II

Locales

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This chapter is an introduction to the basic concepts, constructions, and results concerning locales. Locales (frames) are the object of study of the so called point-free topology. They sufficiently resemble the lattices of open sets of topological spaces to allow the treatment of many topological questions. One motivation for the theory of locales is building topology on the intuition of "places of non-trivial extent" rather than on points. Not the only one; hence it is not surprising that the theory has developed beyond the purely geometric scope. Still, we can think of a locale as of a kind of space, more general than the classical one, allowing us to see topological phenomena in a new perspective. Other aspects are, for instance, connections with domain theory [53, 52], continuous lattices [5, 31], logic [65, 20] and topos theory [42, 20].

Modern topology originates, in principle, from Hausdorff's "*Mengenlehre*" [30] in 1914. One year earlier there was a paper by Caratheodory [23] containing the idea of a point as an entity localized by a special system of diminishing sets; this is also of relevance for the modern point-free thinking. In the twenties and thirties the importance of (the lattice of) open sets (which are, typically, "places of non-trivial extent") became gradually more and more apparent (see e.g. Alexandroff [1] or Sierpinski [54]). In [57] and [58], Stone presented his famous duality theorem from which it followed that compact zero-dimensional spaces and continuous maps are well represented by the Boolean algebras of closed open sets and lattice homomorphisms. This was certainly an encouragement for those who endeavoured to treat topology other than as a structure on a given system of points (Wallman [66] in 1938, Menger [45] in 1940, McKinsey and Tarski [44] in 1944). In the Ehresmann seminar of the late fifties [28, 19, 46], we encounter the theory of frames (introduced as "local lattices") already in the form that we know today (it should be noted that almost at the same time there appeared independently two important papers, by Bruns [22] and Thron [59], on homeomorphism of spaces with isomorphic lattices of open sets, under weak separation axioms). Then many authors got interested (C.H. Dowker, D. Papert (Strauss), J. Isbell, B. Banaschewski, etc.) and the field started to develop rapidly. The pioneering paper by Isbell [32], which opened several topics and placed specific emphasis on the dual of the category of frames, introducing the term "locale", merits particular mentioning. In 1982, Johnstone published his monograph [34] which is still a primary source of reference.

The notion of a locale can be viewed as an extension of the notion of a (topological) space. Extending or generalizing a notion calls for justification (as Johnstone says in [35], "*there remains the question: why study locales at all?*"). Several questions naturally arise: When abandoning points, do we not lose too much information? Is the broader range of "spaces" we now have desirable at all? That is, is the theory in this context, in whatever sense, more satisfactory? And is it not so that the new techniques obscure the geometric contents? Here are some answers:

1. Starting with very low separation axioms (sobriety, T_D) the point-free representation contains all the information of the original space.
2. The class of Hausdorff locally compact spaces is represented equivalently by distributive continuous lattices, also a very satisfactory fact.
3. The broader context does yield, in some areas, better theory. For instance, the passage from sober spaces to locales is a full embedding of categories which, in general, does not preserve products. This discrepancy between the two products is, however, beneficial in some respects, as it was firstly observed by Isbell in [32]. If one recalls how badly the notion of paracompactness (which is very important in applications of topology) behaves under constructions in the classical context, one starts to be quite happy about the product being changed sometimes: products of paracompact and metric spaces are not necessarily paracompact but this is not the case in the point-free context (the category of paracompact locales is reflective in the category of locales).

   Subobjects in this broader context also behave differently, again with advantages for locales. This is clear from the fact that the intersection of any family of dense sublocales of a given locale is again dense. In the words of Johnstone [35],

   > "*...the single most important fact which distinguishes locales from spaces: the fact that every locale has a smallest dense sublocale. If you want to 'sell' locale theory to a classical topologist, it's a good idea to begin by asking him to imagine a world in which any intersection of dense subspaces would always be dense; once he has contemplated some of the wonderful consequences that would flow from this result, you can tell him that that world is exactly the category of locales.*
   >
   > *(...) It is certainly clear that in order to achieve such a world, we have to abandon the idea of a space as a set of points equipped with some kind of structure; for there will be examples in any category of this type of pairs of dense subspaces of a nontrivial space having no points in common.*"

4. The techniques are sometimes less intuitive than the classical ones; but it can be argued that they are very often simpler. And, perhaps surprisingly, they often yield constructive results where the classical counterparts cannot. For instance, the Tychonoff product theorem is fully constructive (meaning: no choice principle and no excluded middle, see [33, 6]).

   So, locales have characteristics that go beyond the interest they may deserve as generalized topological spaces. In many situations, certain spaces are non-trivial only by virtue of some choice principle, whereas their lattices of open sets already have previous existence, before such assumptions. This means that, in some sense, we always see the lattice of opens, but to see their points may require some additional tool in the form of some choice principle. This idea was nicely expressed by Banaschewski, with the following slogan [7]:

$$\frac{\begin{array}{c}\textit{choice-free localic argument}\\ + \\ \textit{suitable choice principle}\end{array}}{\textit{classical result on spaces}}$$

5. In many of the localic constructions that precede certain familiar spaces, in more general contexts than classical set theory, the aforementioned spaces are not sufficient and the corresponding lattices of open sets take over their place. A typical example is the Gelfand Duality: classically, it is a duality between the category of commutative C^*-algebras and the category of compact Hausdorff spaces; within the constructive context of a Grothendieck topos, the Gelfand Duality takes the form of a dual equivalence between the category of commutative C^*-algebras and the category of compact completely regular locales [15, 16]. This is the real version of Gelfand's Duality, its classical version being an accidental consequence because of the special assumptions assumed: the category of compact regular locales is, in the presence of the Boolean Ultrafilter Theorem, equivalent to the category of compact Hausdorff spaces.

   Thus, locales play an important role in a constructive approach to topology, allowing us to develop topology in an arbitrary topos and other non-classical contexts (see e.g. [14, 16, 61]).

For more information on the history and development of point-free topology we advise the reader to see the excellent survey in Johnstone's [35] or [36] (see also the introduction to [34]).

We assume that the reader is acquainted with basic categorical notions such as adjunctions, limits, colimits, and factorization systems. We also assume that the reader is familiar with the basics of lattice theory. A general reference for categorical concepts is [20] and for lattice theoretical concepts we refer to Chapter I of this volume.

1. Spaces, frames, and locales

1.1. From topological spaces to frames. Take a topological space X viewed as a set $|X|$ endowed with a system $\mathcal{O}X \subseteq \{M \mid M \subseteq |X|\}$ of *open sets*. The system $(\mathcal{O}X, \subseteq)$ is a complete lattice since the union of any family of open sets is again open; evidently the infinite distributive law

$$U \wedge \bigvee_{i \in I} V_i = \bigvee_{i \in I} (U \wedge V_i)$$

is valid in $\mathcal{O}X$, since finite meet $\wedge$ and supremum $\bigvee$ in $\mathcal{O}X$ are given by the usual set theoretical operations of intersection $\cap$ and union $\bigcup$, respectively (note that arbitrary meet $\bigwedge_{i \in I} V_i$ is given by the interior, $int(\bigcap_{i \in I} V_i)$, of the intersection $\bigcap_{i \in I} V_i$).

The question naturally arises how much information of the space X is contained in this lattice considered as an algebraic object (that is, ignoring the fact that and how the elements $U \in \mathcal{O}X$ consist of the points $x \in |X|$).

On the other hand, a continuous mapping $f : X \to Y$ induces a lattice homomorphism $h : \mathcal{O}Y \to \mathcal{O}X$, defined by $h(U) = f^{-1}[U]$, which clearly preserves arbitrary joins and finite meets; again, one may naturally ask how accurately the homomorphism h represents the original continuous map f.

Suppose the answers to these questions are satisfactory. Then we might treat topology, or a considerable part of it, as a part of algebra (perhaps with some advantages of a handy calculus).

Now the answers are indeed fairly pleasing. For instance, if the spaces in question are sober (a separation axiom considerably weaker than the Hausdorff one, see 1.3 below), they can be reconstructed from the lattices $\mathcal{O}X$, and the continuous map f can be reconstructed from the homomorphism h above. (And it turns out that the ensuing calculus does bring surprising advantages.)

Abstracting the properties of the lattices $\mathcal{O}X$ and the mappings $(U \mapsto f^{-1}[U]) : \mathcal{O}Y \to \mathcal{O}X$, one defines a *frame* A as a complete lattice satisfying the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\} \tag{1.1.1}$$

for all $a \in A$ and $S \subseteq A$, and a *frame homomorphism* $h : A \to B$ as a mapping preserving all joins, including the least element 0, and finite meets, including the largest element 1. The resulting category will be denoted by

$$\mathsf{Frm}.$$

Besides the lattices $\mathcal{O}X$ of open subsets of a topological space, obvious examples of frames are the finite distributive lattices, the complete Boolean algebras and the complete chains.

Remarks. (1) Recall from Chapter I that (1.1.1) makes a frame a Heyting algebra. In fact, the frame distributivity law says that for each $a \in A$ the map

$$\begin{array}{rccc} -\wedge a: & A & \to & A \\ & c & \mapsto & c \wedge a \end{array}$$

preserves suprema. Consequently, by a standard fact on posetal categories (Corollary I.3.3), it has a right adjoint; denoting it by $a \to -$ we see that a frame is a complete Heyting algebra, that is, a complete lattice endowed with an extra operation $\to$ satisfying

$$c \wedge a \leq b \quad \text{if and only if} \quad c \leq a \to b.$$

Using the distributivity rule in the frame we immediately see that $a \to b = \bigvee\{x \in A \mid x \wedge a \leq b\}$. In particular,

$$a^c = \bigvee\{x \in A \mid x \wedge a = 0\}$$

is the *pseudo-complement* $a \to 0$ of a (also called *negation* in Chapter I), that is, one has

$$x \wedge a = 0 \text{ if and only if } x \leq a^c. \tag{1.1.2}$$

The following are the standard basic properties of pseudo-complements:

(C1) $a \mapsto a^c$ is antitone, $0^c = 1$ and $1^c = 0$.
(C2) $a \leq a^{cc}$.
(C3) $a^c = a^{ccc}$.
(C4) $a \wedge b = 0$ iff $a^{cc} \wedge b = 0$.
(C5) $(\bigvee a_i)^c = \bigwedge(a_i^c)$.
(C6) $(a \wedge b)^{cc} = a^{cc} \wedge b^{cc}$.

(2) It should be noted right away that not every frame A is isomorphic to an $\mathcal{O}X$ (see 1.8 and 2.14 below for such examples). Thus, viewing frames as representations of spaces, we have "more spaces than before". Such an extension of the scope of the objects considered as spaces may be seen as becoming, and again, it may not. During the development of point-free topology it has turned out that it is unequivocally of advantage.

Exercises.

1. Prove formulas (C1)-(C6) of Remark 1 (cf. Proposition I.3.7).
2. Show that in $\mathcal{O}X$ we have:
   (a) $U \to V = int((X \setminus U) \cup V)$;
   (b) $U^{cc} = int(\overline{U})$, the interior of the closure of U.
   (Compare this with classical logic, where $p \to q \Leftrightarrow \neg p \vee q$ and $\neg\neg p \Leftrightarrow p$.)
3. Show that in a frame, $a \wedge a^c = 0$. Find an example in the locale $\mathcal{O}\mathbb{R}$ of open subsets of the real line, provided with its usual topology, for which $a \vee a^c \neq 1$.
4. Show that the locale $\mathcal{O}\mathbb{R}$ does not satisfy the de Morgan Law $(a \wedge b)^c = a^c \vee b^c$.
5. Show that the lattice $\mathcal{O}X$ of open sets of a topological space X need not satisfy the dual of the infinite distributive law (1.1.1).

6. Give examples to show that the frame homomorphism $f^{-1} : \mathcal{O}Y \to \mathcal{O}X$ need not preserve infinite meets, nor the Heyting implication $\to$, and not even the pseudocomplements U^c.
7. Prove that if $f : X \to Y$ is any open continuous map, then $f^{-1} : \mathcal{O}Y \to \mathcal{O}X$ preserves infinite meets and the Heyting implication.

1.2. Locales. Let Top denote the category of topological spaces and continuous maps. Sending X to $\mathcal{O}X$ and f to f^{-1} yields a contravariant functor $\mathsf{Top} \to \mathsf{Frm}$. To obtain a category which is extending (in a way) that of spaces (we will see shortly that we can think at least of generalized sober spaces) we consider the dual category of Frm. It will be denoted by

$$\mathsf{Loc}$$

and called the category of *locales*. The functor $\mathsf{Top} \to \mathsf{Frm}$ mentioned above becomes now a covariant functor

$$\mathsf{Lc} : \mathsf{Top} \to \mathsf{Loc}.$$

Thus, a locale X (that is, an object of Loc thought of as a space) is the same thing as a frame, but if we wish to emphasize the algebraic (lattice) aspects we often write $\mathcal{O}X$ for the same object (as if it would be the "lattice of open sets" of the "generalized space" X). A morphism of locales (*localic map*) is thought of as represented by a frame homomorphism in the opposite direction. If $f : X \to Y$ is a localic map, the corresponding frame homomorphism will be indicated by $f^* : \mathcal{O}Y \to \mathcal{O}X$, as if it were a left Galois adjoint of f (see I.2.2)—which point of view has a certain substantiation, see 1.5 below. This notation will make it clear whether we wish to think of a given object as sitting in Loc or in Frm. Notice that since f^* preserves arbitrary joins it has a right adjoint. This right adjoint is denoted by f_* and is given by the formula

$$\begin{array}{rccl} f_* : & \mathcal{O}X & \to & \mathcal{O}Y \\ & a & \mapsto & \bigvee\{b \mid f^*(b) \leq a\}. \end{array}$$

Exercise. Let $f : X \to Y$ be a continuous map between topological spaces and let $f_* : \mathcal{O}X \to \mathcal{O}Y$ be the right adjoint of $f^{-1} : \mathcal{O}Y \to \mathcal{O}X$.

(a) Show that $f_*(U) = Y \setminus \overline{f(X \setminus U)}$, for every $U \in \mathcal{O}X$.
(b) Give an example to show that f_* need not preserve joins.

1.3. Sober spaces. An element $a \neq 1$ of a lattice is *meet-irreducible* if $x \wedge y \leq a$ implies $x \leq a$ or $y \leq a$ (or, equivalently, if $x \wedge y = a$ implies $x = a$ or $y = a$). A topological space X is *sober* if it is T_0 and if there are no other meet-irreducibles $U \in \mathcal{O}X$ but the $X \setminus \overline{\{x\}}$.

Exercises.

1. Let X be a topological space, $x \in X$. Show that:
   (a) $X \setminus \overline{\{x\}}$ is a meet-irreducible of $\mathcal{O}X$;
   (b) X is sober if and only if each meet-irreducible $U \neq X$ in $\mathcal{O}X$ is $X \setminus \overline{\{x\}}$ for a unique $x \in X$.

2. Prove that each Hausdorff space is sober.
3. Give examples to show that not every T_1-space is sober, nor is every sober space T_1.
4. Prove that:
   (a) an open subset of a sober space is a sober space;
   (b) a closed subset of a sober space is a sober space.
5. Construct a sober space with a non-sober subspace. (Hint: consider $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ with $[n, \infty]$ as open subsets.)

Lemma. *Let X, Y be topological spaces, Y sober. Then, for each frame homomorphism $h : \mathcal{O}Y \to \mathcal{O}X$, there is exactly one continuous map $f : X \to Y$ such that $h = f^{-1}$.*

Proof. For $x \in X$ consider $\mathcal{F}_x = \{U \in \mathcal{O}Y \mid x \notin h(U)\}$. Since h preserves unions, $F_x = \bigcup \mathcal{F}_x \in \mathcal{F}_x$, hence it is the largest element in $\mathcal{F}_x$ and we have

$$U \in \mathcal{F}_x \text{ if and only if } U \subseteq F_x. \tag{1.3.1}$$

Each F_x is meet-irreducible: if $F_x = U \cap V$, we have $h(F_x) = h(U) \cap h(V)$; but then, say $x \notin h(U)$ and $U \subseteq F_x \subseteq U$. Thus, $F_x = Y \setminus \overline{\{y\}}$ with uniquely defined y (since Y is T_0). Denoting this y by $f(x)$ we obtain, from (1.3.1),

$$\begin{aligned} x \in h(U) \quad &\Leftrightarrow \quad U \nsubseteq Y \setminus \overline{\{f(x)\}} \\ &\Leftrightarrow \quad f(x) \in U \\ &\Leftrightarrow \quad x \in f^{-1}[U]. \end{aligned}$$

Thus, the mapping $f : X \to Y$ is continuous ($f^{-1}[U] = h(U) \in \mathcal{O}X$) and we have $h = f^{-1}$. ☐

Denote by Sob the full subcategory of Top defined by sober spaces. Then:

Corollary. *The restriction* $\mathsf{Lc} : \mathsf{Sob} \to \mathsf{Loc}$ *is a full embedding.*

1.4. The points of a locale. Using Lemma 1.3 we can easily reconstruct a sober space X from the lattice $A = \mathcal{O}X$. The one-point space is the terminal object T in Top, and a point of X can be viewed as a continuous map $T \to X$. Since the two-element Boolean algebra $\mathbf{2} = \{0, 1\}, 0 \neq 1$, is (isomorphic to) $\mathcal{O}T$, we have by Lemma 1.3 the points of X in a natural one-one correspondence with the frame homomorphisms $A \to \mathbf{2}$; the open sets U from the original X are then seen as the $\widetilde{U} \subseteq \{h \mid h : A \to \mathbf{2} \text{ in } \mathsf{Frm}\}$ defined by $h \in \widetilde{U}$ if and only if $h(U) = 1$ (such a h is f_x^{-1} for f_x sending the sole point of T to x, and $x \in U$ if and only if $f_x^{-1}[U]$ is non-void).

This construction may be generalized as follows. For any locale X define a *point* of X to be a localic map $T \to X$ from the terminal object of Loc to X (cf. Definition 1.3 of [41]). Equivalently, in frame terms, set

$$\mathsf{Pt}(X) = \Big(\{h \mid h : \mathcal{O}X \to \mathbf{2} \text{ in } \mathsf{Frm}\}, \{\Sigma_a \mid a \in \mathcal{O}X\}\Big)$$

where $\Sigma_a = \{h \mid h(a) = 1\}$. The following is straightforward:

Remark. $\Sigma_0 = \emptyset$, $\Sigma_1 = \mathsf{Pt}(X)$, $\Sigma_{a\wedge b} = \Sigma_a \cap \Sigma_b$ and $\Sigma_{\bigvee a_i} = \bigcup \Sigma_{a_i}$. Thus, in particular, $\{\Sigma_a \mid a \in \mathcal{O}X\}$ is a topology on $\{h \mid h : \mathcal{O}X \to \mathbf{2}\}$.

The space $\mathsf{Pt}(X)$ is often referred to as the *spectrum* of the locale X.

For a localic map $f : X \to Y$ define $\mathsf{Pt}(f) : \mathsf{Pt}(X) \to \mathsf{Pt}(Y)$ by setting $\mathsf{Pt}(f)(h) = h \cdot f^*$.

Lemma. *We have* $(\mathsf{Pt}(f))^{-1}[\Sigma_a] = \Sigma_{f^*(a)}$*. Thus* $\mathsf{Pt}(f)$ *is a continuous mapping.*

Proof. $(\mathsf{Pt}(f))^{-1}[\Sigma_a] = \{h \mid h \cdot f^* \in \Sigma_a\} = \{h \mid h(f^*(a)) = 1\}$. □

Obviously, $\mathsf{Pt}(1_X) = 1_{\mathsf{Pt}(X)}$ and $\mathsf{Pt}(f \cdot g) = \mathsf{Pt}(f) \cdot \mathsf{Pt}(g)$. Therefore we have a functor

$$\mathsf{Pt} : \mathsf{Loc} \to \mathsf{Top}.$$

Exercise. Let B be a complete Boolean algebra. Show that frame homomorphisms $B \to \mathbf{2}$ correspond bijectively to atoms in B.

1.5. Two alternative representations of the spectrum Pt.

(1) Recall that a *completely prime filter* (briefly, *complete filter*) in a lattice A is a proper filter $F \subseteq A$ such that for any system $\{a_i \mid i \in I\}$,

$$\bigvee_{i\in I} a_i \in F \Rightarrow \exists i, a_i \in F.$$

Complete filters are in a obvious one-one correspondence with frame homomorphisms $h : A \to \mathbf{2}$ ($h(a) = 1$ if and only if $a \in F$) and thus we can represent $\mathsf{Pt}(X)$ as

$$\Big(\{F \mid F \text{ complete filter in } \mathcal{O}X\}, \{\Sigma_a \mid a \in \mathcal{O}X\}\Big),$$

where $F \in \Sigma_a$ if and only if $a \in F$. The mapping $\mathsf{Pt}(f)$ sends F to $(f^*)^{-1}(F)$.

