Problem 12005
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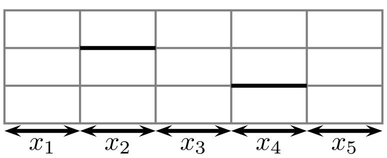
Proposed by D. E. Knuth (USA).

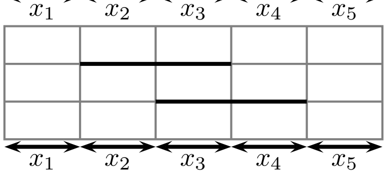
A tight m-by-n paving is a decomposition of an m-by-n rectangle into $m+n-1$ rectangular tiles with integer sides such that each of the $m-1$ horizontal lines and $n-1$ vertical lines within the rectangle is part of the boundary of at least one tile. Let $a_{m,n}$ denote the number of tight m-by-n pavings.
(a) Determine $a_{3,n}$ as a function of n.
(b) Show for $m \geq 3$ that $\lim_{n\to\infty} \frac{a_{m,n}}{m^n}$ exists, and compute its value.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

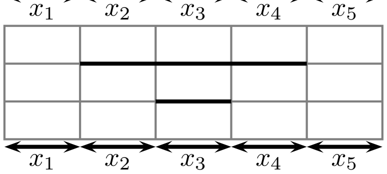
Solution. For (a), we consider several cases with respect to the relative positions of the two horizontal lines. For each case we count the number of tight 3-by-n pavings and the corresponding generating functions. The coefficient before the sum symbol denotes the number of symmetries.

1) 

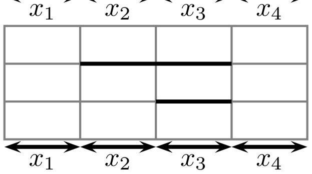
$$2 \cdot \sum_{\substack{x_1+x_2+x_3+x_4+x_5=n \\ x_1\geq 0, x_3\geq 0, x_5 \geq 0 \\ x_2\geq 1, x_4\geq 1}} 2^{x_2-1}2^{x_4-1} = [z^n]\frac{2z^2}{(1-z)^3(1-2z)^2}$$

2) 

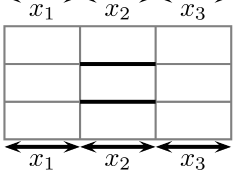
$$2 \cdot \sum_{\substack{x_1+x_2+x_3+x_4+x_5=n \\ x_1\geq 0, x_5 \geq 0 \\ x_2\geq 1, x_3\geq 1, x_4\geq 1}} 2^{x_2-1}3^{x_3-1}2^{x_4-1} = [z^n]\frac{2z^3}{(1-z)^2(1-2z)^2(1-3z)}$$

3) 

$$2 \cdot \sum_{\substack{x_1+x_2+x_3+x_4+x_5=n \\ x_1\geq 0, x_5 \geq 0 \\ x_2\geq 1, x_3\geq 1, x_4\geq 1}} 2^{x_2-1}3^{x_3-1}2^{x_4-1} = [z^n]\frac{2z^3}{(1-z)^2(1-2z)^2(1-3z)}$$

4) 

$$4 \cdot \sum_{\substack{x_1+x_2+x_3+x_4=n \\ x_1\geq 0, x_4 \geq 0 \\ x_2\geq 1, x_3\geq 1}} 2^{x_2-1}3^{x_3-1} = [z^n]\frac{4z^2}{(1-z)^2(1-2z)(1-3z)}$$

5) 

$$\sum_{\substack{x_1+x_2+x_3=n \\ x_1\geq 0, x_3 \geq 0 \\ x_2\geq 1}} 3^{x_2-1} = [z^n]\frac{z}{(1-z)^2(1-3z)}$$

Hence, by adding the five enumerations all together we obtain

$$\begin{aligned} a_{3,n} &= [z^n]\frac{z+3z^2}{(1-z)^3(1-2z)(1-3z)} \\ &= [z^n]\left(\frac{27/4}{1-3z} - \frac{20}{1-2z} + \frac{2}{(1-z)^3} + \frac{7/2}{(1-z)^2} + \frac{31/4}{1-z}\right) \\ &= \frac{27}{4}\cdot 3^n - 20\cdot 2^n + n^2 + \frac{13}{2}\cdot n + \frac{53}{4}. \end{aligned}$$

The first terms of the sequence (see OEIS A285361) are

$$1, 11, 64, 282, 1071, 3729, 12310, 39296, 122773, 378279, 1154988, 3505542, 10598107, 31957661.$$

As regards (b), in a similar way we find that the generating function f_m of $\{a_{m,n}\}_n$ is a rational function with a simple pole at $1/m$, and other poles at $1, 1/2, \dots, 1/(m-1)$. More precisely,

$$f_m(z) = \frac{z}{(1-z)^2(1-2z)^2\cdots(1-(m-1)z)^2(1-mz)} + h_m(z)$$

where h_m is holomorphic in a neighborhood of $1/m$. Hence

$$\lim_{n\to\infty} \frac{a_{m,n}}{m^n} = -m\mathrm{Res}\left(f_m, \frac{1}{m}\right) = \frac{1/m}{(1-1/m)^2(1-2/m)^2\cdots(1-(m-1)/m)^2} = \frac{m^{2m-3}}{((m-1)!)^2}.$$

□

Update August 2020. By applying the same approach, we find that for $n \geq 1$,

$$\begin{aligned}
a_{2,n} &= [z^n]\frac{z}{(1-z)^2(1-2z)} \\
&= 1, 4, 11, 26, 57, 120, 247, 502, 1013, 2036, 4083, 8178, 16369, 32752, 65519, 131054, 262125\dots \\
a_{4,n} &= [z^n]\frac{z+8z^2-47z^3+6z^4+104z^5}{(1-z)^4(1-2z)^2(1-3z)^2(1-4z)} \\
&= 1, 26, 282, 2072, 12279, 63858, 305464, 1382648, 6029325, 25628762, 107026662, 441439944\dots \\
a_{5,n} &= [z^n]\frac{z+26z^2-264z^3+122z^4+4367z^5-11668z^6+3000z^7+11168z^8+160z^9}{(1-z)^5(1-2z)^2(1-3z)^3(1-4z)^2(1-5z)} \\
&= 1, 57, 1071, 12279, 106738, 781458, 5111986, 30980370, 178047831, 985621119, 5311715977\dots \\
a_{6,n} &= [z^n]\frac{p_6(z)}{(1-z)^6(1-2z)^3(1-3z)^4(1-4z)^3(1-5z)^2(1-6z)} \\
&= 1, 120, 3729, 63858, 781458, 7743880, 66679398, 521083252, 3802292847, 26409556208, 176950120591\dots
\end{aligned}$$

where

$$\begin{aligned}
p_6(z) = z &+ 68z^2 - 1253z^3 + 2098z^4 + 89339z^5 - 810592z^6 + 2854037z^7 - 2568982z^8 - 12123516z^9 \\
&+ 40272448z^{10} - 41732272z^{11} + 1566912z^{12} + 15581632z^{13} + 1084480z^{14} - 67200z^{15}.
\end{aligned}$$

See OEIS' sequences: A285357, A336732, A336734.