BORDISM: OLD AND NEW

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What follows are lecture notes from a graduate course given at the University of Texas at Austin in Fall, 2012. The first half covers some classical topics in bordism, leading to the Hirzebruch Signature Theorem. The second half covers some more recent topics, leading to the Galatius-Madsen-Tillmann-Weiss theorem and the cobordism hypothesis. The only prerequisite was our first year course in algebraic and differential topology, which includes some homology theory and basic theorems about transversality but no cohomology or homotopy theory. Therefore, the text is somewhat quirky about what is and what is not explained in detail. While bordism is an organizing principle for the course, I include basics about standard topics such as classifying spaces, characteristic classes, categories, Γ-spaces, sheaves, etc. Many proofs are missing; perhaps some will be filled in if these notes are distributed more formally. I sprinkled exercises throughout the first part of the text, but then switched to writing problem sets during the second half of the course; these are included at the end of the text. I warmly thank the members of the class for their feedback on an earlier version of these notes.

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Lecture 1: Introduction to bordism

sec:1

Overview

Bordism is a notion which can be traced back to Henri Poincaré at the end of the 19th century, but it comes into its own mid-20th century in the hands of Lev Pontrjagin and René Thom [T]. Poincaré originally tried to develop homology theory using smooth manifolds, but eventually simplices were used instead. Recall that a *singular q-chain* in a topological space S is a formal sum of continuous maps $\Delta^q \to S$ from the standard q-simplex. There is a boundary operation ∂ on chains, and a chain c is a *cycle* if $\partial c = 0$; a cycle c is a *boundary* if there exists a $(q+1)$-chain b with $\partial b = c$. If S is a point, then every cycle of positive dimension is a boundary. In other words, abstract chains carry no information. In bordism theory one replaces cycles by *closed*[1] *smooth manifolds* mapping continuously into S. A chain is replaced by a compact smooth manifold X and a continuous map $X \to S$; the boundary of this chain is the restriction $\partial X \to S$ to the boundary. Now there is information even if $S = \mathrm{pt}$. For not every closed smooth manifold is the boundary of a compact smooth manifold. For example, $Y = \mathbb{RP}^2$ is *not* the boundary of a compact 3-manifold. (It *is* the boundary of a noncompact 1-manifold with boundary—which? In fact, show that *every* closed smooth manifold Y is the boundary of a noncompact manifold with boundary.)

A variation is to consider smooth manifolds equipped with a *tangential structure* of a fixed type. One type of a tangential structure you already know is an *orientation*, which we review in Lecture 2. We give a general discussion in a few weeks.

One main idea of the course is to extract various algebraic structures of increasing complexity from smooth manifolds and bordism. Today we will use bordism to construct an equivalence relation, and so construct *sets* of bordism classes of manifolds. We will introduce an algebraic structure to obtain *abelian groups* and even a *commutative ring*. These ideas date from the 1950s. The modern results concern more intricate algebraic gadgets extracted from smooth manifolds and bordism: *categories* and their more complicated cousins. Some of the main theorems in the course identify these algebraic structures explicitly. For example, an easy theorem asserts that the bordism group of oriented 0-manifolds is the free abelian group on a single generator, that is, the infinite cyclic group (isomorphic to $\mathbb{Z}$). One of the recent results which we state in the last lecture, the *cobordism hypothesis* [L1, F1], is a vast generalization of this easy classical theorem.

We will also study *bordism invariants*. These are homomorphisms out of a bordism group or category into an abstract group or category. Such homomorphisms, as all homomorphisms, can be used in two ways: to extract information about the domain or to extract information about the codomain. In the classical case the codomain is typically the integers or another simple number system, so we are typically using bordism invariants to learn about manifolds. A classic example of such an invariant is the *signature* of an oriented manifold, and Hirzebruch's signature theorem

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[1] The word 'closed' modifying manifold means 'compact without boundary'.

equates the signature with another bordism invariant constructed from *characteristic numbers*. On the other hand, a typical application of the cobordism hypothesis is to use the structure of manifolds to learn about the codomain of a homomorphism. Incidentally, a homomorphism out of a bordism category is called a *topological quantum field theory* [A1].

subsec:1.7

(1.1) *Convention.* All manifolds in this course—except for a transient exception in the next section—are smooth, or smooth manifolds with boundary or corners, so we omit the modifier 'smooth' from now on. In bordism theory the manifolds are almost always compact, though we retain that modifier to be clear.

Review of smooth manifolds

thm:1 **Definition 1.2.** A *topological manifold* is a paracompact, Hausdorff topological space X such that every point of X has an open neighborhood which is homeomorphic to an open subset of affine space.

Recall that n-dimensional affine space is

eq:1

$$\mathbb{A}^n = \{(x^1, x^2, \dots, x^n) : x^i \in \mathbb{R}\}. \tag{1.3}$$

The vector space $\mathbb{R}^n$ acts transitively on $\mathbb{A}^n$ by translations. The *dimension* $\dim X \colon X \to \mathbb{Z}^{\ge 0}$ assigns to each point the dimension of the affine space in the definition. (It is independent of the choice of neighborhood and homeomorphism, though that is not trivial.) The function $\dim X$ is constant on components of X. If $\dim X$ has constant value n, we say X is an n-dimensional manifold, or n-manifold for short.

subsec:1.1

(1.4) *Smooth structures.* For $U \subset X$ an open set, a homeomorphism $x \colon U \to \mathbb{A}^n$ is a *coordinate chart*. We write $x = (x^1, \dots, x^n)$, where each $x^i \colon U \to \mathbb{R}$ is a continuous function. To indicate the domain, we write the chart as the pair (U, x). If (U, x) and (V, y) are charts, then there is a *transition map*

eq:2

$$y \circ x^{-1} \colon x(U \cap V) \longrightarrow y(U \cap V), \tag{1.5}$$

which is a continuous map between open sets of $\mathbb{A}^n$. We say the charts are C^∞-*compatible* if the transition function (1.5) is *smooth* ($=C^\infty$).

thm:2 **Definition 1.6.** Let X be a topological manifold. An *atlas* or *smooth structure* on X is a collection of charts such that

(i) the union of the charts is X;
(ii) any two charts are C^∞-compatible; and
(iii) the atlas is maximal with respect to (ii).

A topological manifold equipped with an atlas is called a *smooth manifold*.

We usually omit the atlas from the notation and simply notate the smooth manifold as 'X'.

subsec:1.3 **(1.7)** *Empty set.* The empty set $\emptyset$ is trivially a manifold of any dimension $n \in \mathbb{Z}^{\geq 0}$. We use '$\emptyset^n$' to denote the empty manifold of dimension n.

subsec:1.2 **(1.8)** *Manifolds with boundary.* A simple modification of Definition 1.2 and Definition 1.6 allow for manifolds to have boundaries. Namely, we replace affine space with a closed half-space in affine space. So define

eq:3
$$\mathbb{A}^n_- = \{(x^1, x^2, \dots, x^n) \in \mathbb{A}^n : x^1 \leq 0\} \tag{1.9}$$

and ask that coordinate charts take values in open sets of $\mathbb{A}^n_-$. Then if $p \in X$ satisfies $x^1(p) = 0$ in some coordinate system $(x^1, \dots, x^n)$, that will be true in all coordinate systems. In this way X is partitioned into two disjoint subsets, each of which is a manifold: the *interior* (consisting of points with $x^1 < 0$ in every coordinate system) and the *boundary* ∂X (consisting of points with $x^1 = 0$ in every coordinate system).

thm:3 *Remark* 1.10. I remember the convention on charts by the mnemonic 'ONF', which stands for 'Outward Normal First'. The fact that it also stands for 'One Never Forgets' helps me remember! An outward normal in a coordinate system is represented by the first coordinate vector field $\partial/\partial x^1$, and it points out of the manifold at the boundary.

subsec:1.8 **(1.11)** *Tangent bundle at the boundary.* At any point $p \in \partial X$ of the boundary there is a canonical subspace $T_p(\partial X) \subset T_pX$; the quotient space is a real line ν_p. So over the boundary ∂X there is a short exact sequence

eq:21
$$0 \longrightarrow T(\partial X) \longrightarrow TX \longrightarrow \nu \longrightarrow 0 \tag{1.12}$$

of vector bundles. In any boundary coordinate system the vector $\partial/\partial x^1(p)$ projects to a nonzero element of ν_p, but there is no canonical basis independent of the coordinate system. However, any two such vectors are in the same component of $\nu_p \setminus \{0\}$, which means that ν carries a canonical *orientation*. (We review orientations in Lecture 2.)

thm:4 **Definition 1.13.** Let X be a manifold with boundary. A *collar* of the boundary is an open set $U \subset X$ which contains ∂X and a diffeomorphism $(-\epsilon, 0] \times \partial X \to U$ for some $\epsilon > 0$.

thm:5 **Theorem 1.14.** *The boundary ∂X of a manifold X with boundary has a collar.*

This is not a trivial theorem; you can find a proof in [Hi]. We only need this result when X, hence also ∂X, is compact, in which case it is somewhat simpler.

thm:6 **Exercise 1.15.** Prove Theorem 1.14 assuming X is compact. (Hint: Cover the boundary with a finite number of coordinate charts; use a partition of unity to glue the vector fields $-\partial/\partial x^1$ in each coordinate chart into a smooth vector field; and use the fundamental existence theorem for ODEs, including smooth dependence on initial conditions.)

subsec:1.5

(1.16) *Disjoint union.* Let $\{X_1, X_2, \dots\}$ be a countable collection of manifolds. We can form a new manifold, the *disjoint union* of $X_1, X_2, \dots$, which we denote $X_1 \amalg X_2 \amalg \cdots$. As a set it is the disjoint union of the sets underlying the manifolds $X_1, X_2, \dots$. One may wonder how to define the disjoint union. For example, what is $X \amalg X$? This is ultimately a question of set theory, and we will meet such problems again. One solution is to fix an infinite dimensional affine space $\mathbb{A}^\infty$ and regard all manifolds as embedded in it. (This is no loss of generality by the Whitney Embedding Theorem.) Then we can replace X_i (embedded in $\mathbb{A}^\infty$) by $\{i\} \times X_i$ (embedded in $\mathbb{A}^\infty = \mathbb{A}^1 \times \mathbb{A}^\infty$) and define the disjoint union to be the ordinary union of subsets of $\mathbb{A}^\infty$. Another way out is to characterize the disjoint union by a universal property: a disjoint union of $X_1, X_2, \dots$ is a manifold Z and a collection of smooth maps $\iota_i \colon X_i \to Z$ such that for any manifold Y and any collection $f_i \colon X_i \to Y$ of smooth maps, there exists a unique map $f \colon Z \to Y$ such that for each i the diagram

eq:6 (1.17)

$$\begin{tikzcd} X_i \arrow[r, "\iota_i"] \arrow[dr, "f_i"'] & Z \arrow[d, "f"] \\ & Y \end{tikzcd}$$

commutes. (The last statement means $f \circ \iota_i = f_i$.) If you have not seen universal properties before, you might prove that ι_i is an embedding and that any two choices of $\bigl(Z, \{\iota_i\}\bigr)$ are canonically isomorphic. (You should also spell out what 'canonically isomorphic' means.) We will encounter such categorical notions more later in the course.

subsec:1.4

(1.18) *Terminology.* A manifold is *closed* if it is compact without boundary. By contrast, many use the term 'open manifold' to mean a manifold with no closed components.n

Bordism

We now come to the fundamental definition. Fix an integer $n \geq 0$.

thm:7

Definition 1.19. Let Y_0, Y_1 be closed n-manifolds. A *bordism* $\bigl(X\,,\,p\,,\,\theta_0\,,\,\theta_1\bigr)$ *from* Y_0 *to* Y_1 consists of a compact $(n+1)$-manifold X with boundary, a partition $p \colon \partial X \to \{0,1\}$ of its boundary, and embeddings

eq:4 (1.20)
$$\theta_0 \colon [0,+1) \times Y_0 \longrightarrow X$$

eq:5 (1.21)
$$\theta_1 \colon (-1,0] \times Y_1 \longrightarrow X$$

such that $\theta_i(0, Y_i) = (\partial X)_i$, $i = 0,1$, where $(\partial X)_i = p^{-1}(i)$.

Each of $(\partial X)_0, (\partial X)_1$ is a union of components of ∂X; note that there is a finite number of components since X, and so too ∂X, is compact. The map θ_i is a diffeomorphism onto its image, which is a collar neighborhood of $(\partial X)_i$. The collar neighborhoods are included in the definition to make it easy to glue bordisms. Without them we could as well omit the diffeomorphisms and give a simpler informal definition: a bordism X from Y_0 to Y_1 is a compact $(n+1)$-manifold with

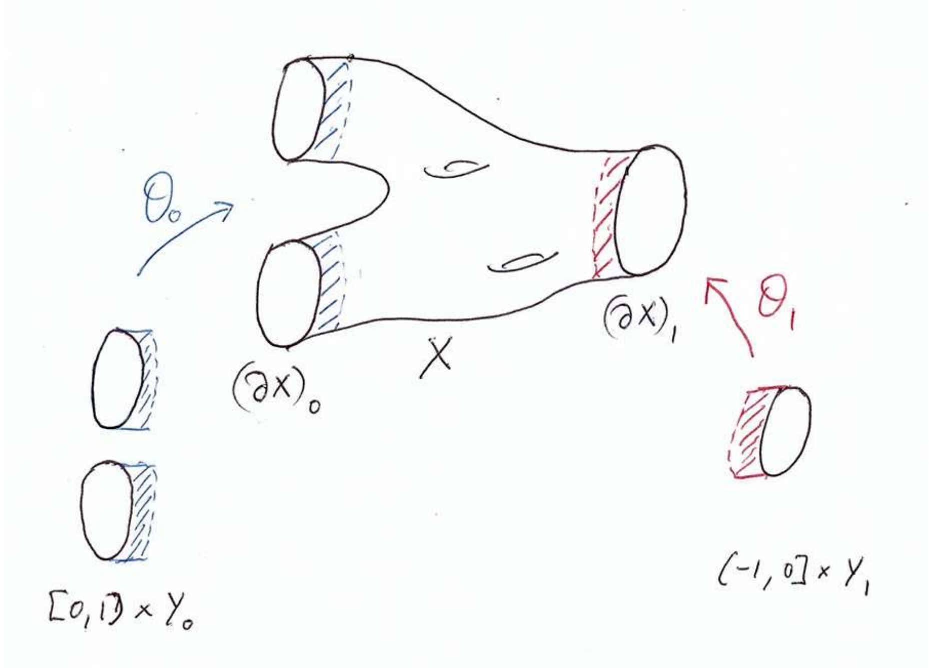

FIGURE 1. X is a bordism from Y_0 to Y_1 fig:1

boundary $Y_0 \amalg Y_1$. But we will keep the slightly more elaborate Definition 1.19. The words 'from' and 'to' in the definition distinguish the roles of Y_0 and Y_1, and indeed the intervals in (1.20) and (1.21) are different. But not that different—for the moment that distinction is only one of semantics and not any mathematics of import. For example, in the informal definition just given the manifolds Y_0, Y_1 play symmetric roles. We picture a bordism in Figure 1. In the older literature a bordism is called a "cobordism". If the context is clear, we notate a bordism $\bigl(X\,,\,p\,,\,\theta_0\,,\,\theta_1\bigr)$ as 'X'.

thm:8 **Definition 1.22.** Let $\bigl(X\,,\,p\,,\,\theta_0\,,\,\theta_1\bigr)$ be a bordism from Y_0 to Y_1. The *dual bordism* from Y_1 to Y_0 is $\bigl(X^\vee\,,\,p^\vee\,,\,\theta_0^\vee\,,\,\theta_1^\vee\bigr)$, where: $X^\vee = X$; the decomposition of the boundary is swapped, so $p^\vee = 1-p$; and

eq:7
$$\begin{aligned} \theta_0^\vee(t,y) &= \theta_1(-t,y), \qquad && t\in[0,+1), \quad y\in Y_1,\\ \theta_1^\vee(t,y) &= \theta_0(-t,y), \qquad && t\in(-1,0], \quad y\in Y_0. \end{aligned} \tag{1.23}$$

More informally, we picture the dual bordism $X^\vee$ as the original bordism X "turned around".

thm:9 *Remark* 1.24. We should view the dual bordism as a bordism from $Y_1^\vee$ to $Y_0^\vee$ where for naked manifolds we set $Y_i^\vee = Y_i$. When we come to manifolds with tangential structure, such as an orientation, we will not necessarily have $Y_i^\vee = Y_i$.

We use Definition 1.19 to extract our first algebraic gadget from compact manifolds: a *set*. Namely, define closed n-manifolds Y_0, Y_1 to be equivalent if there exists a bordism from Y_0 to Y_1.

thm:10 **Lemma 1.25.** *Bordism defines an equivalence relation.*

Proof. For any closed manifold Y, the manifold $X = [0,1] \times Y$ determines a bordism from Y to Y: set $(\partial X)_0 = \{0\} \times Y$, $(\partial X)_1 = \{1\} \times Y$, and use simple diffeomorphisms $[0,1) \to [0,1/3)$ and $(-1,0] \to (2/3,1]$ to construct (1.20) and (1.21). So bordism is a reflexive relation. Definition 1.22 shows that the relation is symmetric: if X is a bordism from Y_0 to Y_1, then $X^\vee$ is a bordism from Y_1 to Y_0. For transitivity, suppose $\bigl(X\,,\,p\,,\,\theta_0\,,\,\theta_1\bigr)$ is a bordism from Y_0 to Y_1 and $\bigl(X'\,,\,p'\,,\,\theta_0'\,,\,\theta_1'\bigr)$ a bordism from Y_1 to Y_2. Then Figure 2 illustrates how to glue X and X' together along Y_1 using θ_1 and θ_0' to obtain a bordism from Y_0 to Y_2. $\square$

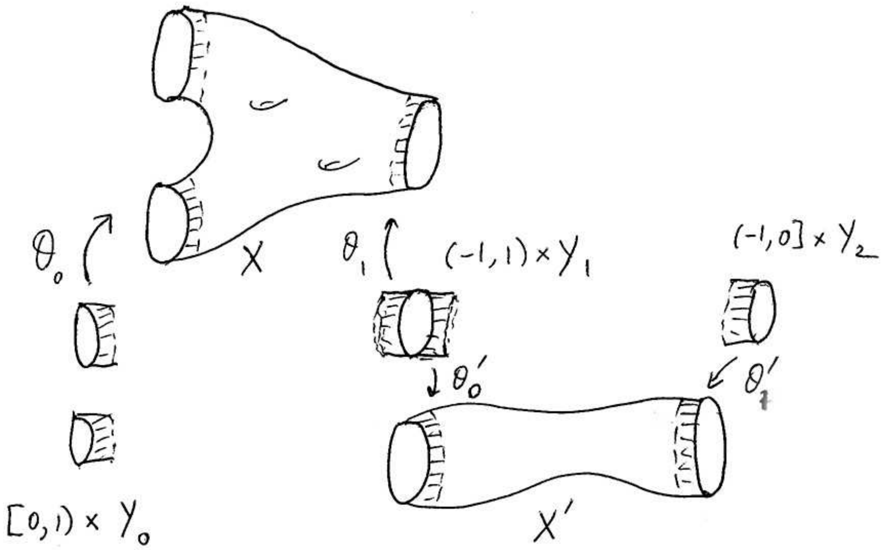

FIGURE 2. Gluing bordisms fig:4

thm:11 **Exercise 1.26.** Write out the details of the gluing argument. Show carefully that the glued space is a manifold with boundary. Note that $\theta_1\bigl(\{0\} \times Y_1\bigr) = \theta_0'\bigl(\{0\} \times Y_1\bigr)$ is a submanifold of the glued manifold, and the maps θ_1 and θ_0' combine to give a diffeomorphism $(-1,1) \times Y_1$ onto an open tubular neighborhood. This is sometimes called a *bi-collaring*.

thm:32 **Exercise 1.27.** Show that diffeomorphic manifolds are bordant.

Let Ω_n denote the set of equivalence classes of closed n-manifolds under the equivalence relation of bordism. We use the term *bordism class* for an element of Ω_n. Note that the empty manifold $\emptyset^0$ is a special element of Ω_n, so we may consider Ω_n as a *pointed set.*

thm:12 *Remark* 1.28. Again there is a set-theoretic worry: is the collection of closed n-manifolds a set? One way to make it so is to consider all manifolds as embedded in $\mathbb{A}^\infty$, as in (**1.16**). We will not make such considerations explicit at this point, but we will use such embeddings to construct a *category* of bordisms in Lecture 20.

Disjoint union and the abelian group structure

Simple operations on manifolds—disjoint union and Cartesian product—give Ω_n more structure.

thm:13 **Definition 1.29.**

(i) A *commutative monoid* is a set with a commutative, associative composition law and identity element.

(ii) An *abelian group* is a commutative monoid in which every element has an inverse.

Typical examples: $\mathbb{Z}^{\geq 0}$ is a commutative monoid; $\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}$ are abelian groups.

Disjoint union is an operation on manifolds which passes to bordism classes: if Y_0 is bordant to Y_0' and Y_1 is bordant to Y_1', then $Y_0 \amalg Y_1$ is bordant to $Y_0' \amalg Y_1'$. So $(\Omega_n, \amalg)$ is a commutative monoid.

thm:14 **Lemma 1.30.** *$(\Omega_n, \amalg)$ is an abelian group. In fact, $Y \amalg Y$ is null-bordant.*

The identity element is represented by $\emptyset^n$. A null bordant manifold is one which is bordant to $\emptyset^n$.

Proof. The manifold $X = [0,1] \times Y$ provides a null bordism: let $p \equiv 0$ and define θ_0, θ_1 appropriately. ☐

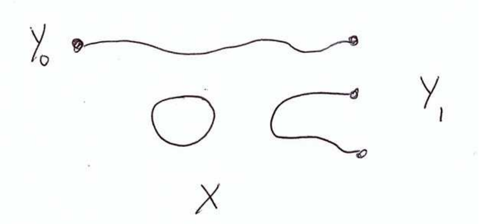

FIGURE 3. 1 point is bordant to 3 points fig:2

It is also true that the abelian group $(\Omega_n, \amalg)$ is finitely generated, though we do not prove that here. It follows that it is isomorphic to a product of cyclic groups of order 2. We denote this abelian group simply by 'Ω_n'.

thm:15 **Proposition 1.31.** *$\Omega_0 \cong \mathbb{Z}/2\mathbb{Z}$ with generator* pt.

Proof. Any 0-manifold has no boundary, and a compact 0-manifold is a finite disjoint union of points. Lemma 1.30 implies that the disjoint union of two points is a boundary, so is zero in Ω_0. It remains to prove that pt is not the boundary of a compact 1-manifold with boundary. That follows from the classification theorem for compact 1-manifolds with boundary [M3]: any such is a finite disjoint union of circles and closed intervals, so its boundary has an even number of points. ☐

The bordism group in dimensions 1,2 can also be computed from elementary theorems.

thm:16 **Proposition 1.32.** *$\Omega_1 = 0$ and $\Omega_2 \cong \mathbb{Z}/2\mathbb{Z}$ with generator the real projective plane $\mathbb{RP}^2$.*

Proof. The first statement follows from the classification theorem in the previous proof: any closed 1-manifold is a finite disjoint union of circles, and a circle is the boundary of a 2-disk, so is null bordant. The second statement follows from the classification theorem for closed 2-manifolds. Recall that there are two connected families. The oriented surfaces are boundaries (of 3-dimensional handlebodies, for example). Any unoriented surface is a *connected sum*[2] of $\mathbb{RP}^2$'s, so it suffices to

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[2] The connected sum is denoted '#'. We do not pause here to define it carefully. The definition depends on choices, but the diffeomorphism class, hence bordism class, does not depend on the choices.

prove that $\mathbb{RP}^2$ does not bound and $\mathbb{RP}^2\#\mathbb{RP}^2$ does bound. A nice argument emerged in lecture for the former. Namely, if X is a compact 3-manifold with boundary $\partial X = \mathbb{RP}^2$, then the *double* $D = X \cup_{\mathbb{RP}^2} X$ has Euler characteristic $2\chi(X) - 1$, which is odd. But D is a closed odd dimensional manifold, so has vanishing Euler characteristic. This contradiction shows X does not exist. We give a different argument in the next lecture. For the latter, recall that $\mathbb{RP}^2\#\mathbb{RP}^2$ is diffeomorphic to a Klein bottle K, which has a map $K \to S^1$ which is a *fiber bundle* with fiber S^1. There is an associated fiber bundle with fiber the disk D^2 which is a compact 3-manifold with boundary K. ☐

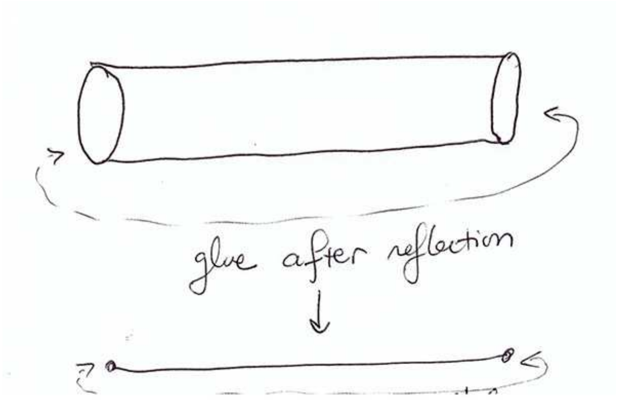

FIGURE 4. Constructing the Klein bottle by gluing fig:3

Recall that we can construct K by gluing together the ends of a cylinder $[0,1] \times S^1$ using a reflection on S^1. Then projection onto the first factor, after gluing, is the map $K \to S^1$. The disk bundle is formed analogously starting with $[0,1] \times D^2$. This is depicted in Figure 4.

Cartesian product and the ring structure

Now we bring in another operation, Cartesian product, which takes an n_1-manifold and an n_2-manifold and produces an $(n_1 + n_2)$-manifold.

thm:17 **Definition 1.33.**

(i) A *commutative ring* R is an abelian group $(+, 0)$ with a second commutative, associative composition law $(\cdot)$ with identity (1) which distributes over the first: $r_1 \cdot (r_2 + r_3) = r_1 \cdot r_2 + r_1 \cdot r_3$ for all $r_1, r_2, r_3 \in R$.

(ii) A $\mathbb{Z}$-*graded commutative ring* is a commutative ring S which as an abelian group is a direct sum

eq:8
$$S = \bigoplus_{n\in\mathbb{Z}} S_n \tag{1.34}$$

of abelian subgroups such that $S_{n_1} \cdot S_{n_2} \subset S_{n_1+n_2}$.

Elements in $S_n \subset S$ are called *homogeneous of degree* n; the general element of S is a finite sum of homogeneous elements.

The integers $\mathbb{Z}$ form a commutative ring, and for any commutative ring R there is a polynomial ring $S = R[x]$ in a single variable which is $\mathbb{Z}$-graded. To define the $\mathbb{Z}$-grading we must assign an integer degree to the indeterminate x. Typically we posit $\deg x = 1$, in which case S_n is the abelian group of homogeneous polynomials of degree n in x. More generally, there is a $\mathbb{Z}$-graded polynomial ring $R[x_1, \dots, x_k]$ in any number of indeterminates with any assigned integer degrees $\deg x_k \in \mathbb{Z}$.

Define

eq:9

$$\Omega = \bigoplus_{n\in\mathbb{Z}^{\ge 0}} \Omega_n. \tag{1.35}$$

We formally define $\Omega_{-m} = 0$ for $m > 0$. The Cartesian product of manifolds is compatible with bordism, so passes to a commutative, associative binary composition law on Ω.

thm:18 **Proposition 1.36.** *$(\Omega, \amalg, \times)$ is a $\mathbb{Z}$-graded ring. A homogeneous element of degree $n \in \mathbb{Z}$ is represented by a closed manifold of dimension n.*

We leave the proof to the reader. The ring Ω is called the *unoriented bordism ring*.

In his Ph.D. thesis Thom [T] proved the following theorem (among many other foundational results).

thm:20 **Theorem 1.37** ([T]). *There is an isomorphism of $\mathbb{Z}$-graded rings*

eq:10

$$\Omega \cong \mathbb{Z}/2\mathbb{Z}[x_2, x_4, x_5, x_6, x_8, \dots] \tag{1.38}$$

where there is a polynomial generator of degree k for each positive integer k not of the form $2^i - 1$.

Furthermore, Thom proved that if k is even, then x_k is represented by the real projective manifold $\mathbb{RP}^k$. Dold later constructed manifolds representing the odd degree generators: they are fiber bundles[3] over $\mathbb{RP}^m$ with fiber $\mathbb{CP}^\ell$.

thm:21 **Exercise 1.39.** Work out Ω_{10}. Find manifolds which represent each bordism class.

Thom proved that the *Stiefel-Whitney numbers* determine the bordism class of a closed manifold. The *Stiefel-Whitney classes* $w_i(Y) \in H^i(Y; \mathbb{Z}/2\mathbb{Z})$ are examples of *characteristic classes* of the tangent bundle. Any closed n-manifold Y has a *fundamental class* $[Y] \in H_n(Y; \mathbb{Z}/2\mathbb{Z})$. If $x \in H^\bullet(Y; \mathbb{Z}/2\mathbb{Z})$, then the pairing $\langle x, [Y]\rangle$ produces a number in $\mathbb{Z}/2\mathbb{Z}$.

thm:22 **Theorem 1.40** ([T]). *The Stiefel-Whitney numbers*

eq:11

$$\langle w_{i_1}(Y) \smile w_{i_2}(Y) \smile \cdots \smile w_{i_k}(Y)\, ,\, [Y]\rangle \in \mathbb{Z}/2\mathbb{Z}, \tag{1.41}$$

determine the bordism class of a closed n-manifold Y.

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[3] They are the quotient of $S^m \times \mathbb{CP}^\ell$ by the free involution which acts as the antipodal map on the sphere and complex conjugation on the complex projective space.

That is, if closed n-manifolds Y_0, Y_1 have the same Stiefel-Whitney numbers, then they are bordant. Notice that not all naively possible nonzero Stiefel-Whitney numbers can be nonzero. For example, $\langle w_1(Y), [Y]\rangle$ vanishes for any closed 1-manifold Y. Also, the theorem implies that a closed n-manifold is the boundary of a compact $(n+1)$-manifold iff all of the Stiefel-Whitney numbers of Y vanish. If it is a boundary, it is immediate that the Stiefel-Whitney numbers vanish; the converse is hardly obvious.

thm:19 *Remark* 1.42. The modern developments in bordism use disjoint union heavily, so generalize the study of classical abelian bordism groups. However, they do not use Cartesian product in the same way.

Lecture 2: Orientations, framings, and the Pontrjagin-Thom construction

sec:2

One of Thom's great contributions was to translate problems in geometric topology—such as the computation (Theorem 1.37) of the unoriented bordism ring—into problems in homotopy theory. The correspondence works in both directions: facts about manifolds can sometimes be used to deduce homotopical information. This lecture ends with a first instance of that principle. The geometric side is the set of *framed* bordism classes of submanifolds of a fixed manifold M; the homotopical side is the set of homotopy classes of maps from M into a sphere. The theorem gives an isomorphism between these two sets. (For framed manifolds it is due to Pontrjagin; Thom's more general statement appears in Lecture 10.) Here we introduce the basic idea; the proof will be given in the next lecture. We will build on these ideas in subsequent lectures and so translate the computation of bordism groups (Lecture 1) into homotopy theory.

Before getting to framed bordism we give a reminder on orientations and introduce the oriented bordism ring. Orientations are an example of a (stable) tangential structure; we will discuss general tangential structures in Lecture 9.

Orientations

subsec:2.4

(2.1) *Orientation of a real vector space.* Let V be a real vector space of dimension $n > 0$. A *basis* of V is a linear isomorphism $b\colon \mathbb{R}^n \to V$. Let $\mathcal{B}(V)$ denote the set of all bases of V. The group $GL_n(\mathbb{R})$ of linear isomorphisms of $\mathbb{R}^n$ acts *simply transitively* on the right of $\mathcal{B}(V)$ by composition: if $b\colon \mathbb{R}^n \to V$ and $g\colon \mathbb{R}^n \to \mathbb{R}^n$ are isomorphisms, then so too is $b \circ g\colon \mathbb{R}^n \to V$. We say that $\mathcal{B}(V)$ is a *right* $GL_n(\mathbb{R})$*-torsor.* For any $b \in \mathcal{B}(V)$ the map $g \mapsto b \circ g$ is a bijection from $GL_n(\mathbb{R})$ to $\mathcal{B}(V)$, and we use it to topologize $\mathcal{B}(V)$. Since $GL_n(\mathbb{R})$ has two components, so does $\mathcal{B}(V)$.

thm:23 **Definition 2.2.** An orientation of V is a choice of component of $\mathcal{B}(V)$.

subsec:2.5

(2.3) *Determinants and orientation.* Recall that the components of $GL_n(\mathbb{R})$ are distinguished by the determinant homomorphism

eq:12
$$\det\colon GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^{\neq 0}; \tag{2.4}$$

the identity component consists of $g \in GL_n(\mathbb{R})$ with $\det(g) > 0$, and the other component consists of g with $\det(g) < 0$. On the other hand, an isomorphism $b\colon \mathbb{R}^n \to V$ does not have a numerical determinant. Rather, its determinant lives in the *determinant line* $\operatorname{Det} V$ of V. Namely, define

eq:13
$$\operatorname{Det} V = \{\epsilon\colon \mathcal{B}(V) \to \mathbb{R} : \epsilon(b \circ g) = \det(g)^{-1}\epsilon(b) \ \text{ for all } b \in \mathcal{B}(V),\ g \in GL_n(\mathbb{R})\}. \tag{2.5}$$

thm:24 **Exercise 2.6.** Prove the following elementary facts about determinants and orientations.

(i) Construct a canonical isomorphism $\operatorname{Det} V \xrightarrow{\cong} \bigwedge^n V$ of the determinant line with the highest exterior power. The latter is often taken as the definition.

(ii) Prove that an orientation is a choice of component of $\operatorname{Det} V \setminus \{0\}$. More precisely, construct a map $\mathcal{B}(V) \to \operatorname{Det} V \setminus \{0\}$ which induces a bijection on components.

(iii) Construct the "determinant" of an arbitrary linear map $b\colon \mathbb{R}^n \to V$ as an element of $\operatorname{Det} V$. Show it is nonzero iff b is invertible.

(iv) More generally, construct the determinant of a linear map $T\colon V \to W$ as a linear map $\det T\colon \operatorname{Det} V \to \operatorname{Det} W$, assuming $\dim V = \dim W$.

(v) Part (ii) gives two descriptions of a canonical $\{\pm 1\}$-torsor[4] (=set of two points) associated to a finite dimensional real vector space. Show that it can also be defined as

eq:17

$$\mathfrak{o}(V) = \{\epsilon\colon \mathcal{B}(V) \to \{\pm 1\} : \epsilon(b \circ g) = \operatorname{sign} \det(g)^{-1} \epsilon(b) \ \text{ for all } b \in \mathcal{B}(V),\ g \in GL_n(\mathbb{R})\}. \tag{2.7}$$

Summary: An orientation of V is a point of $\mathfrak{o}(V)$.

subsec:2.6

(2.8) *Orienting the zero vector space.* There is a unique zero-dimensional vector space 0 consisting of a single element, the zero vector. There is a unique basis—the empty set—and so by (2.5) the determinant line $\operatorname{Det} 0$ is canonically isomorphic to $\mathbb{R}$ and $\mathfrak{o}(V)$ is canonically isomorphic to $\{\pm 1\}$. Note that $\bigwedge^0(0) = \mathbb{R}$ as $\bigwedge^0 V = \mathbb{R}$ for *any* real vector space V. The real line $\mathbb{R}$ has a canonical orientation: the component $\mathbb{R}^{>0} \subset \mathbb{R}^{\neq 0}$. We denote this orientation as '+'. The opposite orientation is denoted '−'.

thm:25

Exercise 2.9 (2-out-of-3)**.** Suppose

eq:14

$$0 \longrightarrow V' \xrightarrow{i} V \xrightarrow{j} V'' \longrightarrow 0 \tag{2.10}$$

is a *short exact sequence* of finite dimensional real vector spaces. Construct a canonical isomorphism

eq:15

$$\operatorname{Det} V'' \otimes \operatorname{Det} V' \longrightarrow \operatorname{Det} V. \tag{2.11}$$

Notice the order: *quotient before sub.* If two out of three of V, V', V'' are oriented, then there is a unique orientation of the third compatible with (2.11). This lemma is quite important in oriented intersection theory.

subsec:2.7

(2.12) *Real vector bundles and orientation.* Now let X be a smooth manifold and $V \to X$ a finite rank real vector bundle. For each $x \in X$ there is associated to the fiber V_x over x a canonical $\{\pm 1\}$-torsor $\mathfrak{o}(V)_x$—a two-element set—which has the two descriptions given in Exercise 2.6(ii).

thm:26

Exercise 2.13. Use local trivializations of $V \to X$ to construct local trivializations of $\mathfrak{o}(V) \to X$, where $\mathfrak{o}(V) = \coprod_{x \in X} \mathfrak{o}(V)_x$.

The 2:1 map $\mathfrak{o}(V) \to X$ is called the *orientation double cover associated to* $V \to X$. In case $V = TX$ is the tangent bundle, it is called the *orientation double cover of* X.

---

[4] $\{\pm 1\}$ is the multiplicative group of square roots of unity, sometimes denoted μ_2.

thm:27 **Definition 2.14.**

(i) An *orientation* of a real vector bundle $V \to X$ is a section of $\mathfrak{o}(V) \to X$.

(ii) If $o\colon X \to \mathfrak{o}(V)$ is an orientation, then the *opposite orientation* is the section $-o\colon X \to \mathfrak{o}(V)$.

(iii) An *orientation* of a manifold X is an orientation of its tangent bundle $TX \to X$.

Orientations may or may not exist, which is to say that a vector bundle $V \to X$ may be orientable or non-orientable. The notation '$-o$' in (ii) uses the fact that $\mathfrak{o}(V) \to X$ is a principal $\{\pm 1\}$-bundle: $-o$ is the result of acting $-1 \in \{\pm 1\}$ on the section o.

thm:28 **Exercise 2.15.** Construct the *determinant line bundle* $\operatorname{Det} V \to X$ by carrying out the determinant construction (2.5) (cf. Exercise 2.6) pointwise and proving local trivializations exist. Show that a *nonzero* section of $\operatorname{Det} V \to X$ determines an orientation.

Our first bordism invariant

This subsection is an extended exercise in which you construct a homomorphism

eq:16
$$\phi\colon \Omega_2 \longrightarrow \mathbb{Z}/2\mathbb{Z} \tag{2.16}$$

and prove that it is an isomorphism. (Recall that we computed $\Omega_2 \cong \mathbb{Z}/2\mathbb{Z}$ in Proposition 1.32, and the proof depends on the fact that $\mathbb{RP}^2$ is not a boundary. In this exercise you will give a different proof of that fact.) An element of Ω_2 is represented by a closed 2-manifold Y. We must (i) define $\phi(Y) \in \mathbb{Z}/2\mathbb{Z}$; (ii) prove that if Y_0 and Y_1 are bordant, then $\phi(Y_0) = \phi(Y_1)$; (iii) prove that ϕ is a homomorphism; and (iv) show that $\phi(\mathbb{RP}^2) \neq 0$. Here is a sketch for you to complete. It relies on elementary differential topology à la Guillemin-Pollack and is a good review of techniques in intersection theory as well as the geometry of projective space.

(i) Choose a section s of $\operatorname{Det} Y \to Y$, where $\operatorname{Det} Y = \operatorname{Det} TY$ is the determinant line bundle of the tangent bundle. Show that we can assume that s is *transverse* to the zero section $Z \subset \operatorname{Det} Y$, where Z is the submanifold of zero vectors. Show that $s^{-1}(Z)$ is a 1-dimensional submanifold of Y. Define $\phi(Y)$ as the mod 2 intersection number of $s^{-1}(Z)$ with itself. Prove that $\phi(Y)$ is independent of the choice of s.

(ii) If X is a bordism from Y_0 to Y_1, show that $\operatorname{Det} X \to X$ restricts on the boundary to the determinant line of the boundary. You may want to use Exercise 2.9 and (1.12). Extend the section s constructed in (i) (for each of Y_0, Y_1) over X so that it is transverse to the zero section. What can you say now about the inverse image of the zero section in X and about its self-intersection?

(iii) This is easy: consider a disjoint union.

(iv) Since $\mathbb{RP}^2$ is the manifold of lines (= one-dimensional subspaces) in $\mathbb{R}^3$, there is a canonical line bundle $L \to \mathbb{RP}^2$ whose fiber at a line $\ell \subset \mathbb{R}^3$ is ℓ. Show that the determinant line bundle of $\mathbb{RP}^2$ is isomorphic to $L \to \mathbb{RP}^2$. (See **(2.17)** below.) Now fix the standard metric on $\mathbb{R}^3$ and define $s(\ell)$ to be the orthogonal projection of the vector $(1, 0, 0)$ onto ℓ. What is $s^{-1}(Z)$?

subsec:2.8

(2.17) *The tangent bundle to projective space.* In (iv) you are asked to "Show that the determinant line bundle of $\mathbb{RP}^2$ is isomorphic to $L \to \mathbb{RP}^2$." For that, let Q_ℓ denote the quotient vector space $\mathbb{R}^3/\ell$ for each line $\ell \subset \mathbb{R}^3$. The 2-dimensional vector spaces Q_ℓ fit together into a vector bundle $Q \to \mathbb{RP}^2$, and there is a short exact sequence

eq:18

$$0 \longrightarrow L \longrightarrow \underline{\mathbb{R}^3} \longrightarrow Q \longrightarrow 0 \tag{2.18}$$

of vector bundles over $\mathbb{RP}^2$, where $\underline{U}$ denotes the vector bundle with constant fiber the vector space U. Claim: There is a natural isomorphism

eq:19

$$T(\mathbb{RP}^2) \xrightarrow{\cong} \mathrm{Hom}(L, Q). \tag{2.19}$$

(There are analogous canonical sub and quotient bundles for any Grassmannian, and the analog of (2.19) is true.) To construct the isomorphism (2.19), fix $\ell \subset \mathbb{R}^3$ and a complementary subspace $W \subset \mathbb{R}^3$. Let ℓ_t, $-\epsilon < t < \epsilon$, be a curve in $\mathbb{RP}^2$ with $\ell_0 = \ell$. For $|t|$ sufficiently small we can write ℓ_t as the graph of a unique linear map $T_t \in \mathrm{Hom}(\ell, W)$. Note $T_0 = 0$. The tangent vector to this curve of linear maps at time 0 is $\dot{T}_0 \in \mathrm{Hom}(\ell, W)$, and its image in $\mathrm{Hom}(\ell, \mathbb{R}^2/\ell)$ after composition with the isomorphism $W \hookrightarrow \mathbb{R}^3 \twoheadrightarrow \mathbb{R}^3/\ell$ is independent of the choice of complement W.

For the rest of (iv) I suggest tensoring (2.18) with L^* and applying the 2-out-of-3 principle (Exercise 2.9). You may also wish to show that the tensor square of a real line bundle is trivializable.

Oriented bordism

We repeat the discussion of unoriented bordism in Lecture 1, beginning with Definition 1.19, for manifolds with orientation. So in Definition 1.19 each of Y_0, Y_1 carries an orientation, as does the bordism X, and the embeddings θ_0, θ_1 are required to be orientation-preserving.

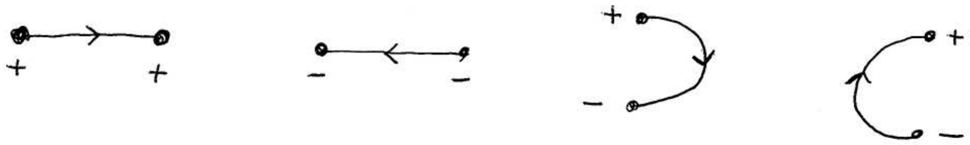

FIGURE 5. Some oriented bordisms of 0-manifolds

fig:5

Figure 5 illustrates four different bordisms in which X is the oriented closed interval. The pictures do not explicitly indicate the decomposition $\partial X = (\partial X)_0 \amalg (\partial X)_1$ of the boundary into incoming and outgoing components, nor do we make explicit the collarings θ_0, θ_1. We make the convention that we read the picture from left to right with the incoming boundary components on the left. Thus, in the first two bordisms the incoming boundary $(\partial X)_0$ and outgoing boundary $(\partial X)_1$ each consist of a single point. In the third bordism the incoming boundary $(\partial X)_0$ consists of two points and the outgoing boundary $(\partial X)_1$ is empty. In the fourth bordism the situation is reversed. Check carefully that (1.20) and (1.21) are orientation-preserving. You will need to think through the orientation of a Cartesian product of manifolds, which amounts to the orientation of a direct sum of vector spaces, which is a special case of Exercise 2.9. (You will also need (**2.8**).)

subsec:2.10

(2.20) *Dual oriented bordism.* There is an important modification to Definition 1.22. Namely, the dual $Y^\vee$ to a closed oriented manifold Y is not equal to Y, as in the unoriented case (see Remark 1.24). Rather,

eq:20

$$Y^\vee = -Y, \tag{2.21}$$

where $-Y$ denotes the manifold Y with the opposite orientation (Definition 2.14(ii)). The reversal of orientation ensures that $\theta_0^\vee$ and $\theta_1^\vee$ in (1.23) are orientation-preserving.

Exercise: Construct the dual to each bordism in Figure 5.

subsec:2.11

(2.22) *Oriented bordism defines an equivalence relation.* Define two closed oriented n-manifolds Y_0, Y_1 to be equivalent if there exists an oriented bordism from Y_0 to Y_1. As in Lemma 1.25 oriented bordism defines an equivalence relation. There is one small, but very important, modification in the proof of symmetry: if X is a bordism from Y_0 to Y_1, then $-X^\vee$ is a bordism from Y_1 to Y_0. (The point is to use the *orientation-reversed* dual.)

subsec:2.12

(2.23) *The oriented bordism ring.* We denote the set of oriented bordism classes of n-manifolds as Ω_n^{SO}. As in (1.35) there is an oriented bordism ring Ω^{SO}.

I will now summarize some facts about Ω^{SO}; see [St, M1, W], [MS, §17] for more details.

thm:29

Theorem 2.24.

(i) [T] *There is an isomorphism*

eq:22

$$\mathbb{Q}[y_4, y_8, y_{12}, \dots] \xrightarrow{\cong} \Omega^{SO} \otimes \mathbb{Q} \tag{2.25}$$

under which y_{4k} maps to the oriented bordism class of the complex projective space $\mathbb{CP}^{2k}$.

(ii) [Av, M2, W] *All torsion in Ω^{SO} is of order 2.*

(iii) [M2, No] *There is an isomorphism*

eq:24

$$\mathbb{Z}[z_4, z_8, z_{12}, \dots] \xrightarrow{\cong} \Omega^{SO}/\text{torsion}. \tag{2.26}$$

(iv) [W] *The Stiefel-Whitney numbers (1.41) and Pontrjagin numbers*

eq:23

$$\langle p_{j_1}(Y) \smile p_{j_2}(Y) \smile \cdots \smile p_{j_k}(Y)\,,\,[Y]\rangle \in \mathbb{Z}, \tag{2.27}$$

determine the oriented bordism class of a closed oriented manifold Y. In particular, Y is the boundary of a compact oriented manifold iff all of the Stiefel-Whitney and Pontrjagin numbers vanish.

The generators in (2.26) are not complex projective spaces, but can be taken to be certain complex manifolds called *Milnor hypersurfaces*. The *Pontrjagin classes* are characteristic classes in integral cohomology, and they live in degrees divisible by 4. The Pontrjagin numbers of an oriented manifold are nonzero only for manifolds whose dimension is divisible by 4.

We will sketch a proof of (i) in Lecture 12 and use it to prove Hirzebruch's signature theorem.

subsec:2.13

(2.28) *Low dimensions.*

$\Omega_0^{SO} \cong \mathbb{Z}$. The generator is an oriented point. Recall from (**2.8**) that a point has two canonical orientations: $+$ and $-$. For definiteness we take the generator to be pt_+, the positively oriented point.

$\Omega_1^{SO} = 0$. Every closed oriented 1-manifold is a finite disjoint union of circles S^1, and $S^1 = \partial D^2$.

$\Omega_2^{SO} = 0$. Every closed oriented surface is a disjoint union of connected sums of 2-tori, and such connected sums bound handlebodies in 3-dimensional space.

$\Omega_3^{SO} = 0$. This is the first theorem which goes beyond classical classification theorems in low dimensions. The general results in Theorem 2.24 imply that Ω_3^{SO} is torsion, but more is needed to prove that it vanishes.

$\Omega_4^{SO} \cong \mathbb{Z}$. The complex projective space $\mathbb{CP}^2$ is a generator. We will see in a subsequent lecture that the *signature* of a closed oriented 4-manifold defines an isomorphism $\Omega_4^{SO} \to \mathbb{Z}$.

$\Omega_5^{SO} \cong \mathbb{Z}/2\mathbb{Z}$. This is the lowest dimensional torsion in the oriented bordism ring. The nonzero element is represented by the Dold manifold Y^5 which is a fiber bundle $Y^5 \to \mathbb{RP}^1 = S^1$ with fiber $\mathbb{CP}^2$. (See the comment after Theorem 1.37.)

$\Omega_6^{SO} = \Omega_7^{SO} = 0$.

$\Omega_8^{SO} \cong \mathbb{Z} \oplus \mathbb{Z}$. It is generated by $\mathbb{CP}^2 \times \mathbb{CP}^2$ and $\mathbb{CP}^4$.

More fun facts: $\Omega_n^{SO} \neq 0$ for all $n \geq 9$. Complex projective spaces and their Cartesian products generate $\Omega_4^{SO}, \Omega_8^{SO}, \Omega_{12}^{SO}$ but not Ω_{16}^{SO}.

thm:45 *Remark* 2.29. The *cobordism hypothesis*, which is a recent theorem about the structure of *multicategories* of manifolds, is a vast generalization of the theorem that Ω_0^{SO} is the free abelian group generated by pt_+.

Framed bordism and the Pontrjagin-Thom construction

Some of this discussion is a bit vague; we give precise definitions and proofs in the next lecture.

Fix a closed m-dimensional manifold M. Let $Y \subset M$ be a submanifold. Recall that on Y there is a short exact sequence of vector bundles

eq:25
$$0 \longrightarrow TY \longrightarrow TM\big|_Y \longrightarrow \nu \longrightarrow 0 \tag{2.30}$$

where ν is defined to be the quotient bundle and is called the *normal bundle of Y in M*.

thm:30 **Definition 2.31.** A *framing* of the submanifold $Y \subset M$ is a trivialization of the normal bundle ν.

Recall that a trivialization of ν is an isomorphism of vector bundles $\underline{\mathbb{R}^q} \to \nu$, where q is the codimension of Y in M. Equivalently, it is a global basis of sections of ν.

Framed submanifolds of M of codimension q arise as follows. Let N be a manifold of dimension q and $f\colon M \to N$ a smooth map. Suppose $p \in N$ is a *regular value* of f and fix a basis $e_1, \dots, e_q$ of T_pN. Then $Y := f^{-1}(p) \subset M$ is a submanifold and the basis $e_1, \dots, e_q$ pulls back to a basis of the normal bundle at each point $y \in Y$. For under the differential f_* at y the subspace $T_yY \subset T_yM$ maps to zero, whence f_* factors down to a map $\nu_y \to T_pN$. The fact that p is a regular value implies that the latter is an isomorphism.

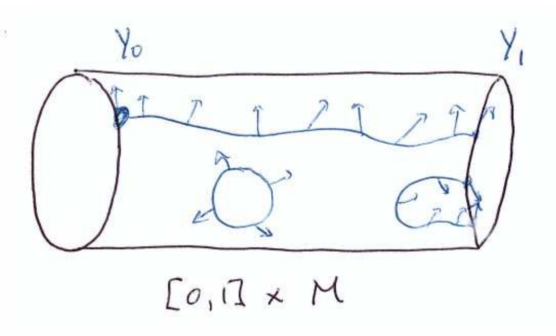

FIGURE 6. A framed bordism in M fig:7

Of course, regular values are not unique. In fact, Sard's theorem asserts that they form an open dense subset of N. If N is connected, then we will see that the inverse images $Y_0 := f^{-1}(p_0)$ and $Y_1 = f^{-1}(p_1)$ of two regular values $p_0, p_1 \in N$ are *framed bordant in* M. (See Figure 6.) This means that there is a framed submanifold with boundary $X \subset [0,1] \times M$ such that $X \cap \bigl(\{i\} \times M\bigr) = Y_i$, $i = 0, 1$, where the framings match at the boundary. While we can transport the framing at p_0 to a framing at p_1 along the path, at least to obtain a homotopy class of framings, we need an *orientation* of N to consistently choose framings at all points of N. In other words, f determines a framed bordism class of framed submanifolds of M of codimension p as long as N is oriented (and connected). Denote the set of these classes as $\Omega^{\text{fr}}_{m-q;M}$. We will also show that homotopic maps lead to the same framed bordism class, so the construction gives a well-defined map

eq:26
$$[M, N] \longrightarrow \Omega^{\text{fr}}_{m-q;M}. \tag{2.32}$$

Here $[M, N]$ denotes the set of homotopy classes of maps from M to N.

From now on suppose $N = S^q$. Then we construct an inverse to (2.32): *Pontrjagin-Thom collapse*. Let $Y \subset M$ be a framed submanifold of codimension q. Recall that any submanifold Y has a *tubular neighborhood*, which is an open neighborhood $U \subset M$ of Y, a submersion $U \to Y$, and an isomorphism $\varphi\colon \nu \to U$ which makes the diagram

eq:27
$$\begin{tikzcd} \nu \arrow[rr, "\varphi"] \arrow[dr] & & U \arrow[dl] \\ & Y & \end{tikzcd} \tag{2.33}$$

commute. The framing of ν then leads to a map $h\colon U \to \mathbb{R}^q$. The collapse map $f_Y : Y \to S^q$ is

eq:28
$$f_Y(x) = \begin{cases} \dfrac{h(x)}{\rho\bigl(|h(x)|\bigr)}, & x \in U; \\ \infty, & x \in N \setminus U. \end{cases} \tag{2.34}$$

Here we write $S^q = \mathbb{R}^q \cup \{\infty\}$ and we fix a cutoff function ρ as depicted in Figure 7. We represent a collapse map in Figure 8.

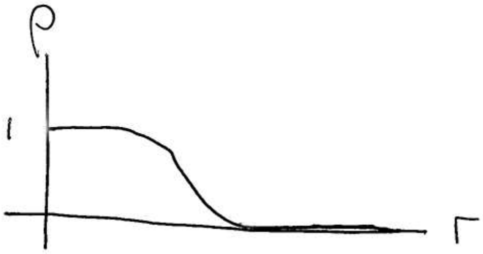

Figure 7. Cutoff function for collapse map

fig:6

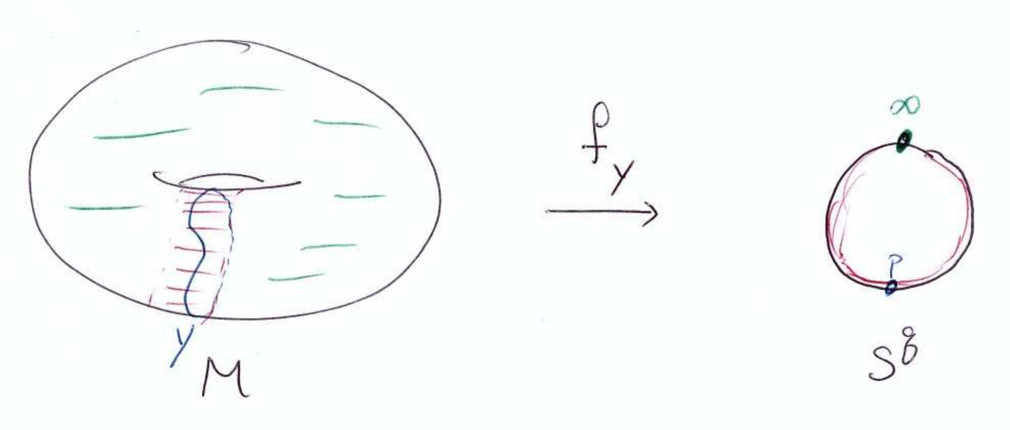

Figure 8. Pontrjagin-Thom collapse

fig:8

thm:31 **Theorem 2.35** (Pontrjagin-Thom)**.** *There is an isomorphism*

eq:29
$$[M, S^q] \longrightarrow \Omega^{\mathrm{fr}}_{m-q;M} \tag{2.36}$$

which takes a map $M \to S^q$ to the inverse image of a regular value. The inverse map is Pontrjagin-Thom collapse.