(2) A slightly less obvious translation of the Pt construction is based on the following result:

Proposition. *The formulas*

$$h \mapsto a = \bigvee \{x \mid h(x) = 0\}$$

$$a \mapsto h, \ \textit{where } h(x) = 1 \ \textit{iff } x \nleq a$$

constitute a one-one correspondence between the set of all frame homomorphisms $h : A \to \mathbf{2}$ *and the set* $\mathsf{irr}(A)$ *of all meet-irreducible elements in* A*.*

Proof. Checking that $\bigvee\{x \mid h(x) = 0\}$ is meet-irreducible and that the above defined h is a homomorphism is immediate. If h is sent to a and a is sent to k as above we have $k(y) = 1$ iff $y \nleq \bigvee\{x \mid h(x) = 0\}$ iff $h(y) = 1$. Finally if $a \mapsto h \mapsto b$ we have $b = \bigvee\{x \mid x \leq a\} = a$. □

Now we can represent $\mathsf{Pt}(X)$ as

$$\Big(\mathsf{irr}(\mathcal{O}X), \{\Sigma_a \mid a \in \mathcal{O}X\}\Big),$$

with $\Sigma_a = \{x \in \mathsf{irr}(\mathcal{O}X) \mid a \nleq x\}$.

Lemma. *Let $h : A \to B$ be a frame homomorphism. Then $h_*[\mathsf{irr}(B)] \subseteq \mathsf{irr}(A)$.*

Proof. If $a \in \mathsf{irr}(B)$ and $x \wedge y \leq h_*(a)$ then $h(x) \wedge h(y) \leq a$ and hence, say, $h(x) \leq a$ and $x \leq h_*(a)$. $\square$

In the original description of Pt we had $\mathsf{Pt}(f)(h) = h{\cdot}f^*$. Represent $a \in \mathsf{irr}(\mathcal{O}X)$ by the h_* above. Then the resulting $\mathsf{Pt}(f)(h)$ corresponds to

$$\bigvee\{x \mid h(f^*(x)) = 0\} = \bigvee\{x \mid f^*(x) \leq a\} = (f^*)_*(a).$$

Thus, if we think, just for the moment, of the localic morphisms as of the right adjoints of f^*, we have now $\mathsf{Pt}(f)$ represented simply as the restriction of f. Note that this is one of the reasons for denoting the algebraic (frame) correspondent of the localic map f by f^*.

1.6. Pt **is right adjoint to** Lc**.**

Proposition. *Each $\mathsf{Pt}(X)$ is sober.*

Proof. We will use the representation from 1.5(2). First, note that we have, since $\Sigma_a = \{x \mid a \nleq x\}$,

$$x \in \overline{\{y\}} \;\Leftrightarrow\; (a \nleq x \Rightarrow a \nleq y) \;\Leftrightarrow\; y \leq x. \tag{1.6.1}$$

Now let Σ_a be meet-irreducible in $\mathcal{O}(\mathsf{Pt}(X))$. Set $b = \bigvee\{c \mid \Sigma_c \subseteq \Sigma_a\}$. Then, obviously, $\Sigma_b = \Sigma_a$, and b is meet-irreducible; indeed, if $x \wedge y \leq b$ then $\Sigma_x \cap \Sigma_y = \Sigma_{x\wedge y} \subseteq \Sigma_b = \Sigma_a$ and hence, say, $\Sigma_x \subseteq \Sigma_a$ and $x \leq b$. We have

$$x \in \Sigma_b \;\Leftrightarrow\; b \nleq x \;\Leftrightarrow\; x \notin \overline{\{b\}} \;\Leftrightarrow\; x \in \mathsf{Pt}(X) \setminus \overline{\{b\}},$$

thus $\Sigma_a = \Sigma_b = \mathsf{Pt}(X) \setminus \overline{\{b\}}$. By (1.6.1), $\mathsf{Pt}(X)$ is T_0. $\square$

Define morphisms $\eta_X : X \to \mathsf{PtLc}(X)$ and $\varepsilon_Y : \mathsf{LcPt}(Y) \to Y$ by setting $\eta_X(x)(U) = 1$ if and only if $x \in U$ and $\varepsilon_Y^*(a) = \Sigma_a$, respectively. It is easy to check that each $\eta_X(x)$ is indeed a homomorphism $\mathsf{Lc}(X) \to \mathbf{2}$. Since

$$\eta_X^{-1}[\Sigma_U] = \{x \mid \eta_X(x)(U) = 1\} = U, \tag{1.6.2}$$

η_X is a continuous mapping. On the other hand, ε_Y is a homomorphism by 1.5.

Lemma. *The systems $\eta = (\eta_X)_{X\in\mathsf{Top}}$ and $\varepsilon = (\varepsilon_Y)_{Y\in\mathsf{Loc}}$ are natural transformations $\eta : 1_{\mathsf{Top}} \xrightarrow{\cdot} \mathsf{PtLc}$ and $\varepsilon : \mathsf{LcPt} \xrightarrow{\cdot} 1_{\mathsf{Loc}}$, respectively.*

Proof. If $f : X \to Y$ is a continuous mapping we have

$$(\mathsf{PtLc}(f) \cdot \eta_X)(x)(U) = (\eta_X(x) \cdot \mathsf{Lc}(f))(U) = \eta_X(x)(\mathsf{Lc}(f)(U)).$$

Therefore

$$\begin{aligned} 1 = (\mathsf{PtLc}(f) \cdot \eta_X)(x)(U) \;&\Leftrightarrow\; x \in \mathsf{Lc}(f)(U) \\ &\Leftrightarrow\; f(x) \in U \\ &\Leftrightarrow\; (\eta_Y \cdot f)(x)(U) = 1. \end{aligned}$$

If $f : X \to Y$ is a localic map we have, by 1.5,

$$\begin{aligned}(\varepsilon_X \cdot \mathsf{LcPt}(f))^*(a) &= (\mathsf{LcPt}(f))^*(\varepsilon_X^*(a)) = (\mathsf{Pt}(f))^{-1}[\Sigma_a] \\ &= \Sigma_{f^*(a)} = \varepsilon_Y^*(f^*(a)) = (f \cdot \varepsilon_Y)^*(a). \qquad \square\end{aligned}$$

Theorem. $\mathsf{Pt} : \mathsf{Loc} \to \mathsf{Top}$ *is right adjoint to* $\mathsf{Lc} : \mathsf{Top} \to \mathsf{Loc}$*, with unit* η *and co-unit* ε *as above.*

Proof. Consider the composition

$$\mathsf{Lc}(X) \xrightarrow{\mathsf{Lc}(\eta_X)} \mathsf{LcPtLc}(X) \xrightarrow{\varepsilon_{\mathsf{Lc}(X)}} \mathsf{Lc}(X).$$

We have

$$(\varepsilon_{\mathsf{Lc}(X)} \cdot \mathsf{Lc}(\eta_X))^*(U) = \eta_X^{-1}[\varepsilon_{\mathsf{Lc}(X)}^{-1}(U)] = \eta_X^{-1}[\Sigma_U] = \{x \mid \eta_X(x)(U) = 1\} = U$$

so that

$$\varepsilon_{\mathsf{Lc}(X)} \cdot \mathsf{Lc}(\eta_X) = 1_{\mathsf{Lc}(X)}. \tag{1.6.3}$$

Consider the composition

$$\mathsf{Pt}(Y) \xrightarrow{\eta_{\mathsf{Pt}(Y)}} \mathsf{PtLcPt}(Y) \xrightarrow{\mathsf{Pt}(\varepsilon_Y)} \mathsf{Pt}(Y).$$

We have

$$((\mathsf{Pt}(\varepsilon_Y) \cdot \eta_{\mathsf{Pt}(Y)})(h))(U) = (\eta_{\mathsf{Pt}(Y)}(h) \cdot \varepsilon_Y^*)(U) = \eta_{\mathsf{Pt}(Y)}(h)[\Sigma_U],$$

thus

$$1 = ((\mathsf{Pt}(\varepsilon_Y) \cdot \eta_{\mathsf{Pt}(Y)})(h))(U) \Leftrightarrow h \in \Sigma_U \Leftrightarrow h(U) = 1,$$

and again $\mathsf{Pt}(\varepsilon_Y) \cdot \eta_{\mathsf{Pt}(Y)} = 1_{\mathsf{Pt}(Y)}$. $\square$

1.7. A reflection of Top onto Sob. The natural transformation η yields a reflection of the category Top onto the subcategory Sob:

Theorem. *The following statements on a space X are equivalent:*

(i) *X is sober;*
(ii) *η_X is one-one and onto;*
(iii) *η_X is a homeomorphism.*

Proof. (i)⇒(ii) is an immediate consequence of Lemma 1.3, (ii)⇒(iii) follows from (1.6.2) (which yields, for an invertible η_X, $\eta_X[U] = \Sigma_U$), and (iii)⇒(i) follows from Proposition 1.6. $\square$

1.8. Spatial locales. A locale X is said to be *spatial* if it is (isomorphic to) $\mathsf{Lc}(Y)$ for some space Y. Here is an easy criterion of spatiality:

Theorem. *The following statements on a locale X are equivalent:*

(i) *X is spatial;*
(ii) *ε_X^* is one-one;*
(iii) *ε_X^* is an isomorphism;*
(iv) *each $a \in \mathcal{O}X$ is a meet of meet-irreducible elements.*

Proof. (i)⇒(ii) since, by (1.6.3),

$$(\mathsf{Lc}(\eta_Y))^* \cdot \varepsilon^*_{\mathsf{Lc}(Y)} = (\varepsilon_{\mathsf{Lc}(Y)} \cdot \mathsf{Lc}(\eta_Y))^* = 1_{\mathsf{Lc}(Y)}$$

for $X = \mathsf{Lc}(Y)$ and therefore $\varepsilon^*_X = \varepsilon^*_{\mathsf{Lc}(Y)}$ is one-one. (ii)⇒(iii) since each ε^*_X is onto, and (iii)⇒(i) is trivial. Assertion (iv) is just a reformulation of (ii) in the representation from 1.5(2). □

All finite distributive lattices and all complete chains are spatial but, by way of contrast, not every Boolean algebra is spatial, showing that locales "considerably transcend topology". In fact the intersection of the class of spatial locales and that of Boolean algebras are only the discrete spaces. We have:

Proposition. *Each meet-irreducible element of a Boolean algebra is a co-atom. Consequently, each spatial Boolean algebra is atomic.*

Proof. Let a be meet-irreducible and let $a < x$. Since $a = (a \vee x) \wedge (a \vee x^c)$, where x^c is the complement of x, we have to have $a = a \vee x^c$ and hence $x^c \leq a < x$. Then $x = x^{cc} \geq a^c$ and hence $1 = a \vee a^c \leq x$ and $x = 1$. □

1.9. The "maximal" equivalence induced by the adjunction $\mathsf{Lc} \dashv \mathsf{Pt}$. Every adjunction induces a "maximal" equivalence between a pair of full subcategories. Here the adjunction $\mathsf{Lc} \dashv \mathsf{Pt}$ gives:

Theorem. *The category of spatial locales is equivalent to the category of sober topological spaces.*

Proof. If X is a topological space, the locale $\mathsf{Lc}(X)$ is spatial. Therefore, by Proposition 1.6, the adjunction of Theorem 1.6 between Top and Loc restricts to the full subcategories of sober spaces and spatial locales. By definition of spatial locale, ε_X is an isomorphism. By Theorem 1.7, if a space X is sober, η_X is an isomorphism. Therefore we get the required equivalence. □

The results in this section show that the category of locales is an appropriate environment in which to develop topology (for more motivation consult [11, 34, 35, 36]). From now on we develop locale theory in this perspective. The topological intuition will be apparent from the use of topological adjectives to describe localic concepts.

2. Sublocales

2.1. Epimorphisms. Consider the frame $\mathbb{S} = \{0 < s < 1\}$. Obviously, for every frame A and every $a \in A$ the mapping $\sigma_a : \mathbb{S} \to A$, sending 0 to 0, 1 to 1 and s to a, is a frame homomorphism. Consequently, if a frame homomorphism $h : A \to B$ is a monomorphism, it must be one-one (if $h(a) = h(b)$ then $h \cdot \sigma_a = h \cdot \sigma_b$). Thus:

Proposition. *The epimorphisms in* Loc *are exactly the f such that f^* is one-one.*

Usually epimorphisms in Loc are called *surjections*.

2.2. Extremal monomorphisms. Recall that a monomorphism μ in a category is said to be *extremal* if, for each factorization $\mu = \nu \cdot \varepsilon$ with ε an epimorphism, ε is an isomorphism.

Proposition. *The extremal monomorphisms in* Loc *are precisely the* f *such that* f^* *is onto.*

Proof. In other words, we should prove that the extremal epimorphisms in Frm are exactly the homomorphisms $h : A \to B$ that are onto. If $h : A \to B$ is an extremal epimorphism and we factor it through its image $h[A]$, the embedding $h[A] \hookrightarrow B$ must be an isomorphism, and h is onto. On the other hand, every onto homomorphism $h : A \to B$ is obviously an epimorphism, and if we have a factorization $h = m \cdot h'$ with a monomorphism m, then m is one-one and onto and hence an isomorphism in Frm. $\square$

The structure of general monomorphisms in Loc is by far not so transparent (see [34, 43]). For instance, there is a locale X such that for each cardinal number α there is a monomorphism $f : Y \to X$ with $|Y| \geq \alpha$ (see 3.9 below).

2.3. Decomposition of morphisms. Every localic map $f : X \to Y$ decomposes as

$$\begin{tikzcd} & Z \arrow[dr, "m"] & \\ X \arrow[ur, two heads, "e"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

with $\mathcal{O}Z = f^*[\mathcal{O}Y]$, $e^* : \mathcal{O}Z \hookrightarrow \mathcal{O}X$ and $m^* = (y \mapsto f^*(y)) : \mathcal{O}Y \twoheadrightarrow \mathcal{O}Z$. Thus, every localic map can be factored as an epimorphism followed by an extremal monomorphism.

We point out that, furthermore, it can be proved that the classes $\mathcal{E}$ of epimorphisms and $\mathcal{M}$ of extremal monomorphisms constitute a factorization system (see III.1.2) in Loc.

2.4. Sublocales. In many everyday life categories (like that of topological spaces, graphs, posets, or general relational systems), extremal monomorphisms represent well the subobjects (as opposed to plain monomorphisms $m : A \to B$ that may not — like for instance the one-one continuous maps — relate the structure of A closely enough to that of B). This point of view is also adopted in Loc and a *sublocale* $j : Y \rightarrowtail X$ of X is defined as a localic map such that the corresponding frame homomorphism $j^* : \mathcal{O}X \to \mathcal{O}Y$ is onto (recall 2.2). It should be noted that in many other categories (like that of Hausdorff spaces, rings, small categories, etc.) choosing extremal monomorphisms for subobjects would be too restrictive. But our situation is closer to that of general spaces, and the definition is also supported by the notion of a subspace Y of a topological space X, where the topology $\mathcal{O}Y$ is defined as $\{U \cap Y \mid U \in \mathcal{O}X\}$, making $j^{-1} : \mathcal{O}X \to \mathcal{O}Y$, for $j : Y \hookrightarrow X$, an onto homomorphism.

On the class of sublocales of X we have the natural preorder $j_1 \sqsubseteq j_2$ if and only if there is a j such that $j_2 \cdot j = j_1$ (note that this j is necessarily again a sublocale). Sublocales $j_i : Y_i \rightarrowtail X$ $(i = 1, 2)$ are *equivalent* if $j_1 \sqsubseteq j_2$ and $j_2 \sqsubseteq j_1$ or, equivalently, if there is an isomorphism $j : Y_1 \to Y_2$ such that $j_2 \cdot j = j_1$. The ensuing partially ordered set will be denoted by $\mathcal{S}(X)$.

We will use the symbol $\sqsubseteq$ also for the corresponding frame homomorphisms. Thus, for frame homomorphisms $h_i : A \twoheadrightarrow B_i$ $(i = 1, 2)$, $h_1 \sqsubseteq h_2$ if there is an h such that $h \cdot h_1 = h_2$, and h_1 and h_2 are considered equivalent if $h_1 \sqsubseteq h_2$ and $h_2 \sqsubseteq h_1$. So, for $j_1, j_2 \in \mathcal{S}(X)$,

$$j_1 \sqsubseteq j_2 \quad \text{iff} \quad j_2^* \sqsubseteq j_1^*. \tag{2.4.1}$$

2.5. Frame congruences. We have the natural correspondence between surjective frame homomorphisms $h : A \twoheadrightarrow B$ and congruences (with respect to finite meets and general joins) on A. More precisely, the formulas

$$h \mapsto C_h = \{(a, b) \mid h(a) = h(b)\}, \quad C \mapsto h_C = \{a \mapsto aC\} : A \to A/C \tag{2.5.1}$$

constitute a one-one correspondence between the set of (the equivalence classes of) the onto homomorphisms $h : A \twoheadrightarrow B$ and the set $\mathcal{C}(A)$ of all congruences on A.

The following is an immediate observation:

In the correspondence (2.5.1) above, we have $h_1 \sqsubseteq h_2$iff $C_{h_1} \subseteq C_{h_2}$. (2.5.2)

Since any intersection of congruences is a congruence, $\mathcal{C}(A)$ is a complete lattice. Therefore, by (2.4.1) and (2.5.2), $\mathcal{S}(X)$ is also a complete lattice, isomorphic to $\mathcal{C}(\mathcal{O}X)^{op}$. The meets and joins in $\mathcal{S}(X)$ will be denoted by

$$j \sqcap k, \bigsqcap_{i \in I} j_i, \text{ etc., resp. } j \sqcup k, \bigsqcup_{i \in I} j_i, \text{ etc.}$$

These symbols will be also used when dealing with the associated frame homomorphisms. Note that $j \sqcap k$ is represented by the pullback

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, "j \sqcap k"] & \cdot \arrow[d, "k"] \\ \cdot \arrow[r, "j"'] & \cdot \end{tikzcd}$$

in Loc.

The initial object in Loc will be denoted by 0; this is the locale that corresponds to the frame $\{0 = 1\}$ (that we denote by $\mathbf{1}$ in analogy with the definition of $\mathbf{2}$). The unique localic map $0_X : 0 \rightarrowtail X$ is the bottom of $\mathcal{S}(X)$. The top element of $\mathcal{S}(X)$ is the (equivalence class of the) identity morphism $1_X : X \to X$ and will be denoted by 1_X or simply by X.

2.6. Open and closed sublocales. We consider now some simple examples of sublocales which resemble open and closed subspaces of a topological space. We have the sublocales given by the frame homomorphisms

$$\begin{array}{rccl} \hat{a} : & \mathcal{O}X & \longrightarrow & {\downarrow}a := \{x \in \mathcal{O}X \mid x \leq a\} \\ & x & \longmapsto & x \wedge a, \end{array} \tag{2.6.1}$$

for every $a \in \mathcal{O}X$, and the sublocales given by the frame homomorphisms

$$\begin{array}{rccl} \check{a}: & \mathcal{O}X & \longrightarrow & \uparrow a := \{x \in \mathcal{O}X \mid x \geq a\} \\ & x & \longmapsto & x \vee a, \end{array} \tag{2.6.2}$$

for every $a \in \mathcal{O}X$. The former will be referred to as *open* sublocales and the latter as *closed* ones. The $\hat{a}$ (resp. $\check{a}$) will also be referred to as open (resp. closed), and similarly one speaks of the corresponding congruences

$$\Delta_a := \{(x,y) \mid x \wedge a = y \wedge b\} \text{ and } \nabla_a := \{(x,y) \mid x \vee a = y \vee b\}.$$

We write X_a for the locale given by the frame $\downarrow a$ and $X\text{-}X_a$ for the locale given by $\uparrow a$. Then (2.6.1) describes a sublocale

$$X_a \rightarrowtail X$$

and (2.6.2) describes a sublocale

$$X\text{-}X_a \rightarrowtail X.$$

Spatially, when we write $X\text{-}X_a \rightarrowtail X$ we are thinking of the closed subspace corresponding to the set theoretic complement of the open a.

Exercises.

1. Prove that a localic map $f : X \to Y$ factors through the open sublocale $Y_a \rightarrowtail Y$ generated by $a \in \mathcal{O}Y$ if and only if $f^*(a) = 1$.
2. Prove that a localic map $f : X \to Y$ factors through the closed sublocale $Y\text{-}Y_a \rightarrowtail Y$ if and only if $f^*(a) = 0$.

We list some properties of open and closed congruences:

Proposition.

(1) $a \leq b \Leftrightarrow \Delta_b \subseteq \Delta_a \Leftrightarrow \nabla_a \subseteq \nabla_b$.
(2) $\nabla_0 = 0 = \Delta_1$ *and* $\nabla_1 = 1 = \Delta_0$.
(3) $\nabla_a \cap \nabla_b = \nabla_{a \wedge b}$.
(4) $\bigvee \nabla_{a_i} = \nabla_{\bigvee a_i}$.
(5) $\Delta_a \vee \Delta_b = \Delta_{a \wedge b}$.
(6) $\bigcap \Delta_{a_i} = \Delta_{\bigvee a_i}$.

