also based on modern particle physics.

In the first few chapters of the thesis, the foundations for the theory of superrings are spelled out. In particular, prime ideals are discussed and studied, which leads to the notion of a superscheme. A superscheme is a locally ringed space with a sheaf of superrings. We also define projective superschemes and show that they are examples of superschemes. We define fibred products and show that arbitrary fibred products exist in the category of superschemes. There is a functor from the category of superschemes to the category of schemes, which does not change the topological space, but only the sheaf. One chapter is devoted to study the concept of dimension of a superring. Since the Krull dimension, which is defined by means of chains of prime ideals, does not give a reasonable concept of dimension for superrings, we first define dimension for local superrings. We show that the dimension of a local regular superring can be read off from its Hilbert function.

After having given the rudiments of a theory of superring, we turn to algebraic supergroups. By restricting ourselves to affine algebraic supergroups, we can reduce the problem of studying algebraic supergroups to the study of super Hopf algebras. We extend the theory of Hopf algebras to super Hopf algebras and deduce elementary properties of affine algebraic supergroups. A fundamental result is that any affine algebraic supergroup is a closed subgroup of the general linear supergroup. The relation between an affine algebraic supergroup and its Lie algebra, which we define as a functor, is extensively studied.

In the last chapter we briefly discuss some aspects of rational supergeometry. As an application we show that the Cayley map, which is a birational morphism between an affine algebraic supergroup and its Lie algebra and that is equivariant with respect to conjugation, exists for the supergroups $\mathrm{Osp}_{p|2q}$.

Zusammenfassung

Diese Dissertation beschäftigt sich mit der Erweiterung der Theorie der kommutativen Ringe zu einer Theorie der Superringe. Das Ziel ist ausreichend viel Theorie der Superringe aufzubauen so, dass mit dieser Theorie algebraische Geometrie basierend auf Superringen entwickelt werden kann. Einer geeigneten Definition algebraischer Supergruppen und ihren elementaren Eigenschaften wird im besonderen Aufmerksamkeit gegeben.

Ein Superring A ist ein assoziativer Ring mit Eins, der eine $\mathbb{Z}_2$-Zerlegung zulässt $A = A_{\bar{0}} \oplus A_{\bar{1}}$ so, dass für je zwei homogene Elemente $a \in A_\alpha$ und $b \in A_\beta$, mit $\alpha, \beta \in \mathbb{Z}_2$, gilt, dass $ab = (-1)^{\alpha\beta} ba$. Im Besonderen sind Superringe generell nicht kommutativ und jedes ungerade Element ist nilpotent. Die Nilpotenz der ungeraden Elemente garantiert, dass sich viele Theoreme aus der kommutativen Algebra zu Superringen erweitern lassen. Das Ziel dieser Dissertation ist es, das Programm des Erweiterns von möglichst vielen Theoremen aus der kommutativen Algebra durchzuführen, einen Rahmen für algebraische Geometrie basierend auf der Menge der Primideale eines Superringes zu finden und algebraische Supergruppen zu definieren und zu besprechen.

Die Motivation Superringe zu studieren und einen Rahmen für Supergeometrie zu entwickeln kommt hauptsächlich aus der Physik. Spiegelsymmetrie und Supersymmetrie spielen eine wichtige Rolle in der modernen theoretischen Physik und sind vielversprechende Forschungsgebiete, was das Finden tiefer Ergebnisse in Stringtheorie, oder allgemeiner in der Teilchenphysik, und in der Mathematik betrifft. Beide Konzepte benützen algebraische Varietäten mit zusätzlichen 'antikommutierenden Koordinaten', das heißt, die lokalen Koordinaten sind Elemente eines Superringes. Eine andere Motivation ist die Bestrebung in der Mathematik, einen Rahmen für nichtkommutative Geometrie zu entwickeln, eine Bestrebung, die wieder ihre Wurzeln in der modernen Quantenfeldtheorie findet.

In den ersten Kapiteln der Dissertation werden die Grundlagen der Theorie der Superringe genau ausgearbeitet. Im Besonderen werden Primideale besprochen und ihre elementaren Eigenschaften untersucht, was zu dem Begriff der Superschemata führt. Ein Superschema ist ein lokal geringter Raum mit einer Garbe von Superringen. Wir definieren auch projektive Superschemata und zeigen, dass diese Beispiele von Superschemata sind. Weiter definieren wir gefaserte Produkte und zeigen, dass in der Kategorie der Superschemata beliebige gefaserte Produkte existieren. Es gibt einen Funktor von der Kategorie der Superschemata in die Kategorie der Schemata, sodass der topologische Raum erhalten bleibt und nur die Garbe geändert wird. Ein Kapitel ist dem Begriff der Dimension eines Superringes gewidmet. Weil die Krulldimension, die mittels Ketten von Primidealen definiert ist, sich nicht zu einem aussagekräftigen Begriff für Superringe erweitern lässt, definieren wir zunächst die Dimension für lokale Superringe. Wir zeigen dass die Dimension eines lokalen regulären Superringes aus der zugehörigen Hilbertschen Funktion abgelesen werden kann.

Nachdem wir die Grundlagen einer Theorie der Superringe präsentiert haben, widmen wir uns den algebraischen Supergruppen. Dadurch, dass wir uns auf affine algebraische Supergruppen einschränken, reduziert sich das Problem algebraische Supergruppen zu studieren, auf die Untersuchung von Superhopfalgebren. Wir erweitern die Theorie der Hopfalgebren zu Superhopfalgebren

und leiten elementare Eigenschaften affiner algebraischer Gruppen her. Ein fundamentales Resultat besagt, dass jede affine algebraische Supergruppe eine abgeschlossene Untergruppe der allgemeinen linearen Supergruppe ist. Der Zusammenhang zwischen einer affinen algebraischen Supergruppe und ihrer Lie Algebra, die wir als einen Funktor definieren, ist gründlich untersucht.

Im letzten Kapitel besprechen wir kurz einige Aspekte der rationalen Supergeometrie. Als eine Anwendung zeigen wir, dass die Cayley Abbildung, die ein birationaler Morphismus zwischen einer affinen algebraischen Supergruppe und ihrer Lie Algebra ist und äquivariant bezüglich Konjugation in der Gruppe ist, für die Supergruppen $\mathrm{Osp}_{p|2q}$ existiert.

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2002-2006:	Ph.D. Theoretical Physics, Rijksuniversiteit Groningen
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Activities

2002-2005:	Member of the 'Promoteam Natuurkunde' of the Rijksuniversiteit Groningen
May 2004:	Co-organizer of the '18th Nordic Network Meeting on Fields, Strings and Brances' in Groningen
Nov. 2005:	Co-organizer of the Conference 'From Strings to Cosmic Web' in Groningen
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Publications

	Master Thesis	Scattering Amplitudes in String Theory
	Paper in JHEP	De Sitter solutions in $N = 4$ matter coupled supergravity
	Paper in JHEP	Potential and mass matrix in gauged $N = 4$ supergravity
	Paper in JHEP	Group manifold reduction of dual $N = 1$ $d = 10$ supergravity
	Paper in JHEP	Coset symmetries in dimensionally reduced heterotic supergravity
	Paper in Class.& Qu.Gr.	Dynamics of assisted inflation
	Paper in Class.& Qu.Gr.	Scaling cosmologies in $N = 8$ gauged supergravity
	Ph.D. Thesis	Symmetries in $N = 4$ supergravities
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2002-2005:	Severial tutorials on high schools
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