There are choices (regular value, tubular neighborhood, cutoff function) in these construction. Part of Theorem 2.35 is that the resulting map (2.36) and its inverse are independent of these choices. We prove Theorem 2.35 in the next lecture.

The Hopf degree theorem

As a corollary of Theorem 2.35 we prove the following.

thm:33 **Theorem 2.37** (Hopf). *Let M be a closed connected manifold of dimension m.*

(i) *If M is orientable, then there is an isomorphism*

eq:30
$$[M, S^m] \longrightarrow \mathbb{Z} \tag{2.38}$$

given by the integer degree.

(ii) *If M is not orientable, then there is an isomorphism*

eq:31
$$[M, S^m] \longrightarrow \mathbb{Z}/2\mathbb{Z} \tag{2.39}$$

given by the mod 2 degree.

By Theorem 2.35 homotopy classes of maps $M \to S^m$ are identified with framed bordism classes of framed 0-dimensional submanifolds of M. Now a 0-dimensional submanifold of M is a finite disjoint union of points, and a framed point is a point $y \in M$ together with a basis of T_yM.

We apply an important general principle in geometry: to study an object O introduce the *moduli space* of all objects of that type and formulate questions in terms of the geometry of that moduli space. In this case we are led to introduce the *frame bundle.*

subsec:2.14

(2.40) *The frame bundle.* For any smooth manifold M, define

eq:32
$$\mathcal{B}(M) = \{(y, b) : y \in M,\ b \in \mathcal{B}(T_yM)\}. \tag{2.41}$$

Recall from (**2.1**) that b is an isomorphism $b\colon \mathbb{R}^m \to T_yM$. There is an obvious projection

eq:33
$$\begin{aligned} \pi\colon \mathcal{B}(M) &\longrightarrow M \\ (y, b) &\longmapsto y \end{aligned} \tag{2.42}$$

We claim that (2.42) is a fiber bundle. (See the appendix for a rapid review of fiber bundles.) There is more structure. Recall that each fiber $\mathcal{B}(M)_y = \mathcal{B}(T_yM)$ is a $GL_n(\mathbb{R})$-torsor. That is, the group $GL_n(\mathbb{R})$ acts simply transitively (on the right) on the fiber. So (2.42) is a *principal bundle* with structure group $GL_n(\mathbb{R})$.

thm:39 **Exercise 2.43.** Prove that (2.42) is a fiber bundle. You can use the principal bundle structure to simplify: to construct local trivializations it suffices to construct local sections. Use coordinate charts to do so.

Each fiber of π has two components. Since M is assumed connected, the following is immediate from Definition 2.14 and covering space theory.

thm:40 **Lemma 2.44.** *If M is connected and orientable, then $\mathcal{B}(M)$ has 2 components. If M is connected and non-orientable, then $\mathcal{B}(M)$ is connected.*

Proof. Let $\rho\colon \mathcal{B}(M) \to \mathfrak{o}(M)$ be the map which sends a basis of T_yM to the orientation of T_yM it determines. By Definition 2.2 ρ is surjective. We claim that ρ induces an isomorphism on components, and for that it suffices to check that if o_{y_0} and o_{y_1} are in the same component of $\mathfrak{o}(M)$, and if b_0, b_1 are bases of $T_{y_0}M, T_{y_1}M$ which induce the orientations o_{y_0}, o_{y_1}, then b_0 and b_1 are in the same component of $\mathcal{B}(M)$. Let $\gamma\colon [0,1] \to M$ be a smooth path with $\gamma(0) = y_0$ and $\gamma(1) = y_1$. Lift the vector field $\partial/\partial t$ on $[0,1]$ to a vector field on $\pi'\colon \gamma^*\mathcal{B}(M) \to [0,1]$, which we can do using a partition of unity since the differential of π' is surjective. Find an integral curve of this lifted vector field with initial point b_0. The terminal point of that integral curve lies in the fiber $\mathcal{B}(M)_{y_1}$ and is in the same component of the fiber as b_1, by the assumption that o_{y_0} and o_{y_1} are in the same component of $\mathcal{B}(M)$. $\square$

thm:42 **Lemma 2.45.** *If $Y_0 = (y_0, b_0)$ and $Y_1 = (y_1, b_1)$ are in the same component of $\mathcal{B}(M)$, then the framed points Y_0 and Y_1 are framed bordant in M.*

One special case of interest is where $y_0 = y_1$ and b_0, b_1 belong to the same orientation.

Proof. Let $\gamma\colon [0,1] \to \mathcal{B}(M)$ be a smooth path with $\gamma(i) = (y_i, b_i)$, $i = 1, 2$. Let $X \subset [0,1] \times M$ be the image of the embedding $s \mapsto \bigl(s, \pi\circ\gamma(s)\bigr)$. The normal bundle at $\bigl(s, (\pi\circ\gamma)(s)\bigr)$ can be identified with $T_{\gamma(s)}M$, and we use the framing $\gamma(s)$ to frame X. $\square$

thm:43 **Lemma 2.46.** *Let $B \subset M$ be the image of the open unit ball in some coordinate system on M. Let $Y = \{y_0\} \amalg \{y_1\}$ be the union of disjoint points $y_0, y_1 \in B$ and choose framings which lie in opposite components of $\mathcal{B}(B)$. Then Y is framed bordant to the empty manifold in B.*

Proof. We may as well take B to be the unit ball in $\mathbb{A}^m$, and after a diffeomorphism we may assume $y_0 = (-1/2, 0, \dots, 0)$ and $y_1 = (1/2, 0, \dots, 0)$. We may also reduce to the case where the framings are $\mp\partial/\partial x^1, \partial/\partial x^2, \dots, \partial/\partial x^m$; see the remark following Lemma 2.45. Then let $X \subset [0,1] \times B$ be the image of

eq:34
$$s \longmapsto \Bigl(s(1-s); s - \frac{1}{2}, 0, \dots, 0\Bigr) \tag{2.47}$$

where the framing at time s is

eq:35
$$s(1-s)\frac{\partial}{\partial t} + (2s-1)\frac{\partial}{\partial x^1}\ ,\ \frac{\partial}{\partial x^2}\ ,\ \dots\ ,\ \frac{\partial}{\partial x^m} \tag{2.48}$$

Here t is the coordinate on $[0,1]$. The m vectors in (2.48) project onto a framing of the normal bundle to X in $[0,1] \times M$, as is easily checked. $\square$

thm:44 **Exercise 2.49.** Assemble Lemma 2.44, Lemma 2.45, and Lemma 2.46 into a proof of Theorem 2.37.

thm:38 **Exercise 2.50.** Use Theorem 2.35 to compute $[S^3, S^2]$ and $[S^4, S^3]$. As a warmup you might start with $[S^2, S^1]$, which you can also compute using covering space theory.

Lecture 3: The Pontrjagin-Thom theorem

sec:3

In this lecture we give a proof of Theorem 2.35. You can read an alternative exposition in [M3]. We begin by reviewing some definitions and theorems from differential topology.

Neat submanifolds

Recall the local model (**1.8**) of a manifold with boundary. We now define a robust notion of submanifold for manifolds with boundary.

thm:46

Definition 3.1. Let M be an m-dimensional manifold with boundary. A subset $Y \subset M$ is a *neat submanifold* if about each $y \in Y$ there is a chart (ϕ, U) of M—that is, an open set $U \subset M$ containing y and a homeomorphism $\phi\colon U \to \mathbb{A}^n$ in the atlas defining the smooth structure—such that $\phi(Y) \subset \mathbb{A}^{m-q} \cap \mathbb{A}^m_-$, where $\mathbb{A}^m_-$ is defined in (1.9) and

eq:36

$$\mathbb{A}^{m-q} = \{(x^1, x^2, \dots, x^m) \in \mathbb{A}^m : x^{m-q+1} = \cdots = x^m = 0\}. \tag{3.2}$$

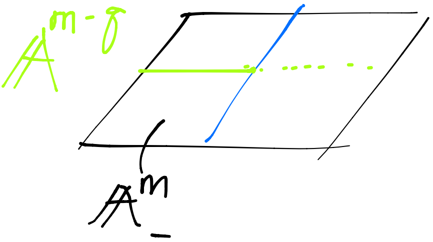

FIGURE 9.

fig:9

The local model induces a smooth structure on Y, so Y is a manifold with boundary, $\partial Y = \partial M \cap Y$, and Y is transverse to ∂M.

subsec:3.1

(3.3) *Normal bundle to neat submanifold.* The neatness condition gives rise to the following diagram of vector bundles over ∂Y:

eq:37 (3.4)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & T(\partial Y) \arrow[r, hook] \arrow[d, hook] & TY\big|_{\partial Y} \arrow[r] \arrow[d, hook] & \mu^Y \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & T(\partial M) \arrow[r, hook] \arrow[d] & TM\big|_{\partial Y} \arrow[r] \arrow[d] & \mu^M \arrow[r] & 0 \\
 & \nu^{\partial} \arrow[r, "\cong"] \arrow[d] & \nu\big|_{\partial Y} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

In this diagram the line bundles μ^Y, μ^M, defined as the indicated horizontal quotients, are the normal bundles to the boundaries of the manifolds Y, M, and the diagram determines an isomorphism between them. Similarly, the vector bundles ν^{∂}, ν, defined as the indicated vertical quotients, are the normal bundles to $\partial Y \subset \partial M$ and $Y \subset M$, respectively; the diagram determines isomorphism between ν^{∂} and the restriction of ν to the boundary of ∂Y.

This shows that there is a well-defined normal bundle $\nu \to Y$ to the neat submanifold $Y \subset M$.

subsec:3.2

(3.5) *Tubular neighborhood of a neat submanifold.* The tubular neighborhood theorem extends to neat submanifolds.

thm:47 **Definition 3.6.** Let M be a manifold with boundary, $Y \subset M$ a neat submanifold, and $\nu \to Y$ its normal bundle. A *tubular neighborhood* is a pair (U, φ) where $U \subset M$ is an open set containing Y and $\varphi\colon \nu \to U$ is a diffeomorphism such that $\varphi\big|_Y = \mathrm{id}_Y$, where we identify $Y \subset \nu$ as the image of the zero section.

thm:48 **Theorem 3.7.** *Tubular neighborhoods exist.*

The proof is easier if Y is compact. In either case one can use Riemannian geometry. Choose a Riemannian metric on M which is a product metric in a collar neighborhood of ∂M. Use the metric to embed $\nu \subset TM\big|_Y$ as the orthogonal complement of TY. Then for an appropriate function $\epsilon\colon TY \setminus Y \to \mathbb{R}^{>0}$ we define $\varphi(\xi)$ to be the time $\epsilon(\xi)$ position of the geodesic with initial position $\pi(\xi)$ and initial velocity $\xi/|\xi|$. Here $\pi\colon \nu \to Y$ is projection and ξ is presumed nonzero.

Proof of Pontrjagin-Thom

thm:49 **Definition 3.8.** Let $f_0, f_1\colon M \to N$ be smooth maps of manifolds. A *smooth homotopy* $F\colon f_0 \to f_1$ is a smooth map $F\colon \Delta^1 \times M \to N$ such that for all $x \in M$ we have $F(0, x) = f_0(x)$ and $F(1, x) = f_1(x)$.

Here $\Delta^1 = [0,1]$ is the 1-simplex. Smooth homotopy is an equivalence relation; the set of equivalence classes is denoted $[M,N]$. This is also the set of homotopy classes of *continuous* maps under continuous homotopy, which can be proved by approximation theorems which show that C^∞ maps are dense in the space of continuous maps.

Recall the definition of $\Omega^{\mathrm{fr}}_{n;M}$ from (2.32); it is the set of framed bordism classes of normally framed n-dimensional closed submanifolds of a smooth manifold M.

thm:50 **Theorem 3.9** (Pontrjagin-Thom)**.** *For any smooth compact m-manifold M there is an isomorphism*

eq:38
$$\phi\colon [M,S^q] \longrightarrow \Omega^{\mathrm{fr}}_{n;M}, \qquad n = m-q. \tag{3.10}$$

The forward map is the inverse image of a regular value; the inverse map is the Pontrjagin-Thom collapse, as illustrated in Figure 8.

Proof. Write $S^q = \mathbb{A}^q \cup \{\infty\}$ (stereographic projection) and fix $p \in \mathbb{A}^q$. Given $f\colon M \to S^q$ use the transversality theorems from differential topology to perturb to a smoothly homotopy $f_0\colon M \to S^q$ such that p is a regular value. Define $\phi\bigl([f]\bigr) = \bigl[(f_0)^{-1}(p)\bigr]$, where $[f]$ is the smooth homotopy class of f and $\bigl[(f_0)^{-1}(p)\bigr]$ is the framed bordism class of the inverse image. Note that $(f_0)^{-1}(p)$ is compact since M is. To see that ϕ is well-defined, suppose $F\colon \Delta^1 \times M \to S^q$ is a smooth homotopy from f_0 to f_1, where p is a simultaneous regular value of f_0, f_1. The transversality theorems imply there is a perturbation F' of F which is transverse to $\{p\}$ and which equals F in a neighborhood of $\{0,1\} \times M \subset \Delta^1 \times M$. Then[5] $(F')^{-1}(p)$ is a framed bordism from $(f_0)^{-1}(p)$ to $(f_1)^{-1}(p)$.

The inverse map

eq:39
$$\psi\colon \Omega^{\mathrm{fr}}_{n;M} \longrightarrow [M,S^q] \tag{3.11}$$

is described in (2.34). That construction depends on a choice of tubular neighborhood (U,φ) and cutoff function (Figure 7). To see it is well-defined, suppose $X \subset \Delta^1 \times M$ is a framed bordism, which in particular is a neat submanifold. We use the existence of tubular neighborhoods (Theorem 3.7) to construct a Pontrjagin-Thom collapse map $\Delta^1 \times M \to S^q$, which is then a smooth homotopy between the Pontrjagin-Thom collapse maps on the boundaries. (We need to know that if we have a tubular neighborhood of $\partial X \subset \partial M$ we can extend that particular tubular neighborhood to one of $X \subset M$. If we construct tubular neighborhoods using geodesics, as indicated in **(3.5)**, then this is a simple matter of extending a Riemannian metric on ∂M to a Riemannian metric on M.)

The composition $\phi \circ \psi$ is clearly the identity. To show that $\psi \circ \phi$ is also the identity, note that if $f_0\colon M \to S^q$ has p as a regular value and we set $Y = (f_0)^{-1}(p)$, then the map $f_1\colon M \to S^q$ representing $(\psi \circ \phi)(f_0)$ also has p as a regular value and $(f_1)^{-1}(p) = Y$. Furthermore, by construction $df_0\big|_Y = df_1\big|_Y$. The desired statement follows from the following lemma.

thm:51 **Lemma 3.12.** *Let M be a closed manifold, $Y \subset M$ a normally framed submanifold, and $f_0, f_2\colon M \to S^q$ such that $(f_0)^{-1}(p) = (f_2)^{-1}(p) = Y$ and $df_0\big|_Y = df_2\big|_Y$, where $p \in \mathbb{A}^q \subset S^q$. Then f_0 is smoothly homotopic to f_2.*

---

[5] This relies on the following theorem: If W is a compact manifold with boundary, $F\colon W \to S$ a smooth map to a manifold S, and $p \in S$ is a regular value of both F and $F\big|_{\partial W}$, then $F^{-1}(p) \subset W$ is a *neat* submanifold.

Proof. We first make a homotopy of f_0 localized in a neighborhood of Y to make f_0 and f_2 agree in a neighborhood of Y. For that choose a tubular neighborhood (U, φ) of Y such that neither f_0 nor f_2 hits $\infty \in S^q$ in U. The framing identifies $U \approx Y \times \mathbb{R}^q$, and under the identification f_0, f_2 correspond to maps $g_0, g_2 \colon Y \times \mathbb{R}^q \to \mathbb{A}^q$. For a cutoff function ρ of the shape of Figure 7 define the homotopy

eq:40
$$(t, y, \xi) \longmapsto g_0(y, \xi) \;+\; t\rho(|\xi|)\bigl(g_2(y, \xi) - g_0(y, \xi)\bigr). \tag{3.13}$$

Let g_1 be the time-one map; it glues to f_0 on the complement of U to give a smooth map $f_1 \colon M \to S^q$. Then $f_1 = f_2$ in a neighborhood $V \subset U$ of Y, and $f_1 = f_0$ on the complement of U. I leave as a calculus exercise to prove that we can adjust the cutoff function (sending it to zero quickly) so that f_1 does not take the value p in $U \setminus V$. This uses the fact that $(dg_0)_{(y,0)} = (dg_1)_{(y,0)}$ for all $y \in Y$.

The second step is to construct a homotopy from f_1 to f_2. For this write $S^q = \mathbb{A}^q \cup \{p\}$, use the fact that both f_1 and f_2 map to the affine part of this decomposition on the complement of V, and then average in that affine space to make the homotopy, as in (3.13). ☐

☐

thm:52 **Exercise 3.14.** Fill in the two missing details in the proof of Lemma 3.12. Namely, first show how to construct a cutoff function ρ so that $f_1(x) \neq p$ for all $x \in U \setminus V$. Construct an example (think low dimensions!) to show that this fails if the normal framings do not agree up to homotopy on Y. Then construct the homotopy in the second step of the proof.

thm:53 **Exercise 3.15.** Show by example that Theorem 3.9 can fail for M noncompact.

thm:54 **Exercise 3.16.** A *framed link* in S^3 is a closed normally framed 1-dimensional submanifold $L \subset S^3$. What can you say about these up to framed bordism, i.e., can you compute $\Omega^{\mathrm{fr}}_{1;S^3}$? Is the framed bordism class of a link an interesting link invariant? How can you compute it?

thm:55 **Exercise 3.17.** A *Lie group* G is a smooth manifold equipped with a point $e \in G$ and smooth maps $\mu \colon G \times G \to G$ and $\iota \colon G \to G$ such that (G, e, μ, ι) is a group. In other words, it is the marriage of a smooth manifold and a group, with compatible structures. Prove that every Lie group is *parallelizable*, i.e., that there exists a trivialization of the tangent bundle $TG \to G$. In fact, construct a canonical trivialization.

thm:56 **Exercise 3.18.** Show that the complex numbers of unit norm form a Lie group $\mathbb{T} \subset \mathbb{C}$. What is the underlying smooth manifold? Do the analogous exercise for the unit quaternions $Sp(1) \subset \mathbb{H}$. The notation $Sp(1)$ suggests that there is also a Lie group $Sp(n)$ for any positive integer n. There is! Construct it.

Lecture 4: Stabilization

`sec:4`

There are many stabilization processes in topology, and often matters simplify in a stable limit. As a first example, consider the sequence of inclusions

`eq:41`
$$S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow S^3 \hookrightarrow \cdots \tag{4.1}$$

where each sphere is included in the next as the equator. If we fix a nonnegative integer n and apply π_n to (4.1), then we obtain a sequence of groups with homomorphisms between them:

`eq:42`
$$\pi_n S^0 \longrightarrow \pi_n S^1 \longrightarrow \pi_n S^2 \longrightarrow \cdots . \tag{4.2}$$

Here the homotopy group $\pi_n(X)$ of a topological space X is the set[6] of homotopy classes of maps $[S^n, X]$, and we must use basepoints, as described below. This sequence stabilizes in a trivial sense: for $m > n$ the group $\pi_n S^m$ is trivial. In this lecture we encounter a different sequence

`eq:43`
$$\pi_n S^0 \longrightarrow \pi_{n+1} S^1 \longrightarrow \pi_{n+2} S^2 \longrightarrow \cdots \tag{4.3}$$

whose stabilization is nontrivial. Here 'stabilization' means that with finitely many exceptions every homomorphism in (4.3) is an isomorphism. The groups thus computed are central in stable homotopy theory: the *stable homotopy groups of spheres.*

One reference for this lecture is [DK, Chapter 8].

Pointed Spaces

This is a quick review; look in any algebraic topology book for details.

`thm:57` **Definition 4.4.**

(i) A *pointed space* is a pair (X, x) where X is a topological space and $x \in X$.
(ii) A *map* $f\colon (X,x) \to (Y,y)$ of pointed spaces is a continuous map $f\colon X \to Y$ such that $f(x) = y$.
(iii) A *homotopy* $F\colon \Delta^1 \times (X,x) \to (Y,y)$ of maps of pointed spaces is a continous map $F\colon \Delta^1 \times X \to Y$ such that $F(t,x) = y$ for all $t \in \Delta^1 = [0,1]$.

The set of homotopy classes of maps between pointed spaces is denoted $\bigl[(X,x),(Y,y)\bigr]$, or if basepoints need not be specified by $[X,Y]_*$.

`thm:58` **Definition 4.5.** Let $(X_i, *_i)$ be pointed spaces, $i = 1, 2$.

---

[6] It is a group for $n \geq 1$, and is an abelian group if $n \geq 2$.

(i) The *wedge* is the identification space

eq:44
$$X_1 \vee X_2 = X_1 \amalg X_2 \;\big/\; *_1 \amalg *_2. \tag{4.6}$$

(ii) The *smash* is the identification space

eq:45
$$X_1 \wedge X_2 = X_1 \times X_2 \;\big/\; X_1 \vee X_2. \tag{4.7}$$

(iii) The *suspension* of X is

eq:46
$$\Sigma X = S^1 \wedge X. \tag{4.8}$$

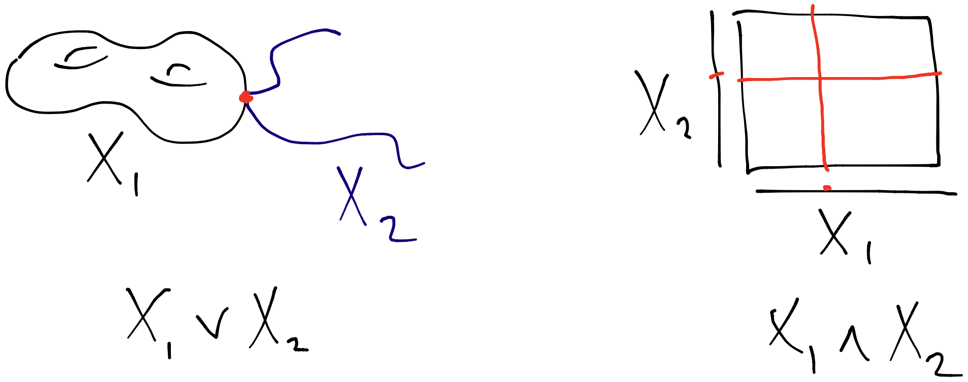

Figure 10. The wedge and the smash fig:10

For the suspension it is convenient to write S^1 as the quotient $D^1/\partial D^1$ of the 1-disk $[-1,1] \subset \mathbb{A}^1$ by its boundary $\{-1,1\}$.

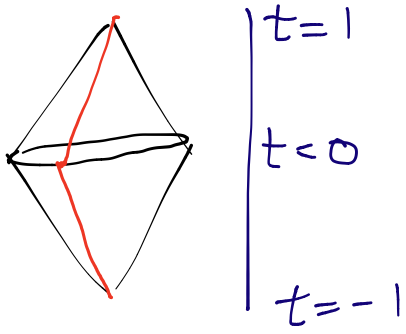

Figure 11. The suspension fig:12

thm:59 **Exercise 4.9.** Construct a homeomorphism $S^k \wedge S^\ell \simeq S^{k+\ell}$. You may find it convenient to write the k-sphere as the quotient of the Cartesian product $(D^1)^{\times k}$ by its boundary.

`thm:60` **Exercise 4.10.** Suppose $f_i\colon X_i \to Y_i$ are maps of pointed spaces, $i = 1, 2$. Construct induced maps

`eq:47`
$$\begin{aligned} f_1 \vee f_2 \colon X_1 \vee X_2 &\longrightarrow Y_1 \vee Y_2 \\ f_1 \wedge f_2 \colon X_1 \wedge X_2 &\longrightarrow Y_1 \wedge Y_2 \end{aligned} \tag{4.11}$$

Suppose all spaces are standard spheres and the maps f_i are smooth maps. Is the map $f_1 \wedge f_2$ smooth? Proof or counterexample.

Note that the suspension of a sphere is a smooth manifold, but in general the suspension of a manifold is not smooth at the basepoint.

`thm:61` **Definition 4.12.** Let $(X, *)$ be a pointed space and $n \in \mathbb{Z}^{\ge 0}$. The n^{th} *homotopy group* $\pi_n(X, *)$ of $(X, *)$ is the set of pointed homotopy classes of maps $\bigl[(S^n, *), (X, *)\bigr]$.

If we write S^n as the quotient $D^n/\partial D^n$ (or as the quotient of $(D^1)^{\times n}$ by its boundary), then it has a natural basepoint. We often overload the notation and use 'X' to denote the pair $(X, *)$. As the terminology suggests, the homotopy set of maps out of a sphere is a group, except for the 0-sphere. Precisely, $\pi_n X$ is a group if $n \ge 1$, and is an abelian group if $n \ge 2$. Figure 12 illustrates the composition in $\pi_n X$, as the composition of a "squeezing map" $S^n \to S^n \vee S^n$ and the wedge $f_1 \vee f_2$.

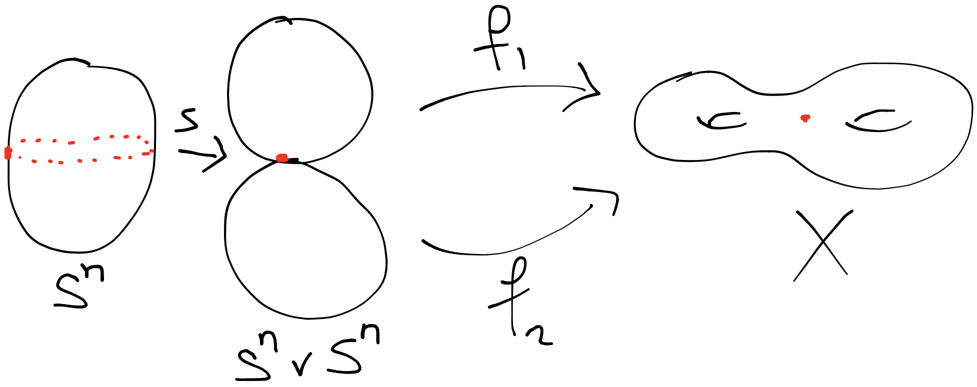

FIGURE 12. Composition in $\pi_n X$ `fig:11`

We refer to standard texts for the proof that this composition is associative, that the constant map is the identity, that there are inverses, and that the composition is commutative if $n \ge 2$.

Stabilization of homotopy groups of spheres

We now study the Pontrjagin-Thom Theorem 3.9 in case $M = S^m$ is a sphere. First apply suspension, to both spaces and maps using Exercise 4.9 and Exercise 4.10, to construct a sequence of group homomorphisms

`eq:48`
$$[S^m, S^q] \xrightarrow{\Sigma} [S^{m+1}, S^{q+1}] \xrightarrow{\Sigma} [S^{m+2}, S^{q+2}] \xrightarrow{\Sigma} \cdots, \tag{4.13}$$

where $m \ge q$ are positive integers.

`thm:62` **Theorem 4.14** (Freudenthal)**.** *The sequence* (4.13) *stabilizes in the sense that all but finitely many maps are isomorphisms.*

The Freudenthal suspension theorem was proved in the late '30s. There are purely algebro-topological proofs. We prove it as a corollary of Theorem 4.44 below and the Pontrjagin-Thom theorem.

`subsec:4.3`

(4.15) *Basepoints.* We can introduce basepoints without changing the groups in (4.13).

`thm:66` **Lemma 4.16.** *If $m, q \geq 1$, then*

`eq:54`
$$[S^m, S^q]_* = [S^m, S^q]. \tag{4.17}$$

Proof. There is an obvious map $[S^m, S^q]_* \to [S^m, S^q]$ since a basepoint-preserving map is, in particular, a map. It is surjective since if $f\colon S^m \to S^q$, then we can compose f with a path R_t of rotations from the identity R_0 to a rotation R_1 which maps $f(*) \in S^q$ to $* \in S^q$. It is injective since if $F\colon D^{m+1} \to S^q$ is a null homotopy of a pointed map $f\colon S^m \to S^q$, then we precompose F with a homotopy equivalence $D^{m+1} \to D^{m+1}$ which maps the radial line segment connecting the center with the basepoint in S^m to the basepoint; see Figure 13. ☐

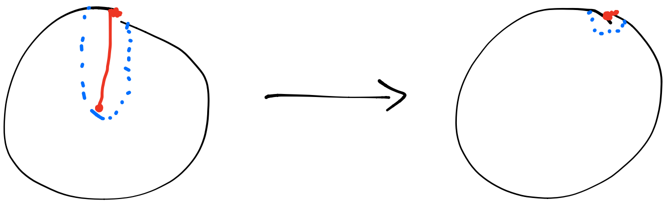

FIGURE 13. Homotopy equivalence of balls `fig:13`

So we can rewrite (4.13) as a sequence of homomorphisms of homotopy groups:

`eq:55`
$$\pi_m S^q \xrightarrow{\Sigma} \pi_{m+1} S^{q+1} \xrightarrow{\Sigma} \pi_{m+2} S^{q+2} \xrightarrow{\Sigma} \cdots \tag{4.18}$$

`subsec:4.1`

(4.19) *A limiting group.* It is natural to ask if there is a group we can assign as the "limit" of (4.18). In calculus we learn about limits, first inside the real numbers and then in arbitrary metric spaces, or in more general topological spaces. Here we want not a limit of elements of a set, but rather a limit of sets. So it is a very different—algebraic—limiting process. The proper setting for such limits is inside a mathematical object whose "elements" are sets, and this is a *category*. We will introduce these in due course, and then the limit we want is, in this case, a *colimit*.[7] We simply give an explicit construction here, in the form of an exercise.

---

[7] Older terminology: *direct limit* or *inductive limit*.

`thm:63` **Exercise 4.20.** Let

`eq:49`

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \tag{4.21}$$

be a sequence of homomorphisms of abelian groups. Define

`eq:50`

$$A = \operatorname*{colim}_{q\to\infty} A_q = \bigoplus_{q=1}^{\infty} A_q \,/\, S \tag{4.22}$$

where S is the subgroup of the direct sum generated by

`eq:99`

$$(f_\ell \circ \cdots \circ f_k)(a_k) \;-\; a_k, \qquad a_k \in A_k, \quad \ell \geq k. \tag{4.23}$$

Prove that A is an abelian group, construct homomorphisms $A_q \to A$, and show they are isomorphisms for $q >> 1$ if the sequence (4.21) stabilizes in the sense that there exists q_0 such that f_q is an isomorphism for all $q \geq q_0$.

`thm:88` **Definition 4.24.** The limiting group of the sequence (4.13) is denoted

`eq:97`

$$\pi_n^s = \operatorname*{colim}_{q\to\infty} \pi_{n+q} S^q \tag{4.25}$$

and is the n^{th} *stable homotopy group of the sphere*, or n^{th} *stable stem*.

Colimits of topological spaces

Question: Is there a pointed space Q so that $\pi_n^s = \pi_n Q$?

There is another construction with pointed spaces which points the way.

`thm:64` **Definition 4.26.** Let $(X, *)$ be a pointed space. The *(based) loop space* of $(X, *)$ is the set of continous maps

`eq:51`

$$\Omega X = \{\gamma\colon S^1 \to X : \gamma(*) = *\}. \tag{4.27}$$

We topologize ΩX using the compact-open topology, and then complete to a compactly generated topology.

`thm:90` **Definition 4.28.** A Hausdorff topological space Z is *compactly generated* if $A \subset Z$ is closed iff $A \cap C$ is closed for every compact subset $C \subset Z$.

Compactly generated Hausdorff spaces are a convenient category in which to work, according to a classic paper of Steenrod [Ste]; see [DK, §6.1] for an exposition. A Hausdorff space Z has a compactly generated completion: declare $A \subset Z$ to be closed in the compactly generated completion iff $A \cap C \subset Z$ is closed in the original topology of Z for all compact subsets $C \subset Z$.

thm:65 **Exercise 4.29.** Let X, Y be pointed spaces. Prove that there is an isomorphism of sets

eq:52
$$\mathrm{Map}_*(\Sigma X, Y) \xrightarrow{\cong} \mathrm{Map}_*(X, \Omega Y). \tag{4.30}$$

Here 'Map$_*$' denotes the set of pointed maps. If X and Y are compactly generated, then the map (4.30) is a homeomorphism of topological spaces, where the mapping spaces have the compactly generated completion of the compact-open topology. Metric spaces, in particular smooth manifolds, are compactly generated. You can find a nice discussion of compactly generated spaces in [DK, §6.1].

Use (4.30) to rewrite (4.18) as

eq:53
$$\pi_n(S^0) \longrightarrow \pi_n(\Omega S^1) \longrightarrow \pi_n(\Omega^2 S^2) \longrightarrow \cdots \tag{4.31}$$

This suggests that the space Q is some sort of limit of the spaces $\Omega^q S^q$ as $q \to \infty$. This is indeed the case.

subsec:4.4

(4.32) *Colimit of a sequence of maps.* Let

eq:93
$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \tag{4.33}$$

be a sequence of continuous inclusions of topological spaces. Then there is a limiting topological space

eq:94
$$X = \operatorname*{colim}_{q\to\infty} X_q = \coprod_{q=1}^{\infty} X_q \,/\!\sim \tag{4.34}$$

equipped with inclusions $g_q\colon X_q \hookrightarrow X$. Here $\sim$ is the equivalence relation generated by setting $x_k \in X_k$ equivalent to $(f_\ell \circ \cdots \circ f_k)(x_k)$ for all $\ell \geq k$. We give X the quotient topology. It is the strongest (finest) topology so that the maps g_q are continuous. More concretely, a set $A \subset X$ is closed iff $A \cap X_q \subset X_q$ is closed for all q. Then X is called the *colimit* of the sequence (4.33).

thm:86 **Exercise 4.35.** Construct S^∞ as the colimit of (4.1). Prove that S^∞ is *weakly contractible*: for $n \in \mathbb{Z}^{\geq 0}$ any map $S^n \to S^\infty$ is null homotopic.

thm:91 **Exercise 4.36.** Show that if each space X_q in (4.33) is Hausdorff compactly generated and f_q is a *closed* inclusion, then the colimit (4.34) is also compactly generated. Furthermore, every compact subset of the colimit is contained in X_q for some q.

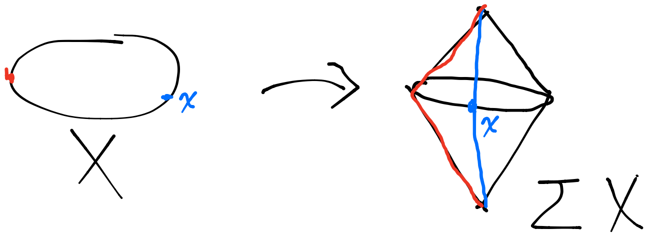

FIGURE 14. The inclusion $X \hookrightarrow \Omega\Sigma X$ fig:15

subsec:4.5

(4.37) *The space QS^0.* Now apply **(4.32)** to the sequence

eq:95

$$S^0 \longrightarrow \Omega S^1 \longrightarrow \Omega^2 S^2 \longrightarrow \cdots \tag{4.38}$$

This is, in fact, a sequence of inclusions of the form $X \hookrightarrow \Omega\Sigma X$, as illustrated in Figure 14. The limiting space of the sequence (4.38) is

eq:96

$$QS^0 := \operatorname*{colim}_{q\to\infty} \Omega^q S^q \tag{4.39}$$

and is the 0-space of the *sphere spectrum*.

thm:87

Proposition 4.40. $\pi_n^s = \pi_n(QS^0)$.

Proof. More generally, for a sequence of closed inclusions of compactly generated Hausdorff spaces (4.33) we prove

eq:98

$$\pi_n(\operatorname*{colim}_{q\to\infty} X_q) \cong \operatorname*{colim}_{q\to\infty} \pi_n X_q. \tag{4.41}$$

Let $X = \operatorname{colim}_{q\to\infty} X_q$. A class in $\pi_n X$ is represented by a continuous map $f\colon S^n \to X$, and by the last assertion you proved in Exercise 4.36 f factors through a map $\tilde f\colon S^n \to X_q$ for some q. This shows that the natural map $\operatorname{colim}_{q\to\infty} \pi_n X_q \to \pi_n X$ is surjective. Similarly, a null homotopy of the composite $S^n \xrightarrow{\tilde f} X_q \hookrightarrow X$ factors through some X_r, $r \ge q$, and this proves that this natural map is also injective. ☐

Stabilization of framed submanifolds

By Theorem 3.9 we can rewrite (4.13) as a sequence of maps

eq:56

$$\Omega^{\mathrm{fr}}_{n;S^m} \xrightarrow{\sigma} \Omega^{\mathrm{fr}}_{n;S^{m+1}} \xrightarrow{\sigma} \Omega^{\mathrm{fr}}_{n;S^{m+2}} \xrightarrow{\sigma} \cdots \tag{4.42}$$

Representatives of these framed bordism groups are submanifolds of S^m. Write $S^m = \mathbb{A}^m \cup \{\infty\}$, and recall from the proof in Lecture §3 that each framed bordism class is represented by a framed submanifold $Y \subset \mathbb{A}^m$; we can arrange $\infty \notin Y$. This is the analog of passing to a sequence of pointed maps, as in Lemma 4.16.

We make two immediate deductions from the identification with (4.13). First, we must have that each $\Omega^{\mathrm{fr}}_{n;S^m}$ is an abelian group. The abelian group law is the disjoint union of submanifolds of $\mathbb{A}^m$, effected by writing $\mathbb{A}^m = \mathbb{A}^m \amalg \mathbb{A}^m$ (similar to the collapse map in Figure 12). Second, the stabilization map σ in (4.42) is the map

eq:57

$$(Y \subset \mathbb{A}^m) \longmapsto (0 \times Y \subset \mathbb{A}^1 \times \mathbb{A}^m) \tag{4.43}$$

and the new normal framing prepends the constant vector field $\partial/\partial x^1$ to the given normal framing of Y.

We can now state the stabilization theorem.

thm:67 **Theorem 4.44.** *The map* $\sigma\colon \Omega^{\mathrm{fr}}_{n;S^m} \to \Omega^{\mathrm{fr}}_{n;S^{m+1}}$ *is an isomorphism for* $m \geq 2n+2$.

As a corollary we obtain a precise estimate on the Freudenthal isomorphism, using the Pontrjagin-Thom identification.

thm:68 **Corollary 4.45.** *The map* $\Sigma\colon \pi_m S^q \to \pi_{m+1} S^{q+1}$ *is an isomorphism for* $m \leq 2q-2$.

We will not prove the precise estimate in Theorem 4.44, and so not the precise estimate in Corollary 4.45 either. Rather, we only prove Theorem 4.44 for sufficiently large m, where sufficiently large depends on n. This suffices to prove the stabilization.

thm:92 **Exercise 4.46.** Show that the bound in Theorem 4.44 is optimal for $n = 1$.

The proof of Theorem 4.44 is based on the Whitney Embedding Theorem. We restrict to compact manifolds. Recall that for compact manifolds embeddings are easier to handle since they are injective immersions. An *isotopy of embeddings* $Y \hookrightarrow \mathbb{A}^N$ is a smooth map

eq:100

$$\Delta^1 \times Y \longrightarrow \mathbb{A}^N \tag{4.47}$$

so that the restriction to $\{t\} \times Y$ is an embedding for all $t \in \Delta^1$. In other words, an isotopy of embeddings is a path of embeddings.

thm:69 **Theorem 4.48.** *Let* Y *be a smooth compact* n*-manifold.*

(i) *There exists an embedding* $i\colon Y \hookrightarrow \mathbb{A}^{2n+1}$. *Furthermore, if* $i\colon Y \hookrightarrow \mathbb{A}^N$ *is an embedding with* $N > 2n+1$, *then there is an isotopy of* i *to an embedding into an affine subspace* $\mathbb{A}^{2n+1} \subset \mathbb{A}^N$.

(ii) *If* $i_0, i_1\colon Y \hookrightarrow \mathbb{A}^{2n+1}$ *are embeddings, then their stabilizations*

eq:101

$$\begin{aligned} \tilde{\imath}_k\colon Y &\longrightarrow \mathbb{A}^{2n+1} \times \mathbb{A}^{2n+1} \\ y &\longmapsto \bigl(0, i_k(y)\bigr) \end{aligned} \tag{4.49}$$

($k = 0, 1$*) are isotopic.*

(iii) *Let* X *be a compact* $(n+1)$*-manifold with boundary. Then there is an embedding* $X \hookrightarrow \mathbb{A}^{2n+3}_{-}$ *as a neat submanifold with boundary.*

Assertion (i) is the *easy* Whitney Embedding Theorem, and we refer to [GP] for a proof. The second statement in (i) follows from the proof, which uses linear projection onto an affine subspace to reduce the dimension of the embedding. Statement (iii) is stated as [Hi, Theorem 4.3]; perhaps ⇒ in the mythical next version of these notes I'll supply a proof. [**do it!**] In any case we do not need the statement with 'neat', and without 'neat' the proof is essentially the same as that of (i). We remark that the *hard* Whitney Embedding Theorem asserts that there is an embedding $Y \hookrightarrow \mathbb{A}^{2n}$.

Proof. We prove (ii). The desired isotopy $\tilde{\imath}_0 \to \tilde{\imath}_1$ is constructed as the composition in time of three isotopies:

eq:102

$$\begin{aligned} &(t,y) \longmapsto \big(t\, i_0(y)\ ,\ i_0(y)\big), && 0 \le t \le 1;\\ &(t,y) \longmapsto \big(i_0(y)\ ,\ (2-t)\, i_0(y) + (t-1)\, i_1(y)\big), && 1 \le t \le 2;\\ &(t,y) \longmapsto \big((3-t)\, i_0(y)\ ,\ i_1(y)\big), && 2 \le t \le 3; \end{aligned} \tag{4.50}$$

$\square$ ⇐

thm:94

Exercise 4.51. [**not an embedding? if $i_0(y) = 0$ in first guy**] Check that the map $[0,3] \times Y \to \mathbb{A}^{4n+2}$ defined by (4.50) is an embedding. Now use the technique of the Whitney Embedding Theorem to project onto a subspace of dimension $2n+2$ so that the composition is still an embedding. Can you use this to prove that, in fact, the stabilizations of i_0, i_1 to embeddings $Y \hookrightarrow \mathbb{A}^{2n+2}$ are isotopic? (That statement can be proved using an approximation theorem; see Exercise 10 in [Hi, p. 183].)

thm:93

Exercise 4.52. A *parametrized knot* is an embedding $i\colon S^1 \to \mathbb{A}^3$. Exhibit two parametrized knots which are not isotopic. Can you *prove* that they are not isotopic? The proof above shows that they are isotopic when stabilized to embeddings $\tilde{\imath}\colon S^1 \to \mathbb{A}^6$. Prove that they are isotopic when stabilized to embeddings $\tilde{\imath}\colon S^1 \to \mathbb{A}^4$.

Sketch proof of Theorem 4.44. To show that $\sigma\colon \Omega^{\mathrm{fr}}_{n;S^m} \to \Omega^{\mathrm{fr}}_{n;S^{m+1}}$ is surjective, suppose $i_0\colon Y \hookrightarrow \mathbb{A}^{m+1}$ is an embedding. By Theorem 4.48(i) there is an isotopy $\Delta^1 \times Y \to \mathbb{A}^{m+1}$ to an embedding $i_1\colon Y \hookrightarrow \mathbb{A}^m \subset \mathbb{A}^{m+1}$. To show that σ is injective, suppose $j_0\colon Y \hookrightarrow \mathbb{A}^m$ is an embedding and $k_0\colon X \hookrightarrow \mathbb{A}^{m+1}$ is a null bordism of the composition $Y \xrightarrow{j_0} \mathbb{A}^m \subset \mathbb{A}^{m+1}$. Then Theorem 4.48(iii) implies there is an isotopy $k_t\colon \Delta^1 \times X \longrightarrow \mathbb{A}^{m+1}$ with $k_1(X) \subset \mathbb{A}^m$ a null bordism of j_0. $\square$

There is one problem: we have not discussed the normal framings. Briefly, in both the surjectivity and injectivity arguments there is an isotopy $\Delta^1 \times Z \to \mathbb{A}^{m+1}$, a normal bundle $\nu \to \Delta^1 \times Z$, and a framing of $\nu\big|_{\{0\}\times Z}$. We need two general results to get the desired framing of $\nu\big|_{\{1\}\times Z}$. First, we can extend the given framing over $\{0\} \times Z$ to the entire cylinder $\Delta^1 \times Z$, for example using parallel transport of a connection (so solving an ODE). Second, the restriction of ν to $\{1\} \times Z$ splits off a trivial line bundle, and we can homotop the framing to one which respects this splitting. This follows from a stability statement for homotopy groups of the general linear group. Perhaps these arguments will appear in that mystical future revision… [**do it!**]. ⇐

Lecture 5: More on stabilization

sec:5

In this lecture we continue the introductory discussion of stable topology. Recall that in Lecture §4 we introduced the *stable stem* $\pi^s_\bullet$, the stable homotopy groups of the sphere. We show that there is a ring structure: $\pi^s_\bullet$ is a $\mathbb{Z}$-graded commutative ring (Definition 1.33). The stable Pontrjagin-Thom theorem identifies it with stably normally framed submanifolds of a sphere. Here we see how stable *normal* framings are equivalent to stable *tangential* framings, and so define a ring $\Omega^{\mathrm{fr}}_\bullet$ of stably tangentially framed manifolds with no reference to an embedding. The image of the *J-homomorphism* gives some easy classes in the stable stem from the stable homotopy groups of the orthogonal group. We describe some low degree classes in terms of *Lie groups*.

A reference for this lecture is [DK, Chapter 8].

Ring structure

Recall that elements in the abelian group π^s_n are represented by homotopy classes $\pi_{q+n}S^q$ for q sufficiently large. The multiplication in $\pi^s_\bullet$ is easy to describe. Suppose given classes $a_1 \in \pi^s_{n_1}$ and $a_2 \in \pi^s_{n_2}$, which are represented by maps

eq:59

$$\begin{aligned} f_1\colon S^{q_1+n_1} &\longrightarrow S^{q_1} \\ f_2\colon S^{q_2+n_2} &\longrightarrow S^{q_2} \end{aligned} \tag{5.1}$$

Then the product $a_1 \cdot a_2 \in \pi^s_{n_1+n_2}$ is represented by the smash product (Exercise 4.10)

eq:60

$$f_1 \wedge f_2\colon S^{q_1+n_1} \wedge S^{q_2+n_2} \longrightarrow S^{q_1} \wedge S^{q_2}. \tag{5.2}$$

Recall that the smash product of spheres is a sphere (Exercise 4.9), so $f_1 \wedge f_2$ does represent an element of $\pi^s_{n_1+n_2}$.

There is a corresponding ring structure on framed manifolds, which we will construct in the next section.

Tangential framings

subsec:5.1

(5.3) *Short exact sequences of vector bundles.* Let

eq:61

$$0 \longrightarrow E' \xrightarrow{i} E \xrightarrow{j} E'' \longrightarrow 0 \tag{5.4}$$

be a short exact sequence of vector bundles over a smooth manifold Y.[8] A *splitting* of (5.4) is a linear map $E'' \xrightarrow{s} E$ such that $j \circ s = \mathrm{id}_{E''}$. A splitting determines an isomorphism

eq:66
$$E'' \oplus E' \xrightarrow{s \oplus i} E. \tag{5.5}$$

thm:70 **Lemma 5.6.** *The space of splittings is a nonempty affine space over the vector space* $\mathrm{Hom}(E'', E')$.

Let's deconstruct that statement, and in the process prove parts of it. First, if s_0, s_1 are splittings, then the difference $\phi = s_1 - s_0$ is a linear map $E'' \to E$ such that $j \circ \phi = 0$. The exactness of (5.4) implies that ϕ factors through a map $\tilde{\phi}\colon E'' \to E'$: in other words, $\phi = i \circ \tilde{\phi}$. This, then, is the affine structure. But we must prove that the space of splittings is nonempty. For that we use a partition of unity argument. Remember that partitions of unity can be used to average sections of a fiber bundle whose fibers are convex subsets of affine spaces. Of course, an affine space is a convex subset of itself.

I outline some details in the following exercise.

thm:71 **Exercise 5.7.**

(i) Construct a vector bundle $\underline{Hom}(E'', E') \to Y$ whose sections are homomorphisms $E'' \to E'$. Similarly, construct an *affine bundle* (a fiber bundle whose fibers are affine spaces) whose sections are splittings of (5.4). You will need to use local trivializations of the vector bundles E, E', E'' to construct these fiber bundles.

(ii) Produce the partition of unity argument. You should prove that if $\mathcal{A} \to Y$ is an affine bundle, and $\mathcal{E} \to Y$ is a fiber subbundle whose fibers are convex subsets of $\mathcal{A}$, then there exist sections of $\mathcal{E} \to Y$. Even better, topologize[9] the space of sections and prove that the space of sections is contractible.

(iii) This is a good time to review the partition of unity argument for the existence of Riemannian metrics. Phrase it in terms of sections of a fiber bundle (which?). More generally, prove that any real vector bundle $\nu \to Y$ admits a positive definite metric, i.e., a smoothly varying inner product on each fiber.

subsec:5.4

(5.8) *Stable framings.* Let $E \to Y$ be a vector bundle of rank q. A *stable framing* or *stable trivialization* of $E \to Y$ is an isomorphism $\phi\colon \underline{\mathbb{R}^{k+q}} \xrightarrow{\cong} \underline{\mathbb{R}^k} \oplus E$ for some $k \geq 0$. A *homotopy of stable framings* is a homotopy of the isomorphism ϕ. We identify ϕ with

eq:71
$$\mathrm{id}_{\underline{\mathbb{R}^\ell}} \oplus \phi\colon \underline{\mathbb{R}^{\ell+k+q}} \xrightarrow{\cong} \underline{\mathbb{R}^{\ell+k}} \oplus E \tag{5.9}$$

for any ℓ. With these identifications we define a set of *homotopy classes of stable framings*.

---

[8] These can be real, complex, or quaternionic.

[9] The topological space of a smooth manifold is metrizable; one can use the metric space structure induced from a Riemannian metric, for example. Then you can topologize the space of sections using the *topology of uniform convergence on compact sets*. One needn't use the metrizability and can describe this as the *compact-open topology*.

subsec:5.2

(5.10) *The stable tangent bundle of the sphere.* Let $S^m \in \mathbb{A}^{m+1}$ be the standard unit sphere, defined by the equation

eq:62

$$(x^1)^2 + (x^2)^2 + \cdots + (x^{m+1})^2 = 1. \tag{5.11}$$

Then the vector field

eq:63

$$\sum_i x^i \frac{\partial}{\partial x^i}, \tag{5.12}$$

restricted to S^m, gives a trivialization of the normal bundle ν to $S^m \subset \mathbb{A}^{m+1}$. Recall that the tangent bundle to $\mathbb{A}^{m+1}$ is the trivial bundle $\underline{\mathbb{R}^{m+1}} \to \mathbb{A}^{m+1}$. Then a splitting of the short exact sequence

eq:64

$$0 \longrightarrow TS^m \longrightarrow \underline{\mathbb{R}^{m+1}} \longrightarrow \nu \to 0 \tag{5.13}$$

over S^m gives a stable trivialization

eq:65

$$\underline{\mathbb{R}} \oplus TS^m \cong \underline{\mathbb{R}^{m+1}} \tag{5.14}$$

of the tangent bundle to the sphere.

subsec:5.3

(5.15) *Stable normal and tangential framings.* Now suppose $Y \subset S^m$ is a submanifold of dimension n with a normal framing, which we take to be an isomorphism $\underline{\mathbb{R}^q} \xrightarrow{\cong} \mu$, where μ is the rank $q = m - n$ normal bundle defined by the short exact sequence

eq:67

$$0 \longrightarrow TY \longrightarrow TS^m\big|_Y \longrightarrow \mu \longrightarrow 0 \tag{5.16}$$

This induces a short exact sequence

eq:68

$$0 \longrightarrow TY \longrightarrow \underline{\mathbb{R}} \oplus TS^m\big|_Y \longrightarrow \underline{\mathbb{R}} \oplus \mu \longrightarrow 0 \tag{5.17}$$

Choose a splitting of (5.17) and use the stable trivialization (5.14) and the trivialization of the normal bundle μ to obtain an isomorphism

eq:69

$$\underline{\mathbb{R}^{q+1}} \oplus TY \xrightarrow{\cong} \underline{\mathbb{R}^{m+1}} \tag{5.18}$$

of vector bundles over Y. This is a *stable tangential framing* of Y, and is one step in the proof of the following.

thm:72 **Proposition 5.19.** *Let $Y \subset S^m$ be a submanifold. Then there is a 1:1 correspondence between homotopy classes of stable normal framings of Y and stable tangential framings of Y.*

Proof. The argument before the proposition defines a map from (stable) normal framings to stable tangential framings. Conversely, if $\underline{\mathbb{R}}^k \oplus TY \xrightarrow{\cong} \underline{\mathbb{R}}^{k+n}$ is a stable tangential framing, with $k \geq 1$, then from a splitting of the short exact sequence

eq:70

$$0 \longrightarrow \underline{\mathbb{R}}^k \oplus TY \longrightarrow \underline{\mathbb{R}}^k \oplus TS^m\big|_Y \longrightarrow \mu \longrightarrow 0 \tag{5.20}$$

we obtain a stable normal framing $\mu \oplus \underline{\mathbb{R}}^{k+n} \xrightarrow{\cong} \underline{\mathbb{R}}^{k+m}$. I leave it to you to check that homotopies of one framing induce homotopies of the other, and that the two maps of homotopy classes are inverse. ☐

Application to framed bordism

Recall the stabilization sequence (4.42) of normally framed submanifolds $Y \subset S^m$. The stabilization sits $S^m \subset S^{m+1}$ as the equator and prepends the standard normal vector field $\partial/\partial x^1$ to the framing. By Proposition 5.19 the normal framing induces a stable tangential framing of Y, and the homotopy class of the stable tangential framing is unchanged under the stabilization map σ in the sequence (4.42). Conversely, if Y^n has a stable tangential framing, then by the Whitney embedding theorem we realize $Y \subset S^m$ as a submanifold for some m, and then by Proposition 5.19 there is a stable framing $\underline{\mathbb{R}}^{q+k} \xrightarrow{\cong} \mu$ of the normal bundle. This is then a framing of the normal bundle to $Y \subset S^{m+k}$, which defines an element of $\Omega^{\mathrm{fr}}_{n;S^{m+k}}$. This argument proves

thm:73 **Proposition 5.21.** *The colimit of* (4.42) *is the bordism group* Ω^{fr}_n *of* n*-manifolds with a stable tangential framing.*

A bordism between two stably framed manifolds Y_0, Y_1 is, informally, a compact $(n+1)$-manifold X with boundary $Y_0 \amalg Y_1$ and a stable tangential framing of X which restricts on the boundary to the given stable tangential framings of Y_i. The formal definition follows Definition 1.19.

The following is a corollary to Theorem 3.9.

thm:74 **Corollary 5.22** (stable Pontrjagin-Thom)**.** *There is an isomorphism*

eq:72

$$\phi\colon \pi^s_n \longrightarrow \Omega^{\mathrm{fr}}_n \tag{5.23}$$

for each $n \in \mathbb{Z}^{\geq 0}$.

subsec:5.5

(5.24) *Ring structure.* Letting n vary we obtain an isomorphism $\phi\colon \pi^s_\bullet \to \Omega^{\mathrm{fr}}_\bullet$ of $\mathbb{Z}$-graded abelian groups. We saw at the beginning of this lecture that the domain is a $\mathbb{Z}$-graded *ring*. So there is a corresponding ring structure on codomain. It is given by *Cartesian product*. For recall that we may assume that the representatives f_1, f_2 of two classes a_1, a_2 in the stable stem (see (5.1)) are *pointed*, in the sense they map the basepoint ∞ to ∞, and then these map under ϕ to the submanifolds Y_1, Y_2 defined as the inverse images of $p_i \in S^{q_i}$, where $p_i \neq \infty$. Then $Y_1 \times Y_2$ is the inverse image of $(p_1, p_2) \in S^{q_1} \wedge S^{q_2}$.