Proof. (1) If $\nabla_a \subseteq \nabla_b$ we have, in particular, $(a,0) \in \nabla_b$ and hence $a \vee b = b$. Similarly, if $\Delta_b \subseteq \Delta_a$ then $(b,1) \in \Delta_a$ and $a \wedge b = a$. Obviously, if $a \leq b$ then $x \vee a = y \vee a$ implies $x \vee b = y \vee b$, and similarly with the meet.
(2), (3) and (6) are immediate; we show (4) and (5):

By (1), $\Delta_{a \wedge b} \supseteq \Delta_a, \Delta_b$ and $\nabla_{\bigvee a_i} \supseteq \nabla_{a_i}$ for all i. If C is a congruence such that $C \supseteq \Delta_a, \Delta_b$ and $x \wedge a \wedge b = y \wedge a \wedge b$, then $(x \wedge a, y \wedge a) \in C$, since $C \supseteq \Delta_b$, and $xCx \wedge aCy \wedge aCy$, since $C \supseteq \Delta_a$. Thus, $C \supseteq \Delta_{a \wedge b}$. If $C \supseteq \nabla_{a_i}$ for each i, we have $(0, a_i) \in C$ for each i, hence $(0, \bigvee a_i) \in C$. Thus, if $x \vee \bigvee a_i = y \vee \bigvee a_i$, we have $(x,y) = (x \vee 0, y \vee \bigvee a_i) = (x \vee 0, x \vee \bigvee a_i) \in C$. $\square$

As a consequence, we have:

(1) $a \leq b \Leftrightarrow X_a \sqsubseteq X_b \Leftrightarrow X\text{-}X_b \sqsubseteq X\text{-}X_a$;
(2) $X\text{-}X_0 = X_1 = 1_X$ and $X\text{-}X_1 = X_0 = 0_X$;

(3) $X\text{-}X_a \sqcup X\text{-}X_b = X\text{-}X_{a\wedge b}$;
(4) $\bigsqcap X\text{-}X_{a_i} = X\text{-}X_{\bigvee a_i}$;
(5) $X_a \sqcap X_b = X_{a\wedge b}$;
(6) $\bigsqcup X_{a_i} = X_{\bigvee a_i}$.

2.7. Open and closed sublocales are complemented in $\mathcal{S}(X)$. Open and closed sublocales, corresponding to the same element $a \in \mathcal{O}X$, are complements in $\mathcal{S}(X)$:

Proposition. *$X_a \sqcup X\text{-}X_a = 1_X$ and $X_a \sqcap X\text{-}X_a = 0_X$.*

Proof. The proof will be done in terms of congruences.

If $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$, then $x = x \wedge (a \vee x) = x \wedge (a \vee y) = (x\wedge a)\vee(x\wedge y) = (y\wedge a)\vee(y\wedge x) = y\wedge(a\vee x) = y\wedge(a\vee y) = y$. Thus, $\Delta_a \cap \nabla_a = \{(x,x) \mid x \in \mathcal{O}X\}$, the bottom of $\mathcal{C}(\mathcal{O}X)$. If $C \supseteq \nabla_a, \Delta_a$ we have $(a,0) \in C$ and $(a,1) \in C$, thus $(0,1) \in C$ and hence $x = x \wedge 1Cx \wedge 0 = y \wedge 0Cy \wedge 1 = y$ for every (x,y). Thus, $C = \mathcal{O}X \times \mathcal{O}X$. □

Remark. Unlike subspaces of spaces, however, not every sublocale is complemented in $\mathcal{S}(X)$ (not even the sublocales of spaces that are subspaces).

2.8. A representation of a general sublocale.

Proposition. *Let $j : Y \rightarrowtail X$ be a sublocale of X. Then*

$$j = \bigsqcap\{X_a \sqcup X\text{-}X_b \mid j^*(a) = j^*(b)\}.$$

In the language of frame congruences: for any congruence C on $\mathcal{O}X$ we have

$$C = \bigvee\{\Delta_a \cap \nabla_b \mid (a,b) \in C\}.$$

Proof. The proof will be done for the congruences.

Let $(a,b) \in C$. Then $\Delta_a \cap \nabla_b \subseteq C$; indeed, if $(x,y) \in \Delta_a \cap \nabla_b$, that is, if $x \wedge a = y \wedge a$ and $x \vee b = y \vee b$, we have $x = x \wedge (y \vee b)Cx \wedge (y \vee a) = (x\wedge y)\vee(x\wedge a) = (x\wedge y)\vee(y\wedge a) = y\wedge(x\vee a)Cy\wedge(x\vee b) = y$; thus, $(x,y) \in C$.

On the other hand, let $D \supseteq \Delta_a \cap \nabla_b$ for all $(a,b) \in C$. In particular, we have $(a, a\vee b) \in \Delta_a \cap \nabla_b$ and $(b, a\vee b) \in \Delta_a \cap \nabla_b$, hence $aDa \vee bDb$ and $(a,b) \in D$. □

2.9. Closure. For a localic map $f : Y \to X$ set

$$\mathsf{c}_f = \bigvee\{a \in \mathcal{O}X \mid f^*(a) = 0\}.$$

Proposition. *Let $j : Y \rightarrowtail X$ be a sublocale. Then the sublocale $X\text{-}X_{\mathsf{c}_j} \rightarrowtail X$ is the smallest closed sublocale k such that $j \sqsubseteq k$.*

Proof. Define $h : \uparrow\mathsf{c}_j \to \mathcal{O}Y$ by setting $h(x) = j^*(x)$. Since $j^*(\mathsf{c}_j) = 0$, h is a frame homomorphism (preserving non-void joins and finite meets being trivial), and we have $(h \cdot \check{\mathsf{c}_j})(x) = j^*(\mathsf{c}_j \vee x) = j^*(x)$. Thus, $\check{\mathsf{c}_j} \sqsubseteq j^*$, that is, $j \sqsubseteq X\text{-}X_{\mathsf{c}_j}$. If $j \sqsubseteq X\text{-}X_a$ there is a $\varphi : \uparrow a \to \mathcal{O}Y$ such that $j^*(x) = \varphi(x \vee a)$ and hence, in particular, $j^*(a) = \varphi(a) = 0$ and $a \le \mathsf{c}_j$ so that $X\text{-}X_{\mathsf{c}_j} \sqsubseteq X\text{-}X_a$ by property (1) of 2.6. □

The sublocale map $X\text{-}X_{\mathsf{c}_j} \rightarrowtail X$ will be denoted by $\overline{j}$ and called the *closure* of the sublocale j.

2.10. Closure behaves like in spaces ... We list some properties of the closure:

Proposition.

(1) $\overline{0_X} = 0_X$.
(2) $j \sqsubseteq k \Rightarrow \overline{j} \sqsubseteq \overline{k}$.
(3) $\overline{\overline{j}} = \overline{j}$.
(4) $\overline{j \sqcup k} = \overline{j} \sqcup \overline{k}$.

Proof. (1) $0_X = (1_X)\check{}$. (2) and (3) follow immediately from Proposition 2.9. By (2), $\overline{j \sqcup k} \sqsupseteq \overline{j} \sqcup \overline{k}$. By Proposition 2.6, the join of two closed sublocales is closed and hence, by Proposition 2.9, $\overline{j \sqcup k} \sqsubseteq \overline{j} \sqcup \overline{k}$ and (4) follows. □

2.11. ... but not in all respects. In spaces, the topology is determined by the closures of subsets. Here we have:

Proposition. *Let $\mathcal{O}X$ be a Boolean algebra. Then each sublocale of X is closed.*

Proof. Let $j : Y \rightarrowtail X$ be a sublocale, c_j as in 2.9. We can define $h : X\text{-}X_{\mathsf{c}_j} \to Y$, by setting $h^*(j^*(a)) = \mathsf{c}_j \vee a$, since if $j^*(a) = j^*(b)$ then $\mathsf{c}_j \vee a = \mathsf{c}_j \vee b$ (indeed, $j^*(a) = j^*(b)$ implies $j^*(a \wedge b^c) = 0$, hence $a \wedge b^c \le \mathsf{c}_j$ making $\mathsf{c}_j \vee a \le \mathsf{c}_j \vee b$, and $\mathsf{c}_j \vee b \le \mathsf{c}_j \vee a$ by symmetry). This shows that $\overline{j} = j$. □

Now T_0-spaces with Boolean topology are necessarily discrete. We have, however, rather non-trivial locales X such that $\mathcal{O}X$ is Boolean (see 2.13 below).

Exercise. Prove that, for each locale X:

(a) $\mathcal{O}X$ is a Boolean algebra if and only if every sublocale of X is closed;
(b) $\mathcal{O}X$ is a Boolean algebra if and only if every sublocale of X is open.

2.12. Density. A sublocale $j : Y \rightarrowtail X$ is said to be *dense* if $\overline{j} = X$. The corresponding j^* will also be referred to as dense.

Proposition. *A sublocale $j : Y \rightarrowtail X$ is dense if and only if $j^*(a) = 0$ implies $a = 0$.*

Proof. $\overline{j} = X$ iff $0 = \bigvee\{a \in \mathcal{O}X \mid j^*(a) = 0\}$ iff $a = 0$ whenever $j^*(a) = 0$. □

More generally, a localic map $f : Y \to X$ is said to be *dense* if $f^*(a) = 0$ implies $a = 0$.

For any $f : Y \to X$, $f^*(\mathsf{c}_f) = 0$ and c_f is the largest such element. Since ${\uparrow}\mathsf{c}_f \to \mathcal{O}Y$, given by $x \mapsto f^*(x)$, is clearly a frame homomorphism we have the factorization

$$\begin{tikzcd} & X\text{-}X_{\mathsf{c}_f} \arrow[dr, "h"] & \\ Y \arrow[ur, "g"] \arrow[rr, "f"'] & & X \end{tikzcd}$$

where $g^*(x) = f^*(x \vee c) = f^*(x)$. Evidently, g is a dense map. This gives the so called *dense factorization*, where each f is factorized as a dense map followed by a closed sublocale. Note that denseness is a condition weaker than injectivity.

2.13. Booleanization and Isbell's Density Theorem. Recall, from 1.1, properties (C1)-(C6) of pseudo-complements a^c.

Lemma. *For each locale X, $\{a \in \mathcal{O}X \mid a^{cc} = a\}$ is a complete Boolean algebra.*

Proof. First, it is easy to see that the formula $\bigvee' a_i = (\bigvee a_i)^{cc}$ yields a join in $\{a \in \mathcal{O}X \mid a^{cc} = a\}$; hence, we have a complete lattice. By property (C6), $\wedge$ gives finite meets. Finally, we have $a \wedge a^c = 0$ and $a \vee' a^c = 1$ since $(a \vee a^c)^c = 0$; indeed, if $x \wedge (a \vee a^c) = 0$ we have both $x \wedge a = 0$ (and hence $x \leq a^c$) and $a \wedge a^c = 0$ (and hence $x \leq a^{cc}$) so that $x \leq a^c \wedge a^{cc} = 0$. $\square$

Moreover, using properties (C5) and (C6), it is also easy to see that the map

$$\mathcal{O}X \to \{a \in \mathcal{O}X \mid a^{cc} = a\}$$

defined by $a \mapsto a^{cc}$ is a frame homomorphism, obviously onto and dense.

Therefore, defining $\mathcal{B}X$ by $\mathcal{O}(\mathcal{B}X) = \{a \in \mathcal{O}X \mid a^{cc} = a\}$ we have a dense sublocale $\beta_X : \mathcal{B}X \rightarrowtail X$ given by $\beta_X^*(a) = a^{cc}$. This sublocale is called the *Booleanization* of X [18].

Proposition. *β_X is the least dense sublocale of X.*

Proof. Let $j : Y \rightarrowtail X$ be dense. If $j^*(a) = j^*(b)$ we have $j^*(a \wedge b^c) = j^*(a) \wedge j^*(b^c) \leq j^*(a) \wedge j^*(b)^c = 0$ (by (1.1.2), $h(x^c) \leq h(x)^c$ for any homomorphism). Thus, $a \wedge b^c = 0$ and $a \leq b^{cc}$. By symmetry also $b \leq a^{cc}$ and we see that

$$j^*(a) = j^*(b) \Rightarrow a^{cc} = b^{cc}$$

and that we can define a localic $\varphi : \mathcal{B}X \to Y$, by putting $\varphi^*(j^*(a)) = a^{cc}$, to obtain $\beta_X \sqsubseteq j$. $\square$

This is a new feature of locale theory: it is not the case that all topological spaces have least dense subspaces.

2.14. Sublocales and subspaces. Now let X be a topological space and let $S \subseteq X$ be a subspace. We have the obvious representation of S as a sublocale $\widetilde{S} \rightarrowtail \mathsf{Lc}(X)$, determined by the congruence C_S:

$$(U, V) \in C_S \Leftrightarrow U \cap S = V \cap S \text{ (that is, } j^{-1}[U] = j^{-1}[V] \text{ for the } j : S \hookrightarrow X).$$

This representation is, however, not always satisfactory. We say that a topological space X satisfies the *axiom* T_D if, for every $x \in X$, there is an open $U \ni x$ such that $U \setminus \{x\}$ is also open.

Axiom T_D is stronger than T_0 and (much) weaker than T_1. It is incomparable with sobriety.

Exercises.

1. Prove that $T_1 \Rightarrow T_D \Rightarrow T_0$.
2. Show that a space X is T_D if and only if there is a neighborhood V such that $V \cap \overline{\{x\}} = \{x\}$.

Proposition. *In a spatial locale* $\mathsf{Lc}(X)$ *we have, for subspaces* S, T *of the space* X*, the implication* $(\widetilde{S} = \widetilde{T} \Rightarrow S = T)$ *if and only if* X *satisfies* T_D*.*

Proof. Let X satisfy T_D and let, for any open sets U and V,

$$U \cap S = V \cap S \text{ iff } U \cap T = V \cap T. \tag{2.14.1}$$

Let $S \nsubseteq T$. Choose $x \in S \setminus T$ and U open, $U \ni x$, such that $V = U \setminus \{x\}$ is open. Then $U \cap S \neq V \cap S$ while $U \cap T = V \cap T$, contradicting (2.14.1).

On the other hand, let T_D not hold and let x be a point such that, for U open, $U \ni x$, $V = U \setminus \{x\}$ is never open. Then, for $S = X \setminus \{x\}$, one has $U \cap S = V \cap S$, that is, $U \setminus \{x\} = V \setminus \{x\}$, only if $U = V$, and hence $\widetilde{S} = \widetilde{X}$. □

For a topological space X, $\mathcal{B}X$ (more exactly, $\mathcal{B}(\mathsf{Lc}(X))$) is the Boolean algebra of the *regular open sets* U of X, that is, the U that are equal to $int(\overline{U})$. Thus (recall Proposition 1.8) they are typically not spatial. Hence, sublocales of a space (or, spatial locale) are not necessarily spatial ("we have more sublocales than subspaces").

The Booleanization $\beta : \mathcal{B}X \rightarrowtail X$ also illustrates the fact that the intersection of sublocales need not agree with the intersection of spaces (in the notation above, $\widetilde{S} \sqcap \widetilde{T}$ is not necessarily the same as $\widetilde{S \cap T}$). For instance, if X is the space of reals, the intersection of the subspaces of the rationals and of the irrationals is void; however, by 2.13, the intersection of the respective sublocales contains at least $\mathcal{B}X$, which is a rather large Boolean algebra.

On the other hand, the unions do not bring any surprise. We have $\widetilde{\bigcup_{i\in I} S_i} = \bigsqcup_{i\in I} \widetilde{S_i}$ since, obviously, $U \cap \bigcup_{i\in I} S_i = V \cap \bigcup_{i\in I} S_i$ if and only if $U \cap S_i = V \cap S_i$ for all i.

2.15. Factorization Theorem. We will now be concerned with a technique of producing sublocales by means of extending a relation (identifying elements according to the needs of a construction) to a congruence. During this paragraph, by convenience, we will consistently use the frame language.

Let $R \subseteq A \times A$ be an arbitrary binary relation on a frame A. An element $s \in A$ is *saturated* (more precisely, *R-saturated*) if

$$\forall a, b, c \;\; aRb \Rightarrow (a \wedge c \leq s \text{ iff } b \wedge c \leq s). \tag{2.15.1}$$

In case R is *meet-stable*, that is, if there is a subset $M \subseteq A$ such that

(1) $1 \in M$ and $a = \bigvee\{x \in M \mid x \leq a\}$ for every $a \in A$,
(2) $\forall a, b \in A\ \forall x \in M,\ aRb \Rightarrow a \wedge xRb \wedge x$,

then $s \in A$ is saturated if and only if

$$\forall a, b\ \ aRb \Rightarrow (a \leq s \text{ iff } b \leq s). \tag{2.15.2}$$

In fact, aRb if and only if $a \wedge xRb \wedge x$ for every $x \in M$, and $a \wedge c \leq s$ if and only if $a \vee x \leq s$ for every $x \leq c$.

Obviously, any meet of saturated elements is saturated. Consequently, we have the saturated

$$\nu(a) = \nu_R(a) = \bigwedge\{s \mid s \text{ saturated },a \leq s\}.$$

Recall Remark 1 of 1.1. For the Heyting implication $\rightarrow$ we have:

Lemma. *Let s be saturated. Then each $x \rightarrow s$ is saturated.*

Proof. $a \wedge c \leq x \rightarrow s$ iff $a \wedge (c \wedge x) \leq s$ iff $b \wedge (c \wedge x) \leq s$ iff $b \wedge c \leq x \rightarrow s$. $\square$

We show some properties of mapping ν:

Proposition.

(1) *$\nu : A \rightarrow A$ is monotone.*
(2) *$a \leq \nu(a)$.*
(3) *$\nu\nu(a) = \nu(a)$.*
(4) *$\nu(a \wedge b) = \nu(a) \wedge \nu(b)$.*

Proof. (1), (2) and (3) are obvious.
(4) By the monotonicity, $\nu(a \wedge b) \leq \nu(a) \wedge \nu(b)$. Now since $a \wedge b \leq \nu(a \wedge b)$ we have $a \leq b \rightarrow \nu(a \wedge b)$ and, by the Lemma, $\nu(a) \leq b \rightarrow \nu(a \wedge b)$. Thus, $\nu(a) \wedge b \leq \nu(a \wedge b)$ and repeating the procedure we obtain $\nu(a) \wedge \nu(b) \leq \nu(a \wedge b)$. $\square$

Mappings $\nu : A \rightarrow A$ satisfying properties (1)-(4) from the Proposition are called *nuclei*. Denote by $\mathcal{N}(A)$ the system of all nuclei on A, endowed with the natural order.

Note that we have already encountered a nucleus, namely the $\beta_X^* : a \mapsto a^{cc}$, if viewed as a map $\mathcal{O}X \rightarrow \mathcal{O}X$.

Exercises.

1. For each nucleus ν on A let $A_\nu = \{a \in A \mid \nu(a) = a\}$. Show that:
   (a) A_ν is a frame;
   (b) $\nu : A \rightarrow A_\nu$ is a surjective frame homomorphism whose right adjoint is the inclusion $A_\nu \subseteq A$;
   (c) Conclude that, for any locale X, the partially ordered sets $\mathcal{S}(X)$ and $\mathcal{N}(\mathcal{O}X)^{op}$ are isomorphic.
2. Prove that a subset S of a frame A is equal to some A_ν if and only if
   - S is closed under arbitrary meets in A, and
   - $s \in S$, $a \in A$ implies $a \rightarrow s \in S$.

3. Define a subset S of a frame A as a *sublocale set* if it satisfies the conditions of the preceding exercise, and denote the system of all sublocale sets of A, ordered by inclusion, by $\mathcal{S}'(A)$. Prove that the correspondences $\nu \mapsto A_\nu$ and $S \mapsto \nu_S$, where $\nu_S(a) = \bigwedge\{s \in S \mid a \le s\}$, constitute an isomorphism $\mathcal{S}'(A) \cong \mathcal{N}(A)^{op}$.

Denote the set of all R-saturated elements by A/R and view the mapping ν as restricted to $\nu : A \to A/R$.

Theorem.

(1) *A/R is a frame and the restriction $\nu : A \to A/R$ is a frame surjection. If aRb then $\nu(a) = \nu(b)$, and for every frame homomorphism $h : A \to B$ such that $h(a) = h(b)$ whenever aRb, there is a frame homomorphism $\overline{h} : A/R \to B$ such that $\overline{h} \cdot \nu = h$. Moreover, $\overline{h}(a) = h(a)$ for all $a \in A/R$.*

(2) *If R is meet-stable then, moreover, for every join-preserving $f : A \to B$, there is a join-preserving $\overline{f} : A/R \to B$ such that $\overline{f} \cdot \nu = f$.*

Proof. We have suprema in A/R given by $\bigvee' a_i = \nu(\bigvee a_i)$: if $a = \nu(a)$ and $a \ge a_i$ for all i then $a \ge \bigvee a_i$ and $a = \nu(a) \ge \nu(\bigvee a_i)$. We have, for general $a_i \in A$, $\nu(\bigvee a_i) \le \nu(\bigvee \nu(a_i)) = \bigvee' \nu(a_i)$ and $\bigvee' \nu(a_i) \le \nu(\bigvee a_i)$, by monotonicity. Preservation of finite meets follows from Proposition ($\nu(1) = 1$ by (2)). Thus we have a complete lattice A/R and $\nu : A \to A/R$ preserving all joins and finite meets. Since it is onto, A/R satisfies the distributivity requirement for frames.