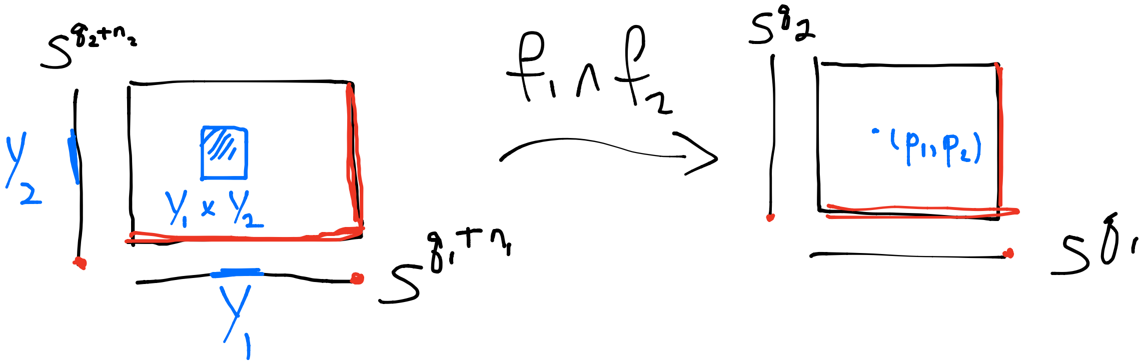

FIGURE 15. Ring structure on $\Omega^{\mathrm{fr}}_{\bullet}$ fig:14

J-homomorphism

subsec:5.6

(5.25) *Twists of framing.* Let $Y \subset M$ be a normally framed submanifold of a smooth manifold M, and suppose its codimension is q. Denote the framing as $\phi\colon \underline{\mathbb{R}^q} \to \nu$, where ν is the normal bundle. Let $g\colon Y \to GL_q\mathbb{R} = GL(\mathbb{R}^q)$ be a smooth map. Then $\phi \circ g$ is a new framing of ν, the *g-twist of ϕ*.

thm:75 *Remark* 5.26. As stated in Exercise 5.7(iii), there is a positive definite metric on the normal bundle $\nu \to Y$, and the metric is a contractible choice. Furthermore, the Gram-Schmidt process gives a deformation retraction of all framings onto the space of *orthonormal* framings. Let

eq:73
$$O(q) = \{g\colon \mathbb{R}^q \to \mathbb{R}^q : g \text{ is an isometry}\} \tag{5.27}$$

denote the *orthogonal group* of $q \times q$ orthogonal matrices. Then we can twist orthonormal framings by a map $g\colon Y \to O(q)$.

thm:76 **Exercise 5.28.** Construct a deformation retraction of $GL_q(\mathbb{R})$ onto $O(q)$. Start with the case $q = 1$ to see what is going on, and you might try $q = 2$ as well. For the general case, you might consider the Gram-Schmidt process.

thm:77 **Exercise 5.29.** Is the space of maps $Y \to O(q)$ contractible? Proof or counterexample.

subsec:5.7

(5.30) *The unstable J-homomorphism.* Specialize to $M = S^m$ and let $Y = S^n \subset S^m$ be an equatorial n-sphere with the canonical normal framing. Explicitly, write $S^m = \mathbb{A}^m \cup \{\infty\}$ as usual, introduce standard affine coordinates $x^1, \dots, x^m$, and let S^n be the unit n-sphere

eq:74
$$S^n = \{(x^1, \dots, x^m) : x^1 = \dots = x^{q-1} = 0, \quad (x^q)^2 + \dots + (x^m)^2 = 1\}. \tag{5.31}$$

We use the framing $\partial/\partial r, \partial/\partial x^1, \dots, \partial/\partial x^{q-1}$, where $\partial/\partial r$ is the outward normal to S^n in the affine subspace $\mathbb{A}^{n+1}$ defined by $x^1 = \dots = x^{q-1} = 0$. Then restricting to pointed maps $g\colon S^n \to O(q)$ we obtain a homomorphism

eq:75
$$J\colon [S^n, O(q)]_* \longrightarrow \Omega^{\mathrm{fr}}_{n;S^m}. \tag{5.32}$$

Applying Pontrjagin-Thom we can rewrite this as

eq:76

$$J\colon \pi_n O(q) \longrightarrow \pi_{n+q} S^q. \tag{5.33}$$

This is the *unstable J-homomorphism.*

thm:80 **Exercise 5.34.** Show that the normally framed S^n in (5.31) is null bordant: it bounds the unit ball D^{n+1} in $\mathbb{A}^{n+1}$ with normal framing $\partial/\partial x^1, \dots, \partial/\partial x^{q-1}$. Then show that (5.32) is indeed a homomorphism.

thm:81 **Exercise 5.35.** Work out some special cases of (5.32) and (5.33) explicitly. Try $n = 0$ first. Then try $n = 1$ and $m = 2, 3$. You should discover that there is a nontrivial map $S^3 \to S^2$, "nontrivial" in the sense that it is not homotopic to a constant map. Here is one explicit, geometric construction. Consider the complex vector space $\mathbb{C}^2$, and restrict scalars to the real numbers $\mathbb{R} \subset \mathbb{C}$. Show that the underlying vector space is isomorphic to $\mathbb{R}^4$. Each unit vector $\xi \in S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$ spans a complex line $\ell(\xi) = \mathbb{C} \cdot \xi \subset \mathbb{C}^2$. The resulting map $S^3 \to \mathbb{P}(\mathbb{C}^2) = \mathbb{CP}^1 \simeq S^2$ is not null homotopic. Prove this by considering the inverse image of a regular value.

subsec:5.9

(5.36) *The stable orthogonal group.* There is a natural sequence of inclusions

eq:77

$$O(1) \xrightarrow{\sigma} O(2) \xrightarrow{\sigma} O(3) \xrightarrow{\sigma} \cdots \tag{5.37}$$

At the end of this lecture we construct the limiting space

eq:78

$$O = \operatorname*{colim}_{q \to \infty} O(q). \tag{5.38}$$

As a set it is the union of the $O(q)$. Its homotopy groups are the (co)limit of the homotopy groups of the finite $O(q)$. More precisely, for each $n \in \mathbb{Z}^{\geq 0}$ the sequence

eq:79

$$\pi_n O(1) \longrightarrow \pi_n O(2) \longrightarrow \pi_n O(3) \longrightarrow \cdots \tag{5.39}$$

stabilizes.

thm:78 **Exercise 5.40.** Prove this. One method is to use the transitive action of $O(q)$ on the sphere S^{q-1}. Check that the stabilizer of a point (which?) is $O(q-1) \subset O(q)$. Use this to construct a fiber bundle with total space $O(q)$ and base S^{q-1}. In fact, this is a principal bundle with structure group $O(q-1)$. Now apply the long exact sequence of homotopy groups for a fiber bundle (more generally, fibration), as explained for example in [H1, §4.2].

The stable homotopy groups of the orthogonal group (as well as the unitary and symplectic groups) were computed by Bott in the late 1950's using Morse theory. The following is known as the *Bott periodicity theorem.*

thm:79 **Theorem 5.41** (Bott)**.** *For all $n \in \mathbb{Z}^{\geq 0}$ there is an isomorphism $\pi_{n+8} O \cong \pi_n O$. The first few homotopy groups are*

eq:81

$$\pi_{\{0,1,2,3,4,5,6,7\}} O \cong \{\mathbb{Z}/2\mathbb{Z}\,,\ \mathbb{Z}/2\mathbb{Z}\,,\ 0\,,\ \mathbb{Z}\,,\ 0\,,\ 0\,,\ 0\,,\ \mathbb{Z}\}. \tag{5.42}$$

A vocal rendition of the right hand side of (5.42) is known as the *Bott song.*

subsec:5.10

(5.43) *The stable J-homomorphism.* The (co)limit $q \to \infty$ in (5.33) gives the *stable J-homomorphism*

eq:80

$$J\colon \pi_n O \longrightarrow \pi_n^s. \tag{5.44}$$

Lie groups

thm:82 **Definition 5.45.** A *Lie group* is a quartet (G, e, μ, ι) consisting of a smooth manifold G, a basepoint $e \in G$, and smooth maps $\mu\colon G \times G \to G$ and $\iota\colon G \to G$ such that the underlying set G and the map μ define a group with identity element e and inverse map ι.

It is often fruitful in mathematics to combine concepts from two different areas. A Lie group, the marriage of a group and a smooth manifold, is one of the most fruitful instances.

If you have never encountered Lie groups before, I recommend [War, §3] for an introduction to some basics. We will not review these here, but just give some examples of *compact* Lie groups.

subsec:5.11

(5.46) *Orthogonal groups.* We already introduced the orthogonal group $O(q)$ in (5.27). The identity element e is the identity $q \times q$ matrix. The multiplication μ is matrix multiplication. The inverse $\iota(A)$ of an orthogonal matrix is its transpose. You can check by explicit formulas that μ and ι are smooth. The orthogonal group has two components, distinguished by the determinant homomorphism

eq:82

$$\det\colon O(q) \longrightarrow \{\pm 1\}. \tag{5.47}$$

The identity component—the kernel of (5.47)—is the *special orthogonal group $SO(q)$*.

subsec:5.12

(5.48) *Unitary groups.* There is an analogous story over the complex numbers. The *unitary group* is

eq:83

$$U(q) = \{g\colon \mathbb{C}^q \to \mathbb{C}^q : g \text{ is an isometry}\}, \tag{5.49}$$

where we use the standard hermitian metric on $\mathbb{C}^q$. Again μ is matrix multiplication and now ι is the transpose conjugate. The unitary group $U(1)$ is the group of unit norm complex numbers, which we denote '$\mathbb{T}$'. The kernel of the determinant homomorphism

eq:84

$$\det\colon U(q) \longrightarrow \mathbb{T} \tag{5.50}$$

is the *special unitary group $SU(n)$*.

thm:83 **Exercise 5.51.** Work out the analogous story for the quaternions $\mathbb{H}$. Define a metric on $\mathbb{H}^q$ using quaternionic conjugation. Define the group $Sp(q)$ of isometries of $\mathbb{H}^q$. Now there is no determinant homomorphism. Show that the underlying smooth manifold of the Lie group $Sp(1)$ of unit norm quaternions is diffeomorphic to S^3. Note, then, that $O(1), U(1), Sp(1)$ are diffeomorphic to S^0, S^1, S^3, respectively.

subsec:5.13 **(5.52)** *Parallelization of Lie groups.* Let G be a Lie group. Then any $g \in G$ determines left multiplication

eq:85
$$\begin{aligned} L_g\colon G &\longrightarrow G \\ x &\longmapsto gx \end{aligned} \tag{5.53}$$

which is a diffeomorphism that maps e to g. Its differential is then an isomorphism

eq:86
$$d(L_g)_e\colon T_eG \xrightarrow{\;\cong\;} T_gG. \tag{5.54}$$

This defines a *parallelism* $\underline{T_eG} \xrightarrow{\;\cong\;} TG$, a trivialization of the tangent bundle of G. There is a similar, but if G is nonabelian different, parallelism using right translation. A parallelism determines a homotopy class of stable tangential framings. Thus we have shown

thm:84 **Proposition 5.55.** *The left invariant parallelism of a compact Lie group G determine a class $[G] \in \Omega^{\mathrm{fr}}_\bullet \cong \pi^s_\bullet$ in the stable stem.*

Low dimensions

The first several stable homotopy groups of spheres are

eq:87
$$\pi^s_{\{0,1,2,3,4,5,6,7,8\}} \cong \{\mathbb{Z}\,,\, \mathbb{Z}/2\mathbb{Z}\,,\, \mathbb{Z}/2\mathbb{Z}\,,\, \mathbb{Z}/24\mathbb{Z}\,,\, 0\,,\, 0\,,\, \mathbb{Z}/2\mathbb{Z}\,,\, \mathbb{Z}/240\mathbb{Z}\,,\, \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\}. \tag{5.56}$$

It is interesting to ask what part of this is in the image of the stable J-homomorphism (5.44). Not much: compare (5.42) and (5.56). Throwing out π^s_0 we have that J is surjective on π^s_n for $n = 1, 3, 7$. The first class which fails to be in the image of J is the generator of π^s_2.

We have more luck looking for classes represented by compact Lie groups in the left invariant framing. Lie groups do represent the generators of the first several groups, starting in degree one:

eq:88
$$\mathbb{T}\;,\; \mathbb{T}\times\mathbb{T}\;,\; Sp(1)\;,\; -\;,\; -\;,\; Sp(1)\times Sp(1) \tag{5.57}$$

There is no compact Lie group which represents the generator of $\pi^s_7 \cong \mathbb{Z}/240\mathbb{Z}$, but that class is represented by a Hopf map

eq:89
$$S^{15} \longrightarrow S^8, \tag{5.58}$$

analogous to the Hopf map $S^3 \to S^2$ described in Exercise 5.35.

thm:85 **Exercise 5.59.** As an intermediary construct the Hopf map $S^7 \to S^4$ by realizing S^7 as the unit sphere in $\mathbb{C}^4 \cong \mathbb{H}^2$ and S^4 as the quaternionic projective line $\mathbb{HP}^1$. Now use the quaternions and octonions to construct (5.58).

Returning to the stable stem, the 8-dimensional Lie group $SU(3)$ represents the generator of π^s_8. For more discussion of the stable stem in low degrees, see [Ho].

subsec:5.14 **(5.60)** π_3^s *and the K3 surface.* As stated in (5.57) the generator of $\pi_3^s \cong \mathbb{Z}/24\mathbb{Z}$ is represented by $Sp(1) \cong SU(2)$ in the left invariant framing. Recall that the underlying manifold is the 3-sphere S^3. The following argument, often attributed to Atiyah, proves that 24 times the class of S^3 vanishes. It does not prove that any smaller multiple does not vanish, but perhaps we will prove that later in the course by constructing a bordism invariant

eq:90
$$\Omega_3^{\mathrm{fr}} \longrightarrow \mathbb{Z}/24\mathbb{Z} \tag{5.61}$$

which is an isomorphism. To prove that 24 times this class vanishes we construct a compact 4-manifold X with a parallelism (framing of the tangent bundle) whose boundary has 24 components, each diffeomorphic to S^3, and such that the framing restricts to a stabilization of the Lie group framing. The argument combines ideas from algebraic geometry, geometric PDE, and algebraic and differential topology. We will only give a brief sketch.

First, let $W \subset \mathbb{CP}^3$ be the smooth complex surface cut out by the quartic equation

eq:91
$$(z^0)^4 + (z^1)^4 + (z^2)^4 + (z^3)^4 = 0, \tag{5.62}$$

where z^0, z^1, z^2, z^3 are homogeneous coordinates on $\mathbb{CP}^3$. Then W is a compact (real) 4-manifold. Characteristic class computations, which we will learn in a few lectures, can be used to prove that the Euler characteristic of W is 24. Further computation and theorems of Lefschetz prove that W is simply connected and has vanishing first Chern class. Now a deep theorem of Yau—his proof of the Calabi conjecture—constructs a *hyperkähler metric* on W. This in particular gives a quaternionic structure on each tangent space. In other words, there are global endomorphisms

eq:92
$$I, J, K \colon TW \longrightarrow TW \tag{5.63}$$

which satisfy the algebraic relations $I^2 = J^2 = K^2 = -\,\mathrm{id}_{TW}$, $IJ = -JI$, etc.

Let $\xi \colon W \to TW$ be a smooth vector field on W which is transverse to the zero section and has exactly 24 simple zeros. Let X be the manifold W with open balls excised about the zeros of ξ, and deform ξ so that it is the outward normal vector field at the boundary ∂X. Then the global vector fields $\xi, I\xi, J\xi, K\xi$ provide the desired parallelism.

Lecture 6: Classifying spaces

sec:6

A vector bundle $E \to M$ is a family of vector spaces parametrized by a smooth manifold M. We ask: Is there a *universal* such family? In other words, is there a vector bundle $E^{\text{univ}} \to B$ such that any vector bundle $E \to M$ is obtained from $E^{\text{univ}} \to B$ by *pullback*? If so, what is this universal parameter space B for vector spaces? This is an example of a *moduli problem*. In geometry there are many interesting spaces which are universal parameter spaces for geometric objects. In this lecture we study universal parameter spaces for linear algebraic objects: *Grassmannians*, named after the 19th century mathematician Hermann Grassmann. We will see that there is no finite dimensional manifold which is a universal parameter space B. This is typical: to solve a moduli problem we often have to expand the notion of "space" with which we begin. Here there are several choices, one of which is to use an infinite dimensional manifold. Another is to use a colimit of finite dimensional manifolds, as in (**4.32**). Yet another is to pass to *simplicial sheaves*, but we do not pursue that here.

The universal parameter space B is called a *classifying space*: it classifies vector bundles. Classifying spaces are important in bordism theory. We use them to define *tangential structures*, which are important in both the classical and modern contexts.

For much of this lecture we do not specify whether the vector bundles are real, complex, or quaternion. All are allowed. In the last part of the lecture we discuss classifying spaces for *principal bundles*, a more general notion.

One excellent reference for some of this and the following lecture is [BT, Chapter IV].

Grassmannians

Let V be a finite dimensional vector space and k an integer such that $0 \leq k \leq \dim V$.

thm:95 **Definition 6.1.** The *Grassmannian* $Gr_k(V)$ is the collection

eq:103
$$Gr_k(V) = \{W \subset V : \dim W = k\} \tag{6.2}$$

of all linear subspaces of V of dimension k. Similarly, we define the Grassmannian

eq:104
$$Gr_{-k}(V) = \{W \subset V : \dim W + k = \dim V\} \tag{6.3}$$

of codimension k linear subspaces of V.

We remark that the notation in (6.3) is nonstandard. The Grassmannian is more than a set: it can be given the structure of a smooth manifold. The following exercise is a guide to defining this.

thm:96 **Exercise 6.4.**

(i) Introduce a locally Euclidean topology on $Gr_k(V)$. Here is one way to do so: Suppose $W \in Gr_k(V)$ is a k-dimensional subspace and C an $(n-k)$-dimensional subspace such that $W \oplus C = V$. (We say that C is a complement to W in V.) Then define a subset $\mathcal{O}_{W,C} \subset Gr_k(V)$ by

eq:206
$$\mathcal{O}_{W,C} = \{W' \subset V : W' \text{ is the graph of a linear map } W \to C\}. \tag{6.5}$$

Show that $\mathcal{O}_{W,C}$ is a vector space, so has a natural topology. Prove that it is consistent to define a subset $U \subset Gr_k(V)$ to be open if and only if $U \cap \mathcal{O}_{W,C}$ is open for all W, C. Note that $\{\mathcal{O}_{W,C}\}$ is a cover of $Gr_k(V)$. (For example, show that $W \in \mathcal{O}_{W,C}$.)

(ii) Use the open sets $\mathcal{O}_{W,C}$ to construct an atlas on $Gr_k(V)$. That is, check that the transition functions are smooth. (Hint: You may first want to check it for two charts with the same W but different complements. Then it suffices to check for two different W which are transverse, using the same complement for both.)

(iii) Prove that $GL(V)$ acts smoothly and transitively on $Gr_k(V)$. What is the subgroup which fixes $W \in Gr_k(V)$?

thm:102 **Exercise 6.6.** Introduce an inner product on V and construct a diffeomorphism $Gr_k(V) \to Gr_{-k}(V)$.

thm:108 **Exercise 6.7.** Be sure you are familiar with the projective spaces $Gr_1(V) = \mathbb{P}V$ for $\dim V = 2$. (What about $\dim V = 1$?) Do this over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.

subsec:6.1

(6.8) *Universal vector bundles over the Grassmannian.* There is a tautological exact sequence

eq:105
$$0 \longrightarrow S \longrightarrow \underline{V} \longrightarrow Q \longrightarrow 0 \tag{6.9}$$

of vector bundles over the Grassmannian $Gr_k(V)$. The fiber of the *universal subbundle* S at $W \in Gr_k(V)$ is W, and the fiber of the *universal quotient bundle* Q at $W \in Gr_k(V)$ is the quotient V/W. The points of $Gr_k(V)$ are vector spaces—subspaces of V—and the universal subbundle is the family of vector spaces parametrized by $Gr_k(V)$.

thm:97 **Exercise 6.10.** For $k = 1$ we denote $Gr_k(V)$ as $\mathbb{P}V$; it is called the *projective space* of V. Construct a tautological linear map

eq:106
$$V^* \longrightarrow \Gamma(\mathbb{P}V; S^*) \tag{6.11}$$

where the codomain is the space of sections of the *hyperplane bundle* $S^* \to \mathbb{P}V$. This bundle is often denoted $\mathcal{O}(1) \to \mathbb{P}V$.

Pullbacks and classifying maps

subsec:6.2

(6.12) *Pullbacks of vector bundles.* Just as functions and differential forms pullback under smooth maps—they are *contravariant objects* on a smooth manifold—so too do vector bundles.

thm:98 **Definition 6.13.** Let $f\colon M' \to M$ be a smooth map and $\pi\colon E \to M$ a smooth vector bundle. The *pullback* $\pi'\colon f^*E \to M'$ is the vector bundle whose total space is

eq:107
$$f^*E = \{(m', e) \in M' \times E : f(m') = \pi(e)\}; \tag{6.14}$$

the projection $\pi'\colon f^*E \to M'$ is the restriction of projection $M' \times E \to M'$ onto the first factor. So we have a canonical isomorphism of fibers

eq:108
$$(f^*E)_{p'} = E_{f(p')}, \qquad p' \in M'. \tag{6.15}$$

Projection $M' \times E \to E$ onto the second factor restricts to the map $\tilde{f}$ in the *pullback diagram*

eq:109 (6.16)
$$\begin{tikzcd} f^*E \arrow[r, "\tilde{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$

Quite generally, if $E' \to M'$ is any vector bundle, then a commutative diagram of the form

eq:110 (6.17)
$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$

in which $\tilde{f}$ is a linear isomorphism on each fiber expresses $E' \to M'$ as the pullback of $E \to M$ via f: it defines an isomorphism $E' \to f^*E$.

Vector bundles may simplify under pullback; they can't become more "twisted".

thm:99 **Exercise 6.18.** Consider the Hopf map $f\colon S^3 \to S^2$, which you constructed in Exercise 5.35. Identify S^2 as the complex projective line $\mathbb{CP}^1 = \mathbb{P}(\mathbb{C}^2)$. Let $\pi\colon S \to \mathbb{P}(\mathbb{C}^2)$ be the universal subbundle. It is nontrivial—it does not admit a global trivialization—though we have not yet proved that. Construct a trivialization of the pullback $f^*S \to S^3$. This illustrates the general principle that bundles may untwist under pullback.

subsec:6.3

(6.19) *Classifying maps.* Now we show that any vector bundle $\pi\colon E \to M$ may be expressed as a pullback of the universal quotient bundle[10] over a Grassmannian, at least in case M is compact.

thm:100 **Theorem 6.20.** *Let $\pi\colon E \to M$ be a vector bundle of rank k over a compact manifold M. Then there is a finite dimensional vector space V and a smooth maps $f, \tilde{f}$ which express π as the pullback*

eq:111 (6.21)
$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & Q \arrow[d] \\ M \arrow[r, "f"] & Gr_{-k}(V) \end{tikzcd}$$

---

[10] We can use the universal subbundle instead, but the construction we give makes the universal quotient bundle more natural.

Proof. Since $E \to M$ is locally trivializable and M is compact, there is a finite cover $\{U^\alpha\}_{\alpha \in A}$ of M and a basis $s_1^\alpha, \dots, s_k^\alpha \colon U^\alpha \to E$ of local sections over each U^α. Let $\{\rho^\alpha\}$ be a partition of unity subbordinate to the cover $\{U^\alpha\}$. Then $\tilde{s}_i^\alpha = \rho^\alpha s_i^\alpha$ extend to global sections of E which vanish outside U^α. Define V to be the linear span of the finite set $\{\tilde{s}_i^\alpha\}_{\alpha \in A,\, i=1,\dots,k}$ over the ground field. Then for each $p \in M$ the linear map

eq:112

$$\begin{aligned} ev_p \colon V &\longrightarrow E_p \\ \tilde{s}_i^\alpha &\longmapsto \tilde{s}_i^\alpha(p) \end{aligned} \tag{6.22}$$

is surjective and induces an isomorphism $V/\ker ev_p \xrightarrow{\cong} E_p$. The inverses of these isomorphisms fit together to form the map $\tilde{f}$ in the diagram (6.21), where f is defined by $f(p) = \ker ev_p$. ☐

Classifying spaces

Theorem 6.20 shows that every vector bundle $\pi \colon E \to M$ over a smooth compact manifold is pulled back from the Grassmannian, but it does not provide a single classifying space for *all* vector bundles; the vector space V depends on π. Furthermore, we might like to drop the assumption that M is compact (and even generalize further to continuous vector bundles over nice topological spaces). There are several approaches, and we outline three of them here. For definiteness we work over $\mathbb{R}$; the same arguments apply to $\mathbb{C}$ and $\mathbb{H}$.

subsec:6.4

(6.23) *The infinite Grassmannian as a colimit.* Fix $k \in \mathbb{Z}^{>0}$ and consider the sequence of closed inclusions

eq:113

$$\mathbb{R}^q \longrightarrow \mathbb{R}^{q+1} \longrightarrow \mathbb{R}^{q+1} \longrightarrow \cdots , \tag{6.24}$$

where at each stage the map is $(\xi^1, \xi^2, \dots) \mapsto (0, \xi^1, \xi^2, \dots)$. There is an induced sequence of closed inclusions

eq:114

$$Gr_k(\mathbb{R}^q) \longrightarrow Gr_k(\mathbb{R}^{q+1}) \longrightarrow Gr_k(\mathbb{R}^{q+2}) \longrightarrow \cdots \tag{6.25}$$

where at each stage the map is $W \mapsto 0 \oplus W$. Similarly, there is an induced sequence of closed inclusions

eq:115

$$Gr_{-k}(\mathbb{R}^q) \longrightarrow Gr_{-k}(\mathbb{R}^{q+1}) \longrightarrow Gr_{-k}(\mathbb{R}^{q+2}) \longrightarrow \cdots \tag{6.26}$$

where at each stage the map is $K \mapsto \mathbb{R} \oplus K$. These maps fit together to a lift of (6.26) to pullback maps of the universal quotient bundles:

eq:116

(6.27)
$$\begin{tikzcd} Q_q \arrow[r] \arrow[d] & Q_{q+1} \arrow[r] \arrow[d] & Q_{q+2} \arrow[r] \arrow[d] & \cdots \\ Gr_{-k}(\mathbb{R}^q) \arrow[r] & Gr_{-k}(\mathbb{R}^{q+1}) \arrow[r] & Gr_{-k}(\mathbb{R}^{q+2}) \arrow[r] & \cdots \end{tikzcd}$$

We take the colimit (see (**4.32**)) of this diagram to obtain a vector bundle[11]

eq:117 (6.28)
$$\pi\colon Q^{\text{univ}} \longrightarrow B_k.$$

Now B_k is a topological space—we don't attempt an infinite dimensional smooth manifold structure here—and π is a continous vector bundle. Any classifying map (6.21) for a vector bundle over a compact smooth manifold induces a classifying map into $Q^{\text{univ}} \to B_k$. More is true, but we will not prove this here; see [H2, Theorem 1.16], for example.

thm:101 **Theorem 6.29.** *Let $\pi\colon E \to M$ be a vector bundle over a metrizable space M. Then there is a classifying diagram*

eq:118 (6.30)
$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & Q^{\text{univ}} \arrow[d] \\ M \arrow[r, "f"] & B_k \end{tikzcd}$$

and the map f is unique up to homotopy. Furthermore, the set of homotopy classes of maps $M \to B_k$ is in 1:1 correspondence with the set of isomorphism classes of vector bundles $E \to M$. ⇐

[**Use notation $BO(n)$. Add section about BO as double colimit. Be careful that the two stabilizations commute: the relevant diagram is needed in (9.48), (9.63), (10.19), (10.30), and is written explicitly in (10.37). Some of these need to be adjusted and perhaps moved sooner.**]

subsec:6.5

(6.31) *The infinite Grassmannian as an infinite dimensional manifold.* Let $\mathcal{H}$ be a separable (real, complex, or quaternionic) Hilbert space. Fix $k \in \mathbb{Z}^{>0}$. Define the Grassmannian

eq:119 (6.32)
$$Gr_k(\mathcal{H}) = \{W \subset \mathcal{H} : \dim W = k\}.$$

We can use the technique of Exercise 6.4 to introduce charts and a manifold structure on $Gr_k(\mathcal{H})$, but now the local model is an infinite dimensional Hilbert space.

Digression: Calculus in finite dimensions is developed on (affine spaces over) finite dimensional vector spaces. A topology on the vector space is needed to take the limits necessary to compute derivatives, and there is a unique topology compatible with the vector space structure. It is usually described by a Euclidean metric, i.e., by an inner product on the vector space. In infinite dimensions one also needs a topology compatible with the linear structure, but now there are many different species of topological vector space. By far the easiest, and the closest to the finite dimensional situation, is the topology induced from a Hilbert space structure: a *complete* inner product. That is the topology we use here, and then the main theorems of differential calculus go through almost without change.

We call $Gr_{-k}(\mathcal{H})$ a *Hilbert manifold.*

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[11] Using the standard inner product, as in Exercise 6.6, we can take orthogonal complements to replace codimension k subspaces with dimension k subspaces and the universal quotient bundle with the universal subbundle.

Choose an orthonormal basis $e_1, e_2, \dots$ of $\mathcal{H}$ and so define the subspace $\mathbb{R}^q \subset \mathcal{H}$ as the span of $e_1, e_2, \dots, e_q$. This induces a commutative diagram

eq:120 (6.33)
$$\begin{tikzcd} \cdots \arrow[r] & Gr_k(\mathbb{R}^{q-1}) \arrow[r] \arrow[dr] & Gr_k(\mathbb{R}^q) \arrow[r] \arrow[d] & Gr_k(\mathbb{R}^{q+1}) \arrow[r] \arrow[dl] & \cdots \\ & & Gr_k(\mathcal{H}) & & \end{tikzcd}$$

of inclusions, and so an inclusion of the colimit

eq:121 (6.34)
$$i\colon B_k \longrightarrow Gr_k(\mathcal{H}).$$

thm:103 **Proposition 6.35.** *The map i in (6.34) is a homotopy equivalence.*

One way to prove Proposition 6.35 is to first show that i is a *weak* homotopy equivalence, that is, the induced map $i_*\colon \pi_n B_k \to \pi_n Gr_k(\mathcal{H})$ is an isomorphism for all n. (We must do this for all basepoints $p \in B_k$ and the corresponding $i(p) \in Gr_k(\mathcal{H})$.) Then we would show that the spaces in (6.34) have the homotopy type of CW complexes. For much more general theorems along these lines, see [Pa1]. In any case I include Proposition 6.35 to show that there are different models for the classifying space which are homotopy equivalent.

subsec:6.6

(6.36) *Classifying space as a simplicial sheaf.* We began with the problem of classifying finite rank vector bundles over a compact smooth manifold. We found that the classifying space is not a compact smooth manifold, nor even a finite dimensional manifold. We have constructed two models: a topological space B_k and a smooth manifold $Gr_k(\mathcal{H})$. There is a third possibility which expands the idea of "space" in a more radical way: to a simplicial sheaf on the category of smooth manifolds. This is too much of a digression at this stage, so we will not pursue it. The manuscript [FH] in progress contains expository material along these lines.

Classifying spaces for principal bundles

Recall first the definition.

thm:104 **Definition 6.37.** Let G be a Lie group. A *principal G bundle* is a fiber bundle $\pi\colon P \to M$ over a smooth manifold M equipped with a right G-action $P \times G \to P$ which is simply transitive on each fiber.

The hypothesis that π is a fiber bundle means it admits local trivializations. For a *principal* bundle a local trivialization is equivalent to a local section. In one direction, if $U \subset M$ and $s\colon U \to P$ is a section of $\pi\big|_U\colon P\big|_U \to U$, then there is an induced local trivialization

eq:122 (6.38)
$$\begin{aligned} \varphi\colon U \times G &\longrightarrow P \\ x, g &\longmapsto s(x) \cdot g \end{aligned}$$

where '$\cdot$' denotes the G-action on P.

subsec:6.7

(6.39) *From vector bundles to principal bundles and back.* Let $\pi\colon E\to M$ be a vector bundle of rank k. Assume for definiteness that π is a *real* vector bundle. There is an associated principal $GL_k(\mathbb{R})$-bundle $\mathcal{B}(E)\to M$ whose fiber at $x\in M$ is the spaces of bases $b\colon \mathbb{R}^k\xrightarrow{\cong}E_x$. These fit together into a principal bundle which admits local sections: a local section of the principal bundle $\mathcal{B}(E)\to M$ is a local trivialization of the vector bundle $E\to M$. Conversely, if $P\to M$ is a principal $G=GL_k(\mathbb{R})$-bundle, then there is an *associated* rank k vector bundle $E\to M$ defined as

eq:123
$$E=P\times\mathbb{R}^k/G, \tag{6.40}$$

where the right G-action on $P\times\mathbb{R}^k$ is

eq:124
$$(p,\xi)\cdot g=(p\cdot g,g^{-1}\xi),\qquad p\in P,\quad \xi\in\mathbb{R}^k,\quad g\in G, \tag{6.41}$$

and we use the standard action of $GL_k(\mathbb{R})$ on $\mathbb{R}^k$ to define $g^{-1}\xi$.

subsec:6.8

(6.42) *Fiber bundles with contractible fiber.* We quote the following general proposition in the theory of fiber bundles.

thm:105 **Proposition 6.43.** *Let $\pi\colon\mathcal{E}\to M$ be a fiber bundle whose fiber F is contractible and a metrizable topological manifold, possibly infinite dimensional. Assume that the base M is metrizable. Then π admits a section. Furthermore, if $\mathcal{E},M,F$ all have the homotopy type of a CW complex, then π is a homotopy equivalence.*

See [Pa1] for a proof of the first assertion. The last assertion follows from the long exact sequence of homotopy groups and Whitehead's theorem (**6.53**). [**Put Whitehead earlier; can we make a better statement of this proposition?**] ⇐

subsec:6.9

(6.44) *Classifying maps for principal bundles.* Now we characterize universal principal bundles.

thm:106 **Theorem 6.45.** *Let G be a Lie group. Suppose $\pi^{\mathrm{univ}}\colon P^{\mathrm{univ}}\to B$ is a principal G-bundle and P^{univ} is a contractible metrizable topological manifold.*[12] *Then for any continuous principal G-bundle $P\to M$ with M metrizable, there is a classifying diagram*

eq:125
$$\begin{tikzcd} P \arrow[r, "\tilde\varphi"] \arrow[d] & P^{\mathrm{univ}} \arrow[d] \\ M \arrow[r, "\varphi"] & B \end{tikzcd} \tag{6.46}$$

In the commutative diagram (6.46) the map $\tilde\varphi$ commutes with the G-actions on P,P^{univ}, i.e., it is a map of principal G-bundles.

Proof. A G-map $\tilde\varphi$ is equivalently a section of the associated fiber bundle

eq:126
$$(P\times P^{\mathrm{univ}})/G\to M \tag{6.47}$$

formed by taking the quotient by the diagonal right G-action. The fiber of the bundle (6.47) is P^{univ}. Sections exist by Proposition 6.43, since P^{univ} is contractible. ☐

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[12] We allow an infinite dimensional manifold modeled on a Hilbert space, say.

subsec:6.10

(6.48) *Back to Grassmannians.* The construction in **(6.39)** defines a principal $GL_k(\mathbb{R})$-bundle over the universal Grassmannian, but we can construct it directly and it has a nice geometric meaning. We work in the infinite dimensional manifold model **(6.31)**. Thus let $\mathcal{H}$ be a separable (real) Hilbert space. Introduce the infinite dimensional *Stiefel manifold*

eq:127

$$St_k(\mathcal{H}) = \{b\colon \mathbb{R}^k \to \mathcal{H} : b \text{ is injective}\}. \tag{6.49}$$

It is an open subset of the linear space $\operatorname{Hom}(\mathbb{R}^k, \mathcal{H}) \cong \mathcal{H} \oplus \cdots \oplus \mathcal{H}$, which we give the topology of a Hilbert space. Then the open subset $St_k(\mathcal{H})$ is a Hilbert manifold. There is an obvious projection

eq:128

$$\pi\colon St_k(\mathcal{H}) \longrightarrow Gr_k(\mathcal{H}) \tag{6.50}$$

which maps b to its image $b(\mathbb{R}^k) \subset \mathcal{H}$. We leave the reader to check that π is smooth. In fact, π is a principal bundle with structure group $GL_k(\mathbb{R})$.

thm:107

Theorem 6.51. *$St_k(\mathcal{H})$ is contractible.*

thm:109

Corollary 6.52. *The bundle* (6.50) *is a universal $GL_k(\mathbb{R})$-bundle.*

The corollary is an immediate consequence of Theorem 6.51 and Theorem 6.45. We give the proof of Theorem 6.51 below.

subsec:6.12

(6.53) *Remark on contractibility.* A fundamental theorem of Whitehead asserts that if X, Y are connected[13] pointed topological spaces which have the homotopy type of a CW complex, and $f\colon X \to Y$ is a continuous map which induces an isomorphism $f_*\colon \pi_n X \to \pi_n Y$ for all $n \in \mathbb{Z}^{\geq 0}$, then f is a homotopy equivalence. A map which satisfies the hypothesis of the theorem is called a *weak homotopy equivalence*. An immediate corollary is that if X satisfies the hypotheses and all homotopy groups of X vanish, then X is contractible. For "infinite spaces" with a colimit topology, weak contractibility can often be verified by an inductive argument. That is the case for the Stiefel space $St_k(\mathbb{R}^\infty)$ with a colimit topology, analogous to that for the Grassmannian in (6.25). We prefer instead a more beautiful geometric argument using the Hilbert manifold $St_k(\mathcal{H})$, which is homotopy equivalent (as in Proposition 6.35).

thm:110

Exercise 6.54. Carry out this argument. You will want to consider submersions $St_k(\mathbb{R}^q) \to St_{k-1}(\mathbb{R}^q)$, as we do below. Then you will need the long exact sequence of homotopy groups for a fibration.

subsec:6.13

(6.55) *The unit sphere in Hilbert space.* The Stiefel manifold $St_1(\mathcal{H})$ is the unit sphere $S(\mathcal{H}) \subset \mathcal{H}$, the space of unit norm vectors. As a first case of Theorem 6.51 we prove that this infinite dimensional sphere with the induced topology is contractible, summarizing an elegant argument of Richard Palais [Pa2].

thm:112

Lemma 6.56. *Let X be a normal topological space and $A \subset X$ a closed subspace homeomorphic to $\mathbb{R}$. Then there exists a fixed point free continuous map $f\colon X \to X$.*

---

[13] Whitehead's theorem easily extends to nonconnected spaces.

Proof. The map $x \mapsto x+1$ on $\mathbb{R}$ induced a map $g\colon A \to A$ with no fixed points. By the Tietze extension theorem g extends to a map $\tilde{g}\colon X \to A$. Let f be the extension g followed by the inclusion $A \hookrightarrow X$. ☐

thm:111 **Theorem 6.57.** *$S(\mathcal{H})$ is contractible.*

Proof. Let $\{e_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis of $\mathcal{H}$, set $S = S(\mathcal{H})$ and let $D = \{\xi \in \mathcal{H} : \|\xi\| \le 1\}$ be the closed unit ball in $\mathcal{H}$. Define $i\colon \mathbb{R} \hookrightarrow D$ by letting $i\big|_{[n,n+1]}$ be a curve on S which connects e_n and e_{n+1}, $n \in \mathbb{Z}$. Explicitly, for $t \in [n, n+1]$,

eq:129
$$i\colon t \longmapsto \cos\bigl[(t-n)\pi/2\bigr]e_n + \sin\bigl[(t-n)\pi/2\bigr]e_{n+1}. \tag{6.58}$$

Then by the lemma there is a continous map $f\colon D \to D$ with no fixed points. We use it, as in Hirsh's beautiful proof of the Brouwer fixed point theorem, to construct a deformation retraction $g\colon D \to S$: namely, $g(\xi)$ is the intersection of S with the ray emanating from $\xi \in D$ in the direction $\xi - f(\xi)$. Then g is a homotopy equivalence. On the other hand, there is an easy radial deformation retraction of D to $0 \in D$, and so D is contractible. ☐

Proof of Theorem 6.51. Let $\pi\colon St_k(\mathcal{H}) \to St_{k-1}(\mathcal{H})$ map $b\colon \mathbb{R}^k \to \mathcal{H}$ to the restriction of b to $\mathbb{R}^{k-1} \subset \mathbb{R}^k$. In terms of bases, if b maps the standard basis of $\mathbb{R}^k$ to $\xi_1, \xi_2, \dots, \xi_k$, then $\bar{b} = \pi(b)$ gives the independent vectors $\xi_2, \dots, \xi_k$. The fiber over $\bar{b}$ deformation retracts onto the set of nonzero vectors in the orthogonal complement $\mathcal{H}'$ of the span of $\xi_2, \dots, \xi_k$, which is a closed subspace of $\mathcal{H}$, hence a Hilbert space. Now the set of nonzero vectors in a Hilbert space deformation retracts onto the unit sphere, which by Theorem 6.57 is contractible. Then Proposition 6.43 implies that π is a homotopy equivalence. Now proceed by induction, beginning with the statement that $St_1(\mathcal{H})$ is contractible. ☐

thm:115 *Remark* 6.59. An alternative proof of Theorem 6.51 is based on *Kuiper's theorem*, which states that the Banach Lie group $GL(\mathcal{H})$ of all invertible linear operators $\mathcal{H} \to \mathcal{H}$ in the norm topology is contractible. This group acts transitively on $St_k(\mathcal{H})$ with stabilizer a contractible group. It follows that the quotient is also contractible.

subsec:6.14

(6.60) *Other Lie groups.* Let G be a *compact* Lie group. (Note G need not be connected.) The *Peter-Weyl theorem* asserts that there is an embedding $G \subset U(k) \subset GL_k(\mathbb{C})$ for some $k > 0$. Let $EG = St_k(\mathcal{H})$ be the Stiefel manifold for a *complex* separable Hilbert space $\mathcal{H}$. Then the restriction of the free $GL_k(\mathbb{C})$-action to G is also free; let BG be the quotient. It is a Hilbert manifold, and

eq:130
$$EG \longrightarrow BG \tag{6.61}$$

is a universal principal G-bundle, by Theorem 6.45.

This gives Hilbert manifold models for the classifying space of any compact Lie group.

thm:113 **Exercise 6.62.** What is the classifying Hilbert manifold of $O(1) = \mathbb{Z}/2\mathbb{Z}$? What about $\mathbb{T} = U(1)$? What about the unit quaternions $Sp(1)$? Show that the classifying Hilbert manifold of a finite cyclic group is an infinite dimensional *lens space*.

thm:114

Exercise 6.63. Let G be a connected compact Lie group and $T \subset G$ a maximal torus. Then T acts freely on EG, and there is an induced fiber bundle $BT \to BG$. What is the fiber? Describe both manifolds explicitly for the classical groups $G = O(k)$, $U(k)$, and $Sp(k)$.

Lecture 7: Characteristic classes

sec:7

In this lecture we describe some basic techniques in the theory of characteristic classes, mostly focusing on Chern classes of complex vector bundles. There is lots more to say than we can do in a single lecture. Much of what we say follows the last chapter of [BT], which is posted on the web site, and so these notes are terse on some points which you can read in detail there. I highly encourage you to do so!

I will summarize a few results on the computation of the ring of characteristic classes, but we will not attempt to prove them here. Those proof require more algebraic topology than I can safely assume.

Classifying revisited

In Lecture 6 we sloughed over the classification statement, which appeared in passing in the statement of Theorem 6.29. Here is a definitive version.

thm:116 **Theorem 7.1.** *Let G be a Lie group and $EG \to BG$ a universal principal G-bundle. Then for any manifold M there is a 1:1 correspondence*

eq:131
$$[M, BG] \xrightarrow{\cong} \{\text{isomorphism classes of principal } G\text{-bundles over } M\}. \tag{7.2}$$

To a map $f\colon M \to BG$ we associate the bundle $f^*EG \to M$. We gave some ingredients in the proof. For example, Theorem 6.45 proves that (7.2) is surjective. One idea missing is that if $f_0, f_1\colon M \to BG$ are homotopic, then $f_0^*(EG) \to M$ is isomorphic to $f_1^*(EG) \to M$. We give a proof in case all maps are smooth and we use a Hilbert manifold model for the universal bundle, as in **(6.60)**.

thm:117 **Proposition 7.3.** *Let $P \to \Delta^1 \times M$ be a smooth principal G-bundle. The the restrictions $P\,\big|_{\{0\}\times M} \to M$ and $P\big|_{\{1\}\times M} \to M$ are isomorphic.*

The assertion about homotopic maps is an immediate corollary: if $F\colon \Delta^1 \times M \to BG$ is a homotopy, consider $F^*(EG) \to \Delta^1 \times M$.

The proof uses the existence of a *connection* and the fundamental existence and uniqueness theorem for ordinary differential equations. Let $\pi\colon P \to N$ be a smooth principal G-bundle. Then at each $p \in P$ there is a short exact sequence

eq:132
$$0 \longrightarrow \ker(\pi_*)_p \longrightarrow T_pP \longrightarrow T_{\pi(p)}N \longrightarrow 0 \tag{7.4}$$

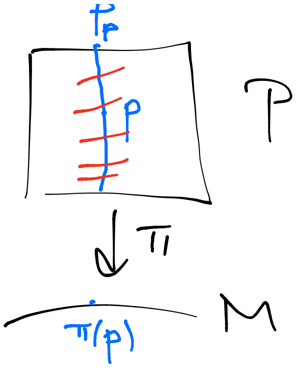

FIGURE 16. A connection fig:16

thm:118 **Definition 7.5.** A *horizontal subspace* at p is a splitting of (7.4). A *connection* is a G-invariant splitting of the sequence of vector bundles

eq:133
$$0 \longrightarrow \ker \pi_* \longrightarrow TP \longrightarrow \pi^*TN \longrightarrow 0 \tag{7.6}$$

over P.

Recall from Lemma 5.6 that splittings form an affine space. Fix $n \in N$. The G-invariant splittings of (7.4) for $p \in \pi^{-1}(n)$ form a finite dimensional affine space. As n varies these glue together into an affine bundle over N. A partition of unity argument (Exercise 5.7) then shows that connections exist.

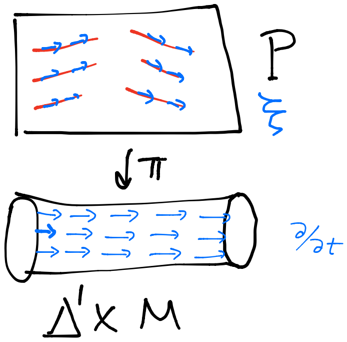

FIGURE 17. Homotopy invariance fig:17

Proof of Proposition 7.3. Let $\partial/\partial t$ denote the vector field on $\Delta^1 \times M$ which is tangent to the $\Delta^1 = [0,1]$ factor. Choose a connection on $\pi\colon P \to \Delta^1 \times M$. The connection determines a G-invariant vector field ξ on P which projects via π to $\partial/\partial t$. The fundamental theorem for ODE gives, for each initial condition $p \in P\big|_{\{0\}\times M}$ an integral curve $\gamma_p\colon [0,1] \to P$ whose composition with $\pi_1 \circ \pi$ is the identity. Here $\pi_1\colon \Delta^1 \times M \to \Delta^1$ is projection onto the first factor. The map $p \mapsto \gamma_p(1)$ is the desired isomorphism of principal bundles. □

thm:140 **Exercise 7.7.** Prove Theorem 7.1.

The idea of characteristic classes

Let X be a topological space. There is an associated chain complex

eq:134

$$C_0 \longleftarrow C_1 \longleftarrow C_2 \longleftarrow \cdots \tag{7.8}$$

of free abelian groups which computes the homology of X. There are several models for the chain complex, depending on the structure of X. A CW structure on X usually leads to the most efficient model, the cellular chain complex. If A is an abelian group, then applying $\mathrm{Hom}(-,A)$ to (7.8) we obtain a cochain complex

eq:135

$$\mathrm{Hom}(C_0,A) \longrightarrow \mathrm{Hom}(C_1,A) \longrightarrow \mathrm{Hom}(C_2,A) \longrightarrow \cdots \tag{7.9}$$

which computes the *cohomology* groups $H^\bullet(X;A)$. If R is a commutative ring, then the cohomology $H^\bullet(X;R)$ is a $\mathbb{Z}$-graded ring—the multiplication is called the *cup product*—and it is commutative in a graded sense. Just as homology is a homotopy invariant, so too is cohomology. There is an important distinction: if $f\colon X \to X'$ is a continuous map, then the induced map on cohomology is by pullback

eq:136

$$f^*\colon H^\bullet(X';A) \longrightarrow H^\bullet(X;A). \tag{7.10}$$

As stated, it is unchanged if f undergoes a homotopy.

Suppose $\alpha \in H^\bullet(BG;A)$ is a cohomology class on the classifying space BG. (Recall that there are different, homotopy equivalent, models for BG; see Proposition 6.35. By the homotopy invariance of cohomology, it won't matter which we use.) Then if $P \to M$ is a principal G-bundle over a manifold M, we define $\alpha(P) \in H^\bullet(M;A)$ by

eq:137

$$\alpha(P) = f_P^*(\alpha) \tag{7.11}$$

where $f_P\colon M \to BG$ is any classifying map. Theorem 7.1 and the homotopy invariance of cohomology guarantee that (7.11) is well-defined. Then $\alpha(P)$ is a *characteristic class* of $P \to M$.

thm:119 **Exercise 7.12.** Suppose $g\colon M' \to M$ is smooth and $P \to M$ is a G-bundle. Prove that

eq:138

$$\alpha(g^*P) = g^*\alpha(P) \tag{7.13}$$

Thus we say that characteristic classes are *natural*.

Cohomology classes in $H^\bullet(BG;A)$ are *universal characteristic classes*, and the problem presents itself to compute the cohomology of BG with various coefficient groups A. We will state a few results at the end of the lecture. First we develop *Chern classes* for complex vector bundles. (Recall from **(6.39)** that this is equivalent to characteristic classes for $G = GL_k(\mathbb{C})$. We will make a contractible choice of a hermitian metric, so may use instead the unitary group $G = U(k)$.)

Complex line bundles

Recall from Corollary 6.52 that a classifying space for complex line bundles is the projective space $\mathbb{P}(\mathcal{H})$ of a complex separable Hilbert space $\mathcal{H}$. To write the chain complex of this space, it is more convenient to use the colimit space $\mathbb{P}(\mathbb{C}^\infty)$, analogous to the discussion in **(6.23)**. That space has a cell decomposition with a single cell in each even dimension, so the cellular chain complex is

eq:139
$$\mathbb{Z} \longleftarrow 0 \longleftarrow \mathbb{Z} \longleftarrow 0 \longleftarrow \mathbb{Z} \longleftarrow 0 \longleftarrow \cdots \tag{7.14}$$

The cochain complex which computes integral cohomology is then

eq:140
$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \tag{7.15}$$

With a bit more work we can prove that the integral cohomology ring of the classifying space is

eq:141
$$H^\bullet\bigl(\mathbb{P}(\mathcal{H});\mathbb{Z}\bigr) \cong \mathbb{Z}[y], \qquad \deg y = 2, \tag{7.16}$$

a polynomial ring on a single generator in degree 2. The generator y is defined by (7.16) only up to sign, and we fix the sign by requiring that

eq:142
$$\langle y, [\mathbb{P}(V)]\rangle = 1, \tag{7.17}$$

where $[\mathbb{P}(V)] \in H_2(\mathbb{P}(\mathcal{H}))$ is the fundamental class of any projective line ($V \in \mathcal{H}$ two-dimensional).

Recall from **(6.8)** the tautological line bundle $S \to \mathbb{P}(\mathcal{H})$.

thm:120 **Definition 7.18.** The *first Chern class* of $S \to \mathbb{P}(\mathcal{H})$ is $-y \in H^2\bigl(\mathbb{P}(\mathcal{H})\bigr)$.

Since $S \to \mathbb{P}(\mathcal{H})$ is a universal line bundle, this defines the first Chern class for all line bundles over any base.

thm:121 **Proposition 7.19.** *Let $L_1, L_2 \to M$ be complex line bundles. Then*

eq:143
$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \in H^2(M). \tag{7.20}$$

Proof. It suffices to prove this universally. Let $\mathcal{H}_1, \mathcal{H}_2$ be infinite dimensional complex separable Hilbert spaces, and $S_i \to \mathbb{P}(\mathcal{H}_i)$ the corresponding tautological line bundles. The *external tensor product* $S_1 \boxtimes S_2$ is classified by the map

eq:144
$$\begin{tikzcd} S_1 \boxtimes S_2 \arrow[r] \arrow[d] & S \arrow[d] \\ \mathbb{P}(\mathcal{H}_1) \times \mathbb{P}(\mathcal{H}_2) \arrow[r, "f"] & \mathbb{P}(\mathcal{H}) \end{tikzcd} \tag{7.21}$$

where $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and if $L_i \in \mathbb{P}(\mathcal{H}_i)$ contain nonzero vectors ξ_i, the line $f(L_1, L_2)$ is the span of $\xi_1 \otimes \xi_2$. (Note that the fiber of $S_1 \boxtimes S_2$ at (L_1, L_2) is $L_1 \otimes L_2$.) If $V_i \subset \mathcal{H}_i$ is 2-dimensional, and if $L_i \in \mathbb{P}(\mathcal{H}_i)$ is a fixed line, then the image of the projective lines $\mathbb{P}(V_1) \times \{L_2\}$ and $\{L_1\} \times \mathbb{P}(V_2)$ are projective lines in $\mathbb{P}(\mathcal{H})$. It follows that $f^*(y) = y_1 + y_2$ in $H^2\bigl(\mathbb{P}(\mathcal{H}_1) \times \mathbb{P}(\mathcal{H}_2)\bigr)$, where y, y_1, y_2 are the properly oriented generators of $H^\bullet\bigl(\mathcal{H}(\mathbb{P})\bigr), H^\bullet\bigl(\mathcal{H}(\mathbb{P}_1)\bigr), H^\bullet\bigl(\mathcal{H}(\mathbb{P}_2)\bigr)$, respectively. $\square$

thm:122 **Corollary 7.22.** *Let $L \to M$ be a complex line bundle. Then*

eq:145
$$c_1(L^*) = -c_1(L). \tag{7.23}$$

This follows since $L \otimes L^* \to M$ is trivializable.

Higher Chern classes

subsec:7.1

(7.24) *The Leray-Hirsch theorem.* As a preliminary we quote the following result in the topology of fiber bundles; see [BT] or [H1, §4.D] for a proof.

thm:123 **Theorem 7.25** (Leray-Hirsch)**.** *Let $F \to \mathcal{E} \to B$ be a fiber bundle and R a commutative ring. Suppose $\alpha_1, \dots, \alpha_N \in H^\bullet(\mathcal{E}; R)$ have the property that $i_b^*\alpha_1, \dots, i_b^*\alpha_N$ freely generate the R-module $H^\bullet(\mathcal{E}_b; R)$ for all $b \in B$. Then $H^\bullet(\mathcal{E}; R)$ is isomorphic to the free $H^\bullet(B; R)$-module with basis $\alpha_1, \dots, \alpha_N$.*

Even though the total space $\mathcal{E}$ is not a product $B \times F$, its cohomology behaves as though it is, at least as an R-module. The ring structure is twisted, however, and we will use that to define the higher Chern classes below.

subsec:7.2

(7.26) *Flag bundles.* Let E be a complex vector space of dimension k with a hermitian metric. There is an associated *flag manifold* $\mathbb{F}(E)$ whose points are orthogonal decompositions

eq:146
$$E = L_1 \oplus \cdots \oplus L_k \tag{7.27}$$

of E as a sum of lines. If $\dim E = 2$, then $\mathbb{F}(E) = \mathbb{P}(E)$ since L_2 is the orthogonal complement of L_1. In general the flag manifold $\mathbb{F}(E)$ has k tautological line bundles $L_j \to \mathbb{F}(E)$, $j = 1, \dots, k$. This functorial construction can be carried out in families. So to a hermitian vector bundle $E \to M$ of rank k over a smooth manifold M there is an associated fiber bundle—the *flag bundle*

eq:147
$$\pi \colon \mathbb{F}(E) \to M \tag{7.28}$$

with typical fiber the flag manifold. There are tautological line bundles $L_j \to \mathbb{F}(E)$, $j = 1, \dots, k$.

thm:124 **Proposition 7.29.** *Polynomials in the cohomology classes $x_j = c_1(L_j) \in H^2\big(\mathbb{F}(E); \mathbb{Z}\big)$ freely generate the integral cohomology of each fiber $\mathbb{F}(E)_p$, $p \in M$, as an abelian group.*

thm:125 **Corollary 7.30.** *The pullback map*

eq:148
$$\pi^* \colon H^\bullet(M; \mathbb{Z}) \longrightarrow H^\bullet\big(\mathbb{F}(E); \mathbb{Z}\big) \tag{7.31}$$

is injective.

Note the choice of sign for x_j; it is opposite to that for y in (7.17). The image of π^* is the subring of symmetric polynomials in x_j with coefficients in the ring $H^\bullet(M; \mathbb{Z})$.

Sketch proof of Proposition 7.29. This is done in [BT], so we only give a rough outline. Consider first the *projective bundle*

eq:149

$$\pi_1 \colon \mathbb{P}(E) \to M \tag{7.32}$$

whose fiber at $p \in M$ is the projectivization $\mathbb{P}(E_x)$ of the fiber E_p. There is a tautological line bundle $S \to \mathbb{P}(E)$ which restricts on each fiber $\mathbb{P}(E)_p$ of π_1 to the tautological line bundle of that projective space. The chain complex of a finite dimensional projective space is a truncation of (7.14), from which it follows that $y = c_1(S^*)$ and its powers generate the cohomology of the fiber of π_1, in the sense of the Leray-Hirsch Theorem 7.25. So

eq:150

$$H^\bullet\bigl(\mathbb{P}(E);\mathbb{Z}\bigr) \cong H^\bullet(M;\mathbb{Z})\{1, y, y^2, \dots, y^{k-1}\} \tag{7.33}$$

as abelian groups. Now consider the projective bundle associated to the quotient bundle[14] $Q \to \mathbb{P}(E)$ and keep iterating. ☐

thm:126

Exercise 7.34. Work out the details of this proof without consulting [BT]!

subsec:7.3

(7.35) *Higher chern classes.* Following Grothendieck we define the Chern classes of E using Theorem 7.25. Namely, the class $y^k \in H^{2k}\bigl(\mathbb{P}(E);\mathbb{Z}\bigr)$ must by (7.33) satisfy a polynomial equation of the form

eq:151

$$y^k + c_1(E)y^{k-1} + c_2(E)y^{k-2} + \cdots + c_k(E) = 0 \tag{7.36}$$

for some unique classes $c_i(E) \in H^{2i}(M;\mathbb{Z})$.

thm:127

Definition 7.37. The class $c_i(E)$ defined by (7.36) is the i^{th} Chern class of $E \to M$.

thm:128

Proposition 7.38. *The pullback $\pi^*c_i(E)$ to the flag bundle (7.28) is the i^{th} elementary symmetric polynomial in $x_1, \dots, x_k$.*

Proof. Define the submersion

eq:152

$$\rho_j \colon \mathbb{F}(E) \longrightarrow \mathbb{P}(E) \tag{7.39}$$

to map the flag $E \cong L_1 \oplus \cdots \oplus L_k$ to the line L_j. It is immediate that $\rho_j^*(y) = -x_i$, where $y = c_1(S^*) \in H^2\bigl(\mathbb{P}(E);\mathbb{Z}\bigr)$ as in (7.33), and $x_j = c_1(L_j)$ as in Proposition 7.29. So each x_j is a root of the polynomial equation

eq:153

$$z^k - \pi^*c_1(E)z^{k-1} + \pi^*c_2(E)z^{k-2} - \cdots + (-1)^k\pi^*c_k(E) = 0 \tag{7.40}$$

in the cohomology of $\mathbb{F}(E)$. The conclusion follows. ☐

thm:131

Exercise 7.41. Prove that the Chern classes of a trivial vector bundle vanish.

---

[14] Since E has a metric, we identify Q as the orthogonal complement to S.

subsec:7.4

(7.42) *The splitting principle.* The pullback $\pi^*E \to \mathbb{F}(E)$ is canonically isomorphic to the sum $L_1 \oplus \cdots \oplus L_k \to \mathbb{F}(E)$ of line bundles. That, combined with Corollary 7.30 and Proposition 7.38, gives a method for computing with Chern classes: one can always assume that a vector bundle is the sum of line bundles. That is not true on the base M, but it is true for the pullback to the flag bundle. Any identity in Chern classes proved there is valid on M, because of the injectivity of the induced map on cohomology. Furthermore, symmetric polynomials in the x_i are polynomials in the Chern classes, by a basic theorem in commutative algebra about polynomial rings, and in particular live on the base M.

As a simple illustration, define the *total Chern class* of $E \to M$ as

eq:154
$$c(E) = 1 + c_1(E) + c_2(E) + \cdots . \tag{7.43}$$

Then we formally write

eq:155
$$c(E) = \prod_{j=1}^{k}(1 + x_j); \tag{7.44}$$

The equation is precisely true on $\mathbb{F}(E)$ for $\pi^*c(E)$. Also, for a smooth manifold M we write

eq:157
$$c(M) = c(TM) \tag{7.45}$$

for the Chern classes of the tangent bundle.

thm:129 **Exercise 7.46.** Prove the *Whitney sum formula*: If $E_1, E_2 \to M$ are complex vector bundles, then

eq:156
$$c(E_1 \oplus E_2) = c(E_1)c(E_2). \tag{7.47}$$

The formula for the tensor product is more complicated. Find a formula for $c_1(E_1 \otimes E_2)$. Can you find a formula for $c(E_1 \otimes E_2)$ in case one of the bundles is a line bundle?

thm:135 **Exercise 7.48.** Define the complex conjugate bundle $\overline{E} \to M$ to a complex vector bundle $E \to M$. Show that a Hermitian metric gives an isomorphism $\overline{E} \xrightarrow{\cong} E^*$. Show that

eq:170
$$c_i(\overline{E}) = (-1)^i c_i(E). \tag{7.49}$$

Some computations

subsec:7.5

(7.50) *The total Chern class of complex projective space.* Consider $\mathbb{CP}^n = \mathbb{P}(\mathbb{C}^{n+1})$. As usual we let $y = c_1(S^*)$ for the tautological line bundle $S \to \mathbb{CP}^n$.

thm:130 **Proposition 7.51.** *The total Chern class of* $\mathbb{CP}^n$ *is*

eq:158
$$c(\mathbb{CP}^n) = (1 + y)^{n+1}. \tag{7.52}$$

This is to be interpreted in the truncated polynomial ring

eq:159
$$H^{\bullet}(\mathbb{CP}^n) \cong \mathbb{Z}[y]/(y^{n+1}). \tag{7.53}$$

Proof. We use the exact sequence

eq:160
$$0 \longrightarrow S \longrightarrow \underline{\mathbb{C}^{n+1}} \longrightarrow Q \longrightarrow 0 \tag{7.54}$$

of vector bundles over $\mathbb{CP}^n$ and the fact (Exercise 7.41) that the Chern classes of a trivial bundle vanish to deduce

eq:161
$$c(S)c(Q) = 1. \tag{7.55}$$

It follows that

eq:163
$$c(Q) = \frac{1}{1-y} = 1 + y + \cdots + y^n. \tag{7.56}$$

There is a canonical isomorphism

eq:162
$$T\mathbb{CP}^n \cong \operatorname{Hom}(S, Q) \cong Q \otimes S^*, \tag{7.57}$$

which was sketched in lecture and is left as a very worthwhile exercise. Using the splitting principle we write (formally, or precisely up on the flag bundle of Q) $Q = L_1 \oplus \cdots \oplus L_n$ and so

eq:164
$$Q \otimes S^* \cong L_1 \otimes S^* \;\oplus\; \cdots \;\oplus\; L_n \otimes S^*. \tag{7.58}$$

Let $x_j = c_1(L_j)$ be the (formal) *Chern roots* of Q. Then

eq:165
$$\begin{aligned} c(\mathbb{CP}^n) = c(Q \otimes S^*) &= \prod_{j=1}^{n} (1 + x_i + y) \\ &= \sum_{j=0}^{n} c_j(Q)(1+y)^{n-j} \\ &= \sum_{j=0}^{n} y^j (1+y)^{n-j} \\ &= (1+y)^{n+1} - y^{n+1} \\ &= (1+y)^{n+1}. \end{aligned} \tag{7.59}$$

□

subsec:7.6

(7.60) *The L-polynomial.* Any symmetric polynomial in $x_1, \dots, x_k$ defines a polynomial in the Chern classes of $E \to M$. So as not to fix the rank or the dimension of the base, we encode these characteristic classes by formal power series in a variable x. For example,

eq:166
$$L = \frac{x}{\tanh x} \tag{7.61}$$

is Hirzebruch's "L-polynomial", introduced in his classic book [Hir], which explains in more detail the yoga for dealing with characteristic classes by "multiplicative sequences". In this case the L-polynomial is actually a power series in x^2, not just in x. This means that L is a characteristic class of *real* vector bundles, as we will see later.

To illustrate, let's write the L-polynomial for a rank two complex vector bundle $E \to M$ where M has dimension four. Let the formal Chern roots of E be x_1, x_2. First, we expand

eq:168
$$\frac{x}{\tanh x} = \frac{x \cosh x}{\sinh x} = \frac{x(1 + x^2/2! + \dots)}{x + x^3/6! + \dots} = 1 + \frac{x^2}{3} + \dots. \tag{7.62}$$

Thus

eq:167
$$L = (1 + \frac{x_1^2}{3})(1 + \frac{x_2^2}{3}) = 1 + \frac{x_1^2 + x_2^2}{3} = 1 + \frac{(x_1 + x_2)^2 - 2x_1x_2}{3} = 1 + \frac{c_1^2 - 2c_2}{3}. \tag{7.63}$$

For example, for $M = \mathbb{CP}^2$ we computed in Proposition 7.51 that $c_1(\mathbb{CP}^2) = 3y$ and $c_2(\mathbb{CP}^2) = 3y^2$, so

eq:169
$$L(\mathbb{CP}^2) = 1 + \frac{9y^2 - 6y^2}{3} = 1 + y^2. \tag{7.64}$$

The pairing with the fundamental class $[\mathbb{CP}^2] \in H_4(\mathbb{CP}^2)$ gives 1.

thm:132 **Exercise 7.65.** Compute the L-polynomial up to degree 8 for any vector bundle of any rank.

thm:133 **Exercise 7.66.** Prove that $\langle L(\mathbb{CP}^n), [\mathbb{CP}^n] \rangle = 1$ for all n.

thm:134 **Exercise 7.67.** Recall the K3 surface $X \subset \mathbb{CP}^3$ defined by a homogeneous quartic polynomial. Compute the total Chern class of X. (There are some hints at the end of Chapter IV of [BT].)

Real vector bundles

We can leverage the Chern classes of a complex bundle to define *Pontrjagin classes* of a real vector bundle. Let $V \to M$ be a real vector bundle of rank k. Define its *complexification* $E = V \otimes_{\mathbb{R}} \mathbb{C} \to M$. Since $E \cong \overline{E}$ we deduce from Exercise 7.48 that the odd Chern classes $c_{2h+1}(E)$ are torsion of order 2. We use the even Chern classes to define the Pontrjagin classes of V:

eq:171
$$p_i(V) = (-1)^i c_{2i}(V \otimes_{\mathbb{R}} \mathbb{C}) \ \in H^{4i}(M; \mathbb{Z}). \tag{7.68}$$

The sign convention is not totally standard, but this is more prevalent. The formal Chern roots of E come in opposite pairs $x, -x$, and taking just one element in each pair we have the formal expression

eq:172
$$p(V) = \prod_j (1 + x_j^2) \tag{7.69}$$

which we usually write simply as $\prod(1 + x^2)$.

thm:136 **Exercise 7.70.** Prove that the total Pontrjagin class of a sphere is trivial: $p(S^n) = 1$.

thm:137 **Exercise 7.71.** Prove that both Chern classes and Pontrjagin classes are *stable* in the sense that they don't change under stabilization of vector bundles (by adding trivial bundles).

Characteristic classes of principal G-bundles

There is much to say about the computation of the cohomology of BG. If G is a finite group, this is reduces to group cohomology à la Eilenberg-MacLane. For a connected compact Lie group G one can use a maximal torus of G to formulate a generalized splitting principle and make computations in terms of Lie theory. The beautiful classic papers of Borel and Hirzebruch [BH1, BH2, BH3] are a fount of useful information derived from this strategy. We just quote one general theorem in this area which determines the *real* cohomology in terms of invariant polynomials on the Lie algebra. ⇒ For simplicity I state it in terms of compact Lie groups. [**in the following assume G is connected.**]

thm:138 **Theorem 7.72.** *Let G be a compact Lie group and $\mathfrak{g}$ its Lie algebra. Then there is a canonical isomorphism of $H^\bullet(BG;\mathbb{R})$ with the ring of Ad-invariant polynomials on $\mathfrak{g}$, where a polynomial of degree i gives a cohomology class of degree $2i$.*

In particular, this is a polynomial ring.

As a special case, we have the following.

thm:139 **Theorem 7.73.** *The real cohomology of the classifying space of the orthogonal group is a polynomial ring on the Pontrjagin classes:*

eq:173
$$H^\bullet(BO(k);\mathbb{R}) \cong \mathbb{R}[p_1, \dots, p_i], \qquad \deg p_i = 4i, \tag{7.74}$$

where i is the greatest positive integer such that $2i \le k$.

Lecture 8: More characteristic classes and the Thom isomorphism

sec:8

We begin this lecture by carrying out a few of the exercises in Lecture 7. We take advantage of the fact that the Chern classes are *stable characteristic classes*, which you proved in Exercise 7.71 from the Whitney sum formula. We also give a few more computations. Then we turn to the *Euler class*, which is decidedly *unstable*. We approach it via the *Thom class* of an oriented real vector bundle. We introduce the *Thom complex* of a real vector bundle. This construction plays an important role in the course.

In lecture I did not prove the existence of the Thom class of an oriented real vector bundle. Here I do so—and directly prove the basic *Thom isomorphism theorem*—when the base is a CW complex. It follows from Morse theory that a smooth manifold is a CW complex. I need to assume the theorem that a vector bundle over a contractible base (in this case a closed ball) is trivializable. For a smooth bundle this follows immediately from Proposition 7.3.

The book [BT] is an excellent reference for this lecture, especially Chapter IV.

Elementary computations with Chern classes

subsec:8.4

(8.1) *Stable tangent bundle of projective space.* We begin with a stronger version of Proposition 7.51. Recall the exact sequence (7.54) of vector bundles over $\mathbb{CP}^n$.

thm:141 **Proposition 8.2.** *The tangent bundle of* $\mathbb{CP}^n$ *is stably equivalent to* $(S^*)^{\oplus(n+1)}$.

Proof. The exact sequence (7.54) shows that $Q \oplus S \cong \underline{\mathbb{C}^{n+1}}$. Tensor with S^* and use (7.57) and the fact that $S \otimes S^*$ is trivializable to deduce that

eq:174
$$T(\mathbb{CP}^n) \oplus \underline{\mathbb{C}} \cong (S^*)^{\oplus(n+1)}. \tag{8.3}$$

$\square$

thm:145 **Exercise 8.4.** Construct a canonical orientation of a complex manifold M. This reduces to a canonical orientation of a (finite dimensional) complex vector space. You may want to review the discussion of orientations in Lecture 2.

subsec:8.1

(8.5) *The L-genus of projective space.* Recall that Hirzebruch's L-class is defined by the power series (7.61). Namely, if a vector bundle $E \to M$ has formal Chern roots $x_1, x_2, \dots, x_k$, then

eq:175
$$L(E) = \prod_{j=1}^{k} \frac{x_j}{\tanh x_j}. \tag{8.6}$$

Each x_j has degree 2, and the term of order $2i$, which is computed by a finite computation, is a symmetric polynomial of degree i in the variables x_j. It is then a polynomial in the elementary symmetric polynomials $c_1, \dots, c_k$, which are the Chern classes of E. The *L-genus* is the pairing of the L-class (8.6) of the tangent bundle of a complex manifold M with its fundamental class $[M]$.

thm:149 *Remark* 8.7 (L-class of a *real* vector bundle). Since $x/\tanh x$ is a power series in x^2, it follows that the L-class is a power series in the Pontrjagin classes of the underlying real vector bundle. So the L-class is defined for a real vector bundle, and the L-genus for a compact oriented real manifold.

thm:142 **Proposition 8.8.** *The L-genus of $\mathbb{CP}^n$ satisfies*

eq:176
$$\langle L(\mathbb{CP}^n), [\mathbb{CP}^n]\rangle = 1 \tag{8.9}$$

if n is even.

Here $[CP^n] \in H_{2n}(\mathbb{CP}^n)$ is the fundamental class, defined using the canonical orientation of a complex manifold. Also, $L(\mathbb{CP}^n)$ is the L-polynomial of the tangent bundle. The degree of each term in the L-class is divisible by 4, so the left hand side of (8.9) vanishes for degree reasons if n is odd.

Proof. By Proposition 8.2 and the fact that the Chern classes are stable, we can replace $T(\mathbb{CP}^n)$ by $(S^*)^{\oplus(n+1)}$. The Chern roots of the latter are not formal—it *is* a sum of line bundles—and each is equal to the positive generator $y \in H_2(\mathbb{CP}^n)$. Since $\langle y^n, [\mathbb{CP}^n]\rangle = 1$, we conclude that the left hand side of (8.9) is the coefficient of y^n in

eq:177
$$L\bigl((S^*)^{\oplus(n+1)}\bigr) = \left(\frac{y}{\tanh y}\right)^{n+1}. \tag{8.10}$$

By the Cauchy integral formula, this equals

eq:178
$$\frac{1}{2\pi i}\int \frac{dy}{y^{n+1}}\left(\frac{y}{\tanh y}\right)^{n+1}, \tag{8.11}$$

where the contour integral is taken over a small circle with center the origin of the complex y-line; the orientation of the circle is counterclockwise. Substitute $z = \tanh y$, and so $dz/(1-z^2) = dy$. Then (8.11) equals

eq:179
$$\frac{1}{2\pi i}\int \frac{dz}{(1-z^2)z^{n+1}} = \frac{1}{2\pi i}\int dz\,\frac{1+z^2+z^4+\dots}{z^{n+1}} = \begin{cases} 1, & n \text{ even};\\ 0, & n \text{ odd}.\end{cases} \tag{8.12}$$

$\square$

subsec:8.2

(8.13) *The Euler characteristic and top Chern class.* We prove the following result at the end of the lecture.

thm:144 **Theorem 8.14.** *Let M be a compact complex manifold of dimension n. Then its Euler characteristic is*

eq:180
$$\chi(M) = \langle c_n(M), [M]\rangle. \tag{8.15}$$

`subsec:8.3`

(8.16) *The genus of a plane curve.* Let C be a complex curve, which means a complex manifold of dimension 1. The underlying real manifold is oriented and has dimension 2. Assume that C is compact and connected. Then, say by the classification of surfaces, we deduce that

`eq:181`
$$H_0(C) \cong \mathbb{Z}, \qquad \dim H_1(C) = 2g(C), \qquad H_2(C) \cong \mathbb{Z} \tag{8.17}$$

for some integer $g(C) \in \mathbb{Z}^{\ge 0}$ called the *genus* of C. The Euler characteristic is

`eq:182`
$$\chi(C) = 2 - 2g(C). \tag{8.18}$$

A *plane curve* is a submanifold $C \subset \mathbb{CP}^2$, and it is cut out by a homogeneous polynomial of degree d for some $d \in \mathbb{Z}^{\ge 1}$. An extension of Exercise 6.10 shows that these polynomials are sections of $(S^*)^{\otimes d} \to \mathbb{CP}^2$, which is the d^{th} power of the hyperplane bundle (and is often denoted $\mathcal{O}(d) \to \mathbb{CP}^2$). We simply assume that C is cut out as the zeros of a transverse section of that bundle.

`thm:146`
Proposition 8.19. *The genus of a smooth plane curve $C \subset \mathbb{CP}^2$ of degree d is*

`eq:183`
$$g(C) = \frac{(d-1)(d-2)}{2}. \tag{8.20}$$

Proof. The normal bundle to $C \subset \mathbb{CP}^2$ is canonically the restriction of $(S^*)^{\otimes d} \to \mathbb{CP}^2$ to C, and so we have the exact sequence (see (2.30))

`eq:184`
$$0 \longrightarrow TC \longrightarrow T(\mathbb{CP}^2)\big|_C \longrightarrow (S^*)^{\otimes d} \longrightarrow 0. \tag{8.21}$$

Since this sequence splits (in C^∞, not necessarily holomorphically), the Whitney sum formula implies that

`eq:185`
$$c(C) = \frac{(1+y)^3}{1+dy} = \frac{1+3y}{1+dy} = 1 + (3-d)y. \tag{8.22}$$

Here we use Proposition 7.51 to obtain the total Chern class of $\mathbb{CP}^2$. Proposition 7.19 together with Corollary 7.22 compute the total Chern class of $(S^*)^{\otimes d}$.

Next, we claim $\langle y, [C]\rangle = d$. One proof is that evaluation of y on a curve in $\mathbb{CP}^2$ is the intersection number of that curve with a generic line, which is the degree of the curve (which is the number of solutions to a polynomial equation of degree d in the complex numbers). Hence by Theorem 8.14 we have

`eq:186`
$$\chi(C) = \langle c_1(C), [C]\rangle = (3-d)d, \tag{8.23}$$

to which we apply (8.18) to deduce (8.20). □

subsec:8.5

(8.24) *The Euler characteristic of the* $K3$ *surface.* A similar computation gives the Euler characteristic of a quartic surface $M \subset \mathbb{CP}^3$ as 24, a fact used in (**5.60**). Do this computation! You will find

eq:187

$$c(M) = \frac{(1+y)^4}{1+4y} = 1 + 6y^2. \tag{8.25}$$

Notice that this also proves that $c_1(M) = 0$.

thm:147

Exercise 8.26. Prove that a degree $(n+1)$ hypersurface $M \subset \mathbb{CP}^n$ has vanishing first Chern class. Such a complex manifold is called *Calabi-Yau.* (In fact, the stronger statement that the complex determinant line bundle $\operatorname{Det} TM \to M$ is holomorphically trivial is true.)

The Thom isomorphism

subsec:8.7

(8.27) *Relative cell complexes.* Let X be a topological space and $A \subset X$ a closed subspace. We write (X, A) for this pair of spaces. A *cell structure on* (X, A) is a cell decomposition of $X \setminus A$. This means that X is obtained from A by successively attaching 0-cells, 1-cells, etc., starting from the space A. The *relative chain complex* of the cell structure is defined analogously to the absolute chain complex (7.8). Cochain complexes which compute cohomology are obtained algebraically from the chain complex, as in (7.9).

Example: The pair (S^k, ∞) has a cell structure with a single k-cell e^k. The chain complex is

eq:190

$$\cdots \longleftarrow 0 \longleftarrow \mathbb{Z}\{e^k\} \longleftarrow 0 \longleftarrow \cdots \tag{8.28}$$

where the nonzero entry is in degree k.

If the pair (X, A) satisfies some reasonable point-set conditions, which are satisfied if it admits a cell structure, then the homology/cohomology of the pair are isomorphic (by excision) to the homology/cohomology of the quotient X/A relative to the basepoint A/A.

subsec:8.6

(8.29) *The cohomology of a real vector space.* Let $\mathbb{V}$ be a real vector space of dimension k. Of course, $\mathbb{V}$ deformation retracts to the origin in $\mathbb{V}$ by scaling, so the cohomology of $\mathbb{V}$ is that of a point. But there is more interesting *relative cohomology*, or *cohomology with compact support.* Suppose $\mathbb{V}$ has an inner product. Let $C_r(\mathbb{V})$ denote the complement of the open ball of radius r about the origin. The pair $\bigl(\mathbb{V}, C_r(\mathbb{V})\bigr)$ has a cell structure with a single k-cell e^k. The chain complex of the pair is then (8.28), and taking $\operatorname{Hom}(-, \mathbb{Z})$ we deduce

eq:188

$$H^q(\mathbb{V}, C_r(\mathbb{V}); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & q = k; \\ 0, & \text{otherwise.} \end{cases} \tag{8.30}$$

The result is, of course, independent of the radius (by the excision property of cohomology). Notice that the quotient $\mathbb{V}/C_r(\mathbb{V})$ is homeomorphic to a k-sphere with a basepoint, so (8.30) is consistent with the example (8.28) above.

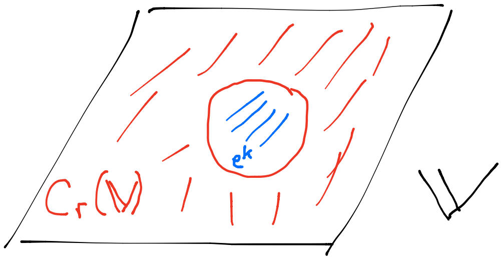

FIGURE 18. The pair $\big(\mathbb{V}, C_r(\mathbb{V})\big)$ fig:18

The isomorphism in (8.30) is determined only up to sign, or rather depends on a precise choice of k-cell e^k. That is, there are two distinguished generators of this cohomology group. These generators form a $\mathbb{Z}/2\mathbb{Z}$-torsor canonically attached to the vector space $\mathbb{V}$.

thm:148 **Lemma 8.31.** *This torsor canonically is $\mathfrak{o}(\mathbb{V})$, as defined in* (2.7).

Proof. Recall that the k-cell is defined by the attaching map, which is a homeomorphism (we can take it to be a diffeomorphism) $f\colon S^{k-1} \to S_r(\mathbb{V})$, where $S^{k-1} = \partial D^k$ is the standard $(k-1)$-sphere and $S_r(\mathbb{V})$ is the sphere of radius r in $\mathbb{V}$ centered at the origin. Given an orientation $o \in \mathfrak{o}(\mathbb{V})$ of $\mathbb{V}$, there is an induced orientation of $S_r(\mathbb{V})$ and so a distinguished homotopy class of orientation-preserving diffeomorphisms f. This singles out a generator in (8.30) and proves the lemma.

Here's an alternative proof. Let $a \in H^k(\mathbb{V}, C_r(\mathbb{V}); \mathbb{Z})$ be a generator. Its image in $H^k(\mathbb{V}, C_r(\mathbb{V}); \mathbb{R})$ can, by the de Rham theorem, be represented by a k-form ω_a on $\mathbb{V}$ whose support is contained in the open ball $B_r(\mathbb{V})$ of radius r centered at the origin. There is a unique orientation of $\mathbb{V}$—a point $o \in \mathfrak{o}(\mathbb{V})$—such that

eq:189
$$\int_{(\mathbb{V},o)} \omega_a = 1. \tag{8.32}$$

(The integral in the opposite orientation is -1.) The isomorphism of the lemma maps $a \mapsto o$. ☐

subsec:8.8

(8.33) *Thom classes.* Let $\pi\colon V \to M$ be a real vector bundle of rank k. Assume it carries an inner product. Consider the pair $\big(V, C_r(V)\big)$, where $C_r(V) \subset V$ is the set of all vectors of norm at least r. Recall also the notion of an *orientation* of a real vector bundle (Definition 2.14), which is a section of the double cover $\mathfrak{o}(V) \to M$.

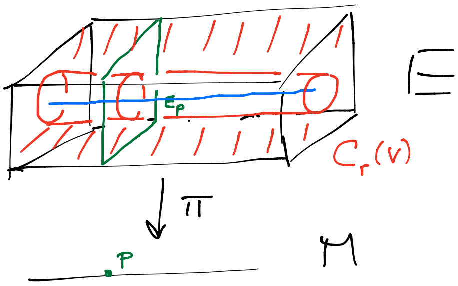

FIGURE 19. The pair $\big(V, C_r(V)\big)$ fig:19

thm:150 **Definition 8.34.** A *Thom class* for $\pi\colon V\to M$ is a cohomology class $U_V\in H^k\bigl(V,C_r(V);\mathbb{Z}\bigr)$ such that $i_p^*U_V$ is a generator of $H^k\bigl(V_p,C_r(V_p);\mathbb{Z}\bigr)$ for all $p\in M$.

It is clear that a Thom class induces an orientation of $V\to M$. The converse is also true.

thm:151 **Proposition 8.35.** *Let $\pi\colon V\to M$ be an oriented real vector bundle. Then there exists a Thom class $U_V\in H^k\bigl(V,C_r(V);\mathbb{Z}\bigr)$.*

We sketch a proof below.

subsec:8.9

(8.36) *Thom isomorphism theorem.* Given the Thom class, we apply the Leray-Hirsch theorem (Theorem 7.25) to the pair $\bigl(V,C_r(V)\bigr)$, which is a fiber bundle over M with typical fiber $\bigl(\mathbb{V},C_r(\mathbb{V})\bigr)$.

⇒ [**This is wrong! Only collapse fiber by fiber?**]

thm:152 **Corollary 8.37.** *Let $\pi\colon V\to M$ be an oriented real vector bundle. Then the integral cohomology of $\bigl(V,C_r(V)\bigr)$ is a free $H^\bullet(M;\mathbb{Z})$-module with a single generator U_V.*

Put differently, the map

eq:191
$$H^\bullet(M;\mathbb{Z})\xrightarrow{U_V\smile\pi^*(-)}H^{k+\bullet}\bigl(V,C_r(V);\mathbb{Z}\bigr)\tag{8.38}$$

is an isomorphism of abelian groups. This map—the *Thom isomorphism*—is pullback from the base followed by multiplication by the Thom class.

It follows immediately from (7.39) that there is a *unique* Thom class compatible with a given orientation.

⇒ [**xref is wrong: find correct one!**]

Proof of Proposition 8.35. As stated earlier a smooth manifold M admits a CW structure, which means it is constructed by iteratively attaching cells, starting with the empty set. Let $\{e_\alpha\}_{\alpha\in A}$ denote the set of cells. For convenience, denote $\mathcal{V}=\bigl(V,C_r(V)\bigr)$. We prove that $\mathcal{V}$ has a cell decomposition with cells $\{f_\alpha\}_{\alpha\in A}$ indexed by the same set A, and $\dim f_\alpha=\dim e_\alpha+k$. Furthermore, the cellular chain complex of $\mathcal{V}$ is the shift of the cellular chain complex of M by k units to the right. The same is then true of cochain complexes derived from these chain complexes. In particular, there is an *isomorphism*

eq:200
$$H^0(M;\mathbb{Z})\xrightarrow{\cong}H^k(\mathcal{V};\mathbb{Z}).\tag{8.39}$$

The image of $1\in H^0(M;\mathbb{Z})$ is the desired Thom class U_V. A bit more argument (using properties of the cup product) shows that the map (8.39) is the map (8.38), and so this gives a proof of the Thom isomorphism.

For each cell e_α there is a continuous map $\Phi_\alpha\colon D_\alpha\to M$, where D_α is a closed ball. Its restriction to the open ball is a homeomorphism onto its image $e_\alpha\subset M$ and M is the disjoint union of these images. The pullback $\Phi_\alpha^*V\to D_\alpha$ is trivializable. Fix a trivialization. This induces a homeomorphism $\Phi_\alpha^*\mathcal{V}\approx D_\alpha\times\bigl(\mathbb{V},C_r(\mathbb{V})\bigr)\approx\bigl(D_\alpha\times\mathbb{V},D_\alpha\times C_r(\mathbb{V})\bigr)$. This pair has a cell structure with a single cell, which is the Cartesian product of D_α and the k-cell described in **(8.29)**. Now the orientation of $V\to M$ induces an orientation of $\Phi_\alpha^*V\to D_\alpha$, and so picks out the k-cell e^k, as in the proof of Lemma 8.31. Define $f_\alpha=e_\alpha\times e^k$. These cells make up a cell decomposition of $\mathcal{V}$. Furthermore, $\partial(f_\alpha)=\partial(e_\alpha)\times e^k$, since $\partial(e^k)=0$. $\square$

Possession of a cell structure for a space is far more valuable than knowledge of its homology or cohomology; the latter can be derived from the former. So you should keep the picture of the cell structure used in the proof.

thm:163 **Exercise 8.40.** Think through the argument in the proof without the assumption that $V \to M$ is oriented. Now there is a sign ambiguity in the definition of e^k. Can you see how to deal with that and what kind of statement you can make?

subsec:8.10

(8.41) *The Thom complex.* As mentioned above, the cohomology of a pair (X, A) is the reduced cohomology of the quotient space X/A with basepoint A/A, at least if certain point-set conditions are satisfied. The quotient $V/C_r(V)$ is called the *Thom complex* of $V \to M$ and is denoted M^V. Figure 19 provides a convenient illustration: imagine the red region collapsed to a point. Note there is no projection from M^V to M: there is no basepoint in M and no distinguished image of the basepoint in M^V. Also, note that the zero section (depicted in blue) induces an inclusion

eq:192
$$i\colon M \longrightarrow M^V. \tag{8.42}$$

thm:156 **Exercise 8.43.** What is the Thom complex of the trivial vector bundle $M \times \mathbb{R}^k \to M$?

thm:157 **Exercise 8.44.** There is a nontrivial real line bundle $V \to S^1$, often called the *Möbius bundle*. What is its Thom complex?

The Euler class

thm:153 **Definition 8.45.** Let $\pi\colon V \to M$ be an oriented real vector bundle of rank k with Thom class U_V. The *Euler class* $e(V) \in H^k(M;\mathbb{Z})$ is defined as

eq:193
$$e(V) = i^*(U_V), \tag{8.46}$$

where i is the zero section (8.42).

thm:154 **Proposition 8.47.** *If $\pi\colon V \to M$ is an oriented real vector bundle which admits a nonvanishing section, then $e(V) = 0$.*

Proof. First, if M is compact, then the norm of the section $s\colon M \to V$ achieves a minimum, and taking r less than that minimum produces a Thom class whose pullback $s^*(U_V)$ by the section vanishes. Since the section is homotopic to the zero section i, the result follows. If M is not compact and the norm of the section does not achieve a minimum, then let $r\colon M \to \mathbb{R}^{>0}$ be a variable function whose value at $p \in M$ is less than $\|s(p)\|$. □

thm:155 **Proposition 8.48.** *Let $L \to M$ be a complex line bundle and $L_{\mathbb{R}} \to M$ the underlying oriented rank 2 real vector bundle. Then*

eq:194
$$e(L_{\mathbb{R}}) = c_1(L) \ \in H^2(M;\mathbb{Z}). \tag{8.49}$$

Proof. Consider the fiber bundle

eq:195
$$\mathbb{P}(L^* \oplus \underline{\mathbb{C}}) \longrightarrow M \tag{8.50}$$

with typical fiber a projective line, or 2-sphere. The dual tautological line bundle $S^* \to \mathbb{P}(L^* \oplus \underline{\mathbb{C}})$ has a first Chern class $\widetilde{U} = c_1(S^*)$ which restricts on each fiber to the positive generator of the cohomology of the projective line. There are two canonical sections of (8.50). The first $j\colon M \to \mathbb{P}(L^* \oplus \underline{\mathbb{C}})$ maps each point of M to the trivial line $\mathbb{C}$; the second $i\colon M \to \mathbb{P}(L^* \oplus \underline{\mathbb{C}})$ maps each point to the line L^*. Note that the complement of the image of j may be identified with L: every line in $L^*_p \oplus \mathbb{C}$ not equal to $\mathbb{C}$ is the graph of a linear functional $L^*_p \to \mathbb{C}$, which can be identified with an element of L_p. Now $j^*(S^*) \to M$ is the trivial line bundle and $i^*(S^*) \to M$ is canonically the line bundle $L \to M$. It follows that $\widetilde{U}$ lifts to a relative class[15] $U \in H^2\bigl(\mathbb{P}(L^* \oplus \mathbb{C}), j(M); \mathbb{Z}\bigr)$, the Thom class of $L_{\mathbb{R}} \to M$. Then

eq:196
$$e(L_{\mathbb{R}}) = i^*(U) = i^*\bigl(c_1(S^*)\bigr) = c_1(i^*S^*) = c_1(L). \tag{8.51}$$

□

I leave the proof of the next assertion as an exercise.

thm:158 **Proposition 8.52.** *Let $V_1, V_2 \to M$ be oriented real vector bundles. Then*

eq:197
$$e(V_1 \oplus V_2) = e(V_1)e(V_2). \tag{8.53}$$

thm:159 **Exercise 8.54.** Prove Proposition 8.52.

thm:160 **Corollary 8.55.** *Let $E \to M$ be a rank k complex vector bundle. Then*

eq:198
$$c_k(E) = e(E_{\mathbb{R}}). \tag{8.56}$$

thm:161 **Exercise 8.57.** Prove Corollary 8.55. Use Proposition 8.48 and the Whitney sum formulas Proposition 8.52 and Exercise 7.46.

subsec:8.11

(8.58) *The Euler characteristic.*

thm:162 **Proposition 8.59.** *Let M be a compact oriented n-manifold. Then its Euler characteristic is*

eq:199
$$\chi(M) = \langle e(M), [M] \rangle. \tag{8.60}$$

Proof. I will sketch a proof which relies on a relative version of the de Rham theorem: If M is a smooth manifold and $A \subset M$ a closed subset, then the de Rham complex of smooth differential forms on M supported in $M \setminus A$ computes the real relative cohomology $H^\bullet(M, A; \mathbb{R})$. We also use the fact that the integer on the right hand side of (8.60) can be computed from the pairing

---

[15] Use excision to push to the pair $\bigl(L, C_r(L)\bigr)$ considered above.

of $e_{\mathbb{R}}(M) \in H^n(M;\mathbb{R})$ with the fundamental class, and that—again, by the de Rham theorem—if ω is a closed n-form which represents $e_{\mathbb{R}}(M)$, then that pairing is $\int_M \omega$. Here $e_{\mathbb{R}}$ is the image of the (integer) Euler class in real cohomology by extension of scalars $\mathbb{Z} \to \mathbb{R}$.

Now for the proof: Recall that the Euler characteristic of M is the self-intersection number of the diagonal in $M \times M$, or equivalently the self-intersection number of the zero section of $TM \to M$. It is computed by choosing a section $\xi\colon M \to TM$—that is, a vector field—which is transverse to the zero section. The intersection number is the sum of local intersection numbers at the zeros of ξ, and each local intersection number is ± 1. Choose a local framing of M on a neighborhood N_i about each zero $p_i \in M$ of ξ—that is, a local trivialization of $TM \to M$ restricted to N_i. By transversality and the inverse function theorem we can cut down the neighborhoods N_i so that $\xi\colon N_i \to \mathbb{R}^n$ (relative to the trivialization) is a diffeomorphism onto its image. Fix a Riemannian metric on M and suppose $\|\xi\| > r$ on the complement of the union of the N_i. Let $\omega \in \Omega^n(TM)$ be a closed differential form with support in $TM \setminus C_r(TM)$ which represents the real Thom class $U_{M;\mathbb{R}} \in H^n\bigl(TM, C_r(TM);\mathbb{R}\bigr)$. Since the section $\xi\colon M \to TM$ is homotopic to the zero section i, we have

eq:201

$$\chi(M) = \int_M \xi^*\omega. \tag{8.61}$$

Because of the support condition on ω, the integral is equal to the sum of integrals over the neighborhoods N_i. Under the local trivialization ω represents the integral generator of $H^n\bigl(\mathbb{R}^n, C_r(\mathbb{R}^n);\mathbb{R}\bigr)$—this by the definition (Definition 8.34) of the Thom class—and so $\int_{N_i} \xi^*\omega = \pm 1$. I leave you to check that the sign is the local intersection number. ☐

Lecture 9: Tangential structures

sec:9

We begin with some examples of *tangential structures* on a smooth manifold. In fact, despite the name—which is appropriate to our application to bordism—these are structures on arbitrary real vector bundles over topological spaces; the name comes from the application to the tangent bundle of a smooth manifold. Common examples may be phrased as a reduction of structure group of the tangent bundle. The general definition allows for more exotic possibilities. We move from a geometric description—and an extensive discussion of orientations and spin structures—to a more abstract topological definition. Note there are both stable and unstable tangential structures. The stable version is what is usually studied in classical bordism theory; the unstable version is relevant to the modern developments, such as the cobordism hypothesis.

I suggest you think through this lecture first for a single tangential structure: orientations.

Orientations revisited

subsec:9.2

(9.1) *Existence and uniqueness.* Let $V \to M$ be a real vector bundle of rank n over a manifold M. (In this whole discussion you can replace a manifold by a metrizable topological space.) In Lecture 2 we constructed an associated double cover $\mathfrak{o}(V) \to M$, the orientation double cover of the vector bundle $V \to M$. An *orientation* of the vector bundle is a section of $\mathfrak{o}(V) \to M$. There is an existence and uniqueness exercise.

thm:165 **Exercise 9.2.** The obstruction to existence is the isomorphism class of the orientation double cover: orientations exists if and only if $\mathfrak{o}(V) \to M$ is trivializable. Show that this isomorphism class is an element of $H^1(M;\mathbb{Z}/2\mathbb{Z})$. If this class vanishes, show that the set of orientations is a torsor for $H^0(M;\mathbb{Z}/2\mathbb{Z})$, the group of locally constant maps $M \to \mathbb{Z}/2\mathbb{Z}$. Of course, this can be identified with the set of maps $\pi_0 M \to \mathbb{Z}/2\mathbb{Z}$.

thm:166 *Remark* 9.3. The isomorphism class of $\mathfrak{o}(V) \to M$ is the *first Stiefel-Whitney class* $w_1(V) \in H^1(M;\mathbb{Z}/2\mathbb{Z})$. The Stiefel-Whitney classes are characteristic classes of real vector bundles. They live in the cohomology algebra $H^\bullet(BO;\mathbb{Z}/2\mathbb{Z})$.

subsec:9.1

(9.4) *Recollection of frame bundles.* Recall from **(6.39)** that to the vector bundle $V \to M$ is associated a principal $GL_n(\mathbb{R})$-bundle $\mathcal{B}(V) \to M$ of bases, often called the *frame bundle* of $V \to M$. If we endow $V \to M$ with a metric, then we can take orthonormal frames and so construct a principal $O(n)$-bundle of frames $\mathcal{B}_O(V) \to M$.

thm:167 **Exercise 9.5.** Recall the determinant homomorphism

eq:202
$$GL_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^{\neq 0}. \tag{9.6}$$

Let $GL_n^+(\mathbb{R}) \subset GL_n(\mathbb{R})$ denote the subgroup $\det^{-1}(\mathbb{R}^{>0})$. Then $GL_n^+(\mathbb{R})$ acts freely on $\mathcal{B}(V)$. Identify the quotient with $\mathfrak{o}(V)$. What is the analogous statement for orthonormal frames?

subsec:9.3

(9.7) *Reduction of structure group.* Let H, G be Lie groups and $\rho\colon H \to G$ a homomorphism. (For the discussion of orientations this is the inclusion $GL_n^+(\mathbb{R}) \hookrightarrow GL_n(\mathbb{R})$.)

thm:168 **Definition 9.8.**

(i) Let $Q \to M$ be a principal H-bundle. The *associated* principal G-bundle $Q_\rho \to M$ is the quotient

eq:203
$$Q_\rho = (Q \times G)/H, \tag{9.9}$$

where H acts freely on the right of $Q \times G$ by

eq:204
$$(q, g) \cdot h = (q \cdot h, \rho(h)^{-1} g), \qquad q \in Q, \quad g \in G, \quad h \in H. \tag{9.10}$$

(ii) Let $P \to M$ be a principal G-bundle. Then a *reduction to H* is a pair (Q, θ) consisting of a principal H-bundle $Q \to M$ and an isomorphism

eq:205
$$\begin{tikzcd} Q_\rho \arrow[rr, "\theta"] \arrow[dr] & & P \arrow[dl] \\ & M & \end{tikzcd} \tag{9.11}$$

of principal G-bundles.

thm:169 **Exercise 9.12.**

(i) What is the G-action on Q_ρ?

(ii) Define an isomorphism of reductions.

(iii) Suppose $V \to M$ is a real vector bundle of rank n with metric. Let $\rho\colon O(n) \hookrightarrow GL_n(\mathbb{R})$ be the inclusion. What is $\mathcal{B}_O(V)_\rho$?

(iv) Assume that ρ is an inclusion. Show that $Q \subset Q_\rho$ and, using θ, we can identify a reduction to H as a sub-fiber bundle $Q \subset P$. Assuming that H is a closed Lie subgroup, show that reductions are in 1:1 correspondence with sections of the G/H bundle $P/H \to M$.

subsec:9.4

(9.13) *Orientations as reductions of structure group.* The definitions conspire to show that an orientation of a real rank n vector bundle $V \to M$ is a reduction of structure group of $\mathcal{B}(V) \to M$ to the group $GL_n^+(\mathbb{R}) \hookrightarrow GL_n(\mathbb{R})$. In particular, this follows by combining Exercise 9.5 and Exercise 9.12(iv) together with the definition of an orientation.

Spin structures

subsec:9.5

(9.14) *The spin group.* Let $SO(n) \subset O(n)$ be the subgroup of orthogonal matrices of determinant one. In low dimensions these are familiar groups. The group $SO(1)$ is trivial: it just has the identity element. The group $SO(2)$ is the group of rotations in the oriented plane $\mathbb{R}^2$, or more concretely the group of matrices

eq:208
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in \mathbb{R}, \tag{9.15}$$

which has the topology of the circle. Its fundamental group is infinite cyclic. The manifold underlying the group $SO(3)$ is diffeomorphic to $\mathbb{RP}^3$, so has fundamental group isomorphic to $\mathbb{Z}/2\mathbb{Z}$. (I gave an argument for this in a previous lecture.) In fact, $\pi_1 SO(n) \cong \mathbb{Z}/2\mathbb{Z}$ for all $n \geq 3$.

thm:170 **Exercise 9.16.** Prove this as follows. The group $SO(n)$ acts transitively on the sphere S^{n-1}, and the stabilizer of a point is isomorphic to $SO(n-1)$. So there is a fiber bundle $SO(n) \to S^{n-1}$ with typical fiber $SO(n-1)$. (It is a *principal* bundle.) Use the long exact sequence of homotopy groups and induction to deduce the assertion.

thm:171 **Definition 9.17.** The spin group $\mathrm{Spin}(n)$ is the double cover group of $SO(n)$.

Thus $\mathrm{Spin}(1) \cong \mathbb{Z}/2\mathbb{Z}$ is cyclic of order 2. The spin group $\mathrm{Spin}(2)$ is abstractly isomorphic to the circle group: $\mathrm{Spin}(2) \to SO(2)$ is the nontrivial double cover. The manifold underlying the group $\mathrm{Spin}(3)$ is diffeomorphic to S^3, and $\mathrm{Spin}(n)$ is connected and simply connected (also called *1-connected*) for $n \geq 3$. There is an explicit realization of the spin group inside the *Clifford algebra*.

thm:172 *Remark* 9.18. Definition 9.17 relies on a general construction in Lie groups. Namely, if G is a *connected* Lie group, $\pi\colon \widetilde{G} \to G$ a covering space, and $\tilde{e} \in \pi^{-1}(e)$ a basepoint, then there is a unique Lie group structure on $\widetilde{G}$ such that $\tilde{e}$ is the identity element and π is a group homomorphism. If you identify the covering space $\widetilde{G}$ with a space of homotopy classes of paths in G, then you might figure out how to define the multiplication. See [War] for details.

thm:173 **Definition 9.19.** Let $V \to M$ be a real vector bundle of rank n with a metric. A *spin structure* on V is a reduction of structure group of the orthonormal frame bundle $\mathcal{B}_O(V) \to M$ along $\rho\colon \mathrm{Spin}(n) \to O(n)$.

Here ρ is the projection $\mathrm{Spin}(n) \to SO(n)$ followed by the inclusion $SO(n) \to O(n)$. So the reduction can be thought of in two steps: an orientation followed by a lift to the double cover.

thm:174 **Exercise 9.20.** Make this explicit: construct an orientation of $V \to M$ from a spin structure on $V \to M$.

subsec:9.6

(9.21) *Existence for complex bundles.* We will not discuss the general existence problem here, but will instead restrict to important special case and give an example of non-existence.

subsec:9.7

(9.22) *The double cover of the unitary group.* The complex vector space $\mathbb{C}^n$ has as its underlying real vector space $\mathbb{R}^{2n}$. Explicitly, to an n-tuple $(z^1, \dots, z^n)$ of complex numbers we associate the $2n$-tuple $(x^1, y^1, \dots, x^n, y^n)$ of real numbers, where $z^n = x^n + \sqrt{-1}y^n$. The real part of the standard hermitian metric on $\mathbb{C}^n$ is the standard real inner product on $\mathbb{R}^{2n}$. So there is a homomorphism, which is an inclusion,

eq:209
$$U(n) \longrightarrow O(2n) \tag{9.23}$$

of unitary transformations of $\mathbb{C}^n$ into orthogonal transformations of $\mathbb{R}^{2n}$. In fact, the image lies in $SO(2n)$, which follows since complex linear transformations preserve the natural orientation of $\mathbb{C}^n = \mathbb{R}^{2n}$. (Alternatively, $U(n)$ is connected, so the image of (9.23) is a connected subgroup of $O(2n)$.) Define $\widetilde{U}(n)$ to be the pullback Lie group

`eq:210` (9.24)

$$\begin{tikzcd} \widetilde{U}(n) \arrow[r, dashed] \arrow[d, dashed] & \mathrm{Spin}(2n) \arrow[d] \\ U(n) \arrow[r] & SO(2n) \end{tikzcd}$$

It is the unique connected double cover of $U(n)$. (The fundamental group of $U(n)$ is infinite cyclic for all n.)

`subsec:9.8`

(9.25) *Spin structures on a complex vector bundle.* Let $E \to M$ be a rank n complex vector bundle. There is an underlying rank $2n$ *real* vector bundle $E_{\mathbb{R}} \to M$. As manifolds $E_{\mathbb{R}} = E$ and the projection map is the same. What is different is that we forget some of the structure of $E \to M$, namely we forget scalar multiplication by $\sqrt{-1}$ and only remember the real scalar multiplication. Choose a hermitian metric on $E \to M$, a contractible choice which carries no topological information. Then there is an associated principal $U(n)$ bundle $\mathcal{B}_U(E) \to M$, the *unitary* bundle of frames.

`thm:175` **Definition 9.26.** A *spin structure* on $E \to M$ is a reduction of $\mathcal{B}_U(E) \to M$ along $\widetilde{U}(n) \to U(n)$.

This is not really a definition, but rather a consequence of (9.24).

`thm:176` **Exercise 9.27.** Recast Definition 9.26 as a theorem and prove that theorem. (Hint: …is a spin structure on $E_{\mathbb{R}} \to M$ …).

`thm:177` **Proposition 9.28.** *Let $E \to M$ be a complex vector bundle which admits a spin structure. Then there exists $\tilde{c} \in H^2(M;\mathbb{Z})$ such that $2c = c_1(E)$.*

`thm:178` **Corollary 9.29.** *The manifold $\mathbb{CP}^n$ does not admit a spin structure if n is even.*

For according to Proposition 7.51 we have $c_1(\mathbb{CP}^n) = (n+1)y$, where $y \in H^2(\mathbb{CP}^n;\mathbb{Z})$ is the generator.

I outline the proof of Proposition 9.28 in the following exercise, the first part of which should have been part of the lecture on Chern classes.

`thm:179` **Exercise 9.30.**

(i) Let $E \to M$ be a complex vector bundle of rank n. Define the associated *determinant line bundle* $\operatorname{Det} E \to M$. One method is to use complex exterior algebra, analogous to Exercise 2.6(i) in the real case. Another is to use principal bundles and the determinant homomorphism $\det\colon GL_n(\mathbb{C}) \to \mathbb{C}^\times$.

(ii) Use the splitting principle **(7.42)** to prove that $c_1(\operatorname{Det} E) = c_1(E)$.

(iii) Construct the top homomorphism in the commutative diagram of Lie group homomorphisms

`eq:211` (9.31)
$$\begin{tikzcd} \widetilde{U}(n) \arrow[r, dashed] \arrow[d] & \mathbb{T} \arrow[d] \\ U(n) \arrow[r, "\det"] & \mathbb{T} \end{tikzcd}$$

in which the right vertical arrow is the squaring map.

(iv) Recall Proposition 7.19 and complete the proof of Proposition 9.28.

`subsec:9.9`

(9.32) *Double covers and uniqueness of spin structures.* We work in the context of **(9.7)**. Let $\rho\colon H \to G$ be a double cover of the Lie group G. We have in mind $G = SO(n)$ and $H = \mathrm{Spin}(n)$. Let $P \to M$ be a principal G-bundle and (Q, θ) a reduction along ρ to a principal H-bundle. Suppose $R \to M$ is a double cover, which may be viewed as a principal $\mathbb{Z}/2\mathbb{Z}$-bundle. Then we can construct a new reduction (Q', θ') by "acting on" the reduction (Q, θ) with the double cover $R \to M$. For this, consider the *fiber product* $Q \times_M R \to M$, which is a principal $(H \times \mathbb{Z}/2\mathbb{Z})$-bundle. The bundle $Q' \to M$ is obtained by dividing out by the diagonal $\mathbb{Z}/2\mathbb{Z} \subset H \to \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z} \subset H$ is the kernel of the covering map ρ. I leave you to construct θ'.

`thm:180` *Remark* 9.33. There is a category of double covers of M, and it has a "product" operation which makes it a categorical analog of a group. That *Picard category* acts on the category of reductions to H.

`thm:181` **Exercise 9.34.** As in Exercise 9.2 the set of isomorphism classes of double covers of M is $H^1(M; \mathbb{Z}/2\mathbb{Z})$. How is its abelian group structure related to Remark 9.33? Show that any two reductions to H are related by a double cover in the manner described. Conclude that $H^1(M; \mathbb{Z}/2\mathbb{Z})$ acts simply transitively on the set of isomorphism classes (Exercise 9.12(ii)) of reductions.

`thm:183` **Exercise 9.35.** Just as an orientation on a manifold M with boundary induces an orientation of the boundary ∂M, show that the same is true of a spin structure. (As a spin structure includes an orientation, the statement about orientations is included.)

`thm:182` **Exercise 9.36.** Show that there are two isomorphism classes of spin structure on S^1. Describe the principal Spin(2)-bundles and the isomorphisms θ explicitly. Which occurs as the boundary of a spin structure on the disk D^2?

Reductions of structure group and classifying spaces

We continue in the context of **(9.7)**, working with an arbitrary homomorphism $\rho\colon H \to G$. As we have only constructed classifying spaces for *compact* Lie groups (in **(6.60)**), we assume H and G are compact. Let $EH \to BH$ be the universal H-bundle. The associated G-bundle has a classifying map

`eq:212` (9.37)
$$\begin{tikzcd} EH \times_\rho G \arrow[r] \arrow[d] & EG \arrow[d] \\ BH \arrow[r, "B\rho"] & BG \end{tikzcd}$$

which we denote[16] $B\rho$. The top horizontal arrow in (9.37) induces an isomorphism $\theta^{\text{univ}}\colon EH\times_\rho G\xrightarrow{\cong}(B\rho)^*(EG)$. The pair $(EG\times_\rho G,\theta^{\text{univ}})$ is the universal reduction of a G-bundle to an H-bundle.

thm:184 **Proposition 9.38.** *Let $P\to M$ be a principal G-bundle and $f\colon M\to BG$ a classifying map. Then a lift $\tilde f$ in the diagram*

eq:213 (9.39)
$$\begin{tikzcd} & BH \arrow[d] \\ M \arrow[ur, "\tilde f"] \arrow[dr, "f"'] & \\ & BG \end{tikzcd}$$

induces a reduction to H, and conversely a reduction to H induces a lift $\tilde f$. Isomorphism classes of reductions are in 1:1 correspondence with homotopy classes of lifts.

Here a homotopy of lifts is a map $F\colon\Delta^1\times M\to BH$ such that $F(t,-)\colon M\to BH$ is a lift of f for all $t\in\Delta^1$.