If aRb then $b \le \nu(a)$, since $a \le \nu(a)$ and $\nu(a)$ is saturated. Hence $\nu(b) \le \nu(a)$, and by symmetry $\nu(b) = \nu(a)$.

Let $h : A \to B$ be such that $h(x) = h(y)$ whenever xRy. For $a \in A$ set $\tau(a) = \bigvee\{a' \in A \mid h(a') \le h(a)\}$. Then

$$a \le \tau(a) \text{ and } h\tau(a) = h(a). \tag{2.15.3}$$

Let xRy and $x \wedge z \le \tau(a)$. Then $h(y \wedge z) = h(x \wedge z) \le h\tau(a) = h(a)$ and hence $y \wedge z \le \tau(a)$. Thus, τ is saturated. Note that if R is meet-stable we can use (2.15.2) and do not need h to preserve the meet. Using (2.15.3) we see that $a \le \nu(a) \le \tau(a)$ and hence $h(a) \le h\nu(a) \le h\tau(a) = h(a)$. Thus, we can define $\overline{h} : A/R \to B$ by $\overline{h}(a) = h(a)$ to obtain $\overline{h} \cdot \nu = h$. □

2.16. The coframe structure of $\mathcal{S}(X)$. So far we met three equivalent ways of representing sublocales of a locale X, given by three different complete lattices that are isomorphic to $\mathcal{S}(X)$: $\mathcal{C}(\mathcal{O}X)^{op}$, in 2.5, and $\mathcal{N}(\mathcal{O}X)^{op}$ and $\mathcal{S}'(\mathcal{O}X)$, in 2.15. The lattice structure of these partially ordered sets is particularly transparent in $\mathcal{S}'(\mathcal{O}X)$: meets are simply intersections (since any intersection of sublocale sets is a sublocale set) and joins are given by

$$\bigvee_{i \in I} S_i = \Big\{\bigwedge T \mid T \subseteq \bigcup_{i \in I} S_i\Big\}$$

(indeed, a sublocale set containing all S_i necessarily contains $\{\bigwedge T \mid T \subseteq \bigcup_{i\in I} S_i\}$; on the other hand, this set is clearly closed under meets, and for any $a \in \mathcal{O}X$ and $T \subseteq \bigcup_{i\in I} S_i$, $a \to \bigwedge T = \bigwedge\{a \to t \mid t \in T\}$ and each $a \to t$ is in $\bigcup_{i\in I} S_i$).

Furthermore,

$$\bigcap_{i\in I}(S_i \vee T) \subseteq (\bigcap_{i\in I} S_i) \vee T :$$

We may assume $I \neq \emptyset$. If $a \in \bigcap_{i\in I}(S_i \vee T)$ then, for each i, $a = s_i \wedge t_i$ for some $s_i \in S_i$ and $t_i \in T$. Let $t = \bigwedge_{i\in I} t_i \in T$; then $a = s_i \wedge t = (t \to s_i) \wedge t$ for every i. On the other hand, $t \wedge s_i = t \wedge s_j$ for all $i, j \in I$ means that $t \to s_i = t \to s_j$. Therefore $t \to s_i$ does not depend on i; denote it by s (s belongs to $\bigcap_{i\in I} S_i$ since, for each i, $s = t \to s_i \in S_i$). Thus $a = s \wedge t$ with $s \in \bigcap_{i\in I} S_i$ and $t \in T$.

Since the reverse inclusion $(\bigcap_{i\in I} S_i) \vee T \subseteq \bigcap_{i\in I}(S_i \vee T)$ is trivial, we have just proved that $\mathcal{S}'(\mathcal{O}X)$ is a coframe. Consequently, $\mathcal{S}(X)$ is also a coframe and $\mathcal{N}(\mathcal{O}X)$ and $\mathcal{C}(\mathcal{O}X)$ are frames.

2.17. Images. Let $f : X \to Y$ be a localic map. The *image* of a sublocale $j : X' \rightarrowtail X$ under f, denoted by $f[j]$ (or, sometimes, by abuse of notation, by $f[X']$), is the unique m in the $(\mathcal{E}, \mathcal{M})$-factorization of $f \cdot j$ (recall 2.3)

$$\begin{tikzcd} X' \arrow[r, two heads, "e"] \arrow[d, tail, "j"'] & Y' \arrow[d, tail, "m"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Technically, of course, one uses the congruence on $\mathcal{O}Y$ defined by

$$a \sim b \text{ if and only if } j^*(f^*(a)) = j^*(f^*(b)).$$

The definition of the *preimage* needs some more knowledge of the category Loc and is postponed to the next section.

3. Limits and colimits

3.1. Equalizers, products, and coequalizers in Frm. If $h_1, h_2 : A \to B$ are frame homomorphisms then

$$k : \{a \in A \mid h_1(a) = h_2(a)\} \hookrightarrow A$$

is obviously the equalizer of h_1, h_2 in Frm.

If A_i, $i \in I$, is any system of frames, the projections of the (cartesian) product

$$p_j : \prod_{i\in I} A_i \to A_j$$

(with the structure in $\prod A_i$ defined coordinatewise) obviously constitute the product in Frm. Thus,

the category Loc *is cocomplete (the category* Frm *is complete).*

Also, *coequalizers* in Frm (*equalizers* in Loc) are easy. If $h_1, h_2 : A \to B$ are frame homomorphisms consider the relation $R = \{(h_1(a), h_2(a)) \mid a \in A\}$ and the homomorphism $\nu : B \to B/R$ from 2.15. We immediately see that ν is the coequalizer of h_1, h_2 in Frm.

The following immediate consequence of Theorem 2.15(2) will be useful.

Proposition. *Let $h_1, h_2 : A \to B$ be frame homomorphisms and let $g : B \to C$ be their coequalizer. Let $\varphi : B \to D$ be a join-preserving map such that $\varphi(h_1(a) \wedge b) = \varphi(h_2(a) \wedge b)$ for all $a \in A$ and $b \in M$, where M join-generates B. Then there is a join-preserving $\overline{\varphi} : C \to D$ such that $\varphi = \overline{\varphi} \cdot g$.* □

Concerning limits and colimits in Loc, the only problem is the product (coproduct in Frm). The existence of products (and, more generally, of all limits and colimits, as well as the exactness of the category Frm) is known at once by the fact that Frm is monadic over the category Set of sets (see V.2.5(5)). But here, we need to know their structure. The major part of this section will be devoted to construct them. For technical reasons, that will be apparent shortly, we will use the frame language.

3.2. Useful facts about the category of semilattices. From Chapter I, paragraph 5.23, recall the downset functor $\mathfrak{D}$ sending a meet-semilattice S to the frame

$$\mathfrak{D}S = \Big(\{U \subseteq S \mid U = {\downarrow}U\}, \subseteq\Big),$$

where ${\downarrow}U = \{a \in S \mid \exists b(a \le b \text{ and } b \in U)\}$ is the *down-closure* of U (cf. I.4.3). For our purposes, it will be handier to use the modification

$$\mathfrak{D}_0 : \mathsf{SLat}_0 \to \mathsf{Frm}$$

where SLat_0 is the category of *bounded* meet-semilattices (that is, semilattices with bottom 0 and top 1), and (bounded) semilattice homomorphisms (that is, mappings preserving the meet, including the top 1, and 0),

$$\mathfrak{D}_0 S = \Big(\{U \subseteq S \mid \emptyset \ne U = {\downarrow}U\}, \subseteq\Big)$$

and

$$\mathfrak{D}_0 h(U) = {\downarrow}h[U].$$

Note that the joins in $\mathfrak{D}_0 S$ are, again, the unions, with one exception: the join of the void system (the bottom of $\mathfrak{D}_0 S$) is $\{0\}$, not the union (which is $\emptyset$ and is not an element of $\mathfrak{D}_0 S$). Moreover, it is obvious that $\mathfrak{D}_0 h$ preserves the bottom, the top and all unions; it also preserves the meet since

$$\begin{aligned} {\downarrow}h[U] \cap {\downarrow}h[V] &= \{x \mid \exists a \in U, b \in V,\ x \le h(a) \wedge h(b)\} \\ &\subseteq \{x \mid \exists c \in U \cap V,\ x \le h(c)\} \\ &= {\downarrow}h[U \cap V] \\ &\subseteq {\downarrow}h[U] \cap {\downarrow}h[V] \end{aligned}$$

(the last inclusion because of the monotonicity).

Consider the maps $\lambda_S : S \to \mathfrak{D}_0 S$ given by $\lambda_S(a) = \downarrow a$. It is an important fact that the λ_S are the universal bounded semilattice homomorphisms, analogously as in the well known situation with the $\mathfrak{D}$ *including the empty set*:

Proposition. *The mapping λ_S is a morphism in SLat_0 and, for each frame A and each $h : S \to A$ in SLat_0, there is exactly one frame homomorphism $\overline{h} : \mathfrak{D}_0 S \to A$ such that $\overline{h} \cdot \lambda_S = h$.*

Proof. The first statement is obvious, since $\downarrow(a \wedge b) = \downarrow a \cap \downarrow b$, by the definition of $a \wedge b$. If $\overline{h} \cdot \lambda_S = h$ then

$$\overline{h}(U) = \overline{h}(\bigcup\{\downarrow a \mid a \in U\}) = \bigvee\{\overline{h}(\downarrow a) \mid a \in U\} = \bigvee\{h(a) \mid a \in U\}$$

and hence $\overline{h}$ is uniquely determined. On the other hand, the formula $\overline{h}(U) = \bigvee\{h(a) \mid a \in U\}$ determines a frame homomorphism. □

In SLat_0 the coproducts are obtained as follows. Set

$$\prod\nolimits'_{i \in I} S_i = \Big\{(a_i)_{i\in I} \in \prod_{i \in I} S_i \mid a_i = 1 \text{ for all but finitely many } i\Big\} \cup \Big\{(0)_{i \in I}\Big\}$$

and define

$$\gamma_j : S_j \to \prod\nolimits'_{i \in I} S_i \quad \text{by setting} \quad (\gamma_j(a))_i = \begin{cases} a & \text{for } \ i = j, \\ 1 & \text{otherwise.} \end{cases}$$

Obviously, if $h_j : S_j \to T$ are morphisms in SLat_0, we have a uniquely defined $h : \prod' S_i \to T$ such that $h \cdot \gamma_j = h_j$, namely that given by $h((a_i)_{i \in I}) = \bigwedge_{i \in I} h_i(a_i)$ — the meet is finite, all but finitely many $h_i(a_i)$ being 1.

3.3. Coproducts of frames. Let now A_i, $i \in I$, be frames. View them, for a moment, as objects of SLat_0, and take $S = \prod'_{i \in I} A_i$. On the frame $\mathfrak{D}_0(\prod'_{i \in I} A_i)$ consider the relation

$$R = \Big\{\Big(\lambda_S\gamma_j(\bigvee_{m \in M} a_m), \bigvee_{m \in M} \lambda_S\gamma_j(a_m)\Big) \mid j \in I,\ M \text{ any set},\ \ a_m \in A_j\Big\},$$

and set

$$\bigoplus_{i \in I} A_i = \mathfrak{D}_0(\prod\nolimits'_{i \in I} A_i)/R.$$

Let $\nu : \mathfrak{D}_0(\prod'_{i \in I} A_i) \to \bigoplus_{i \in I} A_i$ be the homomorphism from 2.15.

Remark. The $\iota_j = \nu \cdot \lambda \cdot \gamma_j$ are frame homomorphisms. Indeed, 0,1 and $\wedge$ are preserved trivially and

$$\Big(\lambda\gamma_j(\bigvee_{m \in M} a_m), \bigvee_{m \in M} \lambda\gamma_j(a_m)\Big) \in R$$

and hence $\iota_j(\bigvee_{m \in M} a_m) = \bigvee_{m \in M} \iota_j(a_m)$.

Proposition. *The system $(\gamma_j : A_j \to \bigoplus_{i \in I} A_i)_{j \in I}$ is a coproduct in* Frm.

Proof. Consider the diagram

$$\begin{tikzcd}
A_j \arrow[r, "\gamma_j"] \arrow[d, "h_j"'] & \prod'_{i\in I} A_i \arrow[r, "\lambda"] \arrow[d, "h'"'] & \mathfrak{D}_0(\prod'_{i\in I} A_i) \arrow[r, "\nu"] \arrow[d, "h''"'] & \bigoplus_{i\in I} A_i \arrow[d, "h"'] \\
B \arrow[r, equal] & B \arrow[r, equal] & B \arrow[r, equal] & B
\end{tikzcd}$$

where h_j are some frame homomorphisms, h' is the coproduct morphism in SLat_0 and h'' is the frame homomorphism from Proposition 3.2. We have

$$h''(\bigvee_M \lambda\gamma_j(a_m)) = \bigvee\{h'((b_i)_{i\in I}) \mid (b_i)_{i\in I} \le \gamma_j(a_m) \text{ for some } m \in M\} =$$
$$= \bigvee_M h'\gamma_j(a_m) = \bigvee_M h_j(a_m) = h_j(\bigvee_M a_m) = h'\gamma_j(\bigvee_M a_m) = h''\lambda\gamma_j(\bigvee_M a_m).$$

Hence, by Theorem 2.15, there is a frame homomorphism h such that $h\cdot\nu = h''$. Thus, $h\cdot\iota_j = h\cdot\nu\cdot\lambda\cdot\gamma_j = h''\cdot\lambda\cdot\gamma_j = h'\cdot\gamma_j = h_j$. The unicity follows from the obvious fact that all the elements of $\mathfrak{D}_0(\prod'_{i\in I}A_i)$ are joins of finite meets of the $\lambda\gamma_j(a)$ and hence all the elements of $\bigoplus_{i\in I} A_i$ are joins of finite meets of the $\iota_j(a)$ $(j\in I, a\in A_j)$. □

Recalling the equalizers from 3.1 we conclude that

the category Loc *is complete (the category* Frm *is cocomplete).*

For finite systems we write $A\oplus B$, $A_1\oplus A_2\oplus A_3$ etc. to denote frame coproducts.

3.4. More about the coproduct structure. Recalling that the join $\bigvee_{m\in M} U_m$ in $\mathfrak{D}_0 A$ is equal to the union $\bigcup_{m\in M} U_m$ if $M\neq\emptyset$, and to the set $\{0\}$ if the index set M is void, we see that, in particular,

$$\Big(\downarrow\gamma_j(0), \{(0)_{i\in I}\}\Big) \in R \quad \text{for all} \quad j.$$

Set

$$\mathbf{O} = \Big\{(a_i)_{i\in I} \in \prod_{i\in I}{}' A_i \mid \exists i, a_i = 0\Big\}.$$

Obviously, $\bigcup_{m\in M}\downarrow\gamma_j(a_m) = \downarrow\gamma_j(\bigcup_{m\in M} a_m)$ for $M\neq\emptyset$. Then, since $\mathfrak{D}_0(\prod'_{i\in I} A_i)$ is generated by the $\downarrow(b_i)_{i\in I}$, we easily infer (recall 2.15) that

$U\in\mathfrak{D}_0(\prod' L_j)$ is saturated if and only if
(1) $\mathbf{O}\subseteq U$, and
(2) for $M\neq\emptyset$, whenever $x_{im} = x_i$ for $i\neq j$, $x_j = \bigvee_{m\in M} x_{jm}$ and $(x_{im})_{i\in I}\in U$ for all m then $(x_i)_{i\in I}\in U$.

Lemma. *For any $(a_i)_{i\in I}\in\prod'_{i\in I} A_i$, the set $\oplus_{i\in I} a_i := \downarrow(a_i)_{i\in I}\cup\mathbf{O}$ is saturated.*

Proof. Let $(x_{im})_{i\in I}$ and x_j be as in the condition above. If $x_i = 0$ for some i then $(x_i)_{i\in I}\in\oplus_{i\in I} a_i$. Otherwise all $x_{im}\neq 0$ for $i\neq j$ and $\bigvee_{i\in I} x_{jm}\neq 0$. Hence $x_{jn}\neq 0$ for some n, $(x_{in})_{i\in I}$ is not in $\mathbf{O}$ and therefore $x_i\le a_i$ for all $i\neq j$; but then also all $x_{jm}\le a_j$ and $x_j = \bigvee_{i\in I} x_{jm}\le a_j$. □

Corollary. *If $\oplus_{i\in I}a_i \le \oplus_{i\in I}b_i$ and $a_i \ne 0$ for all i, then $a_i \le b_i$ for all i.* $\square$

For finite index sets we write

$$a \oplus b,\ a_1 \oplus a_2 \oplus a_3 \ \text{ etc.}$$

Note that, for $(a_i)_{i\in I} \in \prod'_{i\in I} A_i$, we have $\bigwedge_{i\in I} \iota_i(a_i) = \oplus_{i\in I}a_i$. Thus the set of the elements of the form $\oplus_{i\in I}a_i$ generates $\bigoplus_{i\in I} A_i$ by joins and we have, for each $u \in \bigoplus_{i\in I} A_i$,

$$u = \bigvee\{\oplus_{i\in I}a_i \mid (a_i)_{i\in I} \in u\} = \bigvee\{\oplus_{i\in I}a_i \mid \oplus_{i\in I}a_i \le u\}.$$

Exercises.

1. Show that $\mathcal{P}X \oplus \mathcal{P}Y \cong \mathcal{P}(X \times Y)$, and hence that the product of discrete spatial locales is a discrete spatial locale.
2. For the usual topology on $\mathbb{Q}$, prove that the frame $\mathcal{O}(\mathbb{Q}\times\mathbb{Q})$ of open subsets of $\mathbb{Q}\times\mathbb{Q}$ is not isomorphic to the frame $\mathcal{O}(\mathbb{Q}) \oplus \mathcal{O}(\mathbb{Q})$.

The following technical statement will be used later.

Proposition. *Let $A_i,\ i = 1, 2,$ be frames and $a_i \in A_i$. Then we have ${\downarrow}a_1 \oplus {\downarrow}a_2 = {\downarrow}(a_1 \oplus a_2)$. More precisely, if $\iota_i : A_i \to A_1 \oplus A_2$ are the coproduct injections, then*

$$\begin{array}{rccl} \iota'_i : & {\downarrow}a_i & \to & {\downarrow}(a_1 \oplus a_2) \\ & x & \mapsto & \iota_i(x) \wedge (a_1 \oplus a_2) \end{array}$$

constitute the coproduct of frames ${\downarrow}a_1$ and ${\downarrow}a_2$.

Proof. Let $h_i : {\downarrow}a_i \to B$ be frame homomorphisms. Consider the $g : A_1 \oplus A_2 \to B$ such that $g\cdot\iota_i = h_i\cdot\hat{a_i}$. We have $g(x_1\oplus x_2) = g(\iota_1(x_1)\wedge\iota_2(x_2)) = h_1(x_1\wedge a_1)\wedge h_2(x_2\wedge a_2)$. Hence, if $(x_1\oplus x_2)\wedge(a_1\oplus a_2) = (y_1\oplus y_2)\wedge(a_1\oplus a_2)$, then $g(x_1\oplus x_2) = g(y_1\oplus y_2)$. Thus there is a frame homomorphism $h : {\downarrow}(a_1\oplus a_2) \to B$ such that $h\cdot(a_1\oplus a_2)\hat{} = g$. For $x \in {\downarrow}a_i$ we have $h(\iota'_i(x)) = h(\iota_i(x) \wedge (a_1 \oplus a_2)) = g(\iota_i(x)) = h_i(x)$. The unicity of such an h is obvious. $\square$

For any frame homomorphisms $h_i : A_i \to B_i$, $i = 1, 2$, we write $h_1 \oplus h_2$ for the unique frame homomorphism $A_1 \oplus A_2 \to B_1 \oplus B_2$ that makes the following diagram commutative

$$\begin{tikzcd} A_1 \arrow[r, "\iota_1"] \arrow[d, "h_1"'] & A_1 \oplus A_2 \arrow[d, dashed, "h_1\oplus h_2"] & A_2 \arrow[l, "\iota_2"'] \arrow[d, "h_2"] \\ B_1 \arrow[r, "\jmath_1"] & B_1 \oplus B_2 & B_2 \arrow[l, "\jmath_2"'] \end{tikzcd}$$

Obviously,

$$(h_1 \oplus h_2)(\bigvee_{i\in I}(a_i^1 \oplus a_i^2)) = \bigvee_{i\in I}(h_1(a_i^1) \oplus h_2(a_i^2)). \tag{3.4.1}$$

3.5. Coproducts and join-preserving maps. Making the relation R in 3.3 meet-stable (recall 2.15) is easy but it would obscure the notation. For the coproduct of two frames, it is transparent enough. We can replace R by the relation R' consisting of all

$$\Big(\downarrow(\bigvee_{i\in I} a_i, b), \bigvee_{i\in I} \downarrow(a_i, b)\Big) \text{ and } \Big(\downarrow(a, \bigvee_{i\in I} b_i), \bigvee_{i\in I} \downarrow(a, b_i)\Big).$$

Using the second part of Theorem 2.15 we can then easily deduce that:

Proposition. *Let A_1, A_2, B be frames and let mappings $\varphi_i : A_i \to B$ preserve all joins. Then there is (exactly one) $\varphi : A_1 \oplus A_2 \to B$ preserving all joins such that $\varphi(a_1 \oplus a_2) = \varphi_1(a_1) \wedge \varphi_2(a_2)$.* $\square$

3.6. Preimages. Now, knowing that the category Loc is complete and cocomplete, we can add to the definition of image from 2.17 the definition of preimage. It is a general categorical fact that

(∗) *in a category with pullbacks and pushouts, extremal monomorphisms are stable under pullbacks.*

Hence we can define the *preimage* of the sublocale $j : Y' \rightarrowtail Y$ under $f : X \to Y$, that we denote by $f^{-1}[j]$, (or, sometimes, by abuse of notation, by $f^{-1}[Y']$) as the sublocale $m : X' \rightarrowtail X$ from the pullback

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, tail, "m"'] & Y' \arrow[d, tail, "j"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{3.6.1}$$

Images and preimages satisfy the inequalities

$$j \sqsubseteq f^{-1}[f[j]] \quad \text{and} \quad f[f^{-1}[j]] \sqsubseteq j \tag{3.6.2}$$

(see III.1.6 for more details).