Proof. Given a lift, pull back the universal reduction $(EG\times_\rho G,\theta^{\text{univ}})$ to M. Conversely, any reduced bundle $(Q\to M,\theta)$ has a classifying map of principal H-bundles

eq:214 (9.40)
$$\begin{tikzcd} Q \arrow[r] \arrow[d] & EG \arrow[d] \\ M \arrow[r, "g"] & BH \end{tikzcd}$$

and so a map of principal G-bundles

eq:215 (9.41)
$$\begin{tikzcd} Q\times_\rho G \arrow[r] \arrow[d] & EG\times_\rho G \arrow[d] \\ M \arrow[r, "g"] & BH \end{tikzcd}$$

The isomorphism θ then induces a diagram

eq:216 (9.42)
$$\begin{tikzcd} P \arrow[r] \arrow[d] & (B\rho)^*(EG) \arrow[r] \arrow[d] & EG \arrow[d] \\ M \arrow[r, "g"] & BH \arrow[r, "B\rho"] & BG \end{tikzcd}$$

in which the composition $B\rho\circ g$ is a classifying map, so is necessarily homotopic to f. Construct $\tilde f$ as the endpoint of a homotopy of maps $M\to BH$ which lifts the homotopy $B\rho\circ g\to f$ using the homotopy lifting property of $B\rho$. ☐

thm:185 **Exercise 9.43.** Work out the details of the last argument as well as the proof of the last assertion of Proposition 9.38.

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[16] There is a functorial construction B of classifying spaces which inspires this notation.

General tangential structures

We now generalize reductions of $O(n)$-bundles along homomorphisms $\rho\colon H \to G$ to a more general and flexible notion of a tangential structure.

Recall the construction (6.25) of the classifying space $BO(n)$ as a colimit of finite dimensional Grassmannians. There are closed inclusions $Gr_n(\mathbb{R}^q) \to Gr_{n+1}(\mathbb{R}^{q+1})$ obtained by sending $W \mapsto \mathbb{R} \oplus W$ where we write $\mathbb{R}^{q+1} = \mathbb{R} \oplus \mathbb{R}^q$. These induce maps $BO(n) \to BO(n+1)$, and we define

eq:219 (9.44)
$$BO = \operatorname*{colim}_{n\to\infty} BO(n).$$

It is a classifying space for the infinite orthogonal group O defined in (5.38).

thm:186 **Definition 9.45.** An *n-dimensional tangential structure* is a topological space $\mathcal{X}(n)$ and a fibration $\pi(n)\colon \mathcal{X}(n) \to BO(n)$. A *stable tangential structure* is a topological space $\mathcal{X}$ and a fibration $\pi\colon \mathcal{X} \to BO$. It gives rise to an n-dimensional tangential structure for each $n \in \mathbb{Z}^{\geq 0}$ by letting $\pi(n)\colon \mathcal{X}(n) \to BO(n)$ be the fiber product

eq:217 (9.46)
$$\begin{tikzcd} \mathcal{X}(n) \arrow[r, dashed] \arrow[d, dashed, "\pi(n)"'] & \mathcal{X} \arrow[d, "\pi"] \\ BO(n) \arrow[r] & BO \end{tikzcd}$$

If M is a k-dimensional manifold, then an *$\mathcal{X}(n)$-structure* on M is a lift $M \to \mathcal{X}(n)$ of a classifying map $M \to BO(n)$ of $\widetilde{TM}$, where we have stabilized the tangent bundle TM of the m-dimensional manifold M to the rank n bundle

eq:218 (9.47)
$$\widetilde{TM} := \underline{\mathbb{R}^{n-m}} \oplus TM.$$

An *$\mathcal{X}$-structure* on M is a family of coherent $\mathcal{X}(n)$-structures for n sufficiently large.

⇒

[**Say directly in terms of classifying bundle, so introduce universal bundle earlier.**]

Notice that an n-dimensional tangential structure induces an m-dimensional tangential structure for all $m < n$ by taking the fiber product

eq:220 (9.48)
$$\begin{tikzcd} \mathcal{X}(m) \arrow[r, dashed] \arrow[d, dashed, "\pi(m)"'] & \mathcal{X}(n) \arrow[d, "\pi(n)"] \\ BO(m) \arrow[r] & BO(n) \end{tikzcd}$$

thm:192 **Example 9.49.** The trivial tangential structure has $\mathcal{X} = BO$, so $\mathcal{X}(n) = BO(n)$, and the structure maps $\pi(n)$ are identity maps.

thm:187 **Example 9.50.** A *stable* framing is the tangential structure $\mathcal{X} = EO$, a contractible space with a free O-action. What is $\mathcal{X}(n) \to BO(n)$ in this example? (Hint: It is a principal O-bundle.)

thm:196 **Example 9.51.** An *n-framing* is the n-dimensional *unstable* tangential structure $\mathcal{X}(n) = EO(n)$. An n-framing of an m-manifold M is a trivialization of $\widetilde{TM}$. What is $\mathcal{X}(m)$?

thm:188 **Example 9.52.** An orientation is the stable tangential structure $\mathcal{X} = BSO$, and an orientation of $\widetilde{TM}$ amounts to an orientation of TM since $\underline{\mathbb{R}^{n-m}}$ has a canonical orientation. In this case the space $\mathcal{X}(n)$ in (9.46) is the classifying space $BSO(n)$.

thm:189 **Example 9.53.** A spin structure is also a stable tangential framing; the space $\mathcal{X}(n)$ is the classifying space $BSpin(n)$.

thm:190 **Example 9.54.** There are examples which are not reductions of structure group. For example, if $\mathcal{X}(n) = BO(n) \times B\Gamma$ for some finite group Γ, then an $\mathcal{X}(n)$-structure on M is a principal Γ-bundle over M. We can replace $B\Gamma$ by any space Y. Isomorphism classes of $\mathcal{X}(n)$-structures then track homotopy classes of maps $M \to Y$.

thm:191 **Exercise 9.55.** Following Proposition 9.38, define an isomorphism of $\mathcal{X}(n)$-structures. Formulate the classification of isomorphism classes of $\mathcal{X}(n)$-structures as a problem in homotopy theory.

subsec:9.10

(9.56) *The universal $\mathcal{X}(n)$-bundle.* Let $\pi\colon \mathcal{X} \to BO$ be a stable tangential structure with induced tangential structures $\pi(n)\colon \mathcal{X}(n) \to BO(n)$ for each $n \in \mathbb{Z}^{\geq 0}$. Let

eq:222
$$S(n) \longrightarrow BO(n) \tag{9.57}$$

be the universal real vector bundle of rank n, as in (6.28). Its pullback to $\mathcal{X}(n)$ has a tautological $\mathcal{X}(n)$-structure, the identity map $\mathrm{id}_{\mathcal{X}(n)}$ lifting $\pi(n)$ in

eq:224 (9.58)
$$\begin{tikzcd} \pi(n)^*\bigl(S(n)\bigr) \arrow[r] \arrow[d] & S(n) \arrow[d] \\ \mathcal{X}(n) \arrow[r, "\pi(n)"] & BO(n) \end{tikzcd}$$

so is the universal real rank n bundle with $\mathcal{X}(n)$-structure. By abuse of notation we denote this pullback $\pi(n)^*\bigl(S(n)\bigr)$ as simply

eq:223
$$S(n) \longrightarrow \mathcal{X}(n) \tag{9.59}$$

Suppose M is an m-manifold, $m \leq n$. Then an $\mathcal{X}(n)$-structure on M is an $\mathcal{X}(n)$-structure on its stabilized tangent bundle $\widetilde{TM} \to M$, as stated in Definition 9.45, which is more simply a classifying map

eq:221 (9.60)
$$\begin{tikzcd} \widetilde{TM} \arrow[r] \arrow[d] & S(n) \arrow[d] \\ M \arrow[r] & \mathcal{X}(n) \end{tikzcd}$$

subsec:9.11

(9.61) *Manifolds with boundary.* If M is a manifold with boundary and it is equipped with an $\mathcal{X}(n)$-structure (9.60), then there is an induced $\mathcal{X}(n)$-structure on the boundary. Namely, we just restrict (9.60) to ∂M; recall the exact sequence (1.12) at the boundary, which is split by the discussion in **(5.3)**; and use Definition 9.45 which involves the stabilized tangent bundle (9.47) of the boundary: $\widetilde{T(\partial M)} := \underline{\mathbb{R}} \oplus T(\partial M)$.

$\mathcal{X}$-bordism

subsec:9.14

(9.62) *Involutions.* The classifying space $BO(n)$ is a colimit (6.25) of Grassmannians $Gr_n(\mathbb{R}^q)$. Endow $\mathbb{R}^q$ with the standard inner product. Then the map $W \mapsto W^\perp$ to the orthogonal subspace induces inverse diffeomorphisms

eq:226

$$Gr_n(\mathbb{R}^m) \longleftrightarrow Gr_{m-n}(\mathbb{R}^m) \tag{9.63}$$

which exchange the tautological subbundles S with the tautological quotient bundles Q. The double colimit of (9.63) as $n, m \to \infty$ yields an involution

eq:227

$$\iota\colon BO \longrightarrow BO \tag{9.64}$$

If $\mathcal{X} \to BO$ is a stable tangential structure, we define its pullback by ι to be a new stable tangential structure

eq:228

$$\begin{tikzcd} \mathcal{X}^\perp \arrow[r, dashed] \arrow[d, dashed] & \mathcal{X} \arrow[d] \\ BO \arrow[r, "\iota"] & BO \end{tikzcd} \tag{9.65}$$

If $f\colon M \to BO$ is the stable classifying map of a vector bundle $V \to M$, and there is a complementary bundle $V^\perp \to M$ such that $V \oplus V^\perp \cong \underline{\mathbb{R}^m}$, then $\iota \circ f\colon M \to BO$ is a stable classifying map for $V^\perp \to M$.

subsec:9.13

(9.66) *Stable normal structures from stable tangential structures.* We reconsider the discussion in **(5.15)**, only instead of embedding in the sphere we embed in affine space (which is what we were doing anyhow). Fix a stable tangential structure $\pi\colon \mathcal{X} \to BO$. Let M be a smooth n-manifold. A stable $\mathcal{X}$-structure on M is an $\mathcal{X}(n+q)$-structure on $TM \to M$ for sufficiently large q, i.e., compatible classifying maps

eq:225

$$\begin{tikzcd} & S(n) \arrow[r] \arrow[dd] & S(n) \arrow[r] \arrow[dd] & \cdots \\ TM \arrow[ur] \arrow[dd] & & & \\ & \mathcal{X}(n,n+q) \arrow[r] & \mathcal{X}(n,n+q+1) \arrow[r] & \cdots \\ M \arrow[ur] & & & \end{tikzcd} \tag{9.67}$$

In the diagram we $\mathcal{X}(n, n+q)$ is the pullback of $\mathcal{X}(n) \to BO(n)$ to the Grassmannian $Gr_n(\mathbb{R}^{n+q}) \hookrightarrow BO(n)$. Now suppose $M \hookrightarrow \mathbb{A}^{n+q}$ is an embedding with normal bundle $\nu \to M$ of rank q. We use the Euclidean metric to identify $\nu \cong TM^\perp$ so $TM \oplus \nu \cong \underline{\mathbb{R}^{n+q}}$. Then using the perp map (9.63) we obtain a classifying map

eq:229 (9.68)

$$\begin{tikzcd} \nu \arrow[r] \arrow[d] & Q(q) \arrow[d] \\ M \arrow[r] & \mathcal{X}^\perp(q, n+q) \end{tikzcd}$$

Here $\mathcal{X}^\perp(q, n+q)$ is the pullback of $\mathcal{X}^\perp \to BO$ to the Grassmannian $Gr_q(\mathbb{R}^{n+q})$. Stabilizing we obtain a classifying map of the stable normal bundle. It is simply $\iota \circ f$, where f is the stable classifying map (9.67) of the tangent bundle. Note that $\iota \circ f$ is defined without choosing an embedding.

In this way we pass back and forth between stable tangential $\mathcal{X}$-structures and stable normal $\mathcal{X}^\perp$-structures.

subsec:9.12

(9.69) *$\mathcal{X}$-bordism groups.* We now imitate Definition 1.19 to define a bordism of closed manifolds equipped with an $\mathcal{X}$-structure on the stable tangent bundle, or equivalently an $\mathcal{X}^\perp$-structure on the stable normal bundle. Bordism is an equivalence relation, and we denote the bordism group of closed n-dimensional $\mathcal{X}$-manifolds as $\Omega_n^{\mathcal{X}}$.

thm:194 **Exercise 9.70.** Prove that bordism is an equivalence relation. Pay attention to the symmetry argument: see **(2.20)**.

thm:197 **Exercise 9.71.** Show that for $\mathcal{X} = BSO$, as in Example 9.52, this reproduces the oriented bordism group defined in Lecture 2. Quite generally, if $\mathcal{X} = BG$, then we use the notation $\Omega_\bullet^G$ in place of $\Omega_\bullet^{\mathcal{X}}$.

Lecture 10: Thom spectra and $\mathfrak{X}$-bordism

sec:10

We begin with the definition of a spectrum and its antecedents: prespectra and Ω-prespectra. Spectra are the basic objects of *stable homotopy theory*. We construct a prespectrum—then a spectrum—for each unstable or stable tangential structure. They are built using the Thom complex of vector bundles, so they are known as *Thom spectra*. For stable tangential structures there is a version of the Pontrjagin-Thom construction and then the main theorem identifies $\mathfrak{X}$-bordism groups with the homotopy groups of an appropriate Thom spectrum. We then focus on oriented bordism and summarize the computation of its rational homotopy groups.

Prespectra and spectra

This definition is basic to stable homotopy theory. A good reference is [Ma1]. All spaces in this section are pointed.

Let X, Y be pointed spaces. Recall from Exercise 4.29 that there is an isomorphism of spaces

eq:230

$$\text{Map}_*(\Sigma X, Y) \xrightarrow{\cong} \text{Map}_*(X, \Omega Y) \tag{10.1}$$

if we use the correct topologies. In the following definition we only need (10.1) as an isomorphism of sets.

thm:195

Definition 10.2.

(i) A *prespectrum* $T_\bullet$ is a sequence $\{T_q\}_{q\in\mathbb{Z}^{>0}}$ of pointed spaces and maps $s_q\colon \Sigma T_q \to T_{q+1}$.

(ii) An Ω-*prespectrum* is a prespectrum $T_\bullet$ such that the adjoints $t_q\colon T_q \to \Omega T_{q+1}$ of the structure maps are weak homotopy equivalences.

(iii) A *spectrum* is a prespectrum $T_\bullet$ such that the adjoints $t_q\colon T_q \to \Omega T_{q+1}$ of the structure maps are homeomorphisms.

Obviously a spectrum is an Ω-prespectrum is a prespectrum. We can take the sequence of pointed spaces $T_{q_0}, T_{q_0+1}, T_{q_0+2}, \dots$ to begin at any integer $q_0 \in \mathbb{Z}$. If $T_\bullet$ is a *spectrum* which begins at q_0, then we can extend to a sequence of pointed spaces T_q defined for *all* integers q by setting

eq:231

$$T_q = \Omega^{q_0-q}T_{q_0}, \qquad q < q_0. \tag{10.3}$$

Note that each T_q, in particular T_0, is an *infinite* loop space:

eq:232

$$T_0 \simeq \Omega T_1 \simeq \Omega^2 T_2 \simeq \cdots \tag{10.4}$$

There are shift maps on prespectra, Ω-prespectra, and spectra: simply shift the indexing.

thm:198 **Example 10.5.** Let X be a pointed space. The *suspension prespectrum* of X is defined by setting $T_q = \Sigma^q X$ for $q \geq 0$ and letting the structure maps s_q be the identity maps. In particular, for $X = S^0$ we obtain the sphere prespectrum with $T_q = S^q$.

⇐

[**Suspension (shifts) of a spectrum**]

subsec:10.2

(10.6) *Spectra from prespectra.* Associated to each prespectrum $T_\bullet$ is a spectrum[17] $LT_\bullet$ called its *spectrification*. It is easiest to construct in case the adjoint structure maps $t_q\colon T_q \to \Omega T_{q+1}$ are inclusions. Then set $(LT)_q$ to be the colimit of

eq:233
$$T_q \xrightarrow{t_q} \Omega T_{q+1} \xrightarrow{\Omega t_{q+1}} \Omega^2 T_{q+2} \longrightarrow \cdots \tag{10.7}$$

which is computed as an union; see **(4.32)**. For the suspension spectrum of a pointed space X the 0-space is

eq:234
$$(LT)_0 = \operatorname*{colim}_{\ell\to\infty} \Omega^\ell \Sigma^\ell X, \tag{10.8}$$

which is usually denoted QX; see (4.39) for QS^0.

thm:199 **Exercise 10.9.** Prove that the homotopy groups of QX are the *stable* homotopy groups of X. (Recall Proposition 4.40.)

subsec:10.5

(10.10) *Homotopy and homology of prespectra.* Let $T_\bullet$ be a prespectrum. Define its homotopy groups by

eq:235
$$\pi_n(T) = \operatorname*{colim}_{\ell\to\infty} \pi_{n+\ell} T_\ell, \tag{10.11}$$

where the colimit is over the sequence of maps

eq:240
$$\pi_{n+\ell} T_\ell \xrightarrow{\pi_{n+\ell} t_\ell} \pi_{n+\ell} \Omega T_{\ell+1} \xrightarrow{\text{adjunction}} \pi_{n+\ell+1} T_{\ell+1} \tag{10.12}$$

Similarly, define the homology groups as the colimit

eq:241
$$H_n(T) = \operatorname*{colim}_{\ell\to\infty} \widetilde{H}_{n+\ell} T_\ell, \tag{10.13}$$

where $\widetilde{H}$ denotes the reduced homology of a pointed space. We might be tempted to define the cohomology similarly, but that does not work.[18]

thm:203 **Exercise 10.14.** Compute the homology groups of the sphere spectrum. More generally, compute the homology groups of the suspension spectrum of a pointed space X in terms of the reduced homology groups of X.

thm:202 **Exercise 10.15.** Define maps of prespectra. Construct (in case the adjoint structure maps are inclusions) a map $T \to LT$ of prespectra and prove that it induces an isomorphism on homotopy and homology groups.

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[17] The notation 'L' indicates 'left adjoint'.

[18] Homotopy and homology commute with colimits, but cohomology does not: there is a derived functor $\lim^1$ which measures the deviation.

Thom spectra

subsec:10.3

(10.16) *Pullback of the universal bundle.* There is an inclusion

eq:236

$$i\colon BO(q)\longrightarrow BO(q+1) \tag{10.17}$$

defined as the colimit of the inclusions of Grassmannians which are the vertical arrows

eq:238

$$W\longmapsto \mathbb{R}\oplus W \tag{10.18}$$

in the diagram

eq:237

$$\begin{tikzcd} Gr_q(\mathbb{R}^m) \arrow[r] \arrow[d] & Gr_q(\mathbb{R}^{m+1}) \arrow[r] \arrow[d] & Gr_q(\mathbb{R}^{m+2}) \arrow[r] \arrow[d] & \cdots \\ Gr_{q+1}(\mathbb{R}^{m+1}) \arrow[r] & Gr_{q+1}(\mathbb{R}^{m+2}) \arrow[r] & Gr_{q+1}(\mathbb{R}^{m+3}) \arrow[r] & \cdots \end{tikzcd} \tag{10.19}$$

Recalling the definition of the tautological vector bundle $S(q)\to BO(q)$, as in (6.27), we see that there is a natural isomorphism

eq:239

$$i^*S(q+1)\xrightarrow{\;\cong\;}\underline{\mathbb{R}}\oplus S(q) \tag{10.20}$$

over $BO(q)$.

Let $\mathscr{Y}$ be a stable tangential structure (Definition 9.45). Then we also have maps $i\colon \mathscr{Y}(q)\to \mathscr{Y}(q+1)$ and isomorphisms (10.20) of the pullbacks over $\mathscr{Y}(q)$.

subsec:10.4

(10.21) *Thom complexes and suspension.* Let $V\to Y$ be a real vector bundle, and fix a metric. Recall the Thom complex is the quotient $V/C_r(V)$, where $C_r(V)$ is the complement of the open disk bundle of radius $r>0$. (The choice of radius is immaterial.)

thm:201

Proposition 10.22. *The Thom complex of $\underline{\mathbb{R}}\oplus V\to Y$ is homeomorphic to the suspension of the Thom complex of $V\to Y$.*

Note that the Thom complex of the 0-vector bundle—the identity map $Y\to Y$—is the disjoint union of Y and a single point, which is then the basepoint of the disjoint union. That disjoint union is denoted Y_+. Then Proposition 10.22 implies that the Thom complex of $\underline{\mathbb{R}}\to Y$ is ΣY_+, the suspension of Y_+. Iterating, and using the notation Y^V for the Thom complex of $V\to Y$, we have $Y^{\underline{\mathbb{R}}^\ell}\simeq \Sigma^\ell Y_+$. So the Thom complex is a "twisted suspension" of the base space.

Proof. Up to homeomorphism we can replace the disk bundle of $\underline{\mathbb{R}}\oplus V\to Y$ by the Cartesian product of the unit disk in $\mathbb{R}$ and the disk bundle of $V\to Y$. Crushing the complement in $\mathbb{R}\times V$ to a point is the same crushing which one does to form the suspension of Y^V, as in Figure 20. $\square$

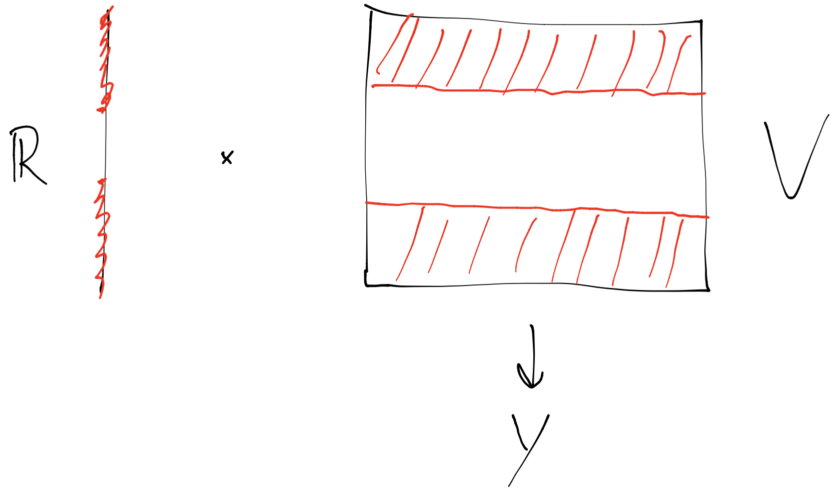

FIGURE 20. The Thom complex of $\underline{\mathbb{R}} \oplus V \to Y$ fig:20

subsec:10.6

(10.23) *The Thom prespectrum.* Let $\mathscr{Y}$ be a stable tangential structure. Consider the diagram

eq:242 (10.24)
$$\begin{tikzcd}
\underline{\mathbb{R}} \oplus S(q) \arrow[r] \arrow[d] & S(q+1) \arrow[d] \\
\mathscr{Y}(q) \arrow[r, "i"] & \mathscr{Y}(q+1)
\end{tikzcd}$$

where we use (10.20). There is an induced map on Thom complexes, and by Proposition 10.22 this is a map

eq:243 (10.25)
$$s_q \colon \Sigma\bigl(\mathscr{Y}(q)^{S(q)}\bigr) \longrightarrow \mathscr{Y}(q+1)^{S(q+1)}.$$

thm:204 **Definition 10.26.**

(i) The *Thom prespectrum* $T\mathscr{Y}_\bullet$ of a stable tangential structure $\mathscr{Y}$ is defined by

eq:244 (10.27)
$$T\mathscr{Y}_q = \mathscr{Y}(q)^{S(q)}$$

and the structure maps (10.25).

(ii) The *Thom spectrum* $M\mathscr{Y}_\bullet$ is $L(T\mathscr{Y}_\bullet)$.

Note that the maps (10.25) are inclusions, so $L(T\mathscr{Y}_\bullet)$ is defined in **(10.6)**.

subsec:10.7

(10.28) *Stable tangential structures from reduction of structure group.* Let $\{G(n)\}_{n\in\mathbb{Z}^{>0}}$ be a sequence of Lie groups and $G(n) \longrightarrow G(n+1)$, $\rho(n)\colon G(n) \to O(n)$ sequences of homomorphisms such that the diagram

eq:245 (10.29)
$$\begin{tikzcd}
\cdots \arrow[r] & G(n) \arrow[r] \arrow[d, "\rho(n)"] & G(n+1) \arrow[r] \arrow[d, "\rho(n+1)"] & \cdots \\
\cdots \arrow[r] & O(n) \arrow[r] & O(n+1) \arrow[r] & \cdots
\end{tikzcd}$$

commutes. There is an stable tangential structure $BG \to BO$ which is the colimit of the induced sequence of maps of classifying spaces

eq:246 (10.30)

$$\begin{tikzcd} \cdots \arrow[r] & BG(n) \arrow[r] \arrow[d, "B\rho(n)"] & BG(n+1) \arrow[r] \arrow[d, "B\rho(n+1)"] & \cdots \\ \cdots \arrow[r] & BO(n) \arrow[r] & BO(n+1) \arrow[r] & \cdots \end{tikzcd}$$

The corresponding bordism groups are denoted $\Omega^G_\bullet$, consistent with the notation in (**2.23**) for $G(n) = SO(n)$ and the obvious inclusion maps.

thm:206 **Exercise 10.31.** Show that the tangential structures in Example 9.49, Example 9.50, Example 9.52, and Example 9.53 are all of the form BG for a suitable $G = \operatorname{colim}_{n\to\infty} G(n)$.

thm:207 **Exercise 10.32.** Show that the Thom spectrum of the stable framing tangential structure (Example 9.50) is the sphere spectrum.

The general Pontrjagin-Thom theorem

This general form of the Pontrjagin-Thom theorem was introduced by Lashof [La]; see [St, §2] for an exposition.

thm:208 **Theorem 10.33.** *Let $\mathcal{X}$ be a stable tangential structure. Then for each $n \in \mathbb{Z}^{\geq 0}$ there is an isomorphism*

eq:247 (10.34)

$$\phi\colon \pi_n(M\mathcal{X}^\perp) \longrightarrow \Omega^{\mathcal{X}}_n.$$

The perp stable tangential structure $\mathcal{X}^\perp$ is defined in (**9.62**) and its Thom spectrum in Definition 10.26. Our notation for the bordism group indicates the stable *tangential* structure, which is not standard in the literature.

thm:209 *Remark* 10.35. I do not know an example in which $\mathcal{X}^\perp \neq \mathcal{X}$. I would like to know one.

thm:210 **Lemma 10.36.** *Let $\mathcal{X} = BSO$ be the stable tangential structure of orientations. Then $\mathcal{X}^\perp = \mathcal{X}$.*

Proof. BSO is a colimit of Grassmannians $Gr^{SO}_n(\mathbb{R}^m)$ of oriented subspaces of $\mathbb{R}^m$. Let the vector space $\mathbb{R}^m$ have its standard orientation. Then the orthogonal complement of an oriented subspace inherits a natural orientation,[19] and this gives a lift

eq:248 (10.37)

$$\begin{tikzcd} Gr^{SO}_n(\mathbb{R}^m) \arrow[r] \arrow[d] & Gr^{SO}_{m-n}(\mathbb{R}^m) \arrow[d] \\ Gr_n(\mathbb{R}^m) \arrow[r] & Gr_{m-n}(\mathbb{R}^m) \end{tikzcd}$$

of (9.63) in which the vertical maps are double covers which forget the orientation. The double colimit of (10.37) gives an equivalence $\mathcal{X}^\perp \approx \mathcal{X}$. ☐

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[19] Check the signs carefully to construct an involution in the following.

thm:213 **Corollary 10.38.** *There is an isomorphism*

eq:249 (10.39)
$$\phi\colon \pi_n(MSO) \longrightarrow \Omega_n^{SO}.$$

In the next lecture we compute the rational vector space obtained by tensoring the left hand side of (10.39) with $\mathbb{Q}$; then $\phi\otimes\mathbb{Q}$ gives an isomorphism to $\Omega_n^{SO}\otimes\mathbb{Q}$.

thm:211 **Exercise 10.40.** Generalize Lemma 10.36 to the tangential structures described in (**10.28**).

thm:212 **Exercise 10.41.** Check that Theorem 10.33 reduces to Corollary 5.22.

Remarks about the proof of Theorem 10.33. The tools from differential topology which go into the proof were all employed in the first lectures for the special case of stably framed manifolds; see especially the proof of Theorem 3.9. So we content ourselves of reminding the reader of the map ϕ and its inverse map ψ.

The map ϕ: A class in $\pi_n(M\mathcal{X}^\perp)$ is represented by

eq:250 (10.42)
$$f\colon S^{n+q} \longrightarrow T\mathcal{X}_q^\perp = \mathcal{X}^\perp(q)^{S(q)}$$

for some $q\in\mathbb{Z}^{>0}$. We choose f so that it is smooth and transverse to the zero section $Z(q)\subset\mathcal{X}^\perp(q)^{S(q)}$. Define $M := f^{-1}\bigl(Z(q)\bigr)\subset S^{n+q}$. The normal bundle $\nu\to M$ to $M\subset S^{n+q}$ is a rank q bundle isomorphic to the pullback of the normal bundle to $Z(q)\subset\mathcal{X}^\perp(q)^{S(q)}$, which is $S(q)\to Z(q)$, so inherits the $\mathcal{X}^\perp$-structure

eq:251 (10.43)
$$M \xrightarrow{f} Z(q)\cong\mathcal{X}^\perp(q)\longrightarrow\mathcal{X}^\perp$$

on its normal bundle, so on its stable normal bundle. By (**9.66**) this is equivalent to an $\mathcal{X}$-structure on the stable tangent bundle to M.

The inverse map ψ: We refer to Figure 21. Suppose M is a closed n-manifold with a stable tangential $\mathcal{X}$-structure, or equivalently a stable normal $\mathcal{X}^\perp$-structure. Choose an embedding $M\hookrightarrow S^{n+q}$ for some $q\in\mathbb{Z}^{>0}$ and a tubular neighborhood $U\subset S^{n+q}$. The normal structure induces—possibly after suspending to increase q—a classifying map

eq:252 (10.44)
$$\begin{tikzcd}
\nu\approx U \arrow[r] \arrow[d] & S(q) \arrow[d] \\
M \arrow[r] & \mathcal{X}^\perp(q)
\end{tikzcd}$$

The Pontrjagin-Thom collapse, which maps the complement of U to the basepoint, induces a map

eq:253 (10.45)
$$S^{n+q}\to\mathcal{X}^\perp(q)^{S(q)}$$

to the Thom complex, and this represents a class in $\pi_n(\mathcal{X}^\perp)$. ∎

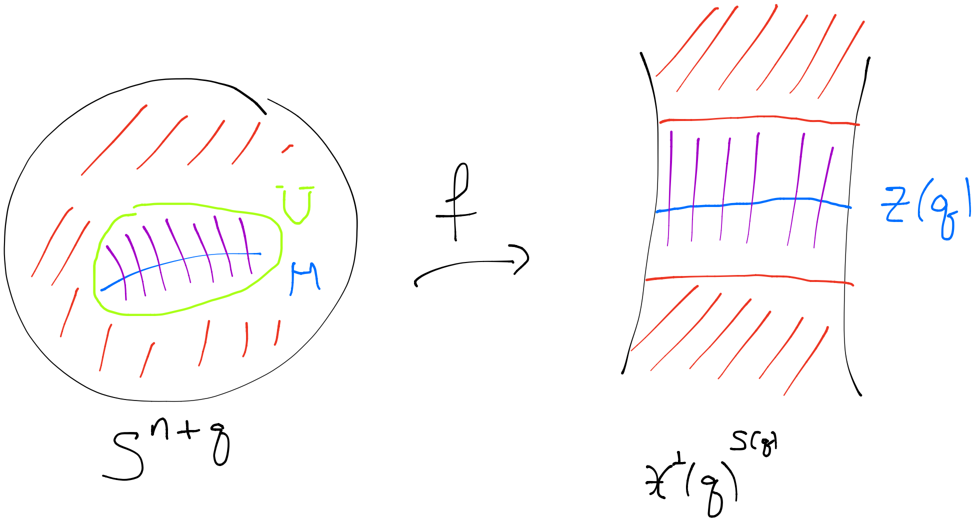

FIGURE 21. The Pontrjagin-Thom collapse

fig:21

Lecture 11: Hirzebruch's signature theorem

sec:11

In this lecture we define the signature of a closed oriented n-manifold for n divisible by four. It is a bordism invariant $\mathrm{Sign}\colon \Omega_n^{SO} \to \mathbb{Z}$. (Recall that we defined a $\mathbb{Z}/2\mathbb{Z}$-valued bordism invariant of non-oriented manifolds in Lecture 2.) The signature is a complete bordism invariant of closed oriented 4-manifolds (see (**2.28**)), as we prove here. It can be determined by tensoring with $\mathbb{Q}$, or even tensoring with $\mathbb{R}$. We use the general Pontrjagin-Thom Theorem 10.33 to convert the computation of this invariant to a homotopy theory problem. We state the theorem that all such bordism invariants can be determined on products of complex projective spaces. In this lecture we illustrate the techniques necessary to compute that $\Omega_4^{SO} \otimes \mathbb{Q}$ is a one-dimensional rational vector space. The general proof will be sketched in the next lecture. Here we also prove Hirzebruch's formula assuming the general result.

We sometimes tensor with $\mathbb{R}$ instead of tensoring with $\mathbb{Q}$. Tensoring with $\mathbb{R}$ has the advantage that real cohomology is represented by differential forms. Also, the computation of the real cohomology of BSO can be related to invariant polynomials on the orthogonal Lie algebra $\mathfrak{so}$.

Definition of signature

subsec:11.1

(11.1) *The fundamental class of an oriented manifold.* Let M be a closed oriented n-manifold for some $n \in \mathbb{Z}^{\ge 0}$. The orientation[20] defines a *fundamental class*

eq:254
$$[M] \in H_n(M). \tag{11.2}$$

Here coefficients in $\mathbb{Z}$ are understood. The fundamental class depends on the orientation: the fundamental class of the oppositely oriented manifold satisfies

eq:255
$$[-M] = -[M]. \tag{11.3}$$

The fundamental class is part of a discussion of duality in homology and cohomology; see [H1, §3.3]. The fundamental class determines a homomorphism

eq:256
$$\begin{aligned} H^n(M;A) &\longrightarrow A \\ c &\longmapsto \langle c, [M]\rangle \end{aligned} \tag{11.4}$$

for any coefficient group A. When $A = \mathbb{R}$ we use the de Rham theorem to represent an element $c \in H^n(M;\mathbb{R})$ by a closed differential n-form ω. Then

eq:257
$$\langle c, [M]\rangle = \int_M \omega. \tag{11.5}$$

(Recall that integration of differential forms depends on an orientation, and is consistent with (11.3).) For that reason the map (11.4) can be thought of as an integration operation no matter the coefficients.

subsec:11.2

(11.6) *The intersection pairing.* Let M be a closed oriented n-manifold and suppose $n = 4k$ for some $k \in \mathbb{Z}^{\ge 0}$. To define the intersection pairing we use the cup product on cohomology. Consider, then, the integer-valued bihomomorphism

eq:258
$$\begin{aligned} I_M\colon H^{2k}(M;\mathbb{Z}) \times H^{2k}(M;\mathbb{Z}) &\longrightarrow \mathbb{Z} \\ c_1, c_2 &\longmapsto \langle c_1 \smile c_2, [M]\rangle \end{aligned} \tag{11.7}$$

This *intersection form* is symmetric, by basic properties of the cup product The abelian group $H^{2k}(M;\mathbb{Z})$ is finitely generated, so has a finite torsion subgroup and a finite rank free quotient; the rank of the free quotient is the second *Betti number* $b_2(M)$.

thm:214 **Exercise 11.8.** Prove that the torsion subgroup is in the kernel of the intersection form (11.7). This means that if c_1 is torsion, then $I(c_1, c_2) = 0$ for all c_2.

It follows that the intersection form drops to a pairing

eq:259
$$\begin{aligned} \overline{I}_M\colon \operatorname{Free} H^{2k}(M;\mathbb{Z}) \times \operatorname{Free} H^{2k}(M;\mathbb{Z}) &\longrightarrow \mathbb{Z} \\ \bar c_1, \bar c_2 &\longmapsto \langle \bar c_1 \smile \bar c_2, [M]\rangle \end{aligned} \tag{11.9}$$

on the free quotient. *Poincaré duality* is the assertion that $\overline{I_M}$ is *nondegenerate*: if $\overline{I}_M(\bar c_1, \bar c_2) = 0$ for all $\bar c_2$, then $\bar c_1 = 0$. See [H1, §3.3] for a discussion.

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[20] We remark that any closed manifold (without orientation, or possibly nonorientable) has a fundamental class in mod 2 homology.

subsec:11.3

(11.10) *Homology interpretation.* Another consequence of Poincaré duality is that there is a dual pairing on Free $H_{2k}(M)$, and it is more geometric. In fact, the name 'intersection pairing' derives from the homology version. To compute it we represent two homology classes in the middle dimension by closed oriented submanifolds $C_1, C_2 \subset M$, wiggle them to be transverse, and define the intersection pairing as the oriented intersection number $I_M(C_1, C_2) \in \mathbb{Z}$.

subsec:11.4

(11.11) *de Rham interpretation.* Let A be a finitely generated abelian group of rank r. Then $A \to A \otimes \mathbb{R}$ has kernel the torsion subgroup of A. The codomain is a real vector space of dimension r, and the image is a full sublattice isomorphic to the free quotient Free A. We apply this to the middle cohomology group. A part of the de Rham theorem asserts that wedge product of closed forms goes over to cup product of real cohomology classes, and so we can represent the intersection pairing $I_M \otimes \mathbb{R}$ in de Rham theory by the pairing

eq:260

$$\begin{aligned} \widehat{I}_M \colon \Omega^{2k}(M) \times \Omega^{2k}(M) &\longrightarrow \mathbb{R} \\ \omega_1, \omega_2 &\longmapsto \int_M \omega_1 \wedge \omega_2 \end{aligned} \tag{11.12}$$

The pairing is symmetric and makes sense for all differential forms.

thm:215 **Exercise 11.13.** Use Stokes' theorem to prove that (11.12) vanishes if one of the forms is closed and the other exact. Conclude that it induces a pairing on de Rham cohomology, hence by the de Rham theorem on real cohomology.

The induced pairing on real cohomology is $I_M \otimes \mathbb{R}$.

thm:216 **Definition 11.14.** The *signature* $\operatorname{Sign}(M)$ is the signature of the symmetric bilinear form $I_M \otimes \mathbb{R}$.

Recall that a symmetric bilinear form B on a real vector space V has three numerical invariants which add up to the dimension of V: the *nullity* and two numbers b_+, b_-. There is a basis $e_1, \dots, e_n$ of V so that

eq:261

$$\begin{aligned} B(e_i, e_j) &= 0, && i \neq j; \\ B(e_i, e_i) &= 1, && i = 1, \dots, b_+; \\ B(e_i, e_i) &= -1, && i = b_+ + 1, \dots, b_+ + b_-; \\ B(e_i, e_i) &= 0, && i = b_+ + b_- + 1, \dots, n. \end{aligned} \tag{11.15}$$

There is a subspace $\ker B \subset V$, the null space of B, whose dimension is the nullity. b_+ is the dimension of the maximal subspace on which B is positive definite; b_- is the dimension of the maximal subspace on which B is negative definite. See [HK], for example. The signature is defined to be the difference $\operatorname{Sign}(B) = b_+ - b_-$. Note B is nondegenerate iff $\ker B = 0$ iff the nullity vanishes.

Examples

The following depends on a knowledge of the cohomology ring in several cases, but you can also use the oriented intersection pairing. We begin with several 4-manifolds.

thm:217 **Example 11.16** (S^4)**.** Since $H^2(S^4;\mathbb{Z}) = 0$, we have $\operatorname{Sign}(S^4) = 0$.

thm:218 **Example 11.17** ($S^2 \times S^2$)**.** The second cohomology $H^2(S^2 \times S^2;\mathbb{Z})$ has rank two. In the standard basis the intersection form is represented by the matrix

eq:262 (11.18)
$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The 'H' stands for 'hyperbolic'. One way to see this is to compute in homology. The submanifolds $S^2 \times \mathrm{pt}$ and $\mathrm{pt} \times S^2$ represent generators of $H_2(S^2 \times S^2)$, each has self-intersection number zero, and the intersection number of one with the other is one. Diagonalize H to check that its signature is zero.

thm:219 **Example 11.19** ($K3$ surface)**.** The $K3$-surface was introduced in (**5.60**). You computed its total Chern class, so its Pontrjagin class, in Exercise 7.67. One can compute (I'm not giving techniques here for doing so) that the intersection form is

eq:263 (11.20)
$$-E_8 \oplus -E_8 \oplus H \oplus H \oplus H,$$

where E_8 is an 8×8 symmetric positive definite matrix of integers derived from the Lie group E_8. Its signature is -16.

The $K3$ surface is spin(able), which follows from the fact that its first Chern class vanishes. (A related statement appears as Proposition 9.28.) The following important theorem of Rohlin applies.

thm:220 **Theorem 11.21** (Rohlin)**.** *Let M^n be a closed oriented manifold with $n \equiv 4 \pmod 8$. Then $\operatorname{Sign} M$ is divisible by 16.*

thm:221 **Example 11.22** ($\mathbb{CP}^2$)**.** The group $H^2(\mathbb{CP}^2;\mathbb{Z})$ is infinite cyclic and a positive generator is Poincareé dual to a projective line $\mathbb{CP}^1 \subset \mathbb{CP}^2$. The self-intersection number of that line is one, whence $\operatorname{Sign} \mathbb{CP}^2 = 1$.

thm:222 **Example 11.23** ($\overline{\mathbb{CP}^2}$)**.** This is the usual notation for the orientation-reversed manifold $-\mathbb{CP}^2$. By (11.3) we find $\operatorname{Sign} \overline{\mathbb{CP}^2} = -1$.

Obviously, neither $\mathbb{CP}^2$ nor $\overline{\mathbb{CP}^2}$ is spinable, as proved in Corollary 9.29 and now also follows from Theorem 11.21.

I leave several important facts to you.

thm:223 **Exercise 11.24.** Prove that $\operatorname{Sign} \mathbb{CP}^{2\ell} = 1$ for all $\ell \in \mathbb{Z}^{>0}$.

thm:224 **Exercise 11.25.** Show that the signature is additive under disjoint union and also connected sum. Prove that if M_1, M_2 have dimensions divisible by 4, then $\operatorname{Sign}(M_1 \times M_2) = \operatorname{Sign}(M_1) \operatorname{Sign}(M_2)$. In fact, the statement is true without restriction on dimension as long as we define $\operatorname{Sign} M = 0$ if $\dim M$ is not divisible by four.

Signature and bordism

We prove that the signature is a bordism invariant: if $M^{4k} = \partial N^{4k+1}$ and N is compact and oriented, then $\operatorname{Sign} M = 0$. We first prove two lemmas. The first should remind you of Stokes' theorem.

thm:225 **Lemma 11.26.** *Let N^{4k+1} be a compact oriented manifold with boundary $i\colon M^{4k} \hookrightarrow N$. Suppose $c \in H^{4k}(N;A)$ for some abelian group A. Then*

eq:264 (11.27)
$$\langle i^*(c), [M]\rangle = 0.$$

Proof. We have

eq:265 (11.28)
$$\langle i^*(c), [M]\rangle = \langle c, i_*[M]\rangle = 0$$

since $i_*[M] = 0$. (This is a property of duality; intuitively, the manifold M *is* a boundary, so too is its fundamental class.) ☐

This can also be proved using differential forms, via the de Rham theorem, if $A \subset \mathbb{R}$. Namely, if ω is a closed $4k$-form on N which represents the real image of c in $H^{4k}(N;\mathbb{R})$, then the pairing $\langle i^*(c), [M]\rangle$ can be computed as

eq:266 (11.29)
$$\int_M i^*(\omega) = \int_N d\omega = 0$$

by Stokes' theorem.

thm:226 **Lemma 11.30.** *Let $B\colon V \times V \to \mathbb{R}$ be a nondegenerate symmetric bilinear form on a real vector space V. Suppose $W \subset V$ is isotropic—$B(w_1, w_2) = 0$ for all $w_1, w_2 \in W$—and $2\dim W = \dim V$. Then* $\operatorname{Sign} B = 0$.

Proof. Let $e_1 \in W$ be nonzero. Since B is nondegenerate there exists $f_1 \in V$ such that $B(e_1, f_1) = 1$. Shifting f_1 by a multiple of e_1 we can arrange that $B(f_1, f_1) = 0$. In other words, the form B on the subspace $\mathbb{R}\{e_1, f_1\} \subset V$ is hyperbolic, so has signature zero. Let V_1 be the orthogonal complement to $\mathbb{R}\{e_1, f_1\} \subset V$ relative to the form B. Since B is nondegenerate we have $V = \mathbb{R}\{e_1, f_1\} \oplus V_1$. Also, $W_1 := W \cap V_1 \subset V_1$ is isotropic and $2\dim W_1 = \dim V_1$. Set $B_1 = B\big|_{V_1}$. Then the data (V_1, B_1, W_1) satisfies the same hypotheses as (V, B, W) and has smaller dimension. So we can repeat and in a finite number of steps write B as a sum of hyperbolic forms. ☐

thm:227 **Theorem 11.31.** *Let N^{4k+1} be a compact oriented manifold with boundary $i\colon M^{4k} \hookrightarrow N$. Then* $\operatorname{Sign} M = 0$.

Proof. Consider the commutative diagram

eq:267 (11.32)
$$\begin{tikzcd}
H^{2k}(N;\mathbb{R}) \arrow[r, "i^*"] \arrow[d, "\cong"'] & H^{2k}(M;\mathbb{R}) \arrow[r] \arrow[d, "\cong"'] & H^{2k+1}(N,M;\mathbb{R}) \arrow[d, "\cong"'] \\
H_{2k+1}(N,M;\mathbb{R}) \arrow[r] & H_{2k}(M;\mathbb{R}) \arrow[r, "i_*"] & H_{2k}(N;\mathbb{R})
\end{tikzcd}$$

The rows are a stretch of the long exact sequences of the pair (N, M) in real cohomology and real homology. The vertical arrows are Poincaré duality isomorphisms. We claim that image(i^*) is isotropic for the real intersection pairing

eq:268

$$I_M \otimes \mathbb{R}\colon H^{2k}(M;\mathbb{R}) \times H^{2k}(M;\mathbb{R}) \longrightarrow \mathbb{R} \tag{11.33}$$

and has dimension $\frac{1}{2}\dim H^{2k}(M;\mathbb{R})$. The isotropy follows immediately from Lemma 11.26. This and the commutativity of (11.32) imply that (i) image(i^*) maps isomorphically to $\ker(i_*)$ under Poincaré duality, and (ii) image(i^*) annihilates $\ker(i_*)$ under the pairing of cohomology and homology. It is an easy exercise that these combine to prove $2\dim \operatorname{image}(i^*) = \dim H^{2k}(M;\mathbb{R})$. Now the theorem follows immediately from Lemma 11.30. ☐

thm:228 **Corollary 11.34.** *For each $k \in \mathbb{Z}^{\ge 0}$ the signature defines a homomorphism*

eq:269

$$\operatorname{Sign}\colon \Omega^{SO}_{4k} \longrightarrow \mathbb{Z}. \tag{11.35}$$

That (11.35) is well-defined follows from Theorem 11.31; that it is a homomorphism follows from Exercise 11.25. In fact, defining the signature to vanish in dimensions not divisible by four, we see from Exercise 11.25 that

eq:270

$$\operatorname{Sign}\colon \Omega^{SO} \longrightarrow \mathbb{Z} \tag{11.36}$$

is a ring homomorphism.

Any manifold with nonzero signature is not null bordant. In particular,

thm:241 **Proposition 11.37.** *$\mathbb{CP}^{2\ell}$ is not null bordant, $l \in \mathbb{Z}^{>0}$.*

thm:242 **Exercise 11.38.** Demonstrate explicitly that $\mathbb{CP}^{2\ell+1}$ is null bordant by exhibiting a null bordism.

Hirzebruch's signature theorem

subsec:11.6

(11.39) *Pontrjagin numbers.* Recall the Pontrjagin classes, defined in (7.68). For a smooth manifold M we have $p_i(M) \in H^{4i}(M;\mathbb{Z})$. Suppose M is closed and oriented. Then for any sequence $(i_1, \dots, i_r)$ of positive integers we define the Pontrjagin number

eq:271

$$p_{i_1,\dots,i_r}(M) = \langle p_{i_1}(M) \smile \cdots \smile p_{i_r}(M), [M]\rangle. \tag{11.40}$$

By degree count, this vanishes unless $4(i_1+\cdots+i_r) = \dim M$. In any case Lemma 11.26 immediately implies the following

thm:229 **Proposition 11.41.** *The Pontrjagin numbers are bordism invariants*

eq:272

$$p_{i_1,\dots,i_r}\colon \Omega^{SO}_n \longrightarrow \mathbb{Z}. \tag{11.42}$$

subsec:11.7

(11.43) *Tensoring with $\mathbb{Q}$.* The following simple observation is crucial: the map $\mathbb{Z} \longrightarrow \mathbb{Z}\otimes\mathbb{Q}$ is injective. For this is merely the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. This means that (11.35) and (11.42) are determined by the linear functionals

eq:273

$$\mathrm{Sign}\colon \Omega^{SO}_{4k}\otimes\mathbb{Q} \longrightarrow \mathbb{Q} \tag{11.44}$$

and

eq:274

$$p_{i_1,\dots,i_r}\colon \Omega^{SO}_{n}\otimes\mathbb{Q} \longrightarrow \mathbb{Q} \tag{11.45}$$

obtained by tensoring with $\mathbb{Q}$. This has the advantage that the vector space $\Omega^{SO}_n\otimes\mathbb{Q}$ is easier to compute than the abelian group Ω^{SO}_n. In fact, we already summarized the main results about Ω^{SO} in Theorem 2.24. These follow by applying the Pontrjagin-Thom theorem of Lecture 10, specifically Corollary 10.38. We recall just the statement we need here and present the proof in the next lecture.

thm:230

Theorem 11.46. *There is an isomorphism*

eq:275

$$\mathbb{Q}[y^1,y^2,y^3,\dots] \xrightarrow{\;\cong\;} \Omega^{SO}\otimes\mathbb{Q} \tag{11.47}$$

under which y^k maps to the oriented bordism class of the complex projective space $\mathbb{CP}^{2k}$.

Assuming Theorem 11.46 for now, we can prove the main theorem of this lecture.

thm:231

Theorem 11.48 (Hirzebruch)**.** *Let M^{4k} be a closed oriented manifold. Then*

eq:276

$$\mathrm{Sign}\,M = \langle L(M),[M]\rangle, \tag{11.49}$$

where $L(M)\in H^\bullet(M;\mathbb{Q})$ is the L-class (7.61).

Proof. It suffices to check the equation (11.49) on a basis of the rational vector space $\Omega^{SO}_{4k}\otimes\mathbb{Q}$. By Theorem 11.46 this is given by a product of projective spaces $M_{k_1,\dots k_r} := \mathbb{CP}^{2k_1}\times\cdots\times\mathbb{CP}^{2k_r}$ for $k_1+\cdots+k_r=k$. By Exercise 11.24 and Exercise 11.25 we see that

eq:281

$$\mathrm{Sign}\,M_{k_1,\dots,k_r} = 1. \tag{11.50}$$

On the other hand, by Proposition 8.8 we have

eq:277

$$\langle L(\mathbb{CP}^{2k_i}),[\mathbb{CP}^{k_i}]\rangle = 1 \tag{11.51}$$

for all i. Since

eq:278

$$L(M_{k_1,\dots,k_r}) = L(\mathbb{CP}^{2k_1})\cdots L(\mathbb{CP}^{2k_r}) \tag{11.52}$$

and

eq:279 (11.53)
$$[\mathbb{CP}^{2k_1}\times\cdots\times\mathbb{CP}^{2k_r}]=[\mathbb{CP}^{2k_1}]\times\cdots\times[\mathbb{CP}^{2k_r}],$$

it follows that

eq:280 (11.54)
$$\langle L(M_{k_1,\dots,k_r}),[M_{k_1,\dots,k_r}]\rangle=1.$$

(The product on the right hand side of (11.53) is the tensor product in the Kunneth theorem for the rational homology vector space $H_{2k}(\mathbb{CP}^{2k_1}\times\cdots\times\mathbb{CP}^{2k_r};\mathbb{Q})$.) The theorem now follows from (11.50) and (11.54). ☐

Integrality

For a 4-manifold M^4 the signature formula (11.49) asserts

eq:282 (11.55)
$$\operatorname{Sign} M=\langle p_1(M)/3,[M]\rangle.$$

In particular, since the left hand side is an integer, so is the right hand side. *A priori* this is far from clear: whereas $p_1(M)$ is an integral cohomology class, $\frac{1}{3}p_1(M)$ is definitely not—it is only a rational class. Also, there exist real vector bundles $V\to M$ over 4-manifolds so that $\langle p_1(V)/3,[M]\rangle$ is not an integer.

thm:232 **Exercise 11.56.** Find an example. Even better, find an example in which M is a spin manifold.

So the integrality is special to the tangent bundle.

This is the tip of an iceberg of integrality theorems.

thm:233 **Exercise 11.57.** Work out the formula for the signature in 8 and 12 dimensions in terms of Pontrjagin numbers. Note that the denominators grow rapidly.

Hurewicz theorems

A basic tool for the computation is the Hurewicz theorem, which relates homotopy and homology groups.

subsec:11.8

(11.58) *The integral Hurewicz theorem.* Let (X,x) be a pointed topological space.[21] The *Hurewicz map*

eq:283 (11.59)
$$\eta_n\colon \pi_nX\longrightarrow H_nX$$

---

[21] I didn't mention earlier the technical issue that the basepoint should be nondegenerate in a certain sense: the inclusion $\{x\}\hookrightarrow X$ should be a *cofibration*. See [Ma1] for details.

sends a homotopy class represented by a pointed map $f\colon S^n \to X$ to the homology class $f_*[S^n]$. You probably proved in the prelim class that for $n = 1$ the Hurewicz map is surjective with kernel the commutator subgroup $[\pi_1X, \pi_1X] \subset \pi_1X$, i.e., H_1X is the abelianization of π_1X. For higher n we have the following. Recall that a pointed space is *k-connected*, $k \in \mathbb{Z}^{>0}$, if it is path connected and if $\pi_iX = 0$ for $i \le k$.

thm:234 **Theorem 11.60** (Hurewicz)**.** *Let X be a pointed space which is $(n-1)$-connected for $n \in \mathbb{Z}^{\ge 2}$. Then the Hurewicz homomorphism η_n is an isomorphism.*

We refer the reader to standard texts (e.g. [H1], [Ma1]) for a proof of the Hurewicz theorem. The following is immediate by induction.

thm:238 **Corollary 11.61.** *Let X be a 1-connected pointed space which satisfies $H_iX = 0$ for $i = 2, 3, \dots, n-1$. Then X is $(n-1)$-connected and (11.59) is an isomorphism.*

subsec:11.11

(11.62) *The rational Hurewicz theorem.* There is also a version of the Hurewicz theorem over $\mathbb{Q}$. We state it here and refer to [KK] for an "elementary" proof. (It truly is more elementary than other proofs!)

thm:243 **Theorem 11.63** ($\mathbb{Q}$-Hurewicz)**.** *Let X be a 1-connected pointed space, and assume that $\pi_iX \otimes \mathbb{Q} = 0$, $2 \le i \le n-1$, for some $n \in \mathbb{Z}^{\ge 2}$. Then the rational Hurewicz map*

eq:292
$$\eta_i \otimes \mathbb{Q}\colon \pi_iX \otimes \mathbb{Q} \longrightarrow H_i(X;\mathbb{Q}) \tag{11.64}$$

is an isomorphism for $1 \le i \le 2n-2$.

It is also true that η_{2n-1} is surjective, but we do not need this.

Computation for 4-manifolds

By Corollary 10.38 there is an isomorphism

eq:288
$$\phi\colon \pi_4(MSO) \longrightarrow \Omega_4^{SO}. \tag{11.65}$$

Recall that $\pi_4(MSO) \cong \pi_{4+q}MSO(q)$ for q sufficiently large. And **(11.43)** it suffices to compute $\pi_4(MSO) \otimes \mathbb{Q}$.

thm:244 **Theorem 11.66.** *If $q \ge 6$, then* $\dim_{\mathbb{Q}} \pi_{4+q}\left(MSO(q) \otimes \mathbb{Q}\right) = 1$.

Proof. Recall that there is a diffeomorphism $SO(3) \simeq \mathbb{RP}^3$, so its rational homotopy groups are isomorphic to those of the double cover S^3, the first few of which are

eq:293
$$\pi_iSO(3) \otimes \mathbb{Q} \cong \begin{cases} 0, & i = 1, 2; \\ \mathbb{Q}, & i = 3. \end{cases} \tag{11.67}$$

Now for any integer $q \geq 3$ the group $SO(q+1)$ acts transitively on S^q with stabilizer of a point in S^q the subgroup $SO(q)$. So there is a fiber bundle $SO(q) \to SO(q+1) \to S^q$, which is in fact a principal $SO(q)$-bundle.[22] The induced long exact sequence of homotopy groups[23] has a stretch

eq:294
$$\pi_{i+1}SO(q+1) \longrightarrow \pi_{i+1}S^q \longrightarrow \pi_i SO(q) \longrightarrow \pi_i SO(q+1) \longrightarrow \pi_i S^q \longrightarrow \pi_{i-1}SO(q) \longrightarrow \cdots \tag{11.68}$$

and it remains exact after tensoring with $\mathbb{Q}$. First use it to show $\pi_2 SO(q) \otimes \mathbb{Q} = 0$ for all[24] $q \geq 3$. Set $q = 3$. Then, using the result that $\pi_4 S^3 \cong \mathbb{Z}/2\mathbb{Z}$, so that $\pi_4 S^3 \otimes \mathbb{Q} = 0$, we deduce that $\pi_3 SO(4) \otimes \mathbb{Q}$ has dimension 2. Now set $q = 4$ and deduce that $\pi_3 SO(5) \otimes \mathbb{Q}$ has dimension 1. You will need to also use the result that $\pi_5 S^3 \otimes \mathbb{Q} = 0$. By induction on $q \geq 5$ we then prove

eq:295
$$\pi_i SO(q) \otimes \mathbb{Q} \cong \begin{cases} 0, & i = 1, 2; \\ \mathbb{Q}, & i = 3 \end{cases} \tag{11.69}$$

for all $q \geq 5$.

Next, use the universal fiber bundle $G \to EG \to BG$ for $G = SO(q)$, $q \geq 5$, which is a special case of (6.61), and the fact that EG is contractible, so has vanishing homotopy groups, to deduce

eq:296
$$\pi_i BSO(q) \otimes \mathbb{Q} \cong \begin{cases} 0, & i = 1, 2, 3; \\ \mathbb{Q}, & i = 4 \end{cases} \tag{11.70}$$

from the long exact sequence of homotopy groups. Then the $\mathbb{Q}$-Hurewicz Theorem 11.63 implies

eq:297
$$H_i\big(BSO(q); \mathbb{Q}\big) \cong \begin{cases} 0, & i = 1, 2, 3; \\ \mathbb{Q}, & i = 4; \\ 0, & i = 5, 6 \end{cases} \tag{11.71}$$

for $q \geq 5$.

The proof of the Thom isomorphism theorem, Proposition 8.35, gives a cell structure for the Thom complex. The resulting Thom isomorphism on homology implies

eq:298
$$H_i\big(MSO(q); \mathbb{Q}\big) \cong \begin{cases} 0, & i = 1, \dots, q-1; \\ \mathbb{Q}, & i = q; \\ 0, & i = 1+q, 2+q, 3+q \\ \mathbb{Q}, & i = 4+q \\ 0, & i = 5+q, 6+q. \end{cases} \tag{11.72}$$

The cell structure also implies that the Thom complex $MSO(q)$ of the universal bundle $S(q) \to BSO(q)$ is $(q-1)$-connected. The $\mathbb{Q}$-Hurewicz theorem then implies that the $\mathbb{Q}$-Hurewicz map

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[22] We construct it here by fixing a point in S^q. Can you construct an isomorphic principal $SO(q)$-bundle without choosing a basepoint? what is the geometric meaning of the total space?

[23] We have used this before; see [H1, Theorem 4.41] or, for a quick review, [BT, §17].

[24] In fact, $\pi_2 G = 0$ for any finite dimensional Lie group G.

$\pi_i MSO(q)\otimes\mathbb{Q}\to H_i\bigl(MSO(q);\mathbb{Q}\bigr)$ is an isomorphism for $1\le i\le 2q-2$, whence if $q\ge 6$ we deduce in particular

eq:299

$$\pi_{4+q}\bigl(MSO(q);\mathbb{Q}\bigr)\cong\mathbb{Q}. \tag{11.73}$$

$\square$

By Proposition 11.37 the class of $\mathbb{CP}^2$ in $\Omega_4^{SO}\otimes\mathbb{Q}$ is nonzero. (We need a bit more: $\mathbb{CP}^2$ has infinite order in Ω_4^{SO} because its signature is nonzero and the signature (11.35) is a homomorphism.) Since $\pi_4(MSO)\otimes\mathbb{Q}$ is one-dimensional, the class of $\mathbb{CP}^2$ is a basis. Finally, we prove (11.55) by checking both sides for $M=\mathbb{CP}^2$ using Example 11.22, Proposition 7.51, and the definition (7.68) of the Pontrjagin classes.

Lecture 12: More on the signature theorem

sec:12

Here we sketch the proof of Theorem 11.46. In the last lecture we indicated most of the techniques involved by proving the theorem for 4-manifolds. There are two additional inputs necessary for the general case. First, we need to know that the rational cohomology of BSO is the polynomial ring on the Pontrjagin classes. We simply quote that result here, but remark that it follows from Theorem 7.72. In fact, all we really end up using is the graded dimension of the rational cohomology—its dimension in each degree. The second input is purely algebraic, to do with symmetric functions. We indicate what the issue is and refer the reader to the literature.

As we are about to leave classical bordism, we begin with a comment—thanks to a student question and off-topic with respect to the signature theorem—which could have been made right at the beginning of the course.

Bordism as a generalized homology theory

The basic building blocks of singular homology theory are continuous maps

eq:300

$$f\colon \Delta^q \longrightarrow X \tag{12.1}$$

from the standard q-simplex Δ^q to a topological space X. Chains are formal sums of such maps, and there is a boundary operator, so a notion of closed chains, or cycles. From this one builds a chain complex and homology. A crucial case is $X = \text{pt}$. Then the homology question comes down to whether a closed simplicial complex is a boundary. It is: one can simply cone off the simplicial complex σ to construct a new simplicial complex $C\sigma$ whose boundary is σ.

In bordism theory—as a generalized homology theory—one replaces (12.1) by continuous maps

eq:301

$$f\colon M^q \longrightarrow X \tag{12.2}$$

out of a closed q-dimensional manifold M. Now rather than defining a formal abelian group of "chains", we define the equivalence relation of bordism: $f_i\colon M_i \to X$, $i = 0, 1$, are equivalent if there exists a compact $(q+1)$-manifold N with $\partial N = M_0 \amalg M_1$ and a continuous map $f\colon N \to X$ whose restriction to the boundary is $f_0 \amalg f_1$. (Of course, we should make a more elaborate definition modeled on Definition 1.19.) The equivalence classes turn out to be an abelian group, which we denote $\Omega_q(X)$. Then the graded abelian group $\Omega_\bullet(X)$ satisfies all of the axioms of homology theory except for the specification of $\Omega_\bullet(\text{pt})$. What we have been studying *is* $\Omega_\bullet(\text{pt})$. But I want you to know that there is an entire homology theory there. See [DK] for one account.

I remark that there is a variation $\Omega^{\mathcal{X}}_\bullet(X)$ for every stable tangential structure $\mathcal{X}$.

Mising steps

We begin with an important result in its own right.

`subsec:12.2`

(12.3) *The cohomology of BSO.* [**summarize Milnor-Stasheff argument with Gysin sequence and induction to compute dimension of the rational cohomology.**]

`thm:246` **Theorem 12.4.** *The rational cohomology ring of the classifying space of the special orthogonal group is the polynomial ring generated by the Pontrjagin classes:*

`eq:302`
$$H^\bullet(BSO;\mathbb{Q})\cong\mathbb{Q}[p_1,p_2,\dots]. \tag{12.5}$$

One proof follows from Theorem 7.72, which identifies real[25] cohomology classes on $BSO(q)$ with invariant polynomials on the orthogonal algebra $\mathfrak{o}(q)$. The latter is the Lie algebra of real skew-symmetric matrices. 'Invariant' means invariant under conjugation by an orthogonal matrix. So for a skew-symmetric matrix A we must produce a polynomial $P(A)\in\mathbb{R}$ so that $P(OAO^{-1})=P(A)$ for every orthogonal matrix O. This is easy to do. Define

`eq:303`
$$Q_t(A)=\det(I-tA)=1+P_1(A)t^2+P_2(A)t^4+\cdots, \tag{12.6}$$

where I is the identity matrix. Then $Q_t(A)$ is a polynomial in t with real coefficients, and by the skew-symmetry of A we can show $Q_{-t}(A)=Q_t(A)$, so only even powers of t occur. (Prove it!) The coefficients P_i are invariant polynomials in A, and up to a factor they correspond to the universal Pontrjagin classes.

`thm:247` **Exercise 12.7** ([Kn]). Here are some hints—using some theory of compact Lie groups—towards a proof of Theorem 7.72.[26] Let $T\subset G$ be a maximal torus, $N\subset G$ its normalizer, and $W=N/T$ the Weyl group. Identify G-invariant polynomials on $\mathfrak{g}$ with W-invariant polynomials on the Lie algebra $\mathfrak{t}$ of T. Consider the iterated fibration $EG/T\to EG/N\to EG/G$, which is $BT\to BN\to BG$. The first map is a finite cover, and induces an isomorphism in rational cohomology. The fiber of $BN\to BG$ is G/N, which has the rational cohomology of a sphere.

`subsec:12.4`

(12.8) *The proof.* Now we sketch a proof of most of Theorem 11.46, which we restate here. The statement about complex projective spaces is deferred to a later subsection.

`thm:245` **Theorem 12.9.** *There is an isomorphism*

`eq:308`
$$\mathbb{Q}[x^1,x^2,x^3,\dots]\xrightarrow{\cong}\Omega^{SO}\otimes\mathbb{Q}. \tag{12.10}$$

All we really need from the statement is that the dimension of $\Omega^{SO}_{4k}\otimes\mathbb{Q}$ is $p(k)$, the number of partitions of k.

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[25] The result over the rationals is stronger, but follows since the Pontrjagin classes are rational.

[26] The Lie group G in the theorem should be assumed connected.

Proof. The rational homology of BSO is the dual vector space to the rational cohomology, so

eq:304 (12.11)
$$H_\bullet(BSO;\mathbb{Q})\cong\mathbb{Q}[p^1,p^2,\dots]$$

for dual homology classes $p^1,p^2,\dots$. The Thom isomorphism theorem, as in the derivation of (11.72), and the definition (10.13) of the homology of a spectrum, imply

eq:305 (12.12)
$$H_\bullet(MSO;\mathbb{Q})\cong\mathbb{Q}[q^1,q^2,\dots]$$

for some classes $q^k\in H_{2k}(MSO;\mathbb{Q})$. Finally, $MSO(q)$ is $(q-1)$-connected, which by $\mathbb{Q}$-Hurewicz implies that the map

eq:306 (12.13)
$$\eta_i\otimes\mathbb{Q}\colon\,\pi_i\bigl(MSO(q)\bigr)\otimes\mathbb{Q}\longrightarrow H_i\bigl(MSO(q);\mathbb{Q}\bigr)$$

is an isomorphism for $1\le i\le 2q-2$. In the limit $q\to\infty$ we obtain an isomorphism for all i. □

subsec:12.5

(12.14) *A very nice exercise.* The following is a great test of your understanding of the Pontrjagin-Thom construction.

thm:248 **Exercise 12.15.** Suppose that M is a closed oriented $4k$-manifold whose rational bordism class is the sum[27] $c_{i_1\cdots i_r}x^{i_1}\cdots x^{i_r}$ under the isomorphism (11.47). Recall the Pontrjagin number (11.42). Prove that $c_{i_1\cdots i_r}$ is the Pontrjagin number $p_{i_1\cdots i_r}$ of the stable normal bundle to M. You will need, of course, to use the generators x^i defined in the proof.

Complex projective spaces as generators

The content of Theorem 12.9 is that $\Omega_{4k}^{SO}\otimes\mathbb{Q}$ is a rational vector space of dimension $p(k)$, the number of *partitions* of k. Recall that a partition of a positive integer k is a finite unordered set $\{i_1,\dots,i_r\}$ of positive integers such that $i_1+\dots+i_r=k$. For example, $\Omega_8^{SO}\otimes\mathbb{Q}$ is 2-dimensional. The remaining statement we must prove is the following.

thm:249 **Proposition 12.16.** *Let $k\in\mathbb{Z}^{\ge1}$. The manifolds $M_{i_1\cdots i_r}:=\mathbb{CP}^{2i_1}\times\cdots\mathbb{CP}^{2i_r}$ form a basis of $\Omega_{4k}^{SO}\otimes\mathbb{Q}$, where $\{i_1,\dots,i_r\}$ ranges over all partitions of k.*

The case $k=1$ is easy, as we used in Lecture 11. For $k=2$ we must show that the classes of $\mathbb{CP}^4$ and $\mathbb{CP}^2\times\mathbb{CP}^2$ are linearly independent. We can use the Pontrjagin numbers p_1^2,p_2 to show that: the matrix

eq:307 (12.17)
$$\begin{pmatrix}25&10\\18&9\end{pmatrix}$$

is nondegenerate. The rows represent the manifolds $\mathbb{CP}^4$, $\mathbb{CP}^2\times\mathbb{CP}^2$ and the colums the Pontrjagin numbers p_1^2, p_2. This sort of argument does not easily generalize. Rather than repeat the necessary algebra of symmetric functions here, we defer to [MS, §16].

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[27] over a basis of polynomials of degree $4k$

Lecture 13: Categories

`sec:13`

We begin again. In Lecture 1 we used bordism to define an equivalence relation on closed manifolds of a fixed dimension n. The set of equivalence classes has an abelian group structure defined by disjoint union of manifolds. Now we extract a more intricate algebraic structure from bordisms. The equivalence relation only remembers the existence of a bordism; now we record the bordism itself. The bordism now has a direction: it is a map from one closed manifold to another. Gluing of bordisms, previously used to prove transitivity of the equivalence relation, is now recorded as a composition law on bordisms. To obtain an associative composition law we remember bordisms only up to diffeomorphism. (In subsequent lectures we will go further and remember the diffeomorphism.) The algebraic structure obtained is a *category* $\mathrm{Bord}_{\langle n-1,n\rangle}$, which here replaces the *set* of equivalence classes Ω_n. The notation for this category suggests more refinements to come later. Disjoint union provides an algebraic operation on $\mathrm{Bord}_{\langle n-1,n\rangle}$, which is then a *symmetric monoidal category*.

In this lecture we introduce categories, homomorphisms, natural transformations, and symmetric monoidal structures. Pay particular attention to the example of the fundamental groupoid (Example 13.14), which shares some features with the bordism category, though with one important difference: the bordism category is not a groupoid.

Categories

`thm:250` **Definition 13.1.** A *category* C consists of a collection of objects, for each pair of objects y_0, y_1 a set of morphisms $C(y_0,y_1)$, for each object y a distinguished morphism $\mathrm{id}_y \in C(y,y)$, and for each triple of objects y_0,y_1,y_2 a composition law

`eq:309`

$$\circ\colon C(y_1,y_2)\times C(y_0,y_1)\longrightarrow C(y_0,y_2) \tag{13.2}$$

such that $\circ$ is associative and id_y is an identity for $\circ$.

The last phrase indicates two conditions: for all $f\in C(y_0,y_1)$ we have

`eq:310`

$$\mathrm{id}_{y_1}\circ f = f\circ \mathrm{id}_{y_0} = f \tag{13.3}$$

and for all $f_1\in C(y_0,y_1)$, $f_2\in C(y_1,y_2)$, and $f_3\in C(y_2,y_3)$ we have

`eq:311`

$$(f_3\circ f_2)\circ f_1 = f_3\circ (f_2\circ f_1). \tag{13.4}$$

We use the notation $y\in C$ for an object of C and $f\colon y_0\to y_1$ for a morphism $f\in C(y_0,y_1)$.

thm:251 *Remark* 13.5 (set theory). The words ‘collection’ and ‘set’ are used deliberately. Russell pointed out that the collection of all sets is not a set, yet we still want to consider a category whose objects are sets. For many categories the objects do form a set. In that case the moniker ‘small category’ is often used. In these lecture we will be sloppy about the underlying set theory and simply talk about a set of objects.

thm:252 **Definition 13.6.** Let C be a category.

(i) A morphism $f \in C(y_0, y_1)$ is *invertible* (or an *isomorphism*) if there exists $g \in C(y_1, y_0)$ such that $g \circ f = \mathrm{id}_{y_0}$ and $f \circ g = \mathrm{id}_{y_1}$.

(ii) If every morphism in C is invertible, then we call C a *groupoid.*

subsec:13.1

(13.7) *Reformulation.* To emphasize that a category is an algebraic structure like any other, we indicate how to formulate the definition in terms of sets[28] and functions. Then a category C consists of a set C_0 of objects, a set C_1 of functions, and structure maps

eq:312

$$\begin{aligned} i\colon C_0 &\longrightarrow C_1 \\ s,t\colon C_1 &\longrightarrow C_0 \\ c\colon C_1 \times_{C_0} C_1 &\longrightarrow C_1 \end{aligned} \tag{13.8}$$

which satisfy certain conditions. The map i attaches to each object y the identity morphism id_y, the maps s, t assign to a morphism $(f\colon y_0 \to y_1) \in C_1$ the source $s(f) = y_0$ and target $t(f) = y_1$, and c is the composition law. The fiber product $C_1 \times_{C_0} C_1$ is the set of pairs $(f_2, f_1) \in C_1 \times C_1$ such that $t(f_1) = s(f_2)$. The conditions (13.3) and (13.4) can be expressed as equations for these maps.

Examples of categories

thm:253 **Example 13.9** (monoid)**.** Let C be a category with a single object, i.e., $C_0 = \{*\}$. Then C_1 is a set with an identity element and an associative composition law. This is called a *monoid.* A groupoid with a single object is a *group.*[29]

thm:254 **Example 13.10** (set)**.** At the other extreme, suppose C is a category with only identity maps, i.e., $i\colon C_0 \to C_1$ is an isomorphism of sets (a 1:1 correspondence). Then C is given canonically by the set C_0 of objects, and we identify the category C as this set.

thm:255 **Example 13.11** (action groupoid)**.** Let S be a set and G a group which acts on S. There is an associated groupoid $C = S/\!/G$ with objects $C_0 = S$ and morphisms $C_1 = G \times S$. The source map is projection to the first factor and the target map is the action $G \times S \to S$. We leave the reader to work out the composition and show that the axioms for a category are a direct consequence of those for a group action. See Figure 22.

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[28] ignoring set-theoretic complications, as in Remark 13.5

[29] So, by analogy, you’d think instead of ‘category’ we’d use ‘monoidoid’!

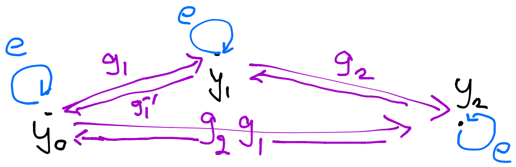

FIGURE 22. The action groupoid $S/\!/G$ fig:22

thm:257 **Example 13.12** (category of sets)**.** Assuming that the set theoretic difficulties alluded to in Remark 13.5 are overcome, there is a category Set whose objects are sets and whose morphisms are functions.

thm:256 **Example 13.13** (subcategories of Set)**.** There is a category Ab of abelian groups. An object $A \in$ Ab is an abelian group and a morphism $f\colon A_0 \to A_1$ is a homomorphism of abelian groups. Similarly, there is a category Vect_k of vector spaces over a field k. There is also a category of rings and a category of R-modules for a fixed ring R. (Note Ab is the special case $R = \mathbb{Z}$.) Each of these categories is special in that the hom-sets are abelian groups. There is also a category Top whose objects are topological spaces Y and in which a morphism $f\colon Y_0 \to Y_1$ is a continuous map.

thm:258 **Example 13.14** (fundamental groupoid)**.** Let Y be a topological space. The simplest invariant is the *set* $\pi_0 Y$. It is defined by imposing an equivalence relation on the set Y underlying the topological space: points y_0 and y_1 in Y are equivalent if there exists a continuous path which connects them, i.e., a continuous map $\gamma\colon [0,1] \to Y$ which satisfy $\gamma(0) = y_0$, $\gamma(1) = y_1$.

The *fundamental groupoid* $C = \pi_{\le 1} Y$ is defined as follows. The objects $C_0 = Y$ are the points of Y. The hom-set $C(y_0, y_1)$ is the set of homotopy classes of maps $\gamma\colon [0,1] \to Y$ which satisfy $\gamma(0) = y_0$, $\gamma(1) = y_1$. The homotopies are taken "rel boundary", which means that the endpoints are fixed in a homotopy. Explicitly, a homotopy is a map

eq:313
$$\Gamma\colon [0,1] \times [0,1] \longrightarrow Y \tag{13.15}$$

such that $\Gamma(s,0) = y_0$ and $\Gamma(s,1) = y_1$ for all $s \in [0,1]$. The composition of homotopy classes of paths is associative, and every morphism is invertible. Note that the *automorphism group* $C(y,y)$ is the fundamental group $\pi_1(Y,y)$. So $\pi_{\le 1} Y$ encodes both $\pi_0 Y$ and all of the fundamental groups.

thm:266 **Exercise 13.16.** Given a groupoid C use the morphisms to define an equivalence relation on the objects and so a set $\pi_0 C$ of equivalence classes. Can you do the same for a category which is not a groupoid?

Functors and natural transformations

thm:259 **Definition 13.17.** Let C, D be categories.

(i) A *functor* or *homomorphism* $F\colon C \to D$ is a pair of maps $F_0\colon C_0 \to D_0$, $F_1\colon C_1 \to D_1$ which commute with the structure maps (13.8).

(ii) Suppose $F, G\colon C \to D$ are functors. A *natural transformation* η from F to G is a map of sets $\eta\colon C_0 \to D_1$ such that for all morphisms $(f\colon y_0 \to y_1) \in C_1$ the diagram

eq:314 (13.18)
$$\begin{tikzcd} Fy_0 \arrow[r, "Ff"] \arrow[d, "\eta(y_0)"'] & Fy_1 \arrow[d, "\eta(y_1)"] \\ Gy_0 \arrow[r, "Gf"] & Gy_1 \end{tikzcd}$$

commutes. We write $\eta\colon F \to G$.

(iii) A natural transformation $\eta\colon F \to G$ is an *isomorphism* if $\eta(y)\colon Fy \to Gy$ is an isomorphism for all $y \in C$.

In (i) the commutation with the structure maps means that F is a homomorphism in the usual sense of algebra: it preserves compositions and takes identities to identities. A natural transformation is often depicted in a diagram

eq:322 (13.19)
$$\begin{tikzcd} C \arrow[r, bend left=50, "G"] \arrow[r, bend right=50, "F"'] \arrow[r, phantom, "\Uparrow \eta"] & D \end{tikzcd}$$

with a double arrow.

thm:260 **Example 13.20** (functor categories)**.** Show that for fixed categories C, D there is a category $\mathrm{Hom}(C, D)$ whose objects are functors and whose morphisms are natural transformations.

thm:261 *Remark* 13.21. Categories have one more layer of structure than sets. Intuitively, elements of a set have no "internal" structure, whereas objects in a category do, as reflected by their self-maps. Numbers have no internal structure, whereas sets do. Try that intuition out on each of the examples above. Anything to do with categories has an extra layer of structure. This is true for homomorphisms of categories: they form a category (Example 13.20) rather than a set. Below we see that when we define a monoidal structure there is an extra layer of data before conditions enter.

thm:262 **Example 13.22.** There is a functor $**\colon \mathrm{Vect} \to \mathrm{Vect}$ which maps a vector space V to its double dual V^{**}. But this is not enough to define it—we must also specify the map on morphisms, which in this case are linear maps. Thus if $f\colon V_0 \to V_1$ is a linear map, there is an induced linear map $f^{**}\colon V_0^{**} \to V_1^{**}$. (Recall that $f^*\colon V_1^* \to V_0^*$ is defined by $\langle f^*(v_1^*), v_0\rangle = \langle v_1^*, f(v_0)\rangle$ for all $v_0 \in V_0$, $V_1^* \in V_1^*$. Then define $f^{**} = (f^*)^*$.) Now there is a natural transformation $\eta\colon \mathrm{id}_{\mathrm{Vect}} \to **$ defined on a vector space V as

eq:315 (13.23)
$$\begin{aligned} \eta(V)\colon V &\longrightarrow V^{**} \\ v &\longmapsto \bigl(v^* \mapsto \langle v^*, v\rangle\bigr) \end{aligned}$$

for all $v^* \in V^*$. I encourage you to check (13.18) carefully.

thm:263 **Example 13.24** (fiber functor)**.** Let Y be a topological space and $\pi\colon Z \to Y$ a covering space. Then there is a functor

eq:316 (13.25)
$$\begin{aligned} F_\pi\colon \pi_{\le 1}Y &\longrightarrow \mathrm{Set} \\ y &\longrightarrow \pi^{-1}(y) \end{aligned}$$

which maps each point of y to the fiber over y. Again, this is not a functor until we tell how morphisms map. For that we need to use the theory of covering spaces. Any path $\gamma\colon [0,1] \to Y$ "lifts" to an isomorphism $\tilde{\gamma}\colon \pi^{-1}(y_0) \to \pi^{-1}(y_1)$, and the isomorphism is unchanged under homotopy. A map

eq:317 (13.26)
$$\begin{tikzcd} Z_0 \arrow[rr, "\varphi"] \arrow[dr, "\pi_0"'] & & Z_1 \arrow[dl, "\pi_1"] \\ & Y & \end{tikzcd}$$

of covering spaces induces a natural transformation $\eta_\varphi\colon F_{\pi_0} \to F_{\pi_1}$.

Symmetric monoidal categories

A category is an enhanced version of a set; a *symmetric monoidal category* is an enhanced version of a commutative monoid. Just as a commutative monoid has data (composition law, identity element) and conditions (associativity, commutativity, identity property), so too does a symmetric monoidal category have data and conditions. Only now the conditions of a commutative monoid become data for a symmetric monoidal category. The conditions are new and numerous. We do not spell them all out, but defer to the references.

subsec:13.2

(13.27) *Product categories.* If C', C'' are categories, then there is a *Cartesian product* category $C = C' \times C''$. The set of objects is the Cartesian product $C_0 = C'_0 \times C''_0$ and the set of objects is likewise the Cartesian product $C_1 = C'_1 \times C''_1$. We leave the reader to work out the structure maps (13.8).

thm:264 **Definition 13.28.** Let C be a category. A *symmetric monoidal structure* on C consists of an object

eq:323 (13.29)
$$1_C \in C,$$

a functor

eq:318 (13.30)
$$\otimes\colon C \times C \longrightarrow C$$

and natural isomorphisms

eq:319 (13.31)
$$\begin{tikzcd} C\times C\times C \arrow[r, bend left, "-\otimes(-\otimes -)"] \arrow[r, bend right, "(-\otimes -)\otimes -"'] \arrow[r, phantom, "\Uparrow \alpha"] & C \end{tikzcd} ,$$

eq:320 (13.32)
$$\begin{tikzcd} C\times C \arrow[r, bend left, "(-\otimes -)\circ\tau"] \arrow[r, bend right, "-\otimes -"'] \arrow[r, phantom, "\Uparrow \sigma"] & C \end{tikzcd} ,$$

and

eq:321 (13.33)
$$\begin{tikzcd} C \arrow[r, bend left, "\mathrm{id}_C"] \arrow[r, bend right, "1_C\otimes -"'] \arrow[r, phantom, "\Uparrow \iota"] & C \end{tikzcd} .$$

The quintuple $(1_C, \otimes, \alpha, \sigma, \iota)$ is required to satisfy the axioms indicated below.

The functor τ in (13.31) is transposition:

eq:328 (13.34)
$$\begin{aligned} \tau\colon C\times C &\longrightarrow C\times C \\ y_1, y_2 &\longmapsto y_2, y_1 \end{aligned}$$

A crucial axiom is that

eq:329 (13.35)
$$\sigma^2 = \mathrm{id}\,.$$

Thus for any $y_1, y_2 \in C$, the composition

eq:330 (13.36)
$$y_1\otimes y_2 \xrightarrow{\ \sigma\ } y_2\otimes y_1 \xrightarrow{\ \sigma\ } y_1\otimes y_2$$

is $\mathrm{id}_{y_1\otimes y_2}$. The other axioms express compatibility conditions among the extra data (13.29)–(13.33). For example, we require that for all $y_1, y_2 \in C$ the diagram

eq:324 (13.37)
$$\begin{tikzcd} & (1_C\otimes y_1)\otimes y_2 \arrow[dl, "\alpha"'] \arrow[dr, "\iota"] & \\ 1_C\otimes(y_1\otimes y_2) \arrow[rr, "\iota"] & & y_1\otimes y_2 \end{tikzcd}$$

commutes. We can state the axioms informally as asserting the equality of any two compositions of maps built by tensoring α, σ, ι with identity maps. These compositions have domain a tensor product of objects $y_1, \dots, y_n$ and any number of identity objects 1_C—ordered and parenthesized arbitrarily—to a tensor product of the same objects, again ordered and parenthesized arbitrarily. Coherence theorems show that there is a small set of conditions which needs to be verified; then arbitrary diagrams of the sort envisioned commute. You can find precise statements and proof in [Mac, JS]

subsec:13.3

(13.38) *Symmetric monoidal functor.* This is a homomorphism between symmetric monoidal categories, but as is typical for categories the fact that the identity maps to the identity and tensor products to tensor products is expressed via data, not as a condition. Then there are higher order conditions.

thm:267 **Definition 13.39.** Let C, D be symmetric monoidal categories. A *symmetric monoidal functor* $F\colon C \to D$ is a functor with two additional pieces of data, namely an isomorphism

eq:325 (13.40)
$$1_D \longrightarrow F(1_C)$$

and a natural isomorphism

eq:326 (13.41)
$$\begin{tikzcd} C \times C \arrow[r, bend left, "F(-\otimes-)"] \arrow[r, bend right, "F(-)\otimes F(-)"'] \arrow[r, phantom, "\Uparrow \psi"] & C\ . \end{tikzcd}$$

There are many conditions on this data.

The first condition expresses compatibility with the associativity morphisms: for all $y_1, y_2, y_3 \in C$ the diagram

eq:327 (13.42)
$$\begin{tikzcd} \big(F(y_1)\otimes F(y_2)\big)\otimes F(y_3) \arrow[r, "\psi"] \arrow[d, "\alpha_D"'] & F(y_1\otimes y_2)\otimes F(y_3) \arrow[d, "\psi"] \\ F(y_1)\otimes \big(F(y_2)\otimes F(y_3)\big) \arrow[d, "\psi"'] & F\big((y_1\otimes y_2)\otimes y_3\big) \arrow[d, "F(\alpha_C)"] \\ F(y_1)\otimes F(y_2\otimes y_3) \arrow[r, "\psi"] & F\big(y_1\otimes (y_2\otimes y_3)\big) \end{tikzcd}$$

is required to commute. Next, there is compatibility with the identity data ι: for all $y \in C$ we requre that

eq:331 (13.43)

eq:331
$$\begin{tikzcd} F(1_C)\otimes F(y) \arrow[r, "F(\psi)"] & F(1_C\otimes y) \arrow[d, "F(\iota)"] \\ 1_D\otimes F(y) \arrow[u, "(13.40)"] \arrow[r, "\iota"] & F(y) \end{tikzcd}$$

commute. The final condition expresses compatibility with the symmetry σ: for all $y_1, y_2 \in C$ the diagram

eq:332 (13.44)

$$\begin{tikzcd}
F(y_1) \otimes F(y_2) \arrow[r, "\sigma_D"] \arrow[d, "\psi"'] & F(y_2) \otimes F(y_1) \arrow[d, "\psi"] \\
F(y_1 \otimes y_2) \arrow[r, "F(\sigma_C)"] & F(y_2 \otimes y_1)
\end{tikzcd}$$

thm:268 **Exercise 13.45.** Define a natural transformation of symmetric monoidal functors.

Lecture 14: Bordism categories

sec:14

The definition

Fix a nonnegative[30] integer n. Recall the basic Definition 1.19 of a bordism $X\colon Y_0 \to Y_1$ whose domain and codomain are closed $(n-1)$-manifolds. A bordism is a quartet $(X, p, \theta_0, \theta_1)$ in which X is a compact manifold with boundary, $p\colon \partial X \to \{0,1\}$ is a partition of the boundary, and θ_0, θ_1 are boundary diffeomorphism. As usual we overload the notation and use 'X' to denote the full quartet of data.

thm:269 **Definition 14.1.** Suppose $X, X'\colon Y_0 \to Y_1$ are bordisms between closed $(n-1)$-manifolds Y_0, Y_1. A *diffeomorphism* $F\colon X \to X'$ is a diffeomorphism of manifolds with boundary which commutes with p, θ_0, θ_1.

So, for example, we have a commutative diagram

eq:333 (14.2)

$$\begin{tikzcd} X \arrow[dr, "p"] \arrow[dd, "F"'] & \\ & \{0,1\} \\ X' \arrow[ur, "p'"'] & \end{tikzcd}$$

and similar commutative diagrams involving the θ's.

thm:270 **Definition 14.3.** Fix $n \in \mathbb{Z}^{\ge 0}$. The *bordism category* $\mathrm{Bord}_{\langle n-1,n\rangle}$ is the symmetric monoidal category defined as follows.

(i) The objects are closed $(n-1)$-manifolds.
(ii) The hom-set $\mathrm{Bord}_{\langle n-1,n\rangle}(Y_0, Y_1)$ is the set of diffeomorphism classes of bordisms $X\colon Y_0 \to Y_1$.
(iii) Composition of morphisms is by gluing (Figure 2).
(iv) For each Y the bordism $[0,1] \times Y$ is $\mathrm{id}_Y\colon Y \to Y$.
(v) The monoidal product is disjoint union.
(vi) The empty manifold $\emptyset^{n-1}$ is the tensor unit (13.29).

The additional data α, σ, ι expresses the associativity and commutativity of disjoint union, which we suppress; but see **(1.16)**. In (iv) the partition of the boundary is projection $p\colon \{0,1\} \times Y \to \{0,1\}$ onto the first factor and the boundary diffeomorphisms are the identity on Y.

---

[30] We allow $n = 0$. Recall that the empty manifold can have any dimension, and we allow $\emptyset^{-1}$ of dimension -1.

subsec:14.1

(14.4) *Isotopy.* Let Diff Y denote the group of smooth diffeomorphisms of a closed manifold Y. It is a topological group[31] if we use the compact-open topology.

thm:271

Definition 14.5.

(i) An *isotopy* is a smooth map $F\colon [0,1]\times Y \to Y$ such that $F(t,-)\colon Y \to Y$ is a diffeomorphism for all $t \in [0,1]$.

(ii) A *pseudoisotopy* is a diffeomorphism $\widetilde{F}\colon [0,1] \times Y \to [0,1] \times Y$ which preserves the submanifolds $\{0\} \times Y$ and $\{1\} \times Y$.

Equivalently,[32] an isotopy is a path in Diff Y. Diffeomorphisms f_0, f_1 are said to be *isotopic* if there exists an isotopy $F\colon f_0 \to f_1$. Isotopy is an equivalence relation. The set of isotopy classes is π_0 Diff Y, which is often called the *mapping class group* of Y. An isotopy induces a pseudoisotopy

eq:334

$$\begin{aligned} \widetilde{F}\colon [0,1] \times Y &\longrightarrow [0,1] \times Y \\ (t,y) &\longmapsto \big(t, F(t,y)\big) \end{aligned} \tag{14.6}$$

We say $\widetilde{F}\colon f_0 \to f_1$ if the induced diffeomorphisms of Y on the boundary of $[0,1] \times Y$ are f_0 and f_1.

thm:272

Exercise 14.7. Prove that pseudoisotopy is an equivalence relation.

thm:273

Remark 14.8. Pseudoisotopy is potentially a courser equivalence relation than isotopy: isotopic diffeomorphisms are pseudoisotopic. The converse is true for simply connected manifolds of dimension ≥ 5 by a theorem of Cerf.

subsec:14.2

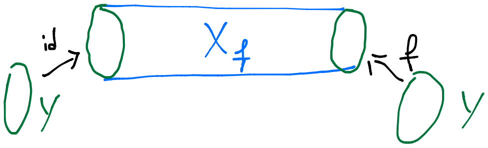

FIGURE 23. The bordism associated to a diffeomorphism

fig:23

(14.9) *Embedding diffeomorphisms in the bordism category.* Let Y be a closed $(n-1)$-manifold and $f\colon Y \to Y$ a diffeomorphism. There is an associated bordism $(X_f, p, \theta_0, \theta_1)$ with (i) $X_f = [0,1] \times Y$, $p\colon \{0,1\} \times Y \to Y$ projection, (iii) $\theta_0 = \mathrm{id}_Y$, and (iv) $\theta_1 = f$, as depicted in Figure 23. If $F\colon f_0 \to f_1$ is an isotopy, then we claim that the bordisms X_{f_0} and X_{f_1} are equal in the hom-set $\mathrm{Bord}_{\langle n-1,n\rangle}(Y,Y)$. For the isotopy F determines a diffeomorphism and the composition of bordisms in the top row of
Figure 24 is X_{f_1}. Of course, Figure 24 shows that *pseudo*isotopic diffeomorphisms determine equal bordisms in $\mathrm{Bord}_{\langle n-1,n\rangle}(Y,Y)$.

---

[31] A *topological group* G is simultaneously and compatibly a topological space and a group: composition and inversion are continuous maps $G \times G \to G$ and $G \to G$.

[32] With what we have introduced we can talk about *continuous* paths in Diff Y, which correspond to maps F which are only continuous in the first variable. But then we can approximate by a smooth map. In any case we can in a different framework discuss *smooth* maps of smooth manifolds into Diff Y.

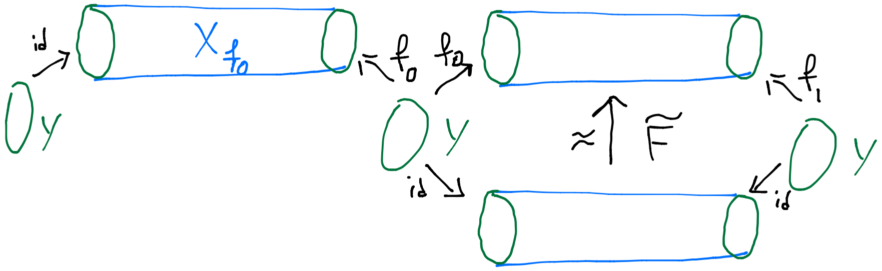

FIGURE 24. Isotopic diffeomorphisms give diffeomorphic bordisms fig:24

thm:274 **Exercise 14.10.** Show that if X_{f_0} and X_{f_1} are equal in $\mathrm{Bord}_{\langle n-1,n\rangle}(Y,Y)$, then f_0 is pseudoisotopic to f_1.

Summarizing, there is a homomorphism

eq:335
$$\pi_0(\mathrm{Diff}\,Y) \longrightarrow \mathrm{Bord}_{\langle n-1,n\rangle}(Y,Y) \tag{14.11}$$

which is not necessarily injective.

thm:278 **Exercise 14.12.** Is (14.11) injective for $n=1$ and $Y = \mathrm{pt} \amalg \mathrm{pt}$?

subsec:14.3

(14.13) *Bordism categories with tangential structures.* Recall Definition 9.45: an n-dimensional tangential structure is a fibration $\mathcal{X}(n) \to BO(n)$. There is a universal rank n bundle $S(n) \to \mathcal{X}(n)$ with $\mathcal{X}(n)$-structure, and an $\mathcal{X}(n)$-structure on a manifold M of dimension $k \le n$ is a commutative diagram

eq:336
$$\begin{tikzcd} \underline{\mathbb{R}^{n-k}} \oplus TM \arrow[r] \arrow[d] & S(n) \arrow[d] \\ M \arrow[r] & \mathcal{X}(n) \end{tikzcd} \tag{14.14}$$

There is a bordism category $\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$ analogous to $\mathrm{Bord}_{\langle n-1,n\rangle}$ as defined in Definition 14.3, but all manifolds Y, X are required to carry $\mathcal{X}(n)$-structures. Examples include *stable* tangential structures, such as orientation and spin, as well as *unstable* tangential structures, such as n-framings. We follow the notational convention of Exercise 9.71.

Examples of bordism categories

thm:275 **Example 14.15** ($\mathrm{Bord}_{\langle -1,0\rangle}$)**.** There is a unique (-1)-dimensional manifold—the empty manifold $\emptyset^{-1}$—so $\mathrm{Bord}_{\langle -1,0\rangle}$ is a category with a single object, hence a monoid (Example 13.9). The monoid is the set of morphisms $\mathrm{Bord}_{\langle -1,0\rangle}(\emptyset^{-1},\emptyset^{-1})$ under composition. In fact, the symmetric monoidal structure gives a second composition law, but it is equal to the first which is necessarily commutative. This follows from general principles, but is easy to see in this case. Namely, the monoid consists of diffeomorphism classes of closed 0-manifolds, so finite unions of points. The set of diffeomorphism classes is $\mathbb{Z}^{\ge 0}$. Composition and the monoidal product are both disjoint union, which induces addition in $\mathbb{Z}^{\ge 0}$.

thm:276 **Example 14.16** ($\mathrm{Bord}^{SO}_{\langle -1,0\rangle}$)**.** Now all manifolds are oriented, so the morphisms are finite unions of pt_+ and pt_-, up to diffeomorphism. Let x_+, x_- denote the diffeomorphism class of $\mathrm{pt}_+, \mathrm{pt}_-$. Then the monoid $\mathrm{Bord}^{SO}_{\langle -1,0\rangle}$ is the free commutative monoid generated by x_+, x_-.

thm:279 **Exercise 14.17.** Prove that $\mathrm{Diff}\, S^1$ has two components, each of which deformation retracts onto a circle.

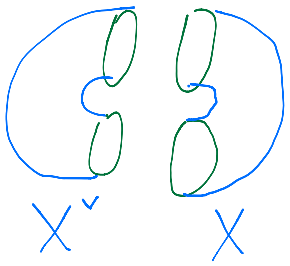

FIGURE 25. Some bordisms in $\mathrm{Bord}_{\langle 1,2\rangle}$ fig:25

thm:277 **Example 14.18** ($\mathrm{Bord}_{\langle 1,2\rangle}$)**.** Objects are closed 1-manifolds, so finite unions of circles. As depicted in Figure 25 the cylinder can be interpreted as a bordism $X\colon (S^1)^{\amalg 2} \to \emptyset^1$; the dual bordism $X^\vee$ (Definition 1.22) is a map $X^\vee\colon \emptyset^1 \to (S^1)^{\amalg 2}$. Let $\rho\colon S^1 \to S^1$ be reflection, $f = 1 \amalg \rho$ the indicated diffeomorphism of $(S^1)^{\amalg 2}$, and X_f the associated bordism **(14.9)**. Then

eq:337
$$\begin{aligned} X \circ X_{\mathrm{id}} \circ X^\vee &\simeq \text{torus} \\ X \circ X_f \circ X^\vee &\simeq \text{Klein bottle} \end{aligned} \tag{14.19}$$

These diffeomorphism become equations in the monoid $\mathrm{Bord}_{\langle 1,2\rangle}(\emptyset^1, \emptyset^1)$ of diffeomorphism classes of closed 2-manifolds.

Topological quantum field theories

Just as we study abstract groups via their representations, so too we study bordism categories via representations. There are linear actions of groups on vector spaces, and also nonlinear actions on more general spaces. Similarly, there are linear and nonlinear representations of bordism categories.

thm:280 **Definition 14.20.** Fix $n \in \mathbb{Z}^{\ge 0}$ and $\mathcal{X}(n)$ an n-dimensional tangential structure. Let C be a symmetric monoidal category. An *n-dimensional topological quantum field theory of $\mathcal{X}(n)$-manifolds with values in C* is a symmetric monoidal functor

eq:338
$$F\colon\ \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle} \longrightarrow C \tag{14.21}$$

Symmetric monoidal functors are defined in **(13.38)**. We use the acronym 'TQFT' for 'topological quantum field theory'. We do not motivate the use of 'quantum field theory' for Definition 14.20 here; see instead the discussion in [F1]. I also strongly recommend the beginning sections of [L1]. The definition originates in the mathematics literature in [A1], which in turn was inspired by [S1]. There is a nice thorough discussion in [Q].

thm:281 *Remark* 14.22. Let Top denote the symmetric monoidal category whose objects are topological spaces and whose morphisms are continuous maps. The monoidal structure is disjoint union. Let Ab denote the category whose objects are abelian groups and whose morphisms are group homomorphisms. Homology theory gives symmetric monoidal functors

eq:339

$$H_q\colon (\mathrm{Top},\amalg) \longrightarrow (\mathrm{Ab},\oplus) \tag{14.23}$$

for all nonnegative integers q. Note that the symmetric monoidal structure on Ab is *direct sum*: the homology of a disjoint union is the direct sum of the homologies. One should think of the direct sum as *classical*; for *quantum* field theories we will use instead tensor product. In vague terms quantization, which is the passage from classical to quantum, is a sort of exponentiation which turns sums to products.

For this reason we keep the 'quantum' in 'TQFT'.

subsec:14.5

(14.24) *Codomain categories.* Typical "linear" choices for C are: (i) the symmetric monoidal category $(\mathrm{Vect}_k,\otimes)$ of vector spaces over a field k, (ii) the symmetric category $({}_R\mathrm{Mod},\otimes)$ of left modules over a commutative ring R, and the special case (iii) the symmetric monoidal category $(\mathrm{Ab},\otimes)$ of abelian groups under tensor product. On the other hand, we can take as codomain a bordism category, which is decidedly nonlinear. For example, if M is a closed k-manifold, then there is a symmetric monoidal functor

eq:340

$$-\times M\colon\ \mathrm{Bord}_{\langle n-1,n\rangle} \longrightarrow \mathrm{Bord}_{\langle n+k-1,n+k\rangle} \tag{14.25}$$

which, I suppose, can be called a TQFT. If $F\colon\ \mathrm{Bord}_{\langle n+k-1,n+k\rangle} \to C$ is any $(n+k)$-dimensional TQFT, then composition with (14.25) gives an n-dimensional TQFT, the *dimensional reduction of F along M*.

Lecture 15: Duality

sec:15

We ended the last lecture by introducing one of the main characters in the remainder of the course, a topological quantum field theory (TQFT). At this point we should, of course, elaborate on the definition and give examples, background, motivation, etc. I will not do so in these notes. Instead I refer you to the expository paper [F1] as well as to the beginning sections of [L1]. There are many other references with great expository material.

In this lecture we explore the finiteness property satisfied by a TQFT, which is encoded via duality in symmetric monoidal categories.

Some categorical preliminaries

We begin with a standard notion which you'll find in any book which contains a chapter on categories, including books on category theory.

thm:282 **Definition 15.1.** Let C, D be categories. A functor $F\colon C \to D$ is an *equivalence* if there exist a functor $G\colon D \to C$, and natural isomorphisms $G \circ F \to \mathrm{id}_C$ and $F \circ G \to \mathrm{id}_D$.

thm:283 **Proposition 15.2.** *A functor $F\colon C \to D$ is an equivalence if and only if it satisfies:*

(i) *For each $d \in D$ there exist $c \in C$ and an isomorphism $\big(f(c) \to d\big) \in D$; and*

(ii) *For each $c_1, c_2 \in C$ the map of hom-sets $F\colon C(c_1, c_2) \to D\big(F(c_1), F(c_2)\big)$ is a bijection.*

If F satisfies (i) it is said to be *essentially surjective* and if it satisfies (ii) it is *fully faithful*.

thm:284 **Exercise 15.3.** Prove Proposition 15.2.

Next we spell out the answer to Exercise 13.45. It is part of the definition of a TQFT.

thm:285 **Definition 15.4.** Let C, D be symmetric monoidal categories and $F, G\colon C \to D$ symmetric monoidal functors. Then a *symmetric monoidal natural transformation* $\eta\colon F \to G$ is a natural transformation such that the diagrams

eq:341 (15.5)
$$\begin{tikzcd} & F(1_C) \arrow[dd, "\eta(1_C)"] \\ 1_D \arrow[ur] \arrow[dr] & \\ & G(1_C) \end{tikzcd}$$

and

eq:342 (15.6)
$$\begin{tikzcd} F(y_1) \otimes F(y_2) \arrow[r, "\psi"] \arrow[d, "\eta\otimes\eta"'] & F(y_1 \otimes y_2) \arrow[d, "\eta"] \\ G(y_1) \otimes G(y_2) \arrow[r, "\psi"] & G(y_1 \otimes y_2) \end{tikzcd}$$

commute for all $y_1, y_2 \in C$.

TQFT's as a symmetric monoidal category

Fix a bordism category $B = \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$ and a symmetric monoidal category C. We now explain that topological quantum field theories $F\colon B \to C$ are objects in a symmetric monoidal category. A morphism $F \to G$ is as defined in Definition 15.4. The monoidal product of theories F_1, F_2 is defined by

eq:343 (15.7)
$$\begin{aligned}(F_1 \otimes F_2)(Y) &= F_1(Y) \otimes F_2(Y)\\ (F_1 \otimes F_2)(X) &= F_1(X) \otimes F_2(X)\end{aligned} \tag{15.7}$$

for all objects $Y \in B$ and morphisms $(X\colon Y_0 \to Y_1) \in B$. The tensor unit **1** is the trivial theory

eq:344 (15.8)
$$\begin{aligned}\mathbf{1}(Y) &= 1_C\\ \mathbf{1}(X) &= \mathrm{id}_{1_C}\end{aligned} \tag{15.8}$$

for all $Y \in B$ and $(X\colon Y_0 \to Y_1) \in B$.

⇒ We denote the symmetric monoidal category of TQFT's as [**Use** $\mathrm{TQFT}_{\langle n-1,n\rangle}$]

eq:345 (15.9)
$$\mathrm{TQFT}_n = \mathrm{TQFT}^{\mathcal{X}(n)}_n[C] = \mathrm{Hom}^{\otimes}(\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}, C). \tag{15.9}$$

The short form of the notation is used if the tangential structure $\mathcal{X}(n)$ and codomain category C are clear.

subsec:15.1

(15.10) *Endomorphisms of the trivial theory.* Suppose $\eta\colon \mathbf{1} \to \mathbf{1}$ in TQFT_n. Then for all $Y \in \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$ we have $\eta(Y) \in C(1_C, 1_C) = \mathrm{End}(1_C)$. Note that if $C = \mathrm{Ab}$ is the category of abelian groups, then $\mathrm{End}(1_C) = \mathbb{Z}$. So η is a numerical invariant of closed $(n-1)$-manifolds. Furthermore, if $X\colon Y_0 \to Y_1$ then by the naturality condition (13.18) we find that $\eta(Y_0) = \eta(Y_1)$. This shows that η factors down to a homomorphism of monoids

eq:346 (15.11)
$$\eta\colon \Omega^{\mathcal{X}(n)}_{n-1} \longrightarrow \mathrm{End}(1_C). \tag{15.11}$$

Now by Lemma 1.30, and its generalization to manifolds with tangential structure, we know that every element of $\Omega^{\mathcal{X}(n)}_{n-1}$ is invertible. It follows that the image of η consists of invertible elements. We say, simply, that η is invertible.

In other words, an endomorphism of **1** is a bordism invariant of the type studied in the first half of the course. A topological quantum field theory, then, is a "categorified" bordism invariant.

thm:286 **Exercise 15.12.** What is a topological quantum field theory whose codomain category has as objects the set of integers and only identity arrows?

The invertibility observed in **(15.10)** is quite general.

thm:287 **Theorem 15.13.** *A morphism* $(\eta\colon F \to G) \in \mathrm{TQFT}_n$ *is invertible.* TQFT_n *is a groupoid.*

The two statements are equivalent. We prove Theorem 15.13 at the end of this lecture.

subsec:15.2

(15.14) *Central problem.* Given a dimension n, a tangential structure $\mathcal{X}(n)$, and a codomain category C we can ask to "compute" the groupoid $\mathrm{TQFT}_n^{\mathcal{X}(n)}[C]$. This is a vague problem whose solution is an equivalent groupoid which is "simpler" than the groupoid of topological quantum field theories. It has a nice answer when $n = 1$. In the oriented case it is a generalization of the theorem that Ω_0^{SO} is the free abelian group with a single generator pt_+. There is also a nice answer in the oriented case for $n = 2$.

Finiteness in TQFT

To motivate the abstract formulation of finiteness in symmetric monoidal categories, we prove the following simple proposition. For simplicity we omit any tangential structure.

thm:288 **Proposition 15.15.** *Let $F\colon \mathrm{Bord}_{\langle n-1,n\rangle} \to \mathrm{Vect}_{\mathbb{C}}$ be a TQFT. Then for all $Y \in \mathrm{Bord}_{\langle n-1,n\rangle}$ the vector space $F(Y)$ is finite dimensional.*

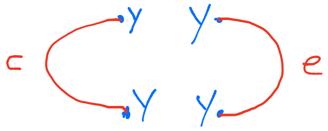

FIGURE 26. Some elementary bordisms fig:26

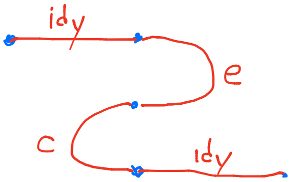

FIGURE 27. The S-diagram fig:27

Proof. Fix $Y \in \mathrm{Bord}_{\langle n-1,n\rangle}$ and let $V = F(Y)$. Let $c\colon \emptyset^{n-1} \to Y \amalg Y$ and $e\colon Y \amalg Y \to \emptyset^{n-1}$ be the bordisms pictured in Figure 26. The manifold Y is depicted as a point, and each bordism has underlying manifold with boundary $[0,1] \times Y$. The composition depicted in Figure 27 is diffeomorphic to the identity bordism $\mathrm{id}_Y\colon Y \to Y$. Under F it maps to $\mathrm{id}_V\colon V \to V$ (see (13.40)). On the other hand, the composition maps to

eq:347

$$V \xrightarrow{\mathrm{id}_V \otimes F(c)} V \otimes V \otimes V \xrightarrow{F(e)\otimes \mathrm{id}_V} V \tag{15.16}$$

Let the value of $F(c)\colon \mathbb{C} \to V \otimes V$ on $1 \in \mathbb{C}$ be $\sum\limits_i v_i' \otimes v_i''$ for some finite set of vectors $v_i', v_i'' \in V$. Then equating (15.16) with the identity map we find that for all $\xi \in V$ we have

eq:348

$$\xi = \sum_i e(\xi, v_i')v_i'', \tag{15.17}$$

and so the finite set of vectors $\{v_i''\}$ spans V. This proves that V is finite dimensional. ☐

thm:289 **Exercise 15.18.** Prove that $F(c)$ and $F(e)$ are inverse bilinear forms.

Duality data and dual morphisms

We abstract the previous argument by singling out those objects in a symmetric monoidal category which obey a finiteness condition analogous to that of a finite dimensional vector space.

thm:290 **Definition 15.19.** Let C be a symmetric monoidal category and $y \in C$.

(i) *Duality data* for y is a triple of data $(y^\vee, c, e)$ in which $y^\vee$ is an object of C and c, e are morphisms $c\colon 1_C \to y \otimes y^\vee$, $e\colon y^\vee \otimes y \to 1_C$. We require that the compositions

eq:349 (15.20)
$$y \xrightarrow{c\otimes \mathrm{id}_y} y \otimes y^\vee \otimes y \xrightarrow{\mathrm{id}_y \otimes e} y$$

and

eq:350 (15.21)
$$y^\vee \xrightarrow{\mathrm{id}_{y^\vee} \otimes c} y^\vee \otimes y \otimes y^\vee \xrightarrow{e\otimes \mathrm{id}_{y^\vee}} y^\vee$$

be identity maps. If duality data exists for y, we say that y is *dualizable*.

(ii) A *morphism of duality data* $(y^\vee, c, e) \to (\tilde{y}^\vee, \tilde{c}, \tilde{e})$ is a morphism $y^\vee \xrightarrow{f} \widetilde{y^\vee}$ such that the diagrams

eq:351 (15.22)
$$\begin{tikzcd}
 & y \otimes y^\vee \arrow[dd, "\mathrm{id}_y \otimes f"] \\
1_C \arrow[ur, "c"] \arrow[dr, "\tilde{c}"'] & \\
 & y \otimes \widetilde{y^\vee}
\end{tikzcd}$$

and

eq:352 (15.23)
$$\begin{tikzcd}
y^\vee \otimes y \arrow[dd, "f\otimes \mathrm{id}_y"'] \arrow[dr, "e"] & \\
 & 1_C \\
\widetilde{y^\vee} \otimes y \arrow[ur, "\tilde{e}"'] &
\end{tikzcd}$$

commute.

c is called *coevaluation* and e is called *evaluation*.

We now express the uniqueness of duality data. As duality data is an object in a category, as defined in Definition 15.19, we cannot say there is a unique object. Rather, here we have the strongest form of uniqueness possible in a category: duality data is unique up to unique isomorphism.

thm:291 **Definition 15.24.** Let C be a category.

(i) If for each pair $y_0, y_1 \in C$ the hom-set $C(y_0, y_1)$ is either empty or contains a unique element, we say that C *is a discrete groupoid.*

(ii) If for each pair $y_0, y_1 \in C$ the hom-set $C(y_0, y_1)$ has a unique element, we say that C *is contractible.*

A discrete groupoid is equivalent to a set (Example 13.10). A contractible groupoid is equivalent to a category with one object and one morphism, the categorical analog of a point.

thm:292 **Proposition 15.25.** *Let C be a symmetric monoidal category and $y \in C$. Then the category of duality data for y is either empty or is contractible.*

The proof is a homework problem (~~Problem Set #2~~).

A morphism between dualizable objects has a dual.

thm:293 **Definition 15.26.** Let $y_0, y_1 \in C$ be dualizable objects in a symmetric monoidal category and $f\colon y_0 \to y_1$ a morphism. The *dual morphism* $f^\vee\colon y_1^\vee \to y_0^\vee$ is the composition

eq:353
$$y_1^\vee \xrightarrow{\mathrm{id}_{y_1^\vee}\otimes c_0} y_1^\vee \otimes y_0 \otimes y_0^\vee \xrightarrow{\mathrm{id}_{y_1^\vee}\otimes f\otimes \mathrm{id}_{y_0^\vee}} y_1^\vee \otimes y_1 \otimes y_0^\vee \xrightarrow{e_1\otimes \mathrm{id}_{y_0^\vee}} y_0^\vee \tag{15.27}$$

In the definition we choose duality data $(y_0^\vee, c_0, e_0)$, $(y_1^\vee, c_1, e_1)$ for y_0, y_1.

thm:294 **Exercise 15.28.** Check that this definition agrees with that of a dual linear map for $C = \mathrm{Vect}$. Also, spell out the consequence of Proposition 15.25 for the dual morphism.

Duality in bordism categories

We already encountered dual manifolds and dual bordisms in Definition 1.22, Remark 1.24, and (**2.20**). In this subsection we prove the following.

thm:295 **Theorem 15.29.** *Every object in a bordism category* $\mathrm{Bord}_{\langle n-1,n\rangle}^{\mathcal{X}(n)}$ *is dualizable.*

Proof. If $\mathcal{X}(n)$ is the trivial tangential structure $BO(n) \to BO(n)$, so $\mathrm{Bord}_{\langle n-1,n\rangle}^{\mathcal{X}(n)} = \mathrm{Bord}_{\langle n-1,n\rangle}$ is the bordism category of (unoriented) manifolds, then for any closed $(n-1)$-manifold Y we have $Y^\vee = Y$ with coevaluation and evaluation as in Figure 26. In the general case, an object $(Y,\theta) \in \mathrm{Bord}_{\langle n-1,n\rangle}^{\mathcal{X}(n)}$ is a closed $(n-1)$-manifold Y equipped with a classifying map

eq:354 (15.30)
$$\begin{tikzcd} \underline{\mathbb{R}} \oplus TY \arrow[r, "\theta"] \arrow[d] & S(n) \arrow[d] \\ Y \arrow[r] & \mathcal{X}(n) \end{tikzcd}$$

to the universal bundle (9.59); cf. (9.60). Its dual $(Y,\theta)^\vee = (Y,\theta^\vee)$ has the same underlying manifold and classifying map $\theta^\vee$ the composition

eq:355 (15.31)
$$\begin{tikzcd} \underline{\mathbb{R}} \oplus TY \arrow[r, "-1\oplus \mathrm{id}_{TY}"] \arrow[d] & \underline{\mathbb{R}} \oplus TY \arrow[r, "\theta"] \arrow[d] & S(n) \arrow[d] \\ Y \arrow[r, "\mathrm{id}_Y"] & Y \arrow[r] & \mathcal{X}(n) \end{tikzcd}$$

The coevaluation and evaluation $(X, p, \theta_0, \theta_1)$ are as depicted in Figure 26. In both cases $X = [0,1] \times Y$. For the coevaluation $p\colon \partial X \to \{0,1\}$ is the constant function 1, and for the evaluation it is the constant function 0. For the evaluation

eq:356

$$\begin{array}{rcl} \theta_0\colon [0,1) \times Y \;\amalg\; [0,1) \times Y & \longrightarrow & [0,1] \times Y \\ \text{(t,y)} \qquad\qquad\qquad & \longmapsto & (t/4, y) \\ \text{(t,y)} & \longmapsto & (1 - t/4, y) \end{array} \tag{15.32}$$

with natural lifts to the $\mathcal{X}(n)$-structures. The formula for θ_1 for the coevaluation is similar. ☐

thm:296 **Exercise 15.33.** Write the map on $\mathcal{X}(n)$-structures explicitly. Note the $-$ sign in the differential of the last formula in (15.32) matches the -1 in the first map of (15.31).

Proof of Theorem 15.13

We first prove the following.

thm:297 **Proposition 15.34.** *Let B, C be symmetric monoidal categories, $F, G\colon B \to C$ symmetric monoidal functors, and $y \in B$ dualizable. Then*

(i) *$F(y) \in C$ is dualizable.*
(ii) *If $\eta\colon F \to G$ is a symmetric monoidal natural transformation, then $\eta(y)\colon F(y) \to G(y)$ is invertible.*

Proof. If $(y^\vee, c, e)$ is duality data for y, then $\big(F(y^\vee), F(c), F(e)\big)$ is duality data for $F(y)$. This proves (i).

For (ii) we claim that $\eta(y^\vee)^\vee$ is inverse to $\eta(y)$. Note that by Definition 15.26, $\eta(y^\vee)^\vee$ is a map $G(y^\vee)^\vee \to F(y^\vee)^\vee$, and since $G(y^\vee) = G(y)^\vee$ it may be interpreted as a map $G(y) \to F(y)$. Let $c\colon 1_B \to y \otimes y^\vee$ and $e\colon y^\vee \otimes y \to 1_B$ be coevaluation and evaluation. Consider the diagram

eq:357

$$\begin{tikzcd} G(y) \arrow[r, "\mathrm{id}\otimes F(c)"] \arrow[rrd, "\mathrm{id}\otimes G(c)"'] & G(y)\otimes F(y^\vee)\otimes F(y) \arrow[r, "\mathrm{id}\otimes\eta(y^\vee)\otimes\mathrm{id}"] \arrow[rd, "\mathrm{id}\otimes\eta(y^\vee)\otimes\eta(y)"] & G(y)\otimes G(y^\vee)\otimes F(y) \arrow[r, "G(e)\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mathrm{id}\otimes\eta(y)"] & F(y) \arrow[d, "\eta(y)"] \\ & & G(y)\otimes G(y^\vee)\otimes G(y) \arrow[r, "G(e)\otimes\mathrm{id}"] & G(y) \end{tikzcd} \tag{15.35}$$

We claim it commutes. The left triangle commutes due to the naturality of η applied to the coevaluation $c\colon 1_B \to y \otimes y^\vee$. The next triangle and the right square commute trivially. Now starting on the left, the composition along the top and then down the right is the composition $\eta(y) \circ \eta(y^\vee)^\vee$. The composition diagonally down followed by the horizontal map is the identity, by G applied to the S-diagram relation (15.20) (and using (13.40)). A similar diagram proves that $\eta(y^\vee)^\vee \circ \eta(y) = \mathrm{id}$. ☐

⇒ [**Corollary:** $\eta(y^\vee) = (\eta(y)^{-1})^\vee = (\eta(y)^\vee)^{-1}$]

Theorem 15.13 is an immediate consequence of Theorem 15.29 and part (ii) of (11.71). Part (i) implies the following.

thm:298

Theorem 15.36. *Let C be a symmetric monoidal category and $F\colon \mathrm{Bord}_{\langle n-1,n\rangle}^{\mathcal{X}(n)} \to C$ be a topological quantum field theory. Then for all $Y \in \mathrm{Bord}_{\langle n-1,n\rangle}^{\mathcal{X}(n)}$, the object $F(Y) \in C$ is dualizable.*

Lecture 16: 1-dimensional TQFTs

sec:16

In this lecture we determine the groupoid of 1-dimensional TQFTs of oriented manifolds with values in any symmetric monoidal category. This is a truncated version of the cobordism hypothesis, but illustrates a few of the basic underlying ideas.

Categorical preliminaries

We need three notions from category theory: a *full subcategory* of an arbitrary category, the *groupoid of units* of an arbitrary category, and the *dimension* of an object in a symmetric monoidal category.

thm:299 **Definition 16.1.** Let C be a category and $C_0' \subset C_0$ a subset of objects. Then the *full subcategory* C' with set of objects C_0' has as hom-sets

eq:358

$$C_1'(y_0, y_1) = C_1(y_0, y_1), \qquad y_0, y_1 \in C_0'. \tag{16.2}$$

There is a natural inclusion $C_0' \to C_0$ which is an isomorphism on hom-sets. We can describe the entire set of morphisms C_1' as a pullback:

eq:359

$$\begin{tikzcd} C_1' \arrow[r, dashed] \arrow[d, dashed] & C_1 \arrow[d, "s\times t"] \\ C_0' \times C_0' \arrow[r, "j\times j"] & C_0 \times C_0 \end{tikzcd} \tag{16.3}$$

where s, t are the source and target maps (13.8) and $j\colon C_0' \hookrightarrow C_0$ is the inclusion.

We need a particular example of a full subcategory.

thm:302 **Definition 16.4.** Let C be a symmetric monoidal category. Define $C^{\mathrm{fd}} \subset C$ as the full subcategory whose objects are the dualizable objects of C.

The notation 'fd' puts in mind 'finite dimensional', which is correct for the category Vect: the dualizable vector spaces are those which are finite dimensional. It also stands for 'fully dualizable'. The 'fully' is not (yet) relevant.

Recall that if M is a monoid, then the *group of units* $M^{\sim} \subset M$ is the subset of invertible elements. For example, if M is the monoid of $n \times n$ matrices under multiplication, then $M^{\sim}$ is the subset of invertible matrices, which form a group.

thm:300 **Definition 16.5.** Let C be a category. Its *groupoid of units*[33] is the groupoid $C^{\sim}$ with same objects $C_0^{\sim} = C_0$ as in the category C and with morphisms $C_1^{\sim} \subset C_1$ the subset of invertible morphisms in C.

---

[33] usually called the *maximal groupoid*

Notice that identity arrows are invertible and compositions of invertible morphisms are invertible, so $C^\sim$ is a category. Obviously, it is a groupoid.

The last definition applies only to symmetric monoidal categories.

thm:301 **Definition 16.6.** Let C be a symmetric monoidal category and $y \in C$ a dualizable object. Then the *dimension* of y, denoted $\dim y \in C(1_C, 1_C)$, is the composition

eq:360
$$\dim y\colon 1_C \xrightarrow{c} y \otimes y^\vee \xrightarrow{\sigma} y^\vee \otimes y \xrightarrow{e} 1_C, \tag{16.7}$$

where $(y^\vee, c, e)$ is duality data for y.

The reader can easily check that $\dim y$ is independent of the choice of duality data (Definition 15.19).

Classification of 1-dimensional oriented TQFTs

Recall from **(2.28)** that the oriented bordism group in dimension zero is the free abelian group on one generator: $\Omega_0^{SO} \cong \mathbb{Z}$. We can restate this in terms of bordism invariants. Let M be any commutative monoid. Then 0-dimensional bordism invariants with values in M is the commutative monoid $\mathrm{Hom}(\Omega_0^{SO}, M)$, where the sum $F + G$ of two bordism invariants is computed elementwise: $(F+G)(Y) = F(Y)+G(Y)$ for all compact 0-manifolds Y. Then $F(Y)$ is automatically invertible, since Ω_0^{SO} is a group.

thm:303 **Theorem 16.8** (cobordism hypothesis—set version)**.** *The map*

eq:361
$$\begin{aligned} \Phi\colon\ \mathrm{Hom}(\Omega_0^{SO}, M) &\longrightarrow M^\sim \\ F &\longmapsto F(\mathrm{pt}_+) \end{aligned} \tag{16.9}$$

is an isomorphism *of abelian groups.*

This is the restatement.

Now we consider 1-dimensional oriented TQFTs.

thm:304 **Theorem 16.10** (cobordism hypothesis—1-categorical version)**.** *Let C be a symmetric monoidal category. Then the map*

eq:362
$$\begin{aligned} \Phi\colon\ \mathrm{TQFT}^{SO}_{\langle 0,1\rangle}(C) &\longrightarrow (C^{\mathrm{fd}})^\sim \\ F &\longmapsto F(\mathrm{pt}_+) \end{aligned} \tag{16.11}$$

is an equivalence of groupoids.

The map Φ is well-defined by Theorem 15.36, which asserts in particular that $F(\mathrm{pt}_+)$ is dualizable. Recall (you shouldn't have forgotten in one page!) Definition 16.4 and Definition 16.5, which give meaning to the subgroupoid $(C^{\mathrm{fd}})^\sim$ of C.

The proof relies on the classification of closed 0-manifolds and compact 1-manifolds with boundary [M3]. Note that if Y_0, Y_1 are closed 0-manifolds which are diffeomorphic, then the set of

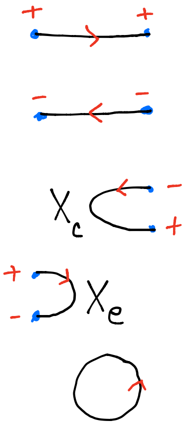

FIGURE 28. The five connected oriented bordisms in $\mathrm{Bord}^{SO}_{\langle 0,1\rangle}$ fig:31

diffeomorphisms $Y_0 \to Y_1$ is a torsor for the group of permutations (of, say, Y_0). A connected compact 1-manifold with boundary is diffeomorphic to a circle or a closed interval, which immediately leads to the classification of connected morphisms in $\mathrm{Bord}^{SO}_{\langle 0,1\rangle}$, as illustrated in Figure 28: every connected oriented bordism is diffeomorphic to one of the five possibilities illustrated there.

Proof. We must show that Φ is fully faithful and essentially surjective. Recall that

First, if F, G are field theories and $\eta_1, \eta_2\colon F \to G$ isomorphisms, and suppose that $\eta_1(\mathrm{pt}_+) = \eta_2(\mathrm{pt}_+)$. Since $\mathrm{pt}_- = \mathrm{pt}_+^\vee$, according to the formula proved in Proposition 15.34 we have $\eta(\mathrm{pt}_-) = \bigl(\eta(\mathrm{pt}_+)^\vee\bigr)^{-1}$ for any natural isomorphism η. It follows that $\eta_1(\mathrm{pt}_-) = \eta_2(\mathrm{pt}_-)$. Since any compact oriented 0-manifold Y is a finite disjoint union of copies of pt_+ and pt_-, it follows that $\eta_1(Y) = \eta_2(Y)$ for all Y, whence $\eta_1 = \eta_2$. This shows that Φ is faithful.

To show Φ is full, given F, G and an isomorphism $f\colon F(\mathrm{pt}_+) \to G(\mathrm{pt}_+)$ we must construct $\eta\colon F \to G$ such that $\eta(\mathrm{pt}_+) = f$. So define $\eta(\mathrm{pt}_+) = f$ and $\eta(\mathrm{pt}_-) = (f^\vee)^{-1}$. Extend using the monoidal structure in C to define $\eta(Y)$ for all compact oriented 0-manifolds Y. This uses the statement given before the proof that any such Y is diffeomorphic to $(\mathrm{pt}_+)^{\amalg n_+} \amalg (\mathrm{pt}_-)^{\amalg n_-}$ for unique $n_+, n_- \in \mathbb{Z}^{\ge 0}$. Also, the diffeomorphism is determined up to permutation, but because of coherence the resulting map $\eta(Y)$ is independent of the chosen diffeomorphism. It remains to show that η is a natural isomorphism, so to verify (13.18) for each morphism in $\mathrm{Bord}^{SO}_{\langle 0,1\rangle}$. It suffices to consider connected bordisms, so each of the morphisms in Figure 28. The first two are identity maps, for which (13.18) is trivial. The commutativity of the diagram

eq:387 (16.12)

$$\begin{tikzcd}
 & F(-)F(+) \arrow[dd, "f\,(f^\vee)^{-1}"] \\
1 \arrow[ur, "F(X_c)"] \arrow[dr, "G(X_c)"'] & \\
 & G(-)G(+)
\end{tikzcd}$$

for coevaluation X_c follows from the commutativity of

eq:386 (16.13)
$$\begin{tikzcd}
1 \arrow[r, "F(X_c)"] \arrow[d, "G(X_c)"'] & F(-)F(+) \arrow[r, "1f"] \arrow[d, "11G(X_c)"] & F(-)G(+) \\
G(-)G(+) \arrow[r, "F(X_c)11"] & F(-)F(+)G(-)G(+) \arrow[r, "1f11"] & F(-)G(+)G(-)G(+) \arrow[u, "1G(X_e)1"']
\end{tikzcd}$$

In these diagrams we use '+' and '−' for 'pt_+' and 'pt_-', and also denote identity maps as '1'. The argument for evaluation X_e is similar, and that for the circle follows since the circle is $X_e \circ \sigma \circ X_c$. Notice that the commutative diagram for the circle S^1 asserts $F(S^1) = G(S^1)$.

Finally, we must show that Φ is essentially surjective. Given $y \in C$ dualizable, we must[34] construct a field theory F with $F(\mathrm{pt}_+) = y$. Let $(y^\vee, c, e)$ be duality data for y. Define $F(\mathrm{pt}_+) = y$, $F(\mathrm{pt}_-) = y^\vee$, and

eq:388 (16.14)
$$F\big((\mathrm{pt}_+)^{\amalg n_+} \amalg (\mathrm{pt}_-)^{\amalg n_-}\big) = y^{\otimes n_+} \otimes (y^\vee)^{\otimes n_-}.$$

Any compact oriented 0-manifold Y is diffeomorphic to some $(\mathrm{pt}_+)^{\amalg n_+} \amalg (\mathrm{pt}_-)^{\amalg n_-}$, and again by coherence the choice of diffeomorphism does not matter. Now any oriented bordism $X\colon Y_0 \to Y_1$ is diffeomorphic to a disjoint union of the bordisms in Figure 28, and for these standard bordisms we define $F(X_c) = c$, $F(X_e) = e$, and $F(S^1) = e \circ \sigma \circ c$; the first two bordisms in the figure are identity maps, which necessarily map to identity maps. We map X to a tensor product of these basic bordisms. It remains to check that F is a functor, i.e., that compositions map to compositions. When composing in $\mathrm{Bord}^{SO}_{\langle 0,1\rangle}$ the only nontrivial compositions are those indicated in Figure 29. The first composition is what we use to define $F(S^1)$. The S-diagram relations (15.20) and (15.21) show that the last compositions are consistent under F. ☐

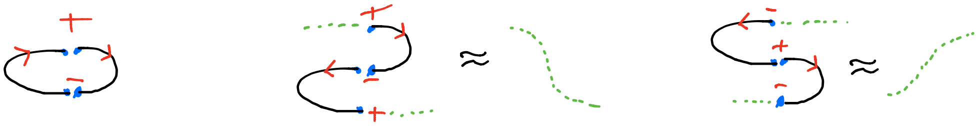

FIGURE 29. Nontrivial compositions in $\mathrm{Bord}^{SO}_{\langle 0,1\rangle}$ fig:32

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[34] In fact, we only need construct F with $F(\mathrm{pt}_+) \cong y$, but we will construct one where $F(\mathrm{pt}_+)$ *equals* y.

Lecture 17: Invertible topological quantum field theories

sec:17

In this lecture we introduce the notion of an *invertible* TQFT. These arise in both topological and non-topological quantum field theory as *anomaly theories*, a topic we might discuss at the end of the course. They are also interesting in homotopy theory, though not terribly much explored to date in that context. As usual, we need some preliminary discussion of algebra.

Group completion and universal properties

subsec:17.1

(17.1) *The group completion of a monoid.* Recall that a monoid M is a set with an associative composition law $M \times M \to M$ and a unit $1 \in M$.

thm:305 **Definition 17.2.** Let M be a monoid. A *group completion* $(|M|, i)$ of M is a group $|M|$ and a homomorphism $i\colon M \to |M|$ of monoids which satisfies the following universal property: If G is a group and $f\colon M \to G$ a homomorphism of monoids, then there exists a unique map $\tilde{f}\colon |M| \to G$ which makes the diagram

eq:363 (17.3)
$$\begin{tikzcd}
M \arrow[rr, "i"] \arrow[dr, "f"'] & & {|M|} \arrow[dl, dashed, "\tilde{f}"] \\
& G &
\end{tikzcd}$$

commute.

The definition does not prove the existence of the group completion—we must provide a proof—but the universal property does imply a strong uniqueness property. Namely, if (H, i) and (H', i') are group completions of M, then there is a unique isomorphism $\phi\colon H \to H'$ of groups which makes the diagram

eq:364 (17.4)
$$\begin{tikzcd}
M \arrow[rr, "i"] \arrow[dr, "i'"'] & & H \arrow[dl, "\phi"] \\
& H' &
\end{tikzcd}$$

commute. The proof, the details of which I leave to the reader, involves four applications of the universal property (to $f = i$ and $f = i'$ to construct the isomorphism and its inverse, and then two more to prove the compositions are identity maps).

thm:306 **Example 17.5.** If $M = \mathbb{Z}^{>0}$ under multiplication, then the group completion is $\mathbb{Q}^{>0}$ under multiplication.

thm:307 **Example 17.6.** If $M = \mathbb{Z}^{\ge 0}$ under multiplication, then the group completion $(|M|, i)$ is the trivial group. For there exists $x \in |M|$ such that $x \cdot i(0) = 1$, and so for any $n \in M$ we have

eq:365
$$i(n) = \bigl(x \cdot i(0)\bigr) \cdot i(n) = x \cdot \bigl(i(0) \cdot i(n)\bigr) = x \cdot i(0 \cdot n) = x \cdot i(0) = 1. \tag{17.7}$$

Now apply *uniqueness* of the factorization.

subsec:17.2

(17.8) *Free groups.* We recall that given a set S, there is a free group $F(S)$ generated by S, and it too is characterized by a universal property. In this setup a free group $\bigl(F(S), i\bigr)$ is a pair consisting of a group $F(S)$ and a map $i\colon S \to F(S)$ of sets such that for any group G and any map $f\colon S \to G$ of sets, there exists a unique homomorphism of groups $\tilde{f}\colon F(S) \to G$ which makes the diagram

eq:366
$$\begin{tikzcd} S \arrow[rr, "i"] \arrow[dr, "f"'] & & F(S) \arrow[dl, dashed, "\hat{f}"] \\ & G & \end{tikzcd} \tag{17.9}$$

commute. Again uniqueness follows immediately. Existence is something you must have seen when discussing van Kampen's theorem, for example. See [H1].

subsec:17.3

(17.10) *Construction of the group completion.* Now we prove that a group completion of a monoid M exists. Let $\bigl(F(M), i\bigr)$ be a free group on the set underlying M. Define N as the normal subgroup of $F(M)$ generated by elements

eq:367
$$i(x_1x_2)i(x_2)^{-1}i(x_1)^{-1}, \qquad x_1, x_2 \in M. \tag{17.11}$$

Given a homomorphism $f\colon M \to G$ of monoids, for all $x_1, x_2 \in M$ we have $f(x_1x_2) = f(x_1)f(x_2)$. But $f = \hat{f}i$ from (17.9), and so it follows that

eq:368
$$\hat{f}\bigl(i(x_1x_2)i(x_2)^{-1}i(x_1)^{-1}\bigr) = 1, \tag{17.12}$$

whence $\hat{f}$ factors down to a unique homomorphism $\tilde{f}\colon F(M)/N \to G$.

thm:308 **Exercise 17.13.** This last step uses a universal property which characterizes a quotient group. What is that universal property?

The groupoid completion of a category

thm:309 **Definition 17.14.** Let C be a category. A *groupoid completion* $(|C|, i)$ of C is a groupoid $|C|$ and a homomorphism $i\colon C \to |C|$ of monoids which satisfies the following universal property: If $\mathcal{G}$ is a groupoid and $f\colon C \to \mathcal{G}$ a functor, then there exists a unique map $\tilde{f}\colon |C| \to \mathcal{G}$ which makes the diagram

eq:369 (17.15)
$$\begin{tikzcd} C \arrow[rr, "i"] \arrow[dr, "f"'] & & {|C|} \arrow[dl, dashed, "\tilde{f}"] \\ & \mathcal{G} & \end{tikzcd}$$

commute.

Intuitively, $|C|$ is obtained from C by "inverting all of the arrows", much in the same way that the group completion of a monoid is constructed. In fact, notice that if C has one object, then Definition 17.14 reduces to Definition 17.2.

We give some examples below; see Theorem 17.41.

subsec:17.4

(17.16) *Uniqueness of $\tilde{f}$.* There is a choice whether to require that $\tilde{f}$ in (17.15) be unique. If so, then you should show that $(|C|, i)$ is unique up to unique isomorphism. We do make that choice. It has a consequence that the map i is an isomorphism $i_0\colon C_0 \to |C|_0$ on objects. For let $\mathcal{G}$ be the groupoid with objects $\mathcal{G}_0 = C_0$ and with a unique morphism between any two objects, so the set of morphisms is $\mathcal{G}_1 = C_0 \times C_0$. There is a unique functor $f\colon C \to \mathcal{G}$ which is the identity on objects, and applying the universal property we deduce that $i_0\colon C_0 \to |C|_0$ is injective. If there exists $y \in |C|_0$ not in the image of i_0, then we argue as follows. Let $\mathcal{G}'$ be the groupoid with two objects a, b and a unique morphisms between any two objects. Let $f\colon C \to \mathcal{G}'$ be the functor which sends all objects to a and all morphisms to id_a. Then the factorization $\tilde{f}$ cannot be unique. For if $\tilde{f}(y) = a$, then define a new factorization with $y \mapsto b$ and adjust all morphisms starting or ending at y accordingly.

subsec:17.8

(17.17) *Sketch of a construction for $|C|$.* Here is a sketch of the existence proof for $|C|$, which follows the argument in **(17.8)**, **(17.10)**. In the next lecture we give a proof using topology. Briefly, given sets C_0, C_1 and maps $s, t\colon C_1, C_0$, there is a *free groupoid* $F(C_0, C_1)$ generated. It has the set C_0 of objects. Let C_1' be the set C_1 equipped with maps $s' = t\colon C_1' \to C_0$, $t' = s\colon C_1' \to C_0$. They are formal inverses of the arrows in C_1. Then a morphism in $F(C_0, C_1)$ is a formal string of composable elements in $C_1 \amalg C_1'$. The composition and inverse operations in $F(C_0, C_1)$ are by amalgamation and order-reversal. If now C is a category, then we take the quotient of $F(C_0, C_1)$ which keeps the same objects C_0 and for every pair of composable arrows g, f in C_1 identifies the amagamation gf with the composition $g \circ f$ in $F(C_0, C_1)$. I didn't try to work out the details of this quotient construction.

Invertibility in symmetric monoidal categories

The following should be compared with Definition 15.19(i).

thm:310 **Definition 17.18.** Let C be a symmetric monoidal category and $y \in C$. Then *invertibility data* for y is a pair (y', θ) consisting of $y' \in C$ and an isomorphism $\theta\colon 1_C \to y \otimes y'$. If invertibility data exists, then we say that y is *invertible*.

There is a category of invertibility data, and it is a contractible groupoid (Definition 15.24). So an inverse to y, if it exists, is unique up to unique isomorphism. We denote any choice of inverse as y^{-1}. Note that the set of invertible objects is closed under the tensor product and it contains the unit object 1_C.

thm:313 *Remark* 17.19. An object $y \in Y$ is invertible if and only if the functor $y \otimes - \colon C \to C$ is an equivalence.

thm:312 **Example 17.20.** Let $C = \mathrm{Vect}_k$ be the category of vector spaces over a field k with symmetric monoidal structure the tensor product. Then $V \in \mathrm{Vect}_k$ is invertible if and only if $\dim V = 1$. In this case we say V is a *line*.

thm:311 **Lemma 17.21.** *Let C be a symmetric monoidal category.*

(i) *If $y \in C$ is invertible, then y is dualizable and y^{-1} is a dual object.*

(ii) *If C is a symmetric monoidal* groupoid *and $y \in C$ is dualizable, then y is invertible with inverse $y^\vee$.*

Proof. Part (ii) is trivial as the coevaluation $c\colon 1_C \to y \otimes y^\vee$ is invertible. For (i) we let $\theta\colon 1_C \to y \otimes y^{-1}$ be coevaluation and evaluation is, up to multiplication by an element of $C(1_C, 1_C)$, the composition

eq:370
$$y^{-1} \otimes y \xrightarrow{\sigma} y \otimes y^{-1} \xrightarrow{\theta^{-1}} 1_C, \tag{17.22}$$

where σ is the symmetry of the symmetric monoidal structure. We leave the details to a homework problem. □

thm:314 **Definition 17.23.** A *Picard groupoid* is a symmetric monoidal category in which all objects and morphisms are invertible.

thm:315 **Example 17.24.** Given a field k, there is a Picard groupoid Line_k whose objects are k-lines and whose morphisms are isomorphisms of k-lines. Given a space X, there are Picard groupoids $\mathrm{Line}_{\mathbb{R}}(X)$ and $\mathrm{Line}_{\mathbb{C}}(X)$ of line bundles over X.

thm:316 **Definition 17.25.** Let C be a symmetric monoidal category. An *underlying Picard groupoid* is a pair $(C^\times, i)$ consisting of a Picard groupoid $C^\times$ and a functor $i\colon C^\times \to C$ which satisfies the universal property: If D is any Picard groupoid and $j\colon D \to C$ a symmetric monoidal functor, then there exists a unique $\tilde{j}\colon D \to C^\times$ such that the diagram

eq:371
$$\begin{tikzcd} C^\times \arrow[rr, "i"] & & C \\ & D \arrow[ul, dashed, "\tilde{j}"] \arrow[ur, "j"'] & \end{tikzcd} \tag{17.26}$$

commutes.

We obtain $C^\times$ from C by discarding all non-invertible objects and non-invertible morphisms. Recall (Definition 16.5) that C contains a subgroupoid $C^\sim \subset C$ of units, obtained by discarding all non-invertible morphisms. So we have $C^\times \subset C^\sim \subset C$.

`subsec:17.7` **(17.27)** *Invariants of a Picard groupoid.* Associated to a Picard groupoid D are abelian groups $\pi_0 D$, $\pi_1 D$ and a *k-invariant*

`eq:385` (17.28)
$$\pi_0 D \otimes \mathbb{Z}/2\mathbb{Z} \to \pi_1 D.$$

Define objects $y_0, y_1 \in D$ to be equivalent if there exists a morphism $y_0 \to y_1$. Then $\pi_0 D$ is the set of equivalence classes. The group law is given by the monoidal structure $\otimes$, and we obtain an abelian group since $\otimes$ is symmetric. Define $\pi_1 D = D(1_D, 1_D)$ as the automorphism group of the tensor unit. If $y \in D$ then there is an isomorphism

`eq:378` (17.29)
$$- \otimes \mathrm{id}_y \colon \operatorname{Aut}(1_D) \longrightarrow \operatorname{Aut}(y)$$

where we write $\operatorname{Aut}(y) = D(y, y)$. The k-invariant on y is the symmetry $\sigma \colon y \otimes y \to y \otimes y$, which is an element of $\operatorname{Aut}(y \otimes y) \cong \operatorname{Aut}(1_D) = \pi_1 D$. We leave the reader to verify that this determines a homomorphism $\pi_0 D \otimes \mathbb{Z}/2\mathbb{Z} \to \pi_1 D$.

Invertible TQFTs

We distinguish the special subset of *invertible* topological quantum field theories.

`thm:317` **Definition 17.30.** Fix a nonnegative integer n, a tangential structure $\mathcal{X}(n)$, and a symmetric monoidal category C. Then a topological quantum field theory $\alpha \colon \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle} \to C$ is *invertible* if it factors through the underlying Picard groupoid of C:

`eq:372` (17.31)
$$\begin{tikzcd}
\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle} \arrow[rr, "\alpha"] \arrow[dr, dashed] & & C \\
& C^{\times} \arrow[ur] &
\end{tikzcd}$$

If α is invertible, it follows from the universal property of the groupoid completion (Definition 17.14) that there is a factorization

`eq:373` (17.32)
$$\begin{tikzcd}
\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle} \arrow[r, "\alpha"] \arrow[d] & C \\
\lvert \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle} \rvert \arrow[r, "\tilde{\alpha}"] & C^{\times} \arrow[u]
\end{tikzcd}$$

We will identify the invertible theory with the map $\tilde{\alpha}$ (and probably omit the tilde.) In Lecture 19 we will see that $\tilde{\alpha}$ can be identified with a map of spectra.

`thm:318` **Lemma 17.33.** *The groupoid completion* $\lvert \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle} \rvert$ *of a bordism category is a Picard groupoid.*

Proof. By Theorem 15.29 an object of $\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$ is dualizable, so by Lemma 17.21(ii) it is also invertible. ☐

`thm:320` *Remark* 17.34. A TQFT α is invertible if and only if it is an invertible object in the symmetric monoidal category (15.9) of TQFTs.

subsec:17.5

(17.35) *Super vector spaces.* We introduce the symmetric monoidal category of *super vector spaces*. For more detail on superalgebra I recommend [DeM]. The word 'super' is a synonym for '$\mathbb{Z}/2\mathbb{Z}$-graded'. A *super vector space* is a pair (V, ϵ) consisting of a vector space (over a field k of characteristic not equal[35] to 2) and an endomorphism $\epsilon\colon V \to V$ such that $\epsilon^2 = \mathrm{id}_V$. The $\pm$-eigenspaces of ϵ provide a decomposition $V = V^0 \oplus V^1$; elements of the subspace V^0 are called *even* and elements of the subspace V^1 are called *odd*. A morphism $(V, \epsilon) \to (V', \epsilon')$ is a linear map $T\colon V \to V'$ such that $T \circ \epsilon = \epsilon' \circ T$. It follows that T maps even elements to even elements and odd elements to odd elements. The monoidal structure is defined as

eq:375

$$(V_1, \epsilon_1) \otimes (V_2, \epsilon_2) = (V_1 \otimes V_2, \epsilon_1 \otimes \epsilon_2) \tag{17.36}$$

What is novel is the symmetry σ. If $v \in V$ is a homogeneous element, define its *parity* $|v| \in \{0, 1\}$ so that $v \in V^{|v|}$. Then for homogeneous elements $v_i \in V_i$ the symmetry is

eq:376

$$\begin{aligned} \sigma\colon (V_1, \epsilon_1) \otimes (V_2, \epsilon_2) &\longrightarrow (V_2, \epsilon_2) \otimes (V_1, \epsilon_1) \\ v_1 \otimes v_2 &\longmapsto (-1)^{|v_1|\,|v_2|}\, v_2 \otimes v_1 \end{aligned} \tag{17.37}$$

This is called the *Koszul sign rule*. Let $s\mathrm{Vect}_k$ denote the symmetric monoidal category of super vector spaces. The obvious forgetful functor $s\mathrm{Vect}_k \to \mathrm{Vect}_k$ is *not* a symmetric monoidal functor, though it is a monoidal functor.

subsec:17.6

(17.38) *Example of an invertible field theory.* According to Theorem 16.10 to define an oriented one-dimensional TQFT

eq:377

$$\alpha\colon \mathrm{Bord}^{SO}_{\langle 0,1\rangle} \to s\mathrm{Vect}_k \tag{17.39}$$

we need only specify $\alpha(\mathrm{pt}_+)$. We let it be the *odd line* $(k, -1)$ whose underlying vector space is the trivial line k (the field as a one-dimensional vector space) viewed as odd: the endomorphism ϵ is multiplication by -1. We leave as a homework problem to prove that α is invertible and that $\alpha(S^1) = -1$.

The groupoid completion of one-dimensional bordism categories

Of course, by Lemma 17.33 the groupoid completion $|B|$ of a bordism category B is a Picard groupoid, so has invariants described in **(17.27)**. We compute them for the bordism categories

eq:379

$$\begin{aligned} B &= \mathrm{Bord}_{\langle 0,1\rangle} \\ B^{SO} &= \mathrm{Bord}^{SO}_{\langle 0,1\rangle} \end{aligned} \tag{17.40}$$

---

[35] We can give a different description in that case.

thm:319 **Theorem 17.41.** *For the group completion of the unoriented bordism category*

eq:380
$$\pi_0|B| \cong \mathbb{Z}/2\mathbb{Z}, \qquad \pi_1|B| = 0 \tag{17.42}$$

and for the group completion of the oriented bordism category

eq:382
$$\pi_0|B^{SO}| \cong \mathbb{Z}, \qquad \pi_1|B^{SO}| \cong \mathbb{Z}/2\mathbb{Z}, \tag{17.43}$$

with nontrivial k-invariant.

Proof. The arguments for π_0 are straightforward and amount to Proposition 1.31 and the assertion $\Omega_0^{SO} \cong \mathbb{Z}$.

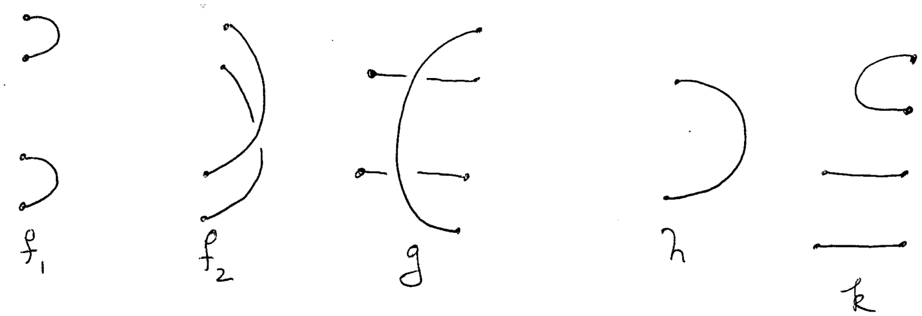

Figure 30. Some unoriented 1-dimensional bordisms fig:28

To compute $\pi_1 B$ we argue as follows. First, $1_B = \emptyset^0$ is the empty 0-manifold, so $\mathrm{End}(1_B) = B(1_B, 1_B)$ consists of diffeomorphism classes of closed 1-manifolds. Therefore, there is an isomorphism of commutative monoids $\mathrm{End}(1_B) \cong \mathbb{Z}^{\ge 0}$ which counts the number of components of a bordism X. Let X_n denote the disjoint union of n circles. Then using the bordisms defined in Figure 30 we have

eq:383
$$f_1 \circ g = f_2 \circ g = h \tag{17.44}$$

as morphisms in B. In the groupoid completion $|B|$ we can compose on the right with the inverse to g to conclude that $i(f_1) = i(f_2)$, where $i\colon B \to |B|$. That implies that in $|B|$ we have

eq:384
$$i(X_1) \circ i(h) = i(f_1) \circ i(k) = i(f_2) \circ i(k) = i(h), \tag{17.45}$$

whence $i(X_1) = i(\emptyset^0) = 1_{|B|}$. It remains to show that every morphism $\emptyset^0 \to \emptyset^0$ in $|B|$ is equivalent to a union of circles and their formal inverses. Observe first that the inverse of the "right elbow" h is the "left elbow", since their composition in one order is the circle, which is equivalent to the identity map. Next, any morphism $\emptyset^0 \to \emptyset^0$ in $|B|$ is the composition of a finite number of morphisms $Y_{2k} \to Y_{2\ell}$ and inverses of such morphisms, where Y_n is the 0-manifold consisting of n points. Furthermore, each such morphism is the disjoint union of circles, identities, right elbows,

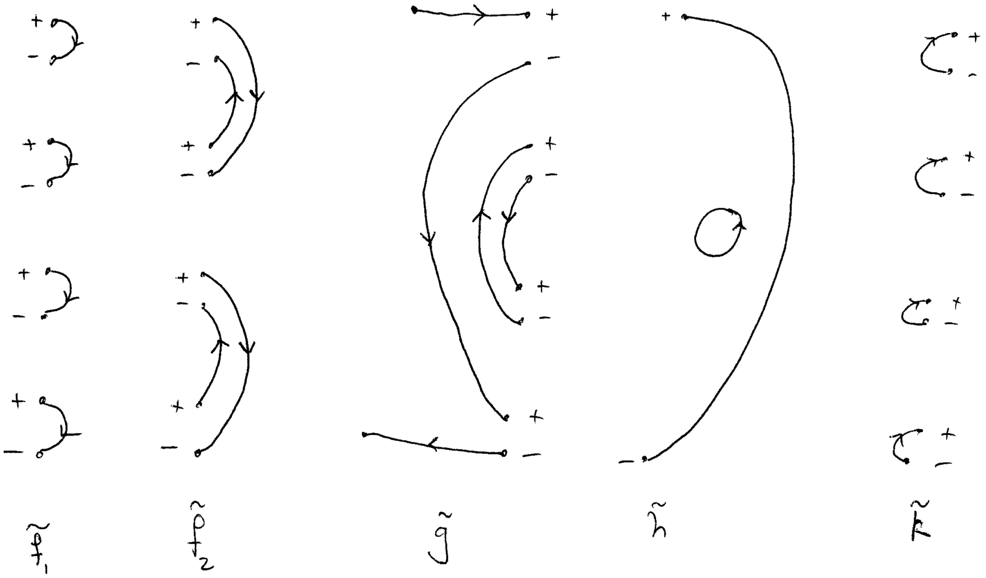

FIGURE 31. Some oriented 1-dimensional bordisms

fig:29

left elbows, and their inverses. Identities are self-inverse and the elbows are each other's inverse, hence carrying out the compositions of elbows and identities we obtain a union of circles and their inverses, as desired. This proves $\pi_1|B|$ is the abelian group with a single element.

To compute $\pi_1|B^{SO}|$ we make a similar argument using the bordisms in Figure 31. Let $\widetilde{X}_n$ be the disjoint union of n *oriented* circles. Note that the circle has a unique orientation up to orientation-preserving diffeomorphism. In this case we conclude that $i(\widetilde{X}_2) = i(\emptyset^0) = 1_{|B^{SO}|}$. To rule out the possibility that $i(\widetilde{X}_1)$ is also the tensor unit, we use the TQFT in **(17.38)**. It maps the oriented circle $\widetilde{X}_1$ to a non-tensor unit (which necessarily has order two).

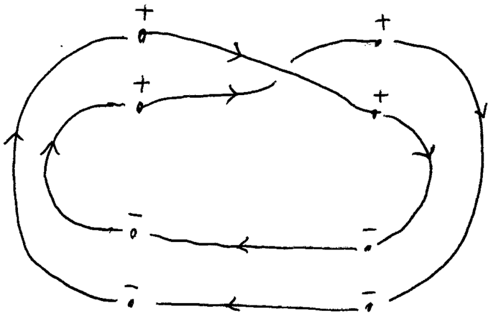

FIGURE 32. The k-invariant of $|\,\mathrm{Bord}^{SO}_{\langle 0,1\rangle}\,|$

fig:30

Figure 32 illustrates the computation of the k-invariant (17.28) of $|B^{SO}|$. The nontrivial element of $\pi_0|B^{SO}|$ is represented by pt_+, and the top part of the diagram is the symmetry $\sigma\colon\ \mathrm{pt}_+ \amalg \mathrm{pt}_+ \to$

$\mathrm{pt}_+ \amalg \mathrm{pt}_+$ in B^{SO}. We then tensor with the identity on the inverse of pt_+, which is pt_-; that is represented by the disjoint union of the top and bottom four strands. The left and right ends implement the isomorphism $1_{|B^{SO}|} \cong \mathrm{pt}_+ \amalg \mathrm{pt}_+ \amalg \mathrm{pt}_- \amalg \mathrm{pt}_-$. The result is the oriented circle $\widetilde{X}_1$, which is the generator of $\pi_1|B^{SO}|$. $\square$

Lecture 18: Groupoids and spaces

sec:18

The simplest algebraic invariant of a topological space T is the set $\pi_0 T$ of path components. The next simplest invariant, which encodes more of the topology, is the fundamental groupoid $\pi_{\le 1}T$. In this lecture we see how to go in the other direction. There is nothing to say for a set T: it is already a discrete topological space. If $\mathcal{G}$ is a groupoid, then we can ask to construct a space $B\mathcal{G}$ whose fundamental groupoid $\pi_{\le 1}B\mathcal{G}$ is equivalent to $\mathcal{G}$. We give such a construction in this section. More generally, for a category C we construct a space BC whose fundamental groupoid $\pi_{\le 1}BC$ is equivalent to the groupoid completion (Definition 17.14) of C. The space BC is called the *classifying space* of the category C. As we will see in the next lecture, if $T_\bullet$ is a spectrum, then its fundamental groupoid $\pi_{\le 1}T_\bullet$ is a Picard groupoid, and conversely the classifying space of a Picard groupoid is a spectrum.

As an intermediate between categories and spaces we introduce *simplicial sets*. These are combinatorial models for spaces, and are familiar in some guise from the first course in topology. We only give a brief introduction and refer to the literature—e.g. [S2, Fr] for details. One important generalization is that we allow *spaces* of simplices rather than simply discrete sets of simplices. In other words, we also consider *simplicial spaces*. This leads naturally to *topological categories*,[36] which we also introduce in this lecture.

In subsequent lectures we will apply these ideas to bordism categories. Lemma 17.33 asserts that the groupoid completion of a bordism category is a Picard groupoid, and we can ask to identify its classifying spectrum. To make the problem more interesting we will yet again extract from smooth manifolds and bordism a more intricate algebraic invariant: a topological category.

Simplices

Let S be a nonempty finite ordered set. For example, we have the set

eq:389

$$[n] = \{0, 1, 2, \dots, n\} \tag{18.1}$$

with the given total order. Any S is uniquely isomorphic to $[n]$, where the cardinality of S is $n+1$. Let $A(S)$ be the affine space generated by S and $\Sigma(S) \subset A(S)$ the simplex with vertex set S. So if $S = \{s_1, s_1, \dots, s_n\}$, then $A(S)$ consists of formal sums

eq:390

$$p = t^0 s_0 + t^1 s_1 + \cdots + t^n s_n, \qquad t^i \in \mathbb{R}, \quad t^0 + t^1 + \cdots + t^n = 1, \tag{18.2}$$

and $\Sigma(S)$ consists of those sums with $t^i \ge 0$. We write $\mathbb{A}^n = A([n])$ and $\Delta^n = \Sigma([n])$. For these standard spaces the point $i \in [n]$ is $(\dots, 0, 1, 0, \dots)$ with 1 in the i^{th} position.

---

[36] This term generates confusion. We follow [S2] in using it to denote an internal category in the category Top of topological spaces. A more common usage is for a category enriched over Top.

Let Δ be the category whose objects are nonempty finite ordered sets and whose morphisms are order-preserving maps (which may be neither injective nor surjective). The category Δ is generated by the morphisms

eq:391 (18.3)
$$\begin{tikzcd}
{[0]} \arrow[r, shift left=2] \arrow[r, shift right=2] & {[1]} \arrow[l, dashed] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & {[2]} \arrow[l, dashed, shift left=2] \arrow[l, dashed, shift right=2] & \cdots
\end{tikzcd}$$

where the right-pointing maps are injective and the left-pointing maps are surjective. For example, the map $d_i\colon [1] \to [2]$, $i = 0, 1, 2$ is the unique injective order-preserving map which does not contain $i \in [2]$ in its image. The map $s_i\colon [2] \to [1]$, $i = 0, 1$, is the unique surjective order-preserving map for which $s_i^{-1}(i)$ has two elements. Any morphism in Δ is a composition of the maps d_i, s_i and identity maps.

Each object $S \in \Delta$ determines a simplex $\Sigma(S)$, as defined above. This assignment extends to a functor

eq:392 (18.4)
$$\Sigma\colon S \longrightarrow \mathrm{Top}$$

to the category of topological spaces and continuous maps. A morphism $\theta\colon S_0 \to S_1$ maps to the affine extension $\theta_*\colon \Sigma(S_0) \to \Sigma(S_1)$ of the map θ on vertices.

Simplicial sets and their geometric realizations

Recall the definition (**13.7**) of a category.

thm:321 **Definition 18.5.** Let C be a category. The *opposite category* C^{op} is defined by

eq:393 (18.6)
$$C_0^{\mathrm{op}} = C_0, \quad C_1^{\mathrm{op}} = C_1, \quad s^{\mathrm{op}} = t, \quad t^{\mathrm{op}} = s, \quad i^{\mathrm{op}} = i,$$

and the composition law is reversed: $g^{\mathrm{op}} \circ f^{\mathrm{op}} = (f \circ g)^{\mathrm{op}}$.

Here recall that C_0 is the set of objects, C_1 the set of morphisms, and $s, t\colon C_1 \to C_0$ the source and target maps. The opposite category has the same objects and morphisms but with the direction of the morphisms reversed.

The following definition is slick, and at first encounter needs unpacking (see [Fr], for example).

thm:322 **Definition 18.7.** A *simplicial set* is a functor

eq:394 (18.8)
$$X\colon \Delta^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

It suffices to specify the sets $X_n = X([n])$ and the basic maps (18.3) between them. Thus we obtain a diagram

eq:395 (18.9)
$$\begin{tikzcd}
X_0 \arrow[r, dashed] & X_1 \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, dashed, shift left=2] \arrow[r, dashed, shift right=2] & X_2 \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] & \cdots
\end{tikzcd}$$

We label the maps d_i and s_i as before. The d_i are called *face maps* and the s_i *degeneracy maps.* The set X_n is a set of abstract simplices. An element of X_n is degenerate if it lies in the image of some s_i.

The morphisms in an abstract simplicial set are gluing instructions for concrete simplices.

thm:323 **Definition 18.10.** Let $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ be a simplicial set. The *geometric realization* is the topological space $|X|$ obtained as the quotient of the disjoint union

eq:396
$$\coprod_S X(S) \times \Sigma(S) \tag{18.11}$$

by the equivalence relation

eq:397
$$(\sigma_1, \theta_* p_0) \sim (\theta^* \sigma_1, p_0), \qquad \theta\colon S_0 \to S_1, \quad \sigma_1 \in X(S_1), \quad p_0 \in \Sigma(S_0). \tag{18.12}$$

The map $\theta_* = \Sigma(\theta)$ is defined after (18.4) and $\theta^* = X(\theta)$ is part of the data of the simplicial set X. Alternatively, the geometric realization map be computed from (18.9) as

eq:398
$$\coprod_n X_n \times \Delta^n \,/\!\sim, \tag{18.13}$$

where the equivalence relation is generated by the face and degeneracy maps.

thm:329 *Remark* 18.14. The geometric realization can be given the structure of a CW complex.

Examples

thm:324 **Example 18.15.** Let X be a simplicial set whose nondegenerate simplices are

eq:399
$$X_0 = \{A, B, C, D\}, \qquad X_1 = \{a, b, c, d\}. \tag{18.16}$$

The face maps are as indicated in Figure 33. For example $d_0(a) = B$, $d_1(a) = A$, etc. (This requires a choice not depicted in Figure 33.) The level 0 and 1 subset of the disjoint union (18.13) is pictured in Figure 34. The 1-simplices a, b, c, d glue to the 0-simplices A, B, C, D to give the space depicted in Figure 33. The red 1-simplices labeled A, B, C, D are degenerate, and they collapse under the equivalence relation (18.12) applied to the degeneracy map s_0.

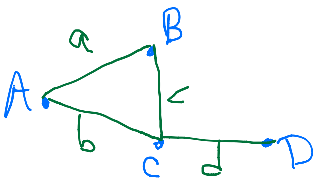

FIGURE 33. The geometric realization of a simplicial set fig:33

thm:325 **Example 18.17.** Let T be a topological space. Then there is a simplicial set $\operatorname{Sing} T$ of singular simplices, defined by

eq:400
$$(\operatorname{Sing} T)(S) = \mathrm{Top}\bigl(\Sigma(S), T\bigr), \tag{18.18}$$

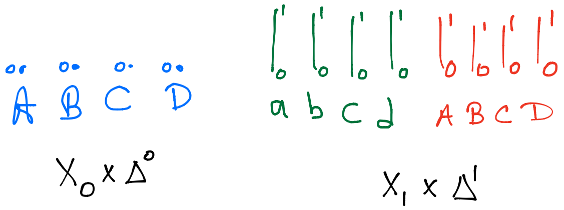

FIGURE 34. Gluing the simplicial set fig:34

where $\mathrm{Top}\bigl(\Sigma(S),T\bigr)$ is the set of continuous maps $\Sigma(S)\to T$, i.e., the set of singular simplices with vertex set S. The boundary maps are the usual ones. The evaluation map

eq:401
$$(\mathrm{Sing}\,T)(S)\times\Sigma(S)=\mathrm{Top}\bigl(\Sigma(S),T\bigr)\times\Sigma(S)\longrightarrow T, \tag{18.19}$$

passes through the equivalence relation to induce a continuous map

eq:402
$$|\,\mathrm{Sing}\,T|\longrightarrow T. \tag{18.20}$$

A basic theorem in the subject asserts that this map is a weak homotopy equivalence.

Categories and simplicial sets

subsec:18.1

(18.21) *The nerve.* Let C be a category, which in part is encoded in the diagram

eq:403
$$\begin{tikzcd} C_0 \arrow[r, dashed, shift left=0] & C_1 \arrow[l, shift right=2] \arrow[l, shift left=2] \end{tikzcd} \tag{18.22}$$

The solid left-pointing arrows are the source s and target t of a morphism; the dashed right-pointing arrow i assigns the identity map to each object. This looks like the start of a simplicial set, and indeed there is a simplicial set NC, the *nerve* of the category C, which begins precisely this way: $NC_0=C_0$, $NC_1=C_1$, $d_0=t$, $d_1=s$, and $s_0=i$. A slick definition runs like this: a finite nonempty ordered set S determines a category with objects S and a unique arrow $s\to s'$ if $s\le s'$ in the order. Then

eq:404
$$NC(S)=\mathrm{Fun}(S,C) \tag{18.23}$$

where $\mathrm{Fun}(-,-)$ denotes the set of functors. As is clear from Figure 35, $NC([n])$ consists of sets of n composable arrows in C. The degeneracy maps in NC insert an identity morphism. The face map d_i omits the i^{th} vertex and composes the morphisms at that spot; if i is an endpoint $i=0$ or $i=n$, then d_i omits one of the morphisms.

thm:326 **Example 18.24.** Let M be a monoid, regarded as a category with a single object. Then

eq:405
$$NM_n=M^{\times n}. \tag{18.25}$$

It is a good exercise to write out the face maps.

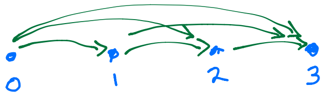

FIGURE 35. A totally ordered set as a category fig:35

thm:327 **Definition 18.26.** Let C be a category. The *classifying space* BC of C is the geometric realization $|NC|$ of the nerve of C.

thm:336 *Remark* 18.27. A homework exercise will explain the nomenclature ‘classifying space’.

thm:328 **Example 18.28.** Suppose $G = \mathbb{Z}/2\mathbb{Z}$ is the cyclic group of order two, viewed as a category with one object. Then NG_n has a single nondegenerate simplex $(g, \dots, g)$ for each n, where $g \in \mathbb{Z}/2\mathbb{Z}$ is the non-identity element. So BG is glued together with a single simplex in each dimension. We leave the reader to verify that in fact $BG \simeq \mathbb{RP}^\infty$.

thm:330 **Theorem 18.29.** *Let M be a monoid. Then $\pi_1 BM$ is the group completion of M.*

The nerve NM has a single 0-simplex, which is the basepoint of BM.

Proof. The fundamental group of a CW complex B is computed from its 2-skeleton B^2. Assuming there is a single 0-cell, the 1-skeleton is a wedge of circles, so its fundamental group is a free group F. The homotopy class of the attaching map $S^1 \to B^1$ of a 2-cell is a word in F, and the fundamental group of B is the quotient F/N, where N is the normal subgroup generated by the words of the attaching maps of 2-cells. For $B = BM$ the set of 1-cells is M, so $\pi_1 BM^1 \cong F(M)$ is the free group generated by the set M. The homotopy class of the 2-cell (m_1, m_2) is the word $(m_1 m_2) m_2^{-1} m_1^{-1}$. By (**17.10**) the quotient $F(M)/N$ is the group completion of M. ☐

We next prove an important proposition [S2].

thm:333 **Proposition 18.30.** *Let $F, G \colon C \to D$ be functors and $\eta \colon F \to G$ a natural transformation. Then the induced maps $|F|, |G| \colon |C| \to |D|$ on the geometric realizations are homotopic.*

Proof. Consider the ordered set $[1]$ as a category, as in Figure 35. Its classifying space is homeomorphic to the closed interval $[0,1]$. Define a functor $H \colon [1] \times C \to D$ which on objects of the form $(0, -)$ is equal to F, on objects of the form $(1, -)$ is equal to G, and which maps the unique morphism $(0 \to 1)$ to the natural transformation η. Then $|H| \colon [0,1] \times |C| \to |D|$ is the desired homotopy. ☐

thm:334 *Remark* 18.31. The proof implicitly uses that the classifying space of a Cartesian product of categories is the Cartesian product of the classifying spaces. That is not strictly true in general; see [S2] for discussion.

thm:332 **Proposition 18.32.** *Let $\mathcal{G}$ be a groupoid. Then the natural functor $i_{\mathcal{G}} \colon \mathcal{G} \to \pi_{\le 1} B\mathcal{G}$ is an equivalence of groupoids.*

The objects of $\mathcal{G}$ are the 0-skeleton of $B\mathcal{G}$, and $i_{\mathcal{G}}$ is the inclusion of the 0-skeleton on objects. The 1-cells of $B\mathcal{G}$ are indexed by the morphisms of $\mathcal{G}$, and imposing a standard parametrization we obtain the desired map $i_{\mathcal{G}}$.

Proof. Any groupoid $\mathcal{G}$ is equivalent to a disjoint union of groups. To construct an equivalence choose a section of the quotient map $\mathcal{G}_0 \to \pi_0\mathcal{G}$ and take the disjoint union of the automorphism groups of the objects in the image of that section. □

thm:331 **Corollary 18.33.** *Let C be a category. The fundamental groupoid $\pi_{\le 1}BC$ is equivalent to the groupoid completion of C.*

Proof. As explained after the statement of Proposition 18.32 there is a natural map $C \xrightarrow{i_C} \pi_{\le 1}BC$. We check the universal property (17.15). Suppose $f\colon C \to \mathcal{G}$ is a functor from C to a groupoid. There is an induced continuous maps $Bf\colon BC \to B\mathcal{G}$ and then an induced functor $\pi_{\le 1}Bf\colon \pi_{\le 1}BC \to \pi_{\le 1}B\mathcal{G}$ such that the diagram

eq:406 (18.34)

$$\begin{tikzcd} C \arrow[r, "i_C"] \arrow[d, "f"'] & \pi_{\le 1}BC \arrow[d, "\pi_{\le 1}Bf"] \\ \mathcal{G} \arrow[r, "i_{\mathcal{G}}"] & \pi_{\le 1}B\mathcal{G} \end{tikzcd}$$

By Proposition 18.32 the map $i_{\mathcal{G}}$ is an equivalence of groupoids, and composition with an inverse equivalence gives the required factorization $\tilde{f}$. □

thm:335 *Remark* 18.35. A *skeleton* of $\pi_{\le 1}BC$ is a groupoid completion as in Definition 17.14; its set of objects is isomorphic to C_0. There is a canonical skeleton: the full subcategory whose set of objects is $i_C(C_0)$.

Simplicial spaces and topological categories

A simplicial set describes a space—its geometric realization—as the gluing of a discrete set of simplices. However, we may also want to glue together a space from continuous families of simplices.

thm:337 **Definition 18.36.** A *simplicial space* is a functor

eq:407 (18.37)

$$X\colon \Delta^{\mathrm{op}} \longrightarrow \mathrm{Top}$$

More concretely, a simplicial space is a sequence $\{X_n\}$ of topological spaces with continuous face and degeneracy maps as in (18.9). The construction of the geometric realization (Definition 18.10) goes through verbatim.

We can also promote the sets and morphisms of a (discrete) category from sets to spaces.

thm:338 **Definition 18.38.** A *topological category* consists of topological spaces C_0, C_1 and continuous maps

eq:408 (18.39)

$$\begin{aligned} i\colon C_0 &\longrightarrow C_1 \\ s,t\colon C_1 &\longrightarrow C_0 \\ c\colon C_1 \times_{C_0} C_1 &\longrightarrow C_1 \end{aligned}$$

which satisfy the algebraic relations of a discrete category.

These are described following (13.8). Thus the partially defined composition law c is associative and $i(y)$ is an identity morphism with respect to the composition.

thm:339 **Example 18.40.** Let M be a topological monoid. So M is both a monoid and a topological space, and the composition law $M \times M \to M$ is continuous. Then M may be regarded as a topological category with a single object.

thm:340 **Example 18.41.** At the other extreme, a topological space T may be regarded as a topological category with only identity morphisms.

thm:341 **Example 18.42.** There is a topological category whose objects are finite dimensional vector spaces and whose spaces of morphisms are spaces of linear maps (with the usual topology).

thm:342 **Example 18.43.** Let M be a smooth manifold and G a Lie group. Then there is a topological category whose objects are principal G-bundles with connection and whose morphisms are flat bundle isomorphisms.

thm:343 **Definition 18.44.** Let C be a topological category. Its nerve NC and classifying space BC are defined as in (**18.21**) and Definition 18.26, *verbatim*.

Notice that the nerve is a simplicial *space*.

Lecture 19: Γ-spaces and deloopings

sec:19

To a topological category C we associate a topological space BC. We saw in (17.32) that an invertible field theory, defined on a discrete bordism category B, factors through the groupoid completion $|B|$ of B. Furthermore, by Corollary 18.33, the groupoid completion is the fundamental groupoid $|B|$ of the classifying space of B. In the next lecture we introduce *topological* bordism categories and a corresponding richer notion of a topological quantum field theory, with values in a symmetric monoidal topological category. In that case we will see that an *invertible* field theory factor through the classifying space of the topological bordism category. Now a topological bordism category has a symmetric monoidal structure, so we can ask what extra structure is reflected on the classifying space. In this lecture we will see that this extra structure is an infinite loop space structure. In other words, *the classifying space BC of a topological symmetric monoidal category is the 0-space of a prespectrum.* (Review Definition 10.2.)

There are many "delooping machines" which build the infinite loop space structure. Here we give an exposition of Segal's Γ-spaces [S2], though we use the observation of Anderson [A] that the opposite category Γ^{op} to Segal's category Γ is the category of finite pointed sets. Further accounts may be found in [BF] and [Sc]. So whereas in Lecture 18 we have the progression

eq:414 (19.1)
$$\text{Topological categories} \longrightarrow \text{Simplicial spaces} \longrightarrow \text{Spaces}$$

in this lecture we make a progression

eq:415 (19.2)
$$\text{Symmetric monoidal topological categories} \longrightarrow \Gamma\text{-spaces} \longrightarrow \text{Prespectra.}$$

In fact, we will only discuss a special type of symmetric monoidal structure, called a *permutative* structure, which is rigid in the sense that the associativity and identity maps (13.31) and (13.33) are equalities. Our treatment follows [Ma2]; see also [EM, §4].

Motivating example: commutative monoids

subsec:19.1

(19.3) *Segal's category.* Segal [S2] defined a category Γ whose opposite (Definition 18.5) is easier to work with.

thm:344 **Definition 19.4.** Γ^{op} is the category whose objects are finite pointed sets and whose morphisms are maps of finite sets which preserve the basepoint.

Any finite pointed set is isomorphic to

eq:409 (19.5)
$$n^+ = \{*, 1, 2, \dots, n\}$$

for some $n \in \mathbb{Z}^{\geq 0}$. We also use the notation

eq:410

$$S^0 = 1^+ = \{*, 1\}. \tag{19.6}$$

There are also categories Set_*, Top_* of pointed sets and pointed topological spaces, and $\Gamma^{\mathrm{op}} \subset \mathrm{Set}_*$ is a subcategory.

subsec:19.2

(19.7) *The Γ-set associated to a commutative monoid.* Let M be a commutative monoid, which we write additively. Forgetting the addition we are left with a pointed set $(M, 0)$. Define the functor

eq:411

$$\begin{aligned} A_M \colon \Gamma^{\mathrm{op}} &\longrightarrow \mathrm{Set}_* \\ S &\longmapsto \mathrm{Set}_*(S, M) \end{aligned} \tag{19.8}$$

This defines A_M on objects: there is a canonical isomorphism $A_M(n^+) = M^{\times n}$. Note in particular that we recover the commutative monoid as

eq:424

$$A_M(S^0) = M. \tag{19.9}$$

Given a map $(S_0 \xrightarrow{\theta} S_1) \in \Gamma^{\mathrm{op}}$, we must produce $\bigl(\mathrm{Set}_*(S_0, M) \xrightarrow{\theta_* = A_M(\theta)} \mathrm{Set}_*(S_1, M)\bigr)$. This is not composition, but rather is a "wrong-way map", or integration. It is defined as

eq:412

$$\theta_*(\mu)(s_1) = \begin{cases} 0, & s_1 = *; \\ \displaystyle\sum_{s_0 \in \theta^{-1}(s_1)} \mu(s_0), & s_1 \neq *, \end{cases} \tag{19.10}$$

where $\mu \colon S_0 \to M$ is a pointed map ($\mu(*) = 0$) and $s_1 \in S_1$. This pushforward map is illustrated in Figure 36. Note that the map $\alpha \colon 2^+ \to 1^+$ with $\alpha(1) = \alpha(2) = 1$ maps to addition $M^{\times 2} \to M$, and said addition is necessarily commutative and associative, which one proves by applying A_M to the commutative diagrams

eq:413

$$\begin{tikzcd} 2^+ \arrow[rr, "\tau"] \arrow[dr, "\alpha"'] & & 2^+ \arrow[dl, "\alpha"] \\ & 1^+ & \end{tikzcd} \qquad \begin{tikzcd} 3^+ \arrow[r] \arrow[d] & 2^+ \arrow[d] \\ 2^+ \arrow[r] & 1^+ \end{tikzcd} \tag{19.11}$$

The functor A_M is a *special* Γ-*set.*

thm:345

Definition 19.12.

(i) A Γ-*set* is a functor $A \colon \Gamma^{\mathrm{op}} \to \mathrm{Set}_*$ such that $A(\{*\}) = \{*\}$.
(ii) A is *special* if the natural map

eq:416

$$A(S_1 \vee S_2) \longrightarrow A(S_1) \times A(S_2) \tag{19.13}$$

is an isomorphism of pointed sets.

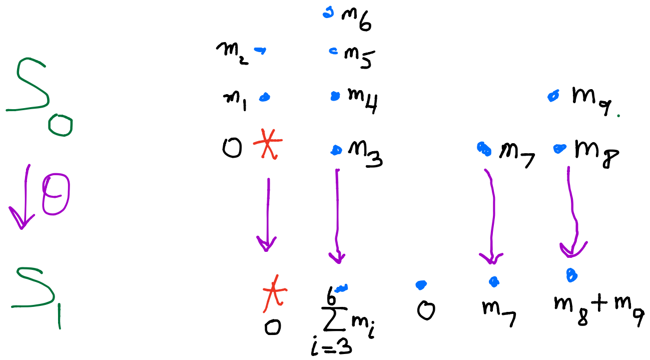

FIGURE 36. The pushforward θ_* associated to $\theta\colon S_0 \to S_1$ fig:36

In (i) the pointed set $\{*\} \in \Gamma^{\mathrm{op}} \subset \mathrm{Set}_*$ is the special object with a single point. A specification of this object makes Γ^{op} and Set_* into *pointed categories*, that is, categories with a distinguished object.[37] So the requirement in (i) is that A be a pointed map of pointed categories. The map (19.13) is induced from the collapse maps

eq:417

$$S_1 \vee S_2 \longrightarrow S_1 \quad \text{and} \quad S_1 \vee S_2 \longrightarrow S_2. \tag{19.14}$$

thm:346 *Remark* 19.15. For any category C a functor $C^{\mathrm{op}} \to \mathrm{Set}$ is called a *presheaf* on C. So a special Γ-set is a pointed presheaf on Γ.

thm:349 *Remark* 19.16. We view a (special) Γ-set A as a set $A(S^0)$ with extra structure. So for $A = A_M$ we have the set M in (19.9) with the extra structure of a basepoint $A(\{*\})$ and a commutative associative composition law $A(2^+ \xrightarrow{\alpha} 1^+)$. A similar picture holds for Γ-spaces below.

thm:348 **Example 19.17.** A *representable* Γ-set is defined by $A(S) = \Gamma^{\mathrm{op}}(T, S)$ for some fixed $T \in \Gamma^{\mathrm{op}}$. Taking $T = S^0$ we have the special Γ-set

eq:418

$$\mathbb{S}(S) = \Gamma^{\mathrm{op}}(S^0, S). \tag{19.18}$$

Notice that $\mathbb{S}(S^0) = S^0$, so that $\mathbb{S}$ is the set S^0 with extra structure. Spoiler alert![38]

At the end of the lecture we give a similar construction (a bit heuristic) in which we replace the commutative monoid M with a symmetric monoidal category C. In that case $\mu\colon S \to C$ assigns an object of C to each element of S and the addition in (19.10) is replaced by the tensor product in C.

Γ-spaces

It is a small leap to generalize Definition 19.12 to spaces. We just need to be careful to replace isomorphisms with weak homotopy equivalences.

---

[37] A standard definition of 'pointed category' also requires that for every object y there be a unique map $* \to y$ and a unique map $y \to *$. We do not make that requirement, though it is true here.

[38] The associated prespectrum is the sphere spectrum, after completing to a spectrum as in **(10.6)**.

thm:347 **Definition 19.19.**

(i) A Γ-*space* is a functor $A\colon \Gamma^{\mathrm{op}} \to \mathrm{Top}_*$ such that $A(\{*\})$ is contractible.

(ii) A is *special* if the natural map

eq:421 (19.20)
$$A(S_1 \vee S_2) \longrightarrow A(S_1) \times A(S_2)$$

is a weak homotopy equivalence of pointed spaces.

Some authors require the stronger condition that $A(\{*\}) = \{*\}$.

Γ and Δ

Recall that Δ is the category of nonempty finite ordered sets and nondecreasing maps. Any object is isomorphic to

eq:419 (19.21)
$$[n] = \{0 < 1 < 2 < \cdots < n\}$$

for some $n \in \mathbb{Z}^{\geq 0}$. We now define a functor

eq:420 (19.22)
$$\kappa\colon \Delta^{\mathrm{op}} \longrightarrow \Gamma^{\mathrm{op}}$$

Composing with κ we obtain a functor from Γ-spaces to simplicial spaces (recall Definition 18.36).

subsec:19.4

(19.23) *Definition of* κ*.* The functor κ on objects is straightforward. If $S \in \Delta$ is a nonempty finite ordered set, let $* \in S$ be the minimum, and consider the pair $\kappa(S) = (S, *)$ as a finite pointed set, forgetting the ordering.

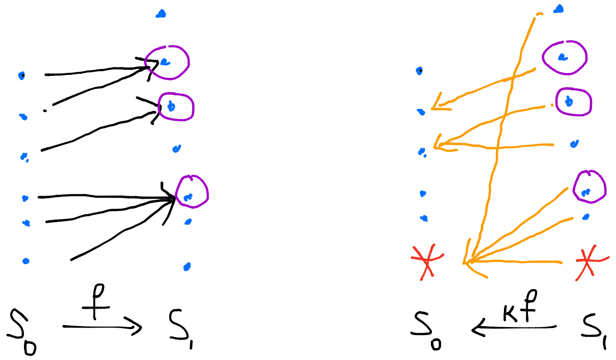

FIGURE 37. The functor $\Delta^{\mathrm{op}} \to \Gamma^{\mathrm{op}}$ on morphisms fig:37

What is trickier is the action of κ on morphisms. We illustrate the general definition in Figure 37. On the left is shown a non-decreasing map $f\colon S_0 \to S_1$ of finite ordered sets. The induced map $\kappa(f)$ of pointed sets maps in the opposite direction. We define it by moving in S_1 from the smallest to the largest element. The smallest element $* \in S_1$ necessarily maps to $* \in S_0$. For each successive element $s_1 \in S_1$ we find the minimal $s_1' \in f(S_0) \subset S_1$ such that $s_1' \geq s_1$; then define $\kappa f(s_{)})$ as the minimal element of $f^{-1}(s_1')$. Finally, if no element $s_1' \geq s_1$ is in the image of f, then set $\kappa f(s_1) = *$.

subsec:19.3

(19.24) *Motivation.* The category Δ is generated by injective/surjective= face/degeneracy maps, as depicted in (18.9). So let's see what κ does on face and degeneracy maps, and we go a step further and apply to the Γ-set A_M defined in **(19.7)**. We leave the reader to check that if $d\colon [n] \to [n+1]$ is the injective map which misses $i \in [n+1]$, then the induced face map $d^*\colon M^{\times(n+1)} \to M^{\times n}$ sends

eq:422

$$(m_1, m_2, \dots m_{n+1}) \longmapsto (m_1, \dots, m_i + m_{i+1}, \dots, m_{n+1}), \tag{19.25}$$

where $m_j \in M$. Similarly, if $s\colon [n] \to [n-1]$ is the surjective map which sends both i and $i+1$ to the same element, then the induced degeneracy map $s^*\colon [n-1] \to [n]$ sends

eq:423

$$(m_1, \dots, m_{n-1}) \longmapsto (m_1, \dots, 0, \dots, m_{n-1}), \tag{19.26}$$

where 0 is inserted in the i^{th} spot. These are the face and degeneracy maps of the nerve of the category with one object whose set of morphisms is M; see Example 18.24.

subsec:19.5

(19.27) *The realization of a Γ-space.* To a Γ-space A is associated a simplicial space $A \circ \kappa$ and so its geometric realization $|A \circ \kappa|$, a topological space. We simply use the notation $|A|$ for this space. Observe that $|A|$ is a *pointed* space. For the set of n-simplices is the pointed space $A(n^+)$, and its basepoint is the degenerate simplex built by successively applying degeneracy maps to the basepoint of $A(0^+)$. The basepoint of $A(0^+)$ gives a distinguished 0-simplex in the geometric realization (18.13), which is then the basepoint of $|A|$. We will now define additional structure on the geometric realization in the form of a Γ-space BA such that $BA(S^0) = |A|$.

The classifying space of a Γ-space

thm:350

Definition 19.28. Let A be a Γ-space. Its *classifying space* BA is the Γ-space

eq:425

$$BA(S) = |T \longmapsto A(S \wedge T)|\,. \tag{19.29}$$

The vertical bars denote the geometric realization of the simplicial space underlying a Γ-space; we prove in the lemma below that the map inside the vertical bars *is* a Γ-space. Note $S, T \in \Gamma^{\text{op}}$. Also, there is a canonical isomorphism

eq:426

$$BA(S^0) = |A|, \tag{19.30}$$

and $BA(\{*\})$ is the basepoint of $|A|$.

thm:351

Remark 19.31. There are modified geometric realizations of a simplicial space which have better technical properties; see the appendix to [S2]. Also, see [D] for another version of geometric realization. Depending on the realization, it may be that $BA(\{*\})$ is a contractible space which contains the basepoint of $|A|$.

thm:352 **Lemma 19.32.** *Let A be a Γ-space and $S \in \Gamma^{\mathrm{op}}$. Then $T \mapsto A(S \wedge T)$ is a Γ-space, special if A is special.*

Proof. Observe that $T \mapsto S \wedge T$ is a functor $\Gamma^{\mathrm{op}} \to \Gamma^{\mathrm{op}}$, and that $0^+ \mapsto S \wedge 0^+ = 0^+$. For the special statement, if $T_1, T_2 \in \Gamma^{\mathrm{op}}$, then

eq:427

$$T_1 \vee T_2 \longmapsto S \wedge (T_1 \vee T_2) = (S \wedge T_1) \vee (S \wedge T_2). \tag{19.33}$$

Now use the special property of A and the fact that the realization of a product is the product of the realizations. ☐

The prespectrum associated to a Γ-space

subsec:19.7

(19.34) *Iteration.* Let A be a Γ-space. We iterate the classifying space construction to obtain a sequence

eq:428

$$A,\ BA,\ B^2A,\ B^3A,\ \dots \tag{19.35}$$

of Γ-spaces, and so too a sequence

eq:429

$$A(S^0),\ BA(S^0),\ B^2A(S^0),\ B^3A(S^0),\ \dots \tag{19.36}$$

of pointed topological spaces.

subsec:19.8

(19.37) *Prespectrum structure.* We define for any Γ-space A a continuous map

eq:430

$$s\colon \Sigma(A(S^0)) \longrightarrow BA(S^0) = |A|. \tag{19.38}$$

Applying this to each space in (19.36) we obtain a prespectrum. The simplicial space associated to A has $A(0^+) = A(\{*\})$ as its space of 0-simplices. Assume for simplicity that $A(\{*\}) = *$; in any case $* \in A(\{*\})$ and the same construction applies. Now the geometric realization of a simplicial space has a natural filtration by subspaces; the q^{th} stage of the filtration is obtained by taking the disjoint union over $n = 0, 1, \dots, q$ in (18.13). Under the hypothesis just made on A, the 0^{th} stage of the filtration is a single point $*$. The 1^{st} stage of the filtration is obtained by gluing on $A(1^+) = A(S^0)$ using the two face maps and single degeneracy map. We leave the reader to check that we exactly obtain the (reduced) suspension $\Sigma(A(S^0))$. Hence the map s is the inclusion of the 1^{st} stage of the filtration of $|A|$.

subsec:19.9 **(19.39)** *Monoid structure on* $\pi_0 A(S^0)$. If A is a special Γ-space, then the composition

eq:431
$$\Gamma^{\mathrm{op}} \xrightarrow{A} \mathrm{Top}_* \xrightarrow{\pi_0} \mathrm{Set}_* \tag{19.40}$$

is a special Γ-set. You will prove in the homework that the Γ-set structure gives $\pi_0 A(S^0)$ the structure of a commutative monoid.

thm:353 **Theorem 19.41** ([S2])**.** *If the commutative monoid* $\pi_0 A(S^0)$ *is an abelian group, then the adjoint*

eq:432
$$t\colon A(S^0) \longrightarrow \Omega B A(S^0) \tag{19.42}$$

is a weak homotopy equivalence.

thm:354 **Corollary 19.43.** *For* $k > 0$ *the space* $B^k A(S^0)$ *is weakly equivalent to* $\Omega B^{k+1} A(S^0)$*.*

Proof. For $B^k A(S^0)$ is the geometric realization of the Γ-space $B^{k-1}A$ which has a contractible space of 0-simplices, and therefore π_0 trivial. ☐

The necessity of the condition in Theorem 19.41 is clear. For if $A(S^0)$ is equivalent to a loop space, then the loop product (on π_0) has additive inverses: reverse the parametrization of the loop. A standard argument, which you encountered encountered in the second problem set, proves that the loop product is equal to the product given by the Γ-space structure. If $\pi_0 A(S^0)$ is an abelian group, then (19.36) is an Ω-prespectrum.

We do not provide a proof of Theorem 19.41 in this version of the notes.

thm:355 **Example 19.44.** Let A be a discrete abelian group. The Ω-prespectrum (19.35) associated to the Γ-set (19.8) defined by A (viewed as a commutative monoid) is an *Eilenberg-MacLane spectrum.*

thm:356 **Example 19.45.** The prespectrum associated to the Γ-set $\mathbb{S}$ is the *sphere spectrum.* (Better: the sphere spectrum is the completion of that Ω-prespectrum to a spectrum.)

Γ-categories

The next definition is analogous to Definition 19.19. Recall that a *pointed category* is a category with a distinguished object. The collection of (small) pointed categories forms a category Cat_*; morphisms are functors and we require associativity on the nose.[39]

thm:357 **Definition 19.46.**

(i) A Γ-*category* is a functor $D\colon \Gamma^{\mathrm{op}} \to \mathrm{Cat}_*$ such that $D(\{*\})$ is equivalent to the trivial category with a single object and the identity morphism.

(ii) D is *special* if the natural map

eq:437
$$D(S_1 \vee S_2) \longrightarrow D(S_1) \times D(S_2) \tag{19.47}$$

is an equivalence of pointed categories.

---

[39] Categories are more naturally objects in a *2-category.* Namely, functors are like sets, and there is an extra layer of structure: natural transformations between functors. So it is rather rigid to demand that composition of functors be associative on the nose.

subsec:19.10

(19.48) *From Γ-categories to Γ-spaces and prespectra.* Let D be a Γ-category. Then composing with the classifying space construction $B\colon \mathrm{Cat}_* \to \mathrm{Top}_*$ we obtain a Γ-space BD and then a prespectrum whose 0-space is $B(D(S^0))$, the classifying space of the category $D(S^0)$.

subsec:19.11

(19.49) *Permutative categories.* We would like to associate a Γ-category to a symmetric monoidal category, but we need to assume additional rigidity to do so. A theorem of Isbell [I] asserts that every symmetric monoidal category is equivalent to a permutative category, so this is not really a loss of generality. A permutative category is a symmetric monoidal category with a strict unit and strict associativity.

thm:358 **Definition 19.50.** A *permutative category* is a quartet $(C, 1_C, \otimes, \sigma)$ consisting of a pointed category $(C, 1_C)$, a functor $\otimes\colon C \times C \to C$, and a natural transformation σ as in (13.32) such that for all $y, y_1, y_2, y_3 \in C$

(i) $1_C \otimes y = y \otimes 1_C = y$;
(ii) $(y_1 \otimes y_2) \otimes y_3 = y_1 \otimes (y_2 \otimes y_3)$;
(iii) the composition $y_1 \otimes y_2 \xrightarrow{\sigma} y_2 \otimes y_1 \xrightarrow{\sigma} y_1 \otimes y_2$ is the identity; and
(iv) the diagrams

eq:433 (19.51)
$$\begin{tikzcd} 1 \otimes y \arrow[rr, "\sigma"] \arrow[dr, "="'] & & y \otimes 1 \arrow[dl, "="] \\ & y & \end{tikzcd} \qquad \begin{tikzcd} y_1 \otimes y_2 \otimes y_2 \arrow[rr, "\sigma"] \arrow[dr, "1\otimes\sigma"'] & & y_3 \otimes y_1 \otimes y_2 \arrow[dl, "\sigma\otimes 1"] \\ & y_1 \otimes y_3 \otimes y_2 & \end{tikzcd}$$

commute.

thm:359 **Example 19.52.** The category Γ^{op} of finite pointed sets has a permutative structure if we take a model in which the set of objects is precisely $\{n^+ : n \in \mathbb{Z}^{\geq 0}\}$. Then define ${n_1}^+ \otimes {n_2}^+ = (n_1 + n_2)^+$. The tensor unit is 0^+ and we leave the reader to define the symmetry σ.

subsec:19.12

(19.53) *The Γ-category associated to a permutative category.* As we said earlier, this construction is analogous to **(19.7)**. We give the basic definitions and leave to the reader the detailed verifications. Let C be a permutative category. We define an associated Γ-category D as follows. For $S \in \Gamma^{\mathrm{op}}$ a finite pointed set let $D(S)$ be the category whose objects are pairs (c, ρ) in which (i) $c(T) \in C$ for each pointed subset $T \subset S$ and (ii) the map

eq:434 (19.54)
$$\rho(T_1, T_2)\colon c(T_1) \otimes c(T_2) \longrightarrow c(T_1 \vee T_2)$$

is an isomorphism for each pair of pointed subsets with $T_1 \cap T_2 = \{*\}$. These data must satisfy several conditions:

(i) $c(\{*\}) = 1_C$;
(ii) $\rho(\{*\}, T) = \mathrm{id}_T$ for all T; and

(iii) for all T_1, T_2, T_3 with correct intersections the diagrams

eq:435 (19.55)

$$\begin{tikzcd} c(T_1) \otimes c(T_2) \arrow[r, "\rho(T_1,T_2)"] \arrow[d, "\sigma"'] & c(T_1 \vee T_2) \arrow[d, equal] \\ c(T_2) \otimes c(T_1) \arrow[r, "\rho(T_2,T_1)"] & c(T_2 \vee T_1) \end{tikzcd}$$

and

eq:436 (19.56)

$$\begin{tikzcd} c(T_1) \otimes c(T_2) \otimes c(T_3) \arrow[r, "\rho(T_1,T_2)\otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \rho(T_2,T_3)"'] & c(T_1 \vee T_2) \otimes c(T_3) \arrow[d, "\rho(T_1 \vee T_2, T_3)"] \\ c(T_1) \otimes c(T_2 \vee T_3) \arrow[r, "\rho(T_1, T_2 \vee T_3)"] & c(T_1 \vee T_2 \vee T_3) \end{tikzcd}$$

commute.

thm:360 **Exercise 19.57.** Define a morphism $\bigl((c,\rho) \to (c',\rho')\bigr) \in D(S)$. The data is, for each pointed $T \subset S$, a morphism $\bigl(c(T) \to c'(T)\bigr) \in C$. What is the condition that these morphisms must satisfy?

This completes the definition of the category $D(S)$ associated to $S \in \Gamma^{\mathrm{op}}$. Now we must define a functor $D(S_0) \xrightarrow{\theta_*} D(S_1)$ for each morphism $(S_1 \xrightarrow{\theta} S_1) \in \Gamma^{\mathrm{op}}$. The definition follows Figure 36. To streamline the notation, for $T \subset S_1$ a pointed subset define the modified inverse image to be the pointed subset

eq:441 (19.58)

$$\widetilde{\theta^{-1}(T)} := \{*\} \cup \theta^{-1}\bigl(T \setminus \{*\}\bigr).$$

Now given $(c,\rho) \in D(S_0)$ define $(c',\rho') = \theta_*(c,\rho)$ by

eq:439 (19.59)

$$c'(T) = c\bigl(\widetilde{\theta^{-1}(T)}\bigr)$$

We leave the reader to supply the definition of ρ' and of θ_* on morphisms.

Observe that there is a natural isomorphism of categories

eq:440 (19.60)

$$D(S^0) \xrightarrow{\cong} C.$$

In this sense the Γ-category D is the category C with extra structure, which encodes its permutative structure.

thm:361 **Exercise 19.61.** Work out the Γ-space associated to the permutative category of Example 19.52. How does it compare to $\mathbb{S}$?

Lecture 20: Topological bordism categories

sec:20

We return to bordism and construct a more complicated “algebraic” invariant than the previous ones: a topological category. Of course, this is not purely algebraic, but rather a mix of algebra and topology. We begin with some preliminaries on the topology of function spaces. The Whitney theorem then gives a model of the classifying space of the diffeomorphism group of a compact manifold in terms of a space of embeddings. Then, following Galatius-Madsen-Tillmann-Weiss (GMTW), we construct the topological category of bordisms. We did not find a symmetric monoidal structure, though morally it should be there. It turns out that in any case the classifying space is the 0-space of a spectrum. For the bordism category with morphisms oriented 2-manifolds, this was first shown in [Ti]; the identity of that spectrum was conjectured in [MT] and first proved in [MW]. The GMTW Theorem is a generalization to all dimensions. As we shall see, it is a generalization of the classical Pontrjagin-Thom Theorem 10.33.

In this lecture we get as far as stating the GMTW Theorem [GMTW]. We discuss the proof in subsequent lectures.

Topology on function spaces

A reference for this subsection is [Hi, Chapter 2]. In particular, Hirsch uses jet spaces to describe the spaces of maps below as subspaces of function spaces with the standard compact-open topology. We pass immediately to C^∞ functions; it is somewhat easier to consider C^r functions for r finite and then take $r \to \infty$.

subsec:20.1

(20.1) *The Whitney topology.* Let Z, M be smooth manifolds with Z closed. We define a topology on the set $C^\infty(Z,M)$ of smooth maps $Z \to M$. The topology is generated[40] by sets $N\bigl(f,(U,z),(V,m),K,\epsilon\bigr)$ where $f\colon Z \to M$ is a smooth function; (U,z) is a chart on Z with $z\colon U \to \mathbb{A}^n$; (V,m) is a chart on M; $K \subset U$ is a compact set such that $f(K) \subset V$; and $\epsilon > 0$. To describe the sets we use multi-index notation $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{Z}^{\ge 0}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and

eq:442

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial z^1)^{\alpha_1} \cdots (\partial z^n)^{\alpha_n}}. \tag{20.2}$$

Then $N\bigl(f,(U,z),(V,m),K,\epsilon\bigr)$ consists of all smooth functions $f'\colon Z \to M$ such that $f'(K) \subset Z$ and for all multi-indices α and all $j = 1, \dots, \dim M$, we have

eq:443

$$\bigl\| D^\alpha(m^j \circ f' \circ z^{-1}) - D^\alpha(m^j \circ f \circ z^{-1}) \bigr\|_{C^0(K)} < \epsilon. \tag{20.3}$$

The $C^0(K)$ norm is the sup norm, which is the maximum of the norm of a continuous function on the compact set K and $m^j\colon V \to \mathbb{R}$ is the coordinate function in the chart.

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[40] In other words, the topology is the smallest topology which contains the sets N.

thm:362 *Remark* 20.4. If Z is noncompact this is called the *weak* Whitney topology; there is also a strong Whitney topology.

subsec:20.2

(20.5) *Embeddings and diffeomorphisms.* Topologize embeddings $\mathrm{Emb}(Z,M) \subset C^\infty(Z,M)$ using the subspace topology. Similarly, topologize the group of diffeomorphisms $\mathrm{Diff}(Z) \subset C^\infty(Z,Z)$ as a subspace. Composition and inversion are continuous, so $Diff(Z)$ is a topological group. For embeddings into affine space define

eq:444
$$\mathrm{Emb}(Z,\mathbb{A}^\infty) = \operatorname*{colim}_{m\to\infty} \mathrm{Emb}(Z,\mathbb{A}^m). \tag{20.6}$$

An element of $\mathrm{Emb}(Z,\mathbb{A}^\infty)$ is an embedding $f\colon Z \to \mathbb{A}^m$ for some m, composed with the inclusion $\mathbb{A}^m \to \mathbb{A}^\infty$.

thm:363 **Theorem 20.7.** $\mathrm{Emb}(Z,\mathbb{A}^\infty)$ *is contractible and* $\mathrm{Diff}(Z)$ *acts freely.*

Notice that a contractible space is nonempty; the nonemptiness is a nontrivial statement. The following argument may be found in [KM, Lemma 44.22].

Proof. $\mathrm{Emb}(Z,\mathbb{A}^\infty)$ is nonempty by Whitney's embedding theorem. The freeness of the diffeomorphism action is clear, since each embedding $f\colon Z\to\mathbb{A}^m$ is injective. For the contractibility consider the homotopy $H_t\colon \mathbb{A}^\infty \to \mathbb{A}^\infty$, $0\le t\le 1$, defined by

eq:445
$$H_t(x^1,x^2,\dots) = \bigl(x^1,\dots,x^{n-1},x^n\cos\theta^n(t),x^n\sin\theta^n(t),x^{n+1}\cos\theta^n(t),x^{n+1}\sin\theta^n(t),\dots\bigr), \tag{20.8}$$

where n is determined by $\frac{1}{n+1}\le t\le\frac{1}{n}$ and

eq:446
$$\theta^n(\frac{1}{n+1}+s) = \rho\bigl(n(n+1)s\bigr)\frac{\pi}{2}. \tag{20.9}$$

Here $\rho\colon [0,1]\to[0,1]$ is a smooth(ing) function with $\rho\bigl([0,\epsilon)\bigr)=0$, $\rho\bigl((1-\epsilon,\epsilon]\bigr)=1$ for some $\epsilon>0$. In fact, n is not uniquely determined if t is the reciprocal of an integer, but the formulas are consistent for the two choices. Since all but finitely many x^i vanish, the map $H\colon [0,1]\times\mathbb{A}^\infty\to\mathbb{A}^\infty$ is smooth. Also, $H_0=\mathrm{id}_{\mathbb{A}^\infty}$ and

eq:447
$$\begin{aligned} H_{1/2}(x^1,x^2,\dots) &= (x^1,0,x^2,0,\dots) \\ H_1(x^1,x^2,\dots) &= (0,x^1,0,x^2,\dots). \end{aligned} \tag{20.10}$$

We use H to construct a contraction. Fix an embedding $i_0\colon Z\to\mathbb{A}^\infty$. Use $H_{0\to1/2}$ to homotop i_0 to an embedding $H_{1/2}\circ i_0$ which lands in $\mathbb{A}^\infty_{\mathrm{odd}}$. Now composition with H_1 is a map $\mathrm{Emb}(Z,\mathbb{A}^\infty)\to \mathrm{Emb}(Z,\mathbb{A}^\infty_{\mathrm{even}}\subset\mathbb{A}^\infty)$. Combining composition with $H_{0\to1}$ with the homotopy

eq:448
$$K_u(z) = (1-u)(H_1\circ i)(z) + u\,(H_{1/2}\circ i_0)(z), \qquad i\in\mathrm{Emb}(Z,\mathbb{A}^\infty), \tag{20.11}$$

we obtain a contraction of $\mathrm{Emb}(Z,\mathbb{A}^\infty)$. ☐

Notice that averaging an embedding into $\mathbb{A}^\infty_{\mathrm{odd}}$ with an embedding into $\mathbb{A}^\infty_{\mathrm{even}}$ yields an embedding.

subsec:20.3

(20.12) *A classifying space for* Diff(Z)*.* Let $B_\infty(Z)$ denote the quotient space of the free Diff(Z)-action on Emb$(Z, \mathbb{A}^\infty)$.

thm:364 **Proposition 20.13** ([BiFi])**.** *The map* π *in*

eq:449 (20.14)

$$\begin{tikzcd} \mathrm{Diff}(Z) \arrow[r] & \mathrm{Emb}(Z, \mathbb{A}^\infty) \arrow[d, "\pi"] \\ & B_\infty(Z) \end{tikzcd}$$

is a topological principal bundle with structure group Diff(Z)*.*

A point of $B_\infty(Z)$ is a submanifold $Y \subset \mathbb{A}^\infty$ which is diffeomorphic to Z. The topology on Emb$(Z, \mathbb{A}^\infty)$ induces a quotient topology on the set $B_\infty(Z)$ via the map π. There are smoothness statements one can make about the fiber bundle (20.14), but we are content with the topological assertion.[41] There is also a generalization for Z noncompact [KM, §44].

Sketch proof. We must prove (20.14) is locally trivial, so produce local sections of π. Fix an embedding $i\colon Z \to \mathbb{A}^\infty$ and let $U \subset \mathbb{A}^\infty$ be a tubular neighborhood around the image $i(Z)$. It is equipped with a submersion $p\colon U \to i(Z) \cong Z$. Then we claim

eq:450 (20.15)

$$\{i' \in \mathrm{Emb}(Z, \mathbb{A}^\infty) : i'(Z) \subset U,\ p \circ i' = i\}$$

is an open subset of Emb$(Z, \mathbb{A}^\infty)$ on which π is injective and whose image under π is an open neighborhood of $\pi(i)$. We defer to the references for the proofs of these claims. ☐

subsec:20.4

(20.16) *The associated bundle.* The topological group Diff(Z) has a left action on Z by evaluation: $f \in$ Diff(Z) acts on $y \in Z$ to give $f(y) \in Z$. We can "mix" this left action with the right Diff(Z)-action in (20.14) to produce the associated fiber bundle

eq:451 (20.17)

$$\begin{tikzcd} Z \arrow[r] & E_\infty(Z) \arrow[d] \\ & B_\infty(Z) \end{tikzcd}$$

in which

eq:452 (20.18)

$$E_\infty(Z) = \mathrm{Emb}(Z, \mathbb{A}^\infty) \times_{\mathrm{Diff}(Z)} Z.$$

Note there is a natural map $E_\infty(Z) \to \mathbb{A}^\infty$ which is an embedding on each fiber. The fiber bundle (20.17) is universal for fiber bundles with fiber (diffeomorphic to) Z embedded in $\mathbb{A}^\infty$. Because of the embedding, the classifying map of such a fiber bundle is unique.

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[41] We should not be complacent, however. In the next lecture we will need to speak of smooth maps into $B_\infty(Z)$, as for example discussed in [KM].

The topological bordism category

The *discrete* category $\mathrm{Bord}^{\mathfrak{X}(n)}_{\langle n-1,n\rangle}$ of Definition 14.3 uses abstract manifolds and bordisms. To define a *topological* category we use manifolds and bordisms which are embedded in affine space. Also, we do not identify diffeomorphic bordisms.

thm:365 **Definition 20.19.** Fix $n \in \mathbb{Z}^{\geq 0}$. The *topological bordism category* ${}^t\mathrm{Bord}_{\langle n-1,n\rangle}$ is defined as follows.

(i) An object is a pair (a, Y) consisting of a real number $a \in \mathbb{R}$ and a closed $(n-1)$-submanifold $Y \subset \mathbb{A}^\infty$;

(ii) A morphism $X\colon (a_0, Y_0) \to (a_1, Y_1)$ is either the identity, if $a_0 = a_1$ and $Y_0 = Y_1$, or if $a_0 < a_1$ a compact n-dimensional neat submanifold $X \subset [a_0, a_1] \times \mathbb{A}^\infty$ such that for some $\delta > 0$ we have

eq:453 (20.20)
$$\begin{aligned} X \cap \bigl([a_0, a_0 + \delta) \times \mathbb{A}^\infty\bigr) &= [a_0, a_0 + \delta) \times Y_0, \\ X \cap \bigl((a_1 - \delta, a_1] \times \mathbb{A}^\infty\bigr) &= (a_1 - \delta, a_1] \times Y_1. \end{aligned}$$

(iii) Composition of non-identity morphisms is the union, as illustrated in Figure 38.

(iv) The set $\coprod_Z \bigl(\mathbb{R} \times B_\infty(Z)\bigr)$ of objects is topologized using the quotient topology on $B_\infty(Z)$, as in **(20.12)**. The disjoint union runs over diffeomorphism types of closed $(n-1)$-manifolds.

(v) There is a similar topology on the set of morphisms, as discussed in [GMTW, §2].

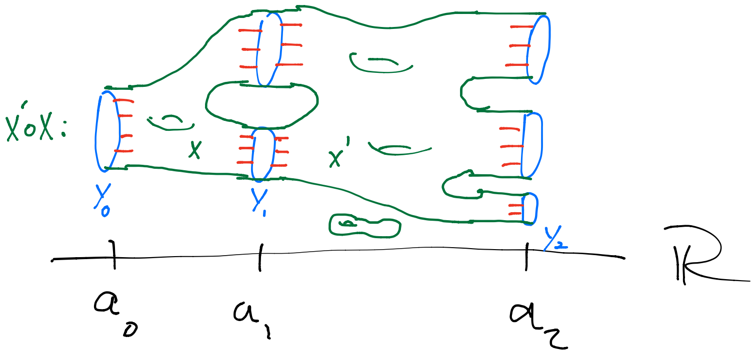

FIGURE 38. Composition of morphisms fig:38

subsec:20.5

(20.21) *Symmetric monoidal structure: discussion.* We would like to introduce a symmetric monoidal structure on ${}^t\mathrm{Bord}_{\langle n-1,n\rangle}$ using disjoint union as usual. Then the discussion in Lecture 19 on Γ-spaces would imply that the classifying space $B\bigl({}^t\mathrm{Bord}_{\langle n-1,n\rangle}\bigr)$ is the 0-space of a spectrum. Unfortunately, we don't see how to introduce that structure, though it *is* true that $B\bigl({}^t\mathrm{Bord}_{\langle n-1,n\rangle}\bigr)$ is an infinite loop space.

Since the manifolds are embedded, we must make the disjoint union concrete. One technique is to introduce the map

eq:454 (20.22)
$$\begin{aligned} m\colon \mathbb{A}^\infty \times \mathbb{A}^\infty &\longrightarrow \mathbb{A}^\infty \\ (x^1, x^2, \dots)\ ,\ (y^1, y^2, \dots) &\longmapsto (x^1, y^1, x^2, y^2, \dots) \end{aligned}$$

and then define the monoidal product as $(a_1, Y_1) \otimes (a_2, Y_2) = \bigl(a_1 + a_2, m(Y_1, Y_2)\bigr)$. The tensor unit is $(0, \emptyset^{n-1})$. Unfortunately, this is not strictly associative nor is the unit strict—in other words, this is not a permutative category—and there are not enough morphisms in ${}^t\mathrm{Bord}_{\langle n-1,n\rangle}$ to define an associator and a map (13.33), much less a symmetry. Naive modifications do not seem to work either. Fortunately, we do not need to use the symmetric monoidal structure to define and identify the classifying space.

We remark that the bordism *multi*-category we will discuss in the last few lectures does have a symmetric monoidal structure[42]

Finally, we would like to define a continuous TQFT as a symmetric monoidal functor from ${}^t\mathrm{Bord}_{\langle n-1,n\rangle}$ into a symmetric monoidal topological category, but absent the symmetric monoidal structure on ${}^t\mathrm{Bord}_{\langle n-1,n\rangle}$ we cannot do so. Nonetheless, we can still motivate interest in the classifying space $B\bigl({}^t\mathrm{Bord}_{\langle n-1,n\rangle}\bigr)$ by asserting that an *invertible* continuous TQFT is a map of topological Picard groupoids $B\bigl({}^t\mathrm{Bord}_{\langle n-1,n\rangle}\bigr) \to C$ for a topological Picard groupoid C.

subsec:20.6

(20.23) $\mathcal{X}(n)$*-structures.* We use $BO(n) = Gr_n(\mathbb{R}^\infty)$ as a model for the classifying space of the orthogonal group **(6.23)**. This is convenient since if $i\colon Y \hookrightarrow \mathbb{A}^\infty$ is an $(n-1)$-dimensional submanifold, then the "Gauss map"

eq:466

$$\begin{tikzcd} TY \arrow[r, "i_*"] \arrow[d] & \underline{\mathbb{R}} \oplus S(n-1) \arrow[r] \arrow[d] & S(n) \arrow[d] \\ Y \arrow[r] & Gr_{n-1}(\mathbb{R}^\infty) \arrow[r] & Gr_n(\mathbb{R}^\infty) \end{tikzcd} \tag{20.24}$$

is a classifying map for the once stabilized tangent bundle. Let $\mathcal{X}(n) \to Gr_n(\mathbb{R}^\infty)$ be an n-dimensional tangential structure. Then an $\mathcal{X}(n)$-structure on $Y \subset \mathbb{A}^\infty$ is a lift of (20.24) to a map

eq:467

$$\begin{tikzcd} TY \arrow[r] \arrow[d] & S(n) \arrow[d] \\ Y \arrow[r] & \mathcal{X}(n) \end{tikzcd} \tag{20.25}$$

The definition of an $\mathcal{X}(n)$-structure on a morphism is similar. There is a topological bordism category ${}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$ whose objects and morphisms are as in Definition 20.19, now with the addition of an $\mathcal{X}(n)$-structure. The equalities in (20.20) now include the $\mathcal{X}(n)$-structure as well. For a fixed $Y \subset \mathbb{A}^\infty$ there is a space of $\mathcal{X}(n)$-structures, and that space enters into the topologization of the sets of objects and morphisms. We refer to [GMTW, §5] for details.

subsec:20.7

(20.26) *The main question.* Identify the classifying space $B\bigl({}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}\bigr)$.

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[42] though the technical details on defining symmetric monoidal (∞, n)-categories may not be written down as of this writing.

Madsen-Tillmann spectra

subsec:20.8

(20.27) *Heuristic definition.* Fix a nonnegative integer n and an n-dimensional tangential structure $\mathcal{X}(n) \to BO(n)$. Recall the universal bundle $S(n) \to \mathcal{X}(n)$, which is pulled back from $BO(n)$.

thm:367 **Definition 20.28.** The *Madsen-Tillmann spectrum* $MT\mathcal{X}(n)$ is the Thom spectrum $\mathcal{X}(n)^{-S(n)}$.

subsec:20.9

(20.29) *Precise definition.* As we have only defined the Thom spectrum associated to $S(n)$ (Definition 10.26), not the *virtual* Thom spectrum associated to $-S(n)$, we need something more concrete. Recall that $BO(n)$ is the colimit **(6.23)** of Grassmannians. Define $\mathcal{X}(n, n+q)$ as the pullback

eq:455 (20.30)

$$\begin{tikzcd} \mathcal{X}(n,n+q) \arrow[r, dashed] \arrow[d, dashed] & \mathcal{X}(n) \arrow[d] \\ Gr_n(\mathbb{R}^{n+q}) \arrow[r] & BO(n) \end{tikzcd}$$

Use the standard metric on $\mathbb{R}^{n+q}$ so that there is a direct sum $\underline{\mathbb{R}^{n+q}} = S(n) \oplus Q(q)$ of vector bundles over $Gr_n(\mathbb{R}^{n+q})$ and, by pullback, over $\mathcal{X}(n, n+q)$.

thm:368 **Definition 20.31.** The *Madsen-Tillmann spectrum* $MT\mathcal{X}(n)$ is the spectrum completion of the prespectrum whose $(n+q)^{\text{th}}$ space is the Thom space of $Q(q) \to \mathcal{X}(n, n+q)$. The structure maps are obtained by applying the Thom space construction to the map

eq:456 (20.32)

$$\begin{tikzcd} \underline{\mathbb{R}} \oplus Q(q) \arrow[r] \arrow[d] & Q(q+1) \arrow[d] \\ \mathcal{X}(n,n+q) \arrow[r] & \mathcal{X}(n,n+q+1) \end{tikzcd}$$

of vector bundles.

This prespectrum has spaces defined for integers $\geq n$, which is allowed; see the remarks following Definition 10.2. The intuition here is that, as formal bundles, $Q(q) = -S(n) + \underline{\mathbb{R}^{n+q}}$, so the Thom space of the vector bundle $Q(q) \to \mathcal{X}(n, n+q)$ represents the 0-space of the $(n+q)^{\text{th}}$ suspension of the spectrum defined in Definition 20.28. The latter is equally the $(n+q)^{\text{th}}$ space of the unsuspended MT spectrum.

subsec:20.10

(20.33) *Notation.* The ‘MT’ notation is due to Mike Hopkins. It not only stands for ‘Madsen-Tillmann’, but also for a Tangential variant of the thoM spectrum. The MT spectra are tangential and unstable; the M-spectra are normal and stable. We will see a precise relationship below. For Madsen-Tillmann spectra constructed from reductions of structure group **(10.28)**, we use the notation $MTG(n)$. For example, the Madsen-Tillmann spectrum for oriented bundles is $MTSO(n)$.

thm:369 **Proposition 20.34.** *There is a homotopy equivalence* $MTSO(1) \simeq S^{-1} = \Sigma^{-1}S^0$.

Here S^0 is the sphere spectrum.

Proof sketch. First, $BSO(1)$ is contractible, since $SO(1)$ is the trivial group with only the identity element. So the formal Definition 20.28 reduces to $MTSO(1) = BSO(1)^{-\mathbb{R}} \simeq \Sigma^{-1}T_{\bullet}$ where $T^{\bullet}$ is the suspension spectrum of a contractible unpointed space, which is the sphere spectrum. (Check that the Thom space $*^{\mathbb{R}}$ of the trivial bundle over a point is the pointed space S^1.) We leave the reader to give the instructive proof based on Definition 20.31. [**check if just get sphere prespectrum on the nose**] ⇐ ☐

subsec:20.11

(20.35) *The perp map.* Now assume that $\mathcal{X}$ is a stable tangential structure (Definition 9.45). There is an induced n-dimensional tangential structure $\mathcal{X}(n)$. Recall from (**9.62**) the perp stable tangential structure $\mathcal{X}^{\perp}$. We now construct a map

eq:457

$$\Sigma^n MT\mathcal{X}(n) \longrightarrow M\mathcal{X}^{\perp} \tag{20.36}$$

from the Madsen-Tillmann spectrum to the Thom spectrum. Namely, the perp map followed by stabilization yields the diagram

eq:458

$$\begin{tikzcd}
Q(q) \arrow[r, "\cong"] \arrow[d] & S(q) \arrow[r] \arrow[d] & S(q) \arrow[d] \\
\mathcal{X}(n,n+q) \arrow[r, "\cong"] \arrow[d] & \mathcal{X}^{\perp}(q,n+q) \arrow[r] \arrow[d] & \mathcal{X}^{\perp}(q) \arrow[d] \\
Gr_n(\mathbb{R}^{n+q}) \arrow[r, "\perp", "\cong"'] & Gr_q(\mathbb{R}^{n+q}) \arrow[r] & Gr_q(\mathbb{R}^{\infty})
\end{tikzcd} \tag{20.37}$$

Vertical

The induced map on the Thom space of the upper left arrow to the Thom space of the upper right arrow is a map $MT\mathcal{X}(n)_{n+q} \to M\mathcal{X}^{\perp}_q$ on the indicated spaces of the spectra. The maps are compatible with the structure maps of the prespectra as q varies, and so we obtain the map (20.36) of spectra.

subsec:20.12

(20.38) *The filtration of the Thom spectrum.* The stabilization map

eq:459

$$\begin{tikzcd}
Q(q) \arrow[r] \arrow[d] & Q(q) \arrow[d] \\
\mathcal{X}(n,n+q) \arrow[r] & \mathcal{X}(n+q,n+1+q)
\end{tikzcd} \tag{20.39}$$

induces a map $\Sigma^n MT\mathcal{X}(n)_q \to \Sigma^{n+1}MT\mathcal{X}(n+1)_q$ on Thom spaces, and iterating with n we obtain a sequence of maps

eq:460

$$MT\mathcal{X}(0) \longrightarrow \Sigma^1 MT\mathcal{X}(1) \longrightarrow \Sigma^2 MT\mathcal{X}(2) \longrightarrow \cdots \tag{20.40}$$

of spectra. Define the colimit to be the stable Madsen-Tillmann spectrum $MT\mathcal{X}$. The perp maps (20.36) induce a map

eq:461

$$MT\mathcal{X} \to M\mathcal{X}^{\perp} \tag{20.41}$$

on the colimit. It is clear from the construction (20.37) that (20.41) is an isomorphism. So the (suitably suspended) Madsen-Tillmann spectra (20.40) give a filtration of the Thom spectrum.

The Galatius-Madsen-Tillmann-Weiss theorem

Now we can state the main theorem.

thm:370 **Theorem 20.42** ([GMTW]). *Let n be a nonnegative integer and $\mathcal{X}(n)$ an n-dimensional tangential structure. Then there is a weak homotopy equivalence*

eq:462
$$B\bigl({}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}\bigr) \simeq \bigl(\Sigma MT\mathcal{X}(n)\bigr)_0. \tag{20.43}$$

In words: The classifying space of the topological bordism category is the 0-space of the suspension of the Madsen-Tillmann spectrum. We sketch the proof in subsequent lectures. The power of the theorem is that the space on the right hand side is constructed from familiar ingredients in algebraic topology, so its invariants are readily calculable.

subsec:20.13

(20.44) *The GMTW theorem on π_0.* Theorem 20.42 is a weak homotopy equivalence of (pointed) spaces. It induces an isomorphism of (pointed) sets by applying π_0. Since both sides are infinite loop spaces, π_0 is an abelian group, and so we obtain an isomorphism of abelian groups. A space is a refinement of the set π_0, and so (20.43) is a refinement of this isomorphism