For the sake of completeness, let us finish this paragraph by proving the statement (∗):

In the pullback (3.6.1), let $m = n \cdot e$ with e an epimorphism. Consider the pushout

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, two heads, "e"'] & Y' \arrow[d, two heads, "e'"] \\ X'' \arrow[r, "f''"] & Y'' \end{tikzcd}$$

Since $(f \cdot n) \cdot e = s \cdot f'$, we have an n' such that $n' \cdot e' = s$ and $n' \cdot f'' = f \cdot n$. Since e' is an epimorphism (epimorphisms are stable under pushouts in any category) and since s is extremal, e' is an isomorphism and we can assume that $e' = 1$. Hence $n' = s$ and $s \cdot f'' = f \cdot n$. From the last equation and the original pullback we obtain a k such that $m \cdot k = n$. Hence $m \cdot k \cdot e = n \cdot e = m$, and since m is a monomorphism, $k \cdot e = 1$. Finally, since e is an epimorphism, we may conclude, using $e \cdot k \cdot e = e$, that also $e \cdot k = 1$. $\square$

3.7. Preimages of open and closed sublocales.

Proposition. *Let $f : X \to Y$ be a localic map and let $a \in \mathcal{O}Y$. Then:*

(1) *The preimage of an open sublocale is open. More precisely,*

$$f^{-1}[Y_a] = X_{f^*(a)};$$

(2) *The preimage of a closed sublocale is closed. More precisely,*

$$f^{-1}[Y\text{-}Y_a] = X\text{-}X_{f^*(a)}.$$

Proof. (1) Let us check that

$$\begin{tikzcd} X_{f^*(a)} \arrow[r, "f'"] \arrow[d, tail] & Y_a \arrow[d, tail] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a pullback in Loc, where $(f')^*(a) = f^*(a)$. It is a commutative square by Exercise 1 of 2.6. Given localic morphisms $g : Z \to X$ and $h : Z \to Y_a$ such that the outer part of the diagram

$$\begin{tikzcd} Z \arrow[dr, dotted, "k"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g"'] & & \\ & X_{f^*(a)} \arrow[r, "f'"] \arrow[d, tail] & Y_a \arrow[d, tail] \\ & X \arrow[r, "f"] & Y \end{tikzcd} \tag{3.7.1}$$

is commutative, define $k : Z \to X_{f^*(a)}$ by $k^*(x) = g^*(x)$ for each $x \leq f^*(a)$ in $\mathcal{O}X$. This is a localic map; indeed k^* preserves binary meets and arbitrary joins, since g^* does, and moreover $k^*(f^*(a)) = g^*(f^*(a)) = h^*(a \wedge a) = h^*(a) = 1$. Furthermore, given $b \in \mathcal{O}X$ and $a' \in \mathcal{O}Y$ with $a \leq a'$,

$$k^*(b \wedge f^*(a)) = g^*(b \wedge f^*(a)) = g^*(b) \wedge g^*(f^*(a)) = g^*(b) \wedge h^*(a) = g^*(b) \wedge 1 = g^*(b)$$

and $k^*(f^*(a')) = g^*(f^*(a')) = h^*(a \wedge a') = h^*(a')$. Thus k is a factorization in diagram (3.7.1). This factorization is unique since $X_{f^*(a)} \rightarrowtail X$ is a monomorphism.
(2) The argument for closed sublocales is similar. ☐

3.8. Preimage as a (co)frame homomorphism. Let $f : X \to Y$ be a localic map. By (3.6.2), $f^{-1} : \mathcal{S}(Y) \to \mathcal{S}(X)$ has a left Galois adjoint $f[-] : \mathcal{S}(X) \to \mathcal{S}(Y)$. Thus,

$$f^{-1} : \mathcal{S}(Y) \to \mathcal{S}(X) \textit{preserves all meets.} \tag{3.8.1}$$

Lemma. $f^{-1}[Y_a \sqcup Y\text{-}Y_b] = X_{f^*(a)} \sqcup X\text{-}X_{f^*(b)}$.

Proof. Since f^{-1} preserves meets, we have

$$\begin{aligned} f^{-1}[Y_a \sqcup Y\text{-}Y_b] \sqcap f^{-1}[Y\text{-}Y_a \sqcap Y_b] &= f^{-1}[(Y_a \sqcup Y\text{-}Y_b) \sqcap (Y\text{-}Y_a \sqcap Y_b)] \\ &= f^{-1}[0_Y] = 0_X, \end{aligned}$$

the zero of $\mathcal{S}(X)$ by 2.5. On the other hand,

$$\begin{aligned} & f^{-1}[Y_a \sqcup Y\text{-}Y_b] \sqcup f^{-1}[Y\text{-}Y_a \sqcap Y_b] \\ = \ & f^{-1}[Y_a \sqcup Y\text{-}Y_b] \sqcup (f^{-1}[Y\text{-}Y_a] \sqcap f^{-1}[Y_b]) \\ \sqsupseteq \ & f^{-1}[Y_a] \sqcup f^{-1}[Y\text{-}Y_b] \sqcup (f^{-1}[Y\text{-}Y_a] \sqcap f^{-1}[Y_b]) \\ = \ & X_{f^*(a)} \sqcup X\text{-}X_{f^*(b)} \sqcup (X\text{-}X_{f^*(a)} \sqcap X_{f^*(b)}) \\ = \ & X, \end{aligned}$$

the top element of $\mathcal{S}(X)$ by (2.5). Thus, $f^{-1}[Y_a \sqcup Y\text{-}Y_b] = (f^{-1}[Y\text{-}Y_a \sqcap Y_b])^c = (f^{-1}[Y\text{-}Y_a] \sqcap f^{-1}[Y_b])^c = (X\text{-}X_{f^*(a)} \sqcap X_{f^*(b)})^c = X_{f^*(a)} \sqcup X\text{-}X_{f^*(b)}$. □

Proposition. $f^{-1} : \mathcal{S}(Y) \to \mathcal{S}(X)$ *is a coframe homomorphism.*

Proof. By (3.8.1) f^{-1} preserves all meets and by Proposition 3.7 $f^{-1}[0_Y] = 0_X$. Thus, it remains to prove that f^{-1} preserves the join $\sqcup$.

By Proposition 2.8 we have, for each sublocale j of Y,

$$j = \bigsqcap \{Y_a \sqcup Y\text{-}Y_b \mid j^*(a) = j^*(b)\}.$$

Using the coframe structure of $\mathcal{S}(Y)$ we obtain

$$\begin{aligned} j_1 \sqcup j_2 &= \bigsqcap \Big\{ Y_{a_1} \sqcup Y\text{-}Y_{b_1} \sqcup Y_{a_2} \sqcup Y\text{-}Y_{b_2} \mid j_i^*(a_i) = j_i^*(b_i), i = 1, 2 \Big\} \\ &= \bigsqcap \Big\{ Y_{a_1 \vee a_2} \sqcup Y\text{-}Y_{b_1 \wedge b_2} \mid j_i^*(a_i) = j_i^*(b_i), i = 1, 2 \Big\}. \end{aligned}$$

Then, by the Lemma, $f^{-1}[j_1 \sqcup j_2]$ is equal to

$$\begin{aligned} & \bigsqcap \Big\{ X_{f^*(a_1 \vee a_2)} \sqcup X\text{-}X_{f^*(b_1 \wedge b_2)} \mid j_i^*(a_i) = j_i^*(b_i), i = 1, 2 \Big\} \\ = \ & \bigsqcap \Big\{ X_{f^*(a_1) \vee f^*(a_2)} \sqcup X\text{-}X_{f^*(b_1) \wedge f^*(b_2)} \mid j_i^*(a_i) = j_i^*(b_i), i = 1, 2 \Big\} \\ = \ & \bigsqcap \Big\{ X_{f^*(a_1)} \sqcup X_{f^*(a_2)} \sqcup X\text{-}X_{f^*(b_1)} \sqcup X\text{-}X_{f^*(b_2)} \mid j_i^*(a_i) = j_i^*(b_i), i = 1, 2 \Big\} \\ = \ & \bigsqcap \Big\{ f^{-1}[Y_{a_1} \sqcup Y\text{-}Y_{b_1}] \sqcup f^{-1}[Y_{a_2} \sqcup Y\text{-}Y_{b_2}] \mid j_i^*(a_i) = j_i^*(b_i), i = 1, 2 \Big\} \\ = \ & f^{-1}[j_1] \sqcup f^{-1}[j_2], \end{aligned}$$

since $f^{-1}[j_i] = \bigsqcap \{f^{-1}[Y_{a_i} \sqcup Y\text{-}Y_{b_i}] \mid j_i^*(a_i) = j_i^*(b_i)\}$. □

Corollary. *The preimage f^{-1} preserves complementarity of sublocales.* □

3.9. $\mathcal{S}$ as a functor $\mathsf{Loc} \to \mathsf{Loc}$. Peculiar monomorphisms. The fact from 3.8 allows us to extend the construction $\mathcal{S}(X)$ from 2.4 (see also 2.16) to a functor

$$\mathsf{S} : \mathsf{Loc} \to \mathsf{Loc},$$

by defining $\mathsf{S}(X) = \mathcal{S}(X)^{op}$ and $\mathsf{S}(f) : \mathcal{S}(X)^{op} \to \mathcal{S}(Y)^{op}$ by $(\mathsf{S}(f))^*(j) = f^{-1}[j]$ for every $j \in \mathcal{S}(Y)^{op}$. In terms of frames, this means that the construction $\mathcal{C}(A)$ from 2.5 gives a functor $\mathcal{C} : \mathsf{Frm} \to \mathsf{Frm}$.

Moreover, we have a natural transformation

$$s : \mathsf{S} \stackrel{\cdot}{\to} 1_{\mathsf{Loc}},$$

defined by $s_X^*(a) = X\text{-}X_a$. In fact we have, for each $f : X \to Y$, $(\mathsf{S}(f))^*(s_Y^*(a)) = f^{-1}(Y_a) = X_{f^*(a)} = s_X^*(f^*(a))$.

Remark. The homomorphisms s_X are monomorphisms in Loc. Indeed, let $f, g : Y \to \mathsf{S}(X)$ such that $s_X \cdot f = s_X \cdot g$. Since a complement, if it exists, is uniquely determined, and since each frame homomorphism preserves complements, homomorphisms $f^*, g^* : \mathcal{O}(\mathsf{S}(X)) \to \mathcal{O}Y$ coinciding on all elements of the form $X\text{-}X_a$ have to coincide on all elements of the form X_a as well, by 2.7. But then, by 2.8, they coincide on all $j \in \mathsf{S}(X)$.

Functor S can be iterated by setting $\mathsf{S}^0 = 1_{\mathsf{Loc}}$, $\mathsf{S}^{\alpha+1}(X) = \mathsf{S}(\mathsf{S}^\alpha(X))$, for non-limit ordinals $\alpha + 1$, and $s_X^{\alpha+1} = s_X^\alpha \cdot s_{\mathsf{S}^\alpha(X)}$, and taking the limit of the obvious diagram in the limit ordinals. More precisely, set $\delta_{\beta\beta} = 1_{\mathsf{S}^\beta(X)} : \mathsf{S}^\beta(X) \to \mathsf{S}^\beta(X)$; if $\delta_{\beta\gamma} : \mathsf{S}^\gamma(X) \to \mathsf{S}^\beta(X)$ are already defined, set $\delta_{\beta,\gamma+1} = \delta_{\beta\gamma} \cdot s_{\mathsf{S}^\gamma(X)}$; for a limit ordinal, if $\mathsf{S}^\beta(X)$ for $\beta < \alpha$ and $\delta_{\beta\gamma}$ for $\beta, \gamma < \alpha$ are already defined, take the limit

$$\Big(\delta_{\beta\alpha} : \mathsf{S}^\alpha(X) \to \mathsf{S}^\beta(X)\Big)_{\beta<\alpha}$$

of the diagram

$$\Big(\delta_{\beta\gamma} : \mathsf{S}^\gamma(X) \to \mathsf{S}^\beta(X)\Big)_{\beta,\gamma<\alpha};$$

finally, set $s_X^\alpha = \delta_{0\alpha}$.

Then

all the $s_X^\alpha : \mathsf{S}^\alpha(X) \to X$ are monomorphisms.

This shows that the structure of monomorphisms in Loc is rather complex. There are locales X for which the iteration $\mathsf{S}^\alpha(X)$ never stops increasing in size ([34], 2.10). Consequently,

there is a locale X such that, for any cardinality α, there exists a locale Y with $|Y| \geq \alpha$, and a monomorphism $f : Y \to X$.

Exercise. If the locale $\mathsf{S}(X)$ is a Boolean algebra, prove that it is the reflection of the locale X in the full subcategory of Boolean locales.

4. Some subcategories of locales

4.1. A very weak separation axiom: subfitness. A locale X is said to be *subfit* ([32], *conjunctive* in [56]) if, for every $a, b \in \mathcal{O}X$,

$$a \nleq b \quad \Rightarrow \quad \exists c \in \mathcal{O}X,\ a \vee c = 1 \neq b \vee c.$$

Exercises.

1. Let X be a topological space. Prove: if X is T_1 then $\mathsf{Lc}(X)$ is subfit, but not conversely.
2. Show that, for T_D-spaces, subfitness coincides with T_1.

4.2. Relations $\prec$ and $\prec\!\prec$. Define also $a \prec b$ if $a^c \vee b = 1$.

Exercises.

1. Prove that $a \prec b$ if and only if there is a c such that $a \wedge c = 0$ and $b \vee c = 1$.
2. Verify that, if X is a space and $U, V \in \mathcal{O}X$, $U \prec V$ if and only if $\overline{U} \subseteq V$.

Lemma. *The relation $\prec$ has the following properties:*

(1) $a \prec b \;\Rightarrow\; a \leq b$;
(2) $a \leq b \prec c \leq d \;\Rightarrow\; a \prec d$;
(3) $a_1, a_2 \prec b \;\Rightarrow\; a_1 \vee a_2 \prec b$;
(4) $a \prec b_1, b_2 \;\Rightarrow\; a \prec b_1 \wedge b_2$;
(5) $a \prec b \;\Rightarrow\; b^c \prec a^c$;
(6) *If $f : X \to Y$ is a localic map and $a \prec b$ in $\mathcal{O}Y$, then $f^*(a) \prec f^*(b)$.*

Proof. (1) If $a^c \vee b = 1$ then $a \wedge b = a \wedge (a^c \vee b) = b$.
(2) It follows from the fact that $a \leq b$ implies $b^c \leq a^c$.
(3) Let $a_i \wedge c_i = 0$ and $a_i \vee b = 1$. Set $c = c_1 \wedge c_2$. Then $c \vee b = (c_1 \vee b) \wedge (c_2 \vee b) = 1$ and $(a_1 \vee a_2) \wedge c = 0$.
(4) Similarly, let $a \wedge c_i = 0$ and $c_i \vee b_i = 1$. Set $c = c_1 \vee c_2$. Then $c \vee (b_1 \wedge b_2) = (c \vee b_1) \wedge (c \vee b_2) = 1$ and $a \wedge c = 0$.
(5) If $a^c \vee b = 1$ then $b^{cc} \vee a^c = 1$.
(6) Since $(f^*(a))^c \geq f^*(a^c)$ we have $(f^*(a))^c \vee f^*(b) \geq f^*(a^c \vee b) = 1$. □

A transitive relation R is *interpolative* if whenever aRb there is a c such that $aRcRb$. It is easy to check that, for each transitive R, there is the largest (transitive) interpolative $\widetilde{R} \subseteq R$, namely the following one. Denote by D the set of dyadic rationals in the unit interval; then

$a\widetilde{R}b$ iff there are a_d, $d \in D$, such that $a = a_0$, $b = a_1$, and $c < d \Rightarrow a_c R a_d$.

The relation $\widetilde{\prec}$ is usually denoted by $\prec\!\prec$. It is easy to see that

Lemma 4.2 holds with $\prec$ replaced by $\prec\!\prec$.

4.3. Regular and completely regular locales. Let X be a locale. For each $a \in \mathcal{O}X$ set

$$\sigma_X(a) = \{x \in \mathcal{O}X \mid x \prec a\} \text{ and } \rho_X(a) = \{x \in \mathcal{O}X \mid x \prec\!\prec a\}.$$

X is said to be *regular* (resp. *completely regular*) if

$$\forall a \in \mathcal{O}X,\ a = \bigvee \sigma(a) \quad (\text{resp. } a = \bigvee \rho(a)).$$

The category of regular (resp. completely regular) locales, and the localic maps between them, will be denoted by

$$\mathsf{RegLoc} \quad (\text{resp.} \quad \mathsf{CRegLoc}).$$

Proposition. RegLoc *and* $\mathsf{CRegLoc}$ *are reflective subcategories of the category* Loc. *Consequently, these categories are complete and cocomplete.*

Proof. For a locale X set

$$R_1(X) = \{a \in \mathcal{O}X \mid a = \bigvee \sigma(a)\}.$$

Obviously, $R_1(X)$ is a subframe of $\mathcal{O}X$. For ordinals α set

$$R_{\alpha+1}(X) = R_1(R_\alpha(X)), \text{ and if } \alpha \text{ is a limit one, } R_\alpha(X) = \bigcap_{\beta<\alpha} R_\beta(X).$$

Now if $f : X \to Y$ is a localic map and Y is regular, $f^*[\mathcal{O}Y] \subseteq R(X)$, by Lemma 4.2(6). Thus, if we set $R_\infty(X) = \bigcap_\alpha R_\alpha(X)$, the epimorphisms $e_X : X \twoheadrightarrow R_\infty(X)$, given by the frame inclusions $e_X^* : R_\infty(X) \hookrightarrow X$, constitute a reflection of Loc onto RegLoc.

Similarly for complete regularity (the situation is in fact simpler: here, the procedure stops after the first step). ☐

Exercises.

1. Let X be a topological space. Prove that:
   (a) X is regular in the classical sense if and only if the locale $\mathsf{Lc}(X)$ is regular in the sense just defined (recall Exercise 1 of 4.2);
   (b) X is completely regular in the classical sense if and only if the locale $\mathsf{Lc}(X)$ is completely regular in the sense just defined. (Hint: use the procedure from the standard proof of the Urysohn Lemma.)
2. Conclude from Lemma 4.2(6) that a sublocale of a (completely) regular locale is (completely) regular.
3. Prove that a product of regular locales is regular.

4.4. Normality. A locale X is said to be *normal* if for any $a, b \in \mathcal{O}X$ such that $a \vee b = 1$ there are $u, v \in \mathcal{O}X$ such that

$$u \vee b = 1, \ a \vee v = 1 \quad \text{and} \quad u \wedge v = 1.$$

This is an immediate translation of the homonymous property of spaces; thus, trivially, a space X is normal in the classical sense if and only if $\mathsf{Lc}(X)$ is normal in the sense just defined.

Lemma. *In a normal locale the relation $\prec$ interpolates (and hence coincides with $\prec\!\prec$).*

Proof. Let $a \prec b$, that is, $a^c \vee b = 1$. Then there are u, v such that $a^c \vee v = 1$, $u \vee b = 1$ and $u \wedge v = 0$ (and hence $u \leq v^c$). Thus, $a \prec v \prec b$. ☐

Proposition. *A subfit normal locale is completely regular.*

Proof. By the Lemma it suffices to prove that it is regular. Let $c \vee a = 1$. We shall prove that then $c \vee \bigvee \sigma(a) = 1$, so that, by subfitness, $a = \bigvee \sigma(a)$. If $c \vee a = 1$ we have u, v such that $u \vee a = 1$, $c \vee v = 1$ and $u \leq v^c$. Thus, $v \leq \bigvee \sigma(a)$ and $c \vee \bigvee \sigma(a) = 1$. ☐

4.5. Hausdorff locales. Recall that a topological space is Hausdorff if and only if the diagonal $\{(x,x) \mid x \in X\}$ is closed in $X \times X$. In analogy with this fact, a locale X is called *Hausdorff* (*strongly Hausdorff* in [32]) if the diagonal

$$\Delta_X : X \to X \times X$$

is a closed sublocale (or, in frame terms, if the codiagonal

$$\nabla_{\mathcal{O}X} = \Delta_X^* : \mathcal{O}(X \times X) = \mathcal{O}X \oplus \mathcal{O}X \to \mathcal{O}X$$

is a closed surjection).

Since, in Frm, $\nabla_{\mathcal{O}X}(a \oplus b) = \nabla_{\mathcal{O}X}(\iota_1(a) \wedge \iota_2(b)) = a \wedge b$, we immediately infer that $\check{d}_{\mathcal{O}X} : \mathcal{O}X \oplus \mathcal{O}X \to \uparrow d_{\mathcal{O}X}$, where

$$d_{\mathcal{O}X} = \bigvee \{a \oplus b \mid a \wedge b = 0\},$$

is the closure of $\nabla_{\mathcal{O}X}$. Hence the Hausdorff condition amounts to the existence of a frame homomorphism $\alpha : \mathcal{O}X \to \uparrow d_{\mathcal{O}X}$ such that

$$\alpha \cdot \nabla_{\mathcal{O}X} = \check{d}_{\mathcal{O}X}. \tag{4.5.1}$$

Remark. Unlike the previous separation axioms, this Hausdorff condition is only an analogy of the classical one. The functor $\mathsf{Lc} : \mathsf{Top} \to \mathsf{Loc}$ does not, in general, preserve products, and a Hausdorff topological space X need not have a closed localic diagonal, that is, it does not necessarily yield a Hausdorff $\mathsf{Lc}(X)$. There are other analogues of the Hausdorff axiom in the literature [26, 40], useful in various contexts. Note that the Hausdorff type axioms presented there are weaker than the Hausdorff property discussed here. From the point of view of categorical topology, the latter (considered first by Isbell in [32]) seems to be, for obvious reasons, of a particular importance.

The category of Hausdorff locales will be denoted by $\mathsf{HausLoc}$.

Lemma. *A locale X is Hausdorff if and only if, for any $a, b \in \mathcal{O}X$,*

$$a \oplus b \le \check{d}_{\mathcal{O}X}\Big((a \wedge b) \oplus (a \wedge b)\Big).$$

Proof. For the α from (4.5.1) we have

$(a \oplus b) \vee d_{\mathcal{O}X} = \alpha(\nabla_{\mathcal{O}X}(a \oplus b)) = \alpha(\nabla_{\mathcal{O}X}((a \wedge b) \oplus (a \wedge b))) = ((a \wedge b) \oplus (a \wedge b)) \vee d_{\mathcal{O}X}$.

On the other hand, if the condition is satisfied set $\alpha(x) = (x \oplus x) \vee d_{\mathcal{O}X}$. As $x_i \oplus x_j \le ((x_i \wedge x_j) \oplus (x_i \wedge x_j)) \vee d_{\mathcal{O}X}$, we have $(x_i \oplus x_j) \vee d_{\mathcal{O}X} \le (x_i \oplus x_i) \vee d_{\mathcal{O}X}$. Hence $\alpha(\bigvee_{i \in I} x_i) = (\bigvee_{i \in I} x_i \oplus \bigvee_{i \in I} x_i) \vee d_{\mathcal{O}X} = \bigvee_{i,j \in I}(x_i \oplus x_j) \vee d_{\mathcal{O}X} = \bigvee_{i \in I}(x_i \oplus x_i) \vee d_{\mathcal{O}X} = \bigvee_{i \in I} \alpha(x_i)$. Trivially, α preserves finite meets and hence $\alpha : \mathcal{O}X \to \uparrow d_{\mathcal{O}X}$ is a frame homomorphism. Since

$$\check{d}_{\mathcal{O}X}(a \oplus b) = (a \oplus b) \vee d_{\mathcal{O}X} = ((a \wedge b) \oplus (a \wedge b)) \vee d_{\mathcal{O}X} = \alpha(\nabla_{\mathcal{O}X}(a \oplus b)),$$

and since the $a \oplus b$ generate $\mathcal{O}X \oplus \mathcal{O}X$, we have $\check{d}_{\mathcal{O}X} = \alpha \cdot \nabla_{\mathcal{O}X}$. $\square$

Proposition. *Each regular locale X is Hausdorff.*

Proof. If $x \prec a$ and $y \prec b$ then

$$\begin{aligned} x \oplus y &= (x \wedge (y^c \vee b)) \oplus (y \wedge (x^c \vee a)) \\ &= ((x \wedge b) \vee (x \wedge y^c)) \oplus ((a \wedge y) \vee (x^c \wedge y)) \\ &\leq ((a \wedge b) \oplus (a \wedge b)) \vee (x \oplus x^c) \vee (y^c \oplus y) \\ &\leq ((a \wedge b) \oplus (a \wedge b)) \vee d_{\mathcal{O}X}. \end{aligned}$$

If X is regular we have

$$\begin{aligned} a \oplus b &= \bigvee\{x \in \mathcal{O}X \mid x \prec a\} \oplus \bigvee\{y \in \mathcal{O}X \mid y \prec b\} \\ &= \bigvee\{x \oplus y \mid x \prec a, y \prec b\} \\ &\leq ((a \wedge b) \oplus (a \wedge b)) \vee c \end{aligned}$$

and, by the Lemma, X is Hausdorff. $\square$

4.6. Some special properties of HausLoc. For the equalizers in HausLoc (and hence in RegLoc and CRegLoc) one has a very simple formula, obtained first by Banaschewski for the regular case in [8]. The Hausdorff case appeared in Chen's thesis [24].

Theorem. *Equalizers in* HausLoc *are closed sublocales. More precisely, if X is Hausdorff and $f_1, f_2 : Y \to X$ are localic maps, then the equalizer of f_1, f_2 is given by*

$$Y\text{-}Y_c \rightarrowtail Y,$$

where c stands for $\bigvee\{f_1^*(a) \wedge f_2^*(b) \mid a \wedge b = 0\}$.

Proof. Recall (3.4.1). Obviously, $c = \nabla_{\mathcal{O}X}((f_1^* \oplus f_2^*)(d_{\mathcal{O}X}))$. For the α from (4.5.1) we have

$$(x \oplus 1) \vee d_{\mathcal{O}X} = \alpha(\nabla_{\mathcal{O}X}(x \oplus 1)) = \alpha(x) = \alpha(\nabla_{\mathcal{O}X}(1 \oplus x)) = (1 \oplus x) \vee d_{\mathcal{O}X}.$$

Therefore

$$\begin{aligned} \check{c}(f_1^*(x)) &= f_1^*(x) \vee c \\ &= \nabla_{\mathcal{O}X}((f_1^* \oplus f_2^*)((x \oplus 1) \vee d_{\mathcal{O}X})) \\ &= \nabla_{\mathcal{O}X}((f_1^* \oplus f_2^*)((1 \oplus x) \vee d_{\mathcal{O}X})) \\ &= f_2^*(x) \vee c = \check{c}(f_2^*(x)). \end{aligned}$$

On the other hand, if $f_1 \cdot \varphi = f_2 \cdot \varphi = f$ for a $\varphi : Z \to Y$, we have

$$\varphi^*(c) = \bigvee\{f^*(x \wedge y) \mid x \wedge y = 0\} = 0$$

and hence we can define $\overline{\varphi} : Z \to Y\text{-}Y_c$ by $\overline{\varphi}^*(x) = \varphi^*(x)$ to obtain $Y\text{-}Y_c \cdot \overline{\varphi} = \varphi$. $\square$

Proposition. *In* HausLoc *(and in* RegLoc *and* CRegLoc*) each dense morphism is an epimorphism.*

Proof. Let f be dense and let $f_1 \cdot f = f_2 \cdot f$. Then, for the c from Theorem 4.6, we have $f^*(c) = 0$ and hence, by density, $c = 0$. Thus, $f_1^*(a) = f_1^*(a) \vee c = f_2^*(a) \vee c = f_2^*(a)$. $\square$

Remark. Conversely, each epimorphism in $\mathsf{HausLoc}$ is dense (the proof is easy but space consuming). Therefore epimorphisms in $\mathsf{HausLoc}$ are precisely the dense morphisms.

4.7. Some special properties of RegLoc**.** In regular locales, congruences are completely described by the congruence classe of the top element 1. Indeed:

Proposition. *Let X be regular and let C_1, C_2 be two congruences on $\mathcal{O}X$ such that the congruence classes $C_1[1], C_2[1]$ coincide. Then $C_1 = C_2$.*

Proof. Let $(a, b) \in C_1$ and let $x \prec a$. Then $x^c \vee a = 1$ and therefore $(x^c \vee b, 1) \in C_2$. Consequently, $(x \wedge b, x) = (x \wedge (x^c \vee a), x) \in C_2$ and $(a \wedge b, a) = ((\bigvee \sigma(a)) \wedge b, a) = (\bigvee\{x \wedge b \mid x \prec a\}, \bigvee\{x \mid x \prec a\}) \in C_2$. Similarly, $(a \wedge b, b) \in C_2$. This shows that $(a, b) \in C_2$. $\square$

A localic map $f : Y \to X$ is said to be *codense* if $f^*(a) = 1$ implies $a = 1$. From the proposition above we immediately obtain:

Corollary. *If X is regular then every codense $f : Y \to X$ is an epimorphism.*

It should be noted that the statement of the Proposition holds, more generally, for *fit* locales, that is, locales X in which

$$a \not\leq b \;\Rightarrow\; \exists c,\; a \vee c = 1 \text{ and } c \to b \neq b,$$

(in fact it characterizes fit locales) and the Corollary holds already for the subfit ones. The relation between fit and subfit is in the following fact:

A locale is fit iff each of its sublocales is subfit.

Exercises.

1. Show that each regular frame is fit.
2. Show that fitness is hereditary, that is, if X is fit and $j : Y \rightarrowtail X$ is a sublocale then Y is fit.
3. Prove that the following statements are equivalent for a locale X:
   (i) X is subfit;
   (ii) $C[1] = \{1\}$ for a congruence C implies that C is trivial;
   (iii) each open sublocale of X is a join of closed sublocales (more exactly, $X_a = \bigsqcup\{X\text{-}X_b \mid b \vee a = 1\}$).
4. Prove that the following statements are equivalent for a locale X:
   (i) X is fit;
   (ii) for any two congruences C_1 and C_2, $C_1[1] = C_2[1]$ implies $C_1 = C_2$;
   (iii) each sublocale of X is a meet of open sublocales (more exactly, $j = \bigsqcap\{X_a \mid j^*(a) = 1\}$);
   (iv) each sublocale of X is subfit.

4.8. First notes on compact locales. Compact locales will have a special section. Here we will just mention a few facts connected with regularity.

A *cover* of a locale X is a subset $A \subseteq \mathcal{O}X$ such that $\bigvee A = 1$, and a locale X is *compact* if each cover contains a finite subcover.

Proposition.

(1) *Each compact regular locale is normal (and hence, by Lemma 4.4, completely regular).*
(2) *Let $f : Y \to X$ be a dense localic map, with Y compact and X regular. Then f is an epimorphism.*
(3) *A compact sublocale of a regular locale is closed.*

Proof. (1) Let $a \vee b = 1$. Thus, $\sigma(a) \cup \sigma(b)$ is a cover and hence there are

$$x_1, \dots, x_n \prec a \text{ and } y_1, \dots, y_m \prec b$$

such that $\bigvee_{i=1}^n x_i \vee \bigvee_{i=1}^m y_i = 1$. Set $x = \bigvee_{i=1}^n x_i$ and $y = \bigvee_{i=1}^m y_i$. Then, by Lemma 4.2, $x^c \vee a = 1$ and $y^c \vee b = 1$. As $x \vee y = 1$ we have $x \vee b = 1 = a \vee y$. Set $u = x \wedge y^c$ and $v = x^c \wedge y$. Then $u \vee b = (x \vee b) \wedge (y^c \vee b) = 1$ and, similarly, $a \vee v = 1$. Trivially, $u \wedge v = 0$.
(2) By Corollary 4.7 it suffices to verify that f is codense. Suppose $f^*(a) = 1$. Consequently $\{f^*(x) \mid x \prec a\}$ is a cover of Y and hence there are $x_1, \dots, x_n \prec a$ such that $\bigvee_{i=1}^n f^*(x_i) = 1$. Set $x = \bigvee_{i=1}^n x_i$. By Lemma 4.2, $x \prec a$. Thus, we have $f^*(x) = 1$ and $x^c \vee a = 1$. Since $f^*(x^c) \leq f^*(x)^c = 1^c = 0$, $x^c = 0$ and finally $a = 1$.
(3) Let $f : Y \rightarrowtail X$ be a compact sublocale of X. For the

$$c = \bigvee\{x \in \mathcal{O}X \mid f^*(x) = 0\}$$

from the closure we have a dense sublocale $f : Y \rightarrowtail X_c$. By (2), f is an epimorphism and hence an isomorphism. □

More generally, one can prove the normality for the regular Lindelöf locales, by the standard procedure imitating the classical proof.

Exercise. Prove that a closed sublocale of a compact locale is compact.

5. Open and closed maps

5.1. Open maps of locales. The Heyting structure (recall Remark 1 of 1.1) will play a crucial role in this section.

A localic map $f : X \to Y$ is said to be *open* (resp. *closed*) if the image of each open (resp. closed) sublocale under f is open (resp. closed).

Proposition. *Let $f : X \to Y$ be a localic map. The following conditions are equivalent:*

(i) *f is open;*

(ii) *f^* is a complete Heyting homomorphism (that is, it preserves all suprema and infima as well as the Heyting operation);*

(iii) *f^* admits a left adjoint $f_!$ that satisfies the "Frobenius Identity"*

$$f_!(a \wedge f^*(b)) = f_!(a) \wedge b,$$

for every $a \in \mathcal{O}X, b \in \mathcal{O}Y$.

Proof. (i)⇒(ii): By the definition of image, f open means that, for each $a \in \mathcal{O}X$, there exists $f_!(a) \in \mathcal{O}Y$ such that

$$b \wedge f_!(a) = c \wedge f_!(a) \quad \text{iff} \quad f^*(b) \wedge a = f^*(c) \wedge a, \tag{5.1.1}$$

or, equivalently,

$$b \wedge f_!(a) \le c \quad \text{iff} \quad f^*(b) \wedge a \le f^*(c). \tag{5.1.2}$$

In particular, for $b = 1$, we obtain $f_!(a) \le c$ if and only if $a \le f^*(c)$. Thus, $f_!$ is a left adjoint of f^* and we see that f^* preserves all meets. Returning to (5.1.2) and using the Heyting formula we obtain

$$\begin{aligned} a \le f^*(b \to c) \quad &\Leftrightarrow \quad f_!(a) \le b \to c \\ &\Leftrightarrow \quad b \wedge f_!(a) \le c \\ &\Leftrightarrow \quad f^*(b) \wedge a \le f^*(c) \\ &\Leftrightarrow \quad a \le f^*(b) \to f^*(c). \end{aligned}$$

Thus, $f^*(b \to c) = f^*(b) \to f^*(c)$, that is, f^* is a Heyting homomorphism.
(ii)⇒(iii): If f^* is a complete Heyting homomorphism, it admits a left adjoint $f_!$. Obviously $f_!(a \wedge f^*(b)) \le f_!(a) \wedge b$ since $f_! \cdot f^* \le 1$. Moreover,

$$\begin{aligned} a \wedge f^*(b) \le f^*(f_!(a \wedge f^*(b))) \quad &\Leftrightarrow \quad a \le f^*(b) \to f^*(f_!(a \wedge f^*(b))) \\ &\Leftrightarrow \quad a \le f^*(b \to f_!(a \wedge f^*(b))) \\ &\Leftrightarrow \quad f_!(a) \le b \to f_!(a \wedge f^*(b)) \\ &\Leftrightarrow \quad b \wedge f_!(a) \le f_!(a \wedge f^*(b)). \end{aligned}$$

(iii)⇒(i): If $f_! \dashv f^*$ and $f_!$ satisfies Frobenius Identity we have

$$b \wedge f_!(a) \le c \;\Leftrightarrow\; f_!(a \wedge f^*(b)) \le c \;\Leftrightarrow\; a \wedge f^*(b) \le f^*(c),$$

so that (5.1.2) is satisfied. □

Remarks. (1) Recall from 4.7 that a congruence on a regular locale is determined by the congruence class of the top element. Thus, in the regular case, the formula (5.1.1) above is equivalent to

$$f_!(a) = c \wedge f_!(a) \quad \text{iff} \quad a = f^*(c) \wedge a,$$

that is, $f_!(a) \le c$ if and only if $a \le f^*(c)$. Hence

if Y is regular, the open localic maps $f : X \to Y$ are those for which f^* is a complete lattice homomorphism.

Note that, hence, each complete lattice homomorphism $h : L \to M$ between frames, with L regular, is automatically Heyting.
(2) If X and Y are spaces and $f : X \to Y$ is an open continuous map, then f is open also as a localic map. The converse is not generally true. It holds for the Y that are T_D; in fact, the coincidence of classical open and localic open maps characterizes property T_D (see [17, 51]).

Exercises.

1. Show that a sublocale $Y \rightarrowtail X$ is open if and only if it is open as a localic map.
2. Let Y be a fit locale. Prove that a localic map $f : X \to Y$ is open if and only if $f^* : \mathcal{O}Y \to \mathcal{O}X$ is a complete lattice homomorphism. (By Exercise 4 of 4.7, this generalizes Remark 1 above.)

5.2. Pullback stability. A straightforward application of the Frobenius Identity yields a characterization of surjections among open maps:

Lemma. *An open $f : X \to Y$ is a surjection if and only if $f_!(1) = 1$.*

Proof. Since f^* is one-one, $f_! \cdot f^* = 1_{\mathcal{O}Y}$. Hence $f_!(1) = f_!(f^*(1)) = 1$.
Conversely, if $f_!(1) = 1$ we have, by the Frobenius Identity, $a = f_!(1) \wedge a = f_!(f^*(a))$, which shows that f^* is one-one. ☐

We may now check pullback stability for open maps and open surjections.

Theorem. *Consider the pullback square*

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in Loc, *where f is open. Then:*

(1) *q is open;*
(2) *for each $a \in \mathcal{O}X$, $g^*(f_!(a)) = q_!(p^*(a))$;*
(3) *q is a surjection whenever f is a surjection.*

Proof. (1) Consider the diagram

$$\begin{tikzcd} \mathcal{O}Y \arrow[rr, "f^*"] \arrow[dd, "g^*"'] & & \mathcal{O}X \arrow[dl, "\iota_X"'] \arrow[dd, "p^*"] \\ & \mathcal{O}Z \oplus \mathcal{O}X \arrow[dr, "\gamma"] & \\ \mathcal{O}Z \arrow[ur, "\iota_Z"] \arrow[rr, "q^*"'] & & \mathcal{O}P \end{tikzcd}$$

in Frm, where the outer diagram is a pushout, constructed in the standard way from the coproduct and the coequalizer of $\iota_Z \cdot g^*$ and $\iota_X \cdot f^*$. By Proposition

5.1, if f is open, f^* has a left adjoint $f_!$ satisfying Frobenius Identity. Define $\varphi : \mathcal{O}Z \oplus \mathcal{O}X \to \mathcal{O}Z$ by $\varphi(z \oplus x) = z \wedge g^*(f_!(x))$ (recall 3.4). Then

$$\begin{aligned}
\varphi(\iota_Z(g^*(a)) \wedge (z \oplus x)) &= \varphi((g^*(a) \wedge z) \oplus x) \\
&= z \wedge g^*(a \wedge f_!(x)) \\
&= z \wedge g^*(f_!(f^*(a) \wedge x)) \\
&= \varphi(z \oplus (f^*(a) \wedge x)) \\
&= \varphi(\iota_X(f^*(a)) \wedge (z \oplus x)),
\end{aligned}$$

and since φ preserves joins, we have a $\overline{\varphi} : \mathcal{O}P \to \mathcal{O}Z$ such that $\overline{\varphi} \cdot \gamma = \varphi$ (recall Proposition 3.1). Then

$$\overline{\varphi}(q^*(z)) = \varphi(\iota_Z(z)) = \varphi(z \oplus 1) = z \wedge g^*(f_!(1)) \leq z$$

and

$$q^*(\overline{\varphi}(\gamma(z \oplus x))) = q^*(z \wedge g^*(f_!(x))) = q^*(z) \wedge p^*(f(f_!(x))) = \gamma(z \oplus f f_!(x)) \geq \gamma(z \oplus x).$$

Thus, $\overline{\varphi} = q_!$, the left Galois adjoint of q^*. Since we have

$$\begin{aligned}
q_!(q^*(a) \wedge \gamma(z \oplus x)) &= q_!(\gamma(\iota_Z(a) \wedge (z \oplus x))) \\
&= \varphi((a \wedge z) \oplus x) \\
&= a \wedge z \wedge g^*(f_!(x)) \\
&= a \wedge q_!(\gamma(z \oplus x)),
\end{aligned}$$

q is open.
(2) For $a \in \mathcal{O}X$ we obtain $q_!(p^*(a)) = \varphi(1 \oplus a) = g^*(f_!(a))$.
(3) If f is a surjection we have $f_!(1) = 1$ by the Lemma. Then, by (2), $q_!(1) = g^*(1) = 1$ and hence q is surjective as well. □

Remark. As is well known, surjections are not stable under pullback in Loc, as the following example [47] shows. Let $\overline{\mathbb{N}}$ with topology

$$\mathcal{O}\overline{\mathbb{N}} = \{U \subseteq \overline{\mathbb{N}} \mid U = \emptyset \text{ or } \overline{\mathbb{N}} \setminus U \text{ is a finite subset of } \mathbb{N}\}$$

and let $\mathbb{N}$ with the discrete topology $\mathcal{P}\mathbb{N}$. The frame homomorphism $i^{-1} : \mathcal{O}\overline{\mathbb{N}} \to \mathcal{P}\mathbb{N}$ given by the continuous map $i : \mathbb{N} \rightarrowtail \overline{\mathbb{N}}$ is a monomorphism but the pushout of i^{-1} along the monomorphism $\mathcal{O}\overline{\mathbb{N}} \rightarrowtail \mathcal{P}\overline{\mathbb{N}}$ is

$$\begin{tikzcd}
\mathcal{O}\overline{\mathbb{N}} \arrow[r] \arrow[d, "i^{-1}"'] & \mathcal{P}\overline{\mathbb{N}} \arrow[d, "i^{-1}"] \\
\mathcal{P}\mathbb{N} \arrow[r, "1"] & \mathcal{P}\mathbb{N}
\end{tikzcd}$$

and $i^{-1} : \mathcal{P}\overline{\mathbb{N}} \to \mathcal{P}\mathbb{N}$ is not a monomorphism.

In [8] Banaschewski characterizes the locales Y for which pullback along every $g : Z \to Y$ preserves surjections: precisely the ones such that $\mathcal{S}(Y)$ is Boolean.

Corollary. *Let $f : X \to Y$ be an open localic map. For every sublocale $j : Y' \rightarrowtail Y$,*

$$f^{-1}[\overline{j}] = \overline{f^{-1}[j]}. \tag{5.2.1}$$

Proof. We know, by 2.9, that $\overline{j}$ is the sublocale $Y\text{-}Y_{\mathsf{c}_j} \rightarrowtail Y$ and $\overline{f^{-1}[j]}$ is the sublocale $X\text{-}X_{\mathsf{c}_{f^{-1}(j)}} \rightarrowtail X$, where

$$\mathsf{c}(j) = \bigvee\{b \in \mathcal{O}Y \mid j^*(b) = 0\}$$

and

$$\mathsf{c}_{f^{-1}(j)} = \bigvee\{a \in \mathcal{O}X \mid (f^{-1}(j))^*(a) = 0\}.$$

Moreover, by Proposition 3.7, $f^{-1}(Y\text{-}Y_{\mathsf{c}_j}) = X\text{-}X_{f^*(\mathsf{c}(j))}$. Therefore it suffices to verify that $f^*(\mathsf{c}(j)) = \mathsf{c}_{f^{-1}(j)}$, that is,

$$\bigvee\{f^*(b) \mid j^*(b) = 0\} = \bigvee\{a \mid (f^{-1}(j))^*(a) = 0\}.$$

The inequality $\bigvee\{f^*(b) \mid j^*(b) = 0\} \le \bigvee\{a \mid (f^{-1}(j))^*(a)\} = 0$ is obvious. Conversely, for each a such that $(f^{-1}(j))^*(a) = 0$ take $b = f_!(a)$. Then $a \le f^*(b)$ and, by condition (2) of the Theorem, $j^*(b) = 0$. $\square$

The converse is not true [37], as the following example due to P. Johnstone shows. Take, for f, the dense embedding $\beta_X : \mathcal{B}X \rightarrowtail X$ from 2.13. Then the condition is trivially satisfied since, in $\mathcal{B}X$, every sublocale is closed, by Proposition 2.11. The localic map β_X is not open, though.

This contrasts with classical topology, where the formula

$$f^{-1}(\overline{A}) = \overline{f^{-1}(A)}, \text{ for any } A \subseteq Y$$

is equivalent to f being open. However, if a localic map f stably has the property (5.2.1) above (that is, if all pullbacks of f satisfy it), then f is necessarily open and we do get a characterization of localic openness (see [37] for a proof; see also III.7.3).

Exercise. For X the unit interval, find an example of a system $\{U_i \in \mathcal{O}X \mid i \in I\}$ such that $\bigwedge_{i\in I} U_i = \emptyset$ and $\bigwedge_{i\in I} \beta_X(U_i) = I$.

5.3. Closed and proper maps of locales. We end this section with a characterization of closed maps via a "co-Frobenius identity".

Proposition. *A localic map $f : X \to Y$ is closed if and only if f_* satisfies*

$$f_*(a \vee f^*(b)) = f_*(a) \vee b.$$

Proof. Similarly like in Proposition 5.1 we see that the closedness amounts to the existence of a map φ such that

$$a \vee f^*(b) = a \vee f^*(c) \quad \text{iff} \quad \varphi(a) \vee b = \varphi(a) \vee c,$$

which is easily seen to be equivalent to

$$f^*(c) \le a \vee f^*(b) \quad \text{iff} \quad c \le \varphi(a) \vee b.$$

Setting $b = 0$ we see that $\varphi = f_*$. Further, the first inequality is equivalent to $c \le f_*(a \vee f^*(b))$ so that we finally transform the condition into the form

$$c \le f_*(a \vee f^*(b)) \quad \text{iff} \quad c \le f_*(a) \vee b,$$

yielding the desired equation. □

Exercise 1 of 5.1 shows that open maps generalize open sublocales and Theorem 5.2 asserts that open maps and open surjections are stable under pullback. In order to have similar results for closed maps one has to restrict the class of closed maps: a localic map $f : X \to Y$ is said to be *proper* if it is closed and f_* preserves directed joins (see [63] and [64] for some alternative descriptions).

Exercise. Observe that proper maps generalize closed sublocales, by proving that a sublocale $Y \rightarrowtail X$ is closed if and only if it is proper as a localic map.

The classes of proper maps and proper surjections are stable under pullback (see [63] for a proof). In particular, if this property is weakened to mention only pullbacks along product projections $Z \times Y \to Y$ one concludes that, for proper $f : X \to Y$, $1_Z \times f : Z \times X \to Z \times Y$ is closed for all locales Z, which corresponds to one of the standard definitions of properness for continuous maps between topological spaces [21]. The converse is also true [64] and so the condition

$$1_Z \times f : Z \times X \to Z \times Y \ \textit{is closed for all locales } Z$$

characterizes the properness property of a map $f : X \to Y$ of locales (similarly to topological spaces).

For more information on open, closed and proper maps consult [38, 60, 63]. In [60] the results about open and proper maps are proved side by side with "parallel proofs for parallel results", showing the similarities between the two classes. For instance the proof that proper maps are stable under pullback is really just a repetition of the proof that open maps are stable under pullback but with "has a left adjoint which is a sup-lattice homomorphism" being replaced with "has a right adjoint which is a preframe homomorphism".

6. Compact locales and compactifications

6.1. Some machinery. The converse of Proposition 4.5 is valid for compact locales and it was first proved constructively by Vermeulen [62]. In order to present it we need a few results of a technical nature, involving binary products of locales.

Let X_1 and X_2 be locales. The maps $\pi_1, \pi_2 : \mathcal{D}(\mathcal{O}X_1 \times \mathcal{O}X_2) \to \mathcal{D}(\mathcal{O}X_1 \times \mathcal{O}X_2)$ defined by

$$\pi_1(U) = \{(\bigvee S, y) \mid S \times \{y\} \subseteq U\}$$

and

$$\pi_2(U) = \{(x, \bigvee S) \mid \{x\} \times S \subseteq U\}$$

are nuclei on the frame $\mathcal{D}(\mathcal{O}X_1 \times \mathcal{O}X_2)$. The map

$$\begin{array}{rccc} \pi_0 : & \mathcal{D}(\mathcal{O}X_1 \times \mathcal{O}X_2) & \rightarrow & \mathcal{D}(\mathcal{O}X_1 \times \mathcal{O}X_2) \\ & U & \mapsto & \pi_1(U) \cup \pi_2(U) \end{array}$$

is a *prenucleus*, that is, for all $U, V \in \mathcal{D}(\mathcal{O}X_1 \times \mathcal{O}X_2)$, $U \subseteq \pi_0(U)$, $\pi_0(U) \cap V \subseteq \pi_0(U \cap V)$ and $\pi_0(U) \subseteq \pi_0(V)$ whenever $U \subseteq V$. But for each prenucleus π_0 there is a unique nucleus π which has the same fixed points as π_0, which is given by

$$\pi(a) = \bigwedge \{b \mid a \leq b, \pi_0(b) = b\} \text{ [6]}.$$

In this case, since

$$Fix(\pi_0) = \{U \in \mathcal{D}(\mathcal{O}X_1 \times \mathcal{O}X_2) \mid \pi_0(U) = U\} = \mathcal{O}X_1 \oplus \mathcal{O}X_2,$$

the associated nucleus π is given by

$$\pi(U) = \bigcap \{V \in \mathcal{O}X_1 \oplus \mathcal{O}X_2 \mid U \subseteq V\}.$$

Furthermore define, for any $U \in \mathcal{D}(\mathcal{O}X_1 \times \mathcal{O}X_2)$,

$$\sigma_0(U) = \{\bigvee D \mid \text{ directed } D \subseteq U\}.$$

This defines a prenucleus. Let σ denote the associated nucleus. Note that $\sigma \leq \pi$, since $\sigma_0(U) \subseteq \pi(U)$ for every U. Indeed, for every directed set $D = \{(c_i, d_i) \mid i \in I\} \subseteq U$, we have $(\bigvee_{i \in F} c_i, \bigvee_{i \in F} d_i) \in U$ for every finite $F \subseteq I$, which implies $(c_i, d_j) \in U$ for every $i, j \in I$. Consequently, $(c_i, \bigvee_{i \in I} d_i) \in \pi(U)$ and finally $\bigvee D = (\bigvee_{i \in I} c_i, \bigvee_{i \in I} d_i) \in \pi(U)$.

As a consequence of this inclusion we have:

Lemma. $\pi = \sigma \cdot \pi_2 \cdot \pi_1$. □

A constructive proof of this result was first provided by Banaschewski [10].

6.2. Compact elements. An element a of a complete lattice A is said to be *compact* (*finite* in [34]) if $a \leq \bigvee S$ for some $S \subseteq A$ implies $a \leq \bigvee F$ for some finite $F \subseteq S$. Clearly a locale X is compact if and only if 1 is a compact element of $\mathcal{O}X$.

Exercises.

1. Let A be a complete lattice, $a \in A$. Prove that a is compact if and only if for every directed subset $D \subseteq A$ with $a \leq \bigvee D$, there exists $d \in D$ with $a \leq d$.
2. Let A be a frame, $a \in A$. Prove that a is compact if and only if for every $S \subseteq A$ with $a = \bigvee S$, there exists a finite $F \subseteq S$ with $\bigvee F = a$.

Lemma. *Let $U \in \mathcal{D}(\mathcal{O}X_1 \times \mathcal{O}X_2)$ and let $a \in \mathcal{O}X_1$ be a compact element. If $a \oplus b \leq \pi(U)$ then $(a, b) \in \pi_2(\pi_1(U))$.*

Proof. Let $S = \pi_2(\pi_1(U)) \in \mathcal{D}(\mathcal{O}X_1 \times \mathcal{O}X_2)$ and

$$\mathcal{W} = \Big\{V \in \mathcal{D}(\mathcal{O}X_1 \times \mathcal{O}X_2) \mid S \subseteq V \subseteq \pi(S), (a, b) \in V \Rightarrow (a, b) \in S\Big\}.$$

Then:

- $S \in \mathcal{W}$;

- $\mathcal{W}$ is σ_0-stable, that is, $V \in \mathcal{W}$ implies $\sigma_0(V) \in \mathcal{W}$. Indeed, take $V \in \mathcal{W}$ and consider $(a,b) = \bigvee D$, for directed $D \subseteq V$. Since a is compact there exists $(c_0,d_0) \in D$ such that $a \le c_0$. For any $(c,d) \in D$ with $(c_0,d_0) \le (c,d)$ we have $(a,d) \in V$, thus $(a,d) \in S$. Since $b = \bigvee\{d \mid (c,d) \in D, (c_0,d_0) \le (c,d)\}$, we get $(a,b) \in \pi_2(S) = S$;
- trivially, $W := \bigcup \mathcal{W} \in \mathcal{W}$.

Thus $\sigma_0(W) \in \mathcal{W}$ and $\sigma_0(W) = W$, which implies $\sigma(W) = W \in \mathcal{W}$. Hence $\sigma(S) \subseteq \sigma(W) = W$ and $\sigma(S) \in \mathcal{W}$. On the other hand, by Lemma 6.1, $\pi(U) \subseteq \sigma(S)$ so $\pi(U) \in \mathcal{W}$. □

6.3. A technical lemma. The following notation will be convenient in the sequel. For locales X_1, X_2 and $I \in \mathcal{O}X_1 \oplus \mathcal{O}X_2$ let

$$I_1[b] = \bigvee\{a \in \mathcal{O}X_1 \mid (a,b) \in I\} \qquad (b \in \mathcal{O}X_2)$$

and

$$I_2[a] = \bigvee\{b \in \mathcal{O}X_2 \mid (a,b) \in I\} \qquad (a \in \mathcal{O}X_1).$$

Note that $(I_1[b], b) \in I$ and $(a, I_2[a]) \in I$ for every $a \in \mathcal{O}X_1$ and $b \in \mathcal{O}X_2$.

Lemma. *Consider locales X_1, X_2 and the projections $p_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$. If X_1 is compact then, for every $a \in \mathcal{O}X_1$ and $I \in \mathcal{O}X_1 \oplus \mathcal{O}X_2$, we have*

$$p_{2_*}(p_1^*(a) \vee I) = \bigvee\{b \in \mathcal{O}X_2 \mid a \vee I_1[b] = 1\}.$$

Proof. Let $b \in \mathcal{O}X_2$ with $a \vee I_1[b] = 1$. In order to show that $b \le p_{2_*}(p_1^*(a) \vee I)$ it suffices to check that $(1,b) \in p_1^*(a) \vee I$, which is true because $(a,b) \in p_1^*(a)$ and $(I_1[b], b) \in I$. So

$$p_{2_*}(p_1^*(a) \vee I) \ge \bigvee\{b \in \mathcal{O}X_2 \mid a \vee I_1[b] = 1\}.$$

Now, for $U = p_1^*(a) \cup I \in \mathcal{D}(\mathcal{O}X_1 \times \mathcal{O}X_2)$ and $u = p_{2_*}(p_1^*(a) \vee I)$, let us prove that $u \le \bigvee\{b \mid a \vee I_1[b] = 1\}$. Since $p_2^*(u) \le p_1^*(a) \vee I$, we have that $(1,u) \in p_1^*(a) \vee I = \pi(U)$. Hence, by Lemma 6.2, $(1,u) \in \pi_2(\pi_1(U))$. But

$$\pi_2(\pi_1(U)) = \Big\{(x,y) \mid y \le \bigvee\{b \mid x \le a \vee I_1[b]\}\Big\}. \tag{6.3.1}$$

Indeed, if $(x,y) \in \pi_2(\pi_1(U))$ then $y = \bigvee S$ with $(x,s) \in \pi_1(U)$ for every $s \in S$. Then $(x,s) = (\bigvee R_s, s)$ where $R_s \times \{s\} \subseteq U$, from which it follows that, for each $s \in S$ and $r \in R_s$, $r \le a \vee I_1[s]$. Therefore $x = \bigvee R_s \le a \vee I_1[s]$ for every $s \in S$, which means that $\bigvee\{b \mid x \le a \vee I_1[b]\} \ge \bigvee S = y$. On the other hand, let (x,y) be such that $y \le \bigvee\{b \mid x \le a \vee I_1[b]\}$. The conclusion that $(x,y) \in \pi_2(\pi_1(U))$ follows immediately from the fact that, for each such b, $(a,b) \in p_1^*(a)$ and $(I_1[b], b) \in I$ so $(a \vee I_1[b], b) \in \pi_1(U)$ and then $(x,b) \in \pi_1(U)$.

Finally, it follows from (6.3.1) that $u \le \bigvee\{b \mid a \vee I_1[b] = 1\}$. □

6.4. Hausdorffness and regularity. We may now prove, at last, the converse of Proposition 4.5 for compact locales.

Theorem. *Each compact Hausdorff locale is regular.*

Proof. Applying Lemma 6.3 in the case $X = X_1 = X_2$ and $I = d_{\mathcal{O}X} = \bigvee\{a \oplus b \mid a \wedge b = 0\}$ we get

$$p_{2_*}(p_1^*(a) \vee d_{\mathcal{O}X}) = \bigvee\{b \in \mathcal{O}X \mid a \vee b^c = 1\} = \bigvee\{b \in \mathcal{O}X \mid b \prec a\}.$$

Since X is Hausdorff, we may conclude, from Theorem 4.6, that the sublocale

$$X \times X\text{-}(X \times X)_{d_{\mathcal{O}X}} \rightarrowtail X$$

is the equalizer of p_1 and p_2, since

$$\bigvee\{p_1^*(a) \wedge p_2^*(b) \mid a \wedge b = 0\} = \bigvee\{a \oplus b \mid a \wedge b = 0\} = d_{\mathcal{O}X}.$$

Therefore $p_{2_*}(p_1^*(a) \vee d_{\mathcal{O}X}) = p_{2_*}(p_2^*(a) \vee d_{\mathcal{O}X}) \geq p_{2_*}(p_2^*(a)) \geq a$, which shows that $\bigvee\{b \in \mathcal{O}X \mid b \prec a\} = a$ for every $a \in \mathcal{O}X$. $\square$

6.5. The Kuratowski-Mrówka Theorem for locales. The Kuratowski-Mrówka Theorem characterizes compact spaces K by the fact that for each X the projection $K \times X \to X$ is closed (see, for example, [29]). Its counterpart for locales is also valid and was first obtained by Pultr and Tozzi [50]. By applying the results on binary coproducts of Sections 6.1 and 6.2 we can present a constructive proof [25].

Theorem. *A locale K is compact if and only if $p_2 : K \times X \to X$ is closed for any locale X.*

Proof. Let K be a compact locale. By Proposition 5.3, p_2 is closed if and only if it satisfies the co-Frobenius Identity

$$p_{2_*}(I \vee p_2^*(x)) = p_{2_*}(I) \vee x \tag{6.5.1}$$

for all $I \in \mathcal{O}(K \times X)$ and $x \in \mathcal{O}X$. Since

$$p_{2_*}(I) = \bigvee\{a \in \mathcal{O}X \mid p_2^*(a) \leq I\} = \bigvee\{a \in \mathcal{O}X \mid (1, a) \in I\},$$

(6.5.1) holds if and only if $(I \vee p_2^*(x))_2[1] = I_2[1] \vee x$ or, equivalently, $(I \vee p_2^*(x))_2[1] \leq I_2[1] \vee x$, that is,

$$(1, y) \in I \vee p_2^*(x) \Rightarrow y \leq I_2[1] \vee x.$$

Let $U = I \cup p_2^*(x) \in \mathcal{D}(\mathcal{O}K \times \mathcal{O}X)$. Then $\pi_2(U) = \{(a, b) \mid b \leq x \vee I_2[a]\}$, and $(1, y) \in \pi_2(U)$ if and only if $y \leq x \vee I_2[1]$. Thus (6.5.1) holds if and only if

$$(1, y) \in \pi(U) \Rightarrow (1, y) \in \pi_2(U).$$

This is true by Lemma 6.1 and the fact that $\pi_1(U) = U$.

Conversely, suppose U is a directed cover of $\mathcal{O}K$. We shall prove that $1 \in U$.

In the set $\mathcal{O}K$ define

$$\mathcal{T}(\mathcal{O}K) = \Big\{S \subseteq \mathcal{O}K \mid 1 \in S \Rightarrow \uparrow u \subseteq S \text{ for some } u \in U\Big\}.$$

This is a topology on $\mathcal{O}K$. Let X be the corresponding locale. By hypothesis, $p_2 : K \times X \to X$ is closed, that is, $p_{2_*}((1 \oplus a) \vee I) = a \vee p_{2_*}(I)$ for all $a \in \mathcal{O}X$, $I \in \mathcal{O}(K \times X)$. Consider $a = K \setminus \{1\} \in \mathcal{O}X$ and $I = \bigvee\{u \oplus \uparrow u \mid u \in U\}$. Then

$$\begin{aligned}(1 \oplus a) \vee I &= \bigvee\{u \oplus a \mid u \in U\} \vee \bigvee\{u \oplus \uparrow u \mid u \in U\} \\ &= \bigvee\{u \oplus (a \cup \uparrow u) \mid u \in U\} \\ &= \{u \oplus 1_{\mathcal{O}X} \mid u \in U\} \\ &= 1_{\mathcal{O}(K \times X)}.\end{aligned}$$

Hence $a \vee p_{2_*}(I) = p_{2_*}((1 \oplus a) \vee I) = p_{2_*}(1) = 1_{\mathcal{O}X} = \mathcal{O}K$, which implies $1 \in p_{2_*}(I)$. By definition of $\mathcal{T}(\mathcal{O}K)$ this means that $\uparrow v \subseteq p_{2_*}(I)$ for some $v \in U$, that is, $p_{2_*}(\uparrow v) \subseteq I$. Then $(1 \oplus \uparrow v) \wedge (1 \oplus \downarrow v) \leq I \wedge (1 \oplus \downarrow v)$ is equivalent to $1 \oplus \{v\} \leq \bigvee\{u \oplus [u, v] \mid u \in U\}$, where $[u, v] = \{k \in \mathcal{O}K \mid u \leq k \leq v\}$. For $W = \{\bigvee V \mid V \subseteq U\}$ let

$$S = \downarrow\{(w, [w, v]) \mid w \in W\} \in \mathcal{D}(\mathcal{O}K \times \mathcal{O}X).$$

It can be easily checked that $\pi_1(S) = S$ and $1 \oplus \{v\} \subseteq \pi(S)$. Since $\{v\}$ is an atom of $\mathcal{O}X$, it is a compact element, and we may apply Lemma 6.2 to conclude that $(1, \{v\}) \in \pi_1(\pi_2(S))$. Again by $\{v\}$ being an atom of $\mathcal{O}X$, this implies that $(1, \{v\}) \in \pi_1(S) = S$. Finally, $(1, \{v\}) \in S$ means that $1 = w$ and $\{v\} = [w, v]$ for some $w \in W$, that is, $1 = w = v \in U$, as required. □

6.6. Regular ideals. In the sequel we will present an easy construction of compact locales starting from general ones. This will give us the compactification of completely regular locales due to Banaschewski and Mulvey [12].

An *ideal* in a frame A is a non-void subset $J \subseteq A$ such that

(1) $a, b \in J \;\Rightarrow\; a \vee b \in J$, and
(2) $a \in J \;\&\; b \leq a \;\Rightarrow\; b \in J$.

It is said to be *regular* if, moreover

(3) for each $a \in J$ there is a $b \in J$ such that $a \prec\!\prec b$.

Examples. The sets $\downarrow a$, or the $\sigma(a)$ from 4.3, are ideals; $\rho(a)$ is a regular ideal.

The collection of all ideals (resp. all regular ideals) in A will be denoted by

$$\mathfrak{J}A \quad (\text{resp. } \mathfrak{R}A).$$

Proposition. *$\mathfrak{J}A$ and $\mathfrak{R}A$ are compact frames with bottom $\{0\}$, top A, intersection for meet and the join defined by*

$$\bigvee_{i \in I} J_i = \Big\{\bigvee F \mid F \text{ finite},\; F \subseteq \bigcup_{i \in I} J_i\Big\}.$$

Proof. $\{0\}$ and A are regular ideals and a finite intersection of a system of (regular) ideals is obviously a (regular) ideal. Also, obviously, $\{\bigvee F \mid F \text{ finite},\; F \subseteq \bigcup_{i \in I} J_i\}$ is an ideal containing all J_i, and if K is an ideal containing all J_i, it has to contain all finite joins of the elements of $\bigcup_{i\in I} J_i$. Thus, $\{\bigvee F \mid F$ finite, $F \subseteq \bigcup_{i\in I} J_i\}$ is the supremum of the system J_i in $\mathfrak{J}A$. If all the J_i are regular and $\{a_1, \ldots, a_n\} \subseteq \bigcup_{i\in I} J_i$ choose b_i, $a_i \prec\!\prec b_i$ in $\bigcup_{i\in I} J_i$; then $a_1 \vee \cdots \vee a_n \prec\!\prec b_1 \vee \cdots \vee b_i$ by Lemma 4.2 and we see that $\bigvee_{i\in I} J_i$ is regular as well. Trivially, $(\bigvee_{i\in I} J_i) \cap K \supseteq \bigvee_{i\in I}(J_i \cap K)$. If $a \in (\bigvee_{i\in I} J_i) \cap K$ we have $a = a_1 \vee \cdots \vee a_n \in K$, $a_j \in \bigcup_{i\in I} J_i$. Since K is an ideal, all the a_i are in K and hence $\{a_1, \ldots, a_n\} \subseteq \bigcup_{i\in I}(J_i \cap K)$ and $a \in \bigvee_{i\in I}(J_i \cap K)$. Thus, $\mathfrak{J}A$ is a frame and $\mathfrak{R}A$ one of its subframes.

Finally, let $\{J_i \mid i \in I\}$ be a cover of $\mathfrak{J}A$. Then $\bigvee_{i\in I} J_i = A \ni 1$ and hence there are $a_1, \ldots, a_n \in \bigcup_{i\in I} J_i$ such that $a_1 \vee \cdots \vee a_n = 1$. Choose J_{i_j} containing a_j. Then $\bigvee_{j=1}^{n} J_{i_j} \ni 1$ and hence $\bigvee_{j=1}^{n} J_{i_j} = A$ by the down-closedness condition (2) of ideals. $\square$

6.7. The Stone-Čech compactification of locales. The constructions $\mathfrak{J}A$ and $\mathfrak{R}A$ can be extended to functors by setting

$$\mathfrak{J}h(J) \text{ (resp. } \mathfrak{R}(h)(J)) = \downarrow h[J].$$

(Checking that $\downarrow h[J]$ is an ideal — a regular one if J is regular — is straightforward and so is the preserving of 0, 1 and joins. Also preserving the meets is an easy exercise.)

Proposition. *If A is completely regular then $\mathfrak{R}A$ is regular. Thus, $\mathfrak{R}$ can be viewed as a functor from* $\mathsf{CRegLoc}$ *to the category* $\mathsf{KRegLoc}$ *of compact regular locales.*

Proof. For a regular ideal J we obviously have $J = \bigvee\{\rho(a) \mid a \in J\}$ $(= \bigcup\{\rho(a) \mid a \in J\})$, and for each a, $\rho(a) = \bigvee\{\rho(b) \mid b \prec\!\prec a\}$. Thus, it suffices to prove that

$$b \prec\!\prec a \text{ in } A \quad\Rightarrow\quad \rho(b) \prec \rho(a) \text{ in } \mathfrak{R}A.$$

Interpolate $b \prec\!\prec x \prec\!\prec y \prec\!\prec a$. Obviously, $\rho(b^c) \cap \rho(b) = \{0\}$ and consequently $\rho(b^c) \subseteq \rho(b)^c$. Then, by 4.2, $x^c \subseteq \rho(b)^c$. Thus, $1 = x^c \vee y \in \rho(b)^c \vee \rho(a)$ and hence $\rho(b)^c \vee \rho(a) = J$. $\square$

For a completely regular locale X define $v_X : X \to \mathfrak{R}X$ by setting $v_X^*(J) = \bigvee J$.

Lemma. *v_X is a dense sublocale.*

Proof. We obviously have

$$v^*(\rho(a)) = a \quad \text{and} \quad \rho(v^*(J)) \supseteq J. \tag{6.7.1}$$

Thus, v^* is a left adjoint and preserves all joins. Furthermore, $v^*(\mathcal{O}X) = 1$ and

$$\begin{aligned} v^*(J) \wedge v^*(K) &= \bigvee J \wedge \bigvee K = \bigvee\{a \wedge b \mid a \in J, b \in K\} \leq \\ &\leq \bigvee\{c \mid c \in J \cap K\} = v^*(J \cap K) \leq v^*(J) \wedge v^*(K), \end{aligned}$$

the last inequality being trivial. Obviously v^* is onto, and if $v^*(J) = \bigvee J \neq 0$ then $J \neq \{0\}$. $\square$

The parallelism with the classical situation is now apparent; we have a reflection of $\mathsf{CRegLoc}$ onto $\mathsf{KRegLoc}$, called by obvious reasons the *Stone-Čech compactification* for locales [12]:

Theorem. *The functor $\mathfrak{R}$ and the system of mappings v_X constitute a reflection of* $\mathsf{CRegLoc}$ *onto* $\mathsf{KRegLoc}$.

Proof. Checking that, for a localic map $f : X \to Y$, $\mathfrak{R}(f) \cdot v_X = v_Y \cdot f$ is immediate. Thus, it remains to be proved that if X is compact, v_X is an isomorphism. According to formulas (6.7.1), it suffices to prove that $\rho(v^*(J)) \subseteq J$. Let $a \in \rho(\bigvee J)$. Then $a^c \vee \bigvee J = 1$ and if X is compact there are $x_1, \dots, x_n \in J$ such that $a^c \vee x_1 \vee \cdots \vee x_n = 1$. Then $x = x_1 \vee \cdots \vee x_n \in J$ and $a \leq x$ so that $a \in J$. $\square$

Remarks. (1) More generally, a *compactification* of a locale X is a dense extremal monomorphism $f : X \to Y$ with compact regular codomain. A locale which has a compactification is called *compactifiable*. For a comprehensive view of compactifications of locales consult [9].
(2) Note that the entire procedure was constructive (no choice principle or the law of the excluded middle was used). Banaschewski and Mulvey presented this construction in [12]. In [13] they presented an alternative construction. Realizing that the fact that a reflective subcategory is closed under limits can be also proved constructively, we can conclude that products of compact regular locales (by Proposition 4.5 and Theorem 6.4 this is the same as compact Hausdorff locales) are compact and this fact does not need non-constructive principles. This is even true in the non-regular case (where a reflection here does not exist), that is, Tychonoff's Theorem is choice-free for locales (the proof is much more difficult, though — see [33] or [6]). This came as a remarkable surprise when Johnstone [33] was able to prove it within Zermelo-Fraenkel axiomatic without choice ($\mathbb{ZF}$). Indeed, this contrasts with the classical case where Tychonoff's Theorem is equivalent to the axiom of choice (in the Hausdorff setting, with the Boolean Ultrafilter Theorem). It turns out that in fact the non-constructive principle is needed for products having enough points rather than for preserving compactness (see III.11.4).

Later refinements of this important result were given by Kříž [39], who uses fewer axioms of $\mathbb{ZF}$ (namely, Kříž's proof does not depend on the non-constructive axiom of replacement as Johnstone's did), and by Vermeulen [61], whose proof is constructively valid in the sense of topos theory (meaning: valid in an arbitrary topos).

7. Locally compact locales

7.1. The "way below" relation and continuous lattices. Recall that in a complete lattice A, a is *well below* b, written

$$a \ll b,$$

if for each directed $D \subseteq A$

$$b \leq \bigvee D \quad \Rightarrow \quad \exists d \in D,\ a \leq d.$$

By Exercise 1 of 6.2, an element a of a complete lattice is compact precisely when $a \ll a$.

A *continuous lattice* [53] is a complete lattice A such that, for every $a \in A$,

$$a = \bigvee \{b \in A \mid b \ll a\}.$$

Lemma. *The relation $\ll$ satisfies the following properties:*

(1) *$0 \ll a$ for all a;*
(2) *if $x \leq a \ll b \leq y$ then $x \ll y$;*
(3) *if $a_1, a_2 \ll b$ then $a_1 \vee a_2 \ll b$. Thus, the set $\{x \mid x \ll a\}$ is always directed;*
(4) *in any frame, $a \prec b \ll 1$ implies $a \ll b$;*
(5) *in any regular (resp. completely regular) frame,*

$$a \ll b \ \Rightarrow \ a \prec b \ (\textit{resp. } a \ll b \ \Rightarrow \ a \prec\!\!\prec b);$$

(6) *in any continuous lattice, the relation $\ll$ interpolates.*

Proof. (1), (2), and (3) are immediate.
(4) Let $b \leq \bigvee D$ for a directed D. Then $1 \leq a^c \vee \bigvee D$ and hence there is a $d \in D$ such that $b \leq a^c \vee d$. Then $a = a \wedge b \leq a \wedge d$, that is, $a \leq d$.
(5) For regular frames we have $b = \bigvee\{x \mid x \prec b\}$ and since the join is directed, there is an $x \prec b$ such that $a \leq b$. For completely regular frames the proof is analogous.
(6) We have $b = \bigvee\{\bigvee\{y \mid y \ll x\} \mid x \ll b\} = \bigvee\{y \mid \exists x,\ y \ll x \ll b\}$ and the join is obviously directed. Thus, if $a \ll b$ there are x, y such that $a \leq y \ll x \ll b$. □

Exercise. Show that, for open subsets U, V of a space X, if there is a compact K such that $U \subseteq K \subseteq V$, then $U \ll V$. Conclude that, for every locally compact space X, the frame $\mathcal{O}X$ is continuous.

7.2. Locally compact locales. Exercise 7.1 above suggests that the continuity may be a good description of local compactness in the localic setting: a locale X is said to be *locally compact* if the frame $\mathcal{O}X$ is a continuous lattice. In fact we will see that it is even better than that (see Theorem 7.4 below).

A general compact locale is not necessarily locally compact. But we have:

Proposition.

(1) *A compact Hausdorff locale is locally compact. More generally, any open sublocale of a compact Hausdorff locale is locally compact. Moreover, in such a case the relations $\ll$, $\prec$ and $\prec\!\!\prec$ coincide.*
(2) *A completely regular locale X is locally compact if and only if it is open in its Stone-Čech compactification $v_X : X \to \mathfrak{R}X$.*

Proof. (1) Let X be a compact Hausdorff locale. Compactness means $1 \ll 1$ and hence, by Lemma 7.1(4), $a \prec b$ implies $a \ll b$ in $\mathcal{O}X$. By Theorem 6.4, X is regular (and hence, by 4.8, completely regular). Therefore, by Lemma 7.1(5), $a \ll b$ implies $a \prec\!\prec b$.
(2) By (1) it suffices to prove that, in case X is locally compact, there is a regular ideal J in $\mathcal{O}X$ such that the congruence associated with v_X^* is the open Δ_J, that is, for any regular ideals J_1 and J_2,

$$v_X(J_1) = v_X(J_2) \text{ if and only if } J_1 \cap J = J_2 \cap J.$$

Consider $J = \{x \in \mathcal{O}X \mid x \ll 1\}$. By Lemma 7.1 it is a regular ideal. By continuity, $v_X^*(J) = \bigvee J = 1$ and hence $J_1 \cap J = J_2 \cap J$ implies $v_X^*(J_1) = v_X^*(J_2)$. On the other hand, let $\bigvee J_1 = \bigvee J_2$, let $a \in J_1 \cap J$ and choose $b \in J_1 \cap J$ such that $a \prec\!\prec b$ (and hence $a \ll b$). Since ideals are directed, we may conclude from $b \leq \bigvee J_2 = \bigvee J_1$ that there is an $x \in J_2$ with $a \leq x$, and hence $a \in J_2$. □

Furthermore, as for compact locales, the converse of Proposition 4.5 is valid for locally compact locales [62] and Hausdorff also means regular. In ([62], Proposition 4.7) it is also proved that, for Hausdorff locales, local compactness can be defined in terms of compact neighborhoods.

7.3. Scott topology. Let A be a lattice. A subset $U \subseteq A$ is said to be *Scott open* if $U = \uparrow U$ and, whenever $D \subseteq A$ is directed and $\bigvee D \in U$, $U \cap D \neq \emptyset$. Obviously the Scott open subsets constitute a topology (the so-called *Scott topology* on A).

Consider now a locale X and represent the spectrum $\mathsf{Pt}(X)$ by complete filters $P \subseteq \mathcal{O}X$ (as in 1.5). One can characterize compact subsets of $\mathsf{Pt}(X)$ in terms of Scott opens:

Lemma. *A subset $K \subseteq \mathsf{Pt}(X)$ is compact if and only if $\bigcap\{P \mid P \in K\}$ is Scott open.*

Proof. Let $\bigcap\{P \mid P \in K\}$ be Scott open and let $K \subseteq \bigcup\{\Sigma_a \mid a \in A\}$. Then $\bigvee A \in \bigcap\{P \mid P \in K\}$ since for each $P \in K$ there is an $a \in A$ such that $a \in P$, and hence $\bigvee A \in P$. Thus, there are $a_1, \ldots, a_n \in A$ with $a_1 \vee \cdots \vee a_n \in \bigcap\{P \mid P \in K\}$ and hence $K \subseteq \Sigma_{a_1 \vee \cdots \vee a_n} = \bigcup_{i=1}^n \Sigma_{a_i}$. If K is compact and $\bigvee A \in \bigcap\{P \mid P \in K\}$ then $K \subseteq \Sigma_{\vee A} = \bigcup\{\Sigma_a \mid a \in A\}$ and there are $a_1, \ldots, a_n \in A$ such that $K \subseteq \Sigma_{a_1 \vee \cdots \vee a_n} = \bigcup_{i=1}^n \Sigma_{a_i}$. Finally $a_1 \vee \cdots \vee a_n \in \bigcap\{P \mid P \in K\}$. □

7.4. The Hofmann-Lawson Duality. On the other hand, prime Scott open filters of $\mathcal{O}X$ give the points of the spectrum $\mathsf{Pt}(X)$:

Lemma. *The elements $P \in \mathsf{Pt}(X)$ are precisely the prime Scott open filters.*

Proof. A completely prime P is obviously Scott open. Now let P be Scott open and prime, and let $\bigvee_{i \in I} a_i \in P$. Since P is open there are $a_{i_1}, \ldots, a_{i_n}$ such that $a_{i_1} \vee \cdots \vee a_{i_n} \in P$. Since it is prime, $a_{i_j} \in P$ for some j. □

Proposition. *Let F be a Scott open filter of $\mathcal{O}X$ such that $a \in F$ and $b \notin F$. Then there is a complete filter $P \supseteq F$ such that $a \in P$ and $b \notin P$. Consequently, each Scott open filter is an intersection of complete filters.*

Proof. This is just the famous Birkhoff's Theorem with the openness added. Using Zorn's Lemma in the standard way (taking into account that unions of open sets are open), we obtain an open filter $P \supseteq F$ maximal with respect to the condition $b \notin P \ni a$. We will prove that it is prime (and hence, by the Lemma, complete). Suppose it is not; then there are $u, v \notin P$ such that $u \vee v \in P$. Set $G = \{x \mid x \vee v \in P\}$. Then G is obviously a Scott open filter and, because of the u, $P \subset G$. Thus, $b \in G$, $b \vee v \in P$ and we can repeat the procedure with $v, b \notin P$, $v \vee b \in P$ and $H = \{x \mid x \vee b \in P\}$ to obtain the contradiction $b = b \vee b \in P$. $\square$

If X is locally compact and $c \ll a$, interpolate inductively

$$a \gg x_1 \gg x_2 \gg \cdots \gg x_n \gg \cdots c,$$

choose the x_n fixedly for each such couple a, c, and set

$$F(a,c) = \{x \mid x \geq x_k \text{ for some } k\}.$$

Then $F(a,c)$ is obviously a Scott open filter. These filters are useful to prove the following theorem, that justifies the definition of locally compact locale.

Theorem. *Each locally compact locale is spatial, and functors* Lc *and* Pt *restrict to an equivalence between the category of sober locally compact spaces and the category of locally compact locales.*

Proof. Let X be a locally compact locale and consider $a, b \in \mathcal{O}X$ with $a \nleq b$. Then there is a $c \ll a$ such that $c \neq b$. Hence $b \notin F(a,c) \ni a$ and, by the Proposition, there exists a complete P such that $b \notin P \ni a$.

Thus, since we already know (Exercise 7.1) that $\mathsf{Lc}(X)$, with locally compact space X, is locally compact, it suffices to show that each $\mathsf{Pt}(X)$, with locally compact locale X, is locally compact.

Let $P \in \Sigma_a$, that is, $a \in P$. Since $a = \bigvee\{x \mid x \ll a\}$ and P is open there is a $c \ll a$, $c \in P$. Set $K = \{Q \in \mathsf{Pt}(X) \mid F(a,c) \subseteq Q\}$. By the Proposition, $\bigcap K = F(a,c)$, and, by Lemma 7.3, K is compact. If $Q \in \Sigma_c$, that is, $c \in Q$, we have $F(a,c) \subseteq Q$, and if $F(a,c) \subseteq Q$ then $a \in Q$. Thus, $P \in \Sigma_c \subseteq K \subseteq \Sigma_a$. $\square$

By Proposition 7.2(1) the equivalence above restricts to an equivalence between the category of compact Hausdorff spaces and $\mathsf{KRegLoc}$. The contravariant version of Theorem 7.4 (in terms of continuous frames) is the well-known Hofmann-Lawson Duality ([31], see also [4]).

7.5. Preservation of products by Lc. The functor Lc is a left adjoint and generally does not preserve products. But we have:

Proposition. *The functor* $\mathsf{Lc} : \mathsf{Top} \to \mathsf{Loc}$ *preserves finite products of sober completely regular locally compact spaces.*

Proof. In view of Theorem 7.4 it suffices to show that finite products in the smaller categories coincide with those in Top resp. Loc. This is obvious for locally compact spaces. Using Proposition 7.2 we infer from 3.4 that a product of two completely regular locally compact locales is locally compact (and completely regular, by Proposition 4.3). □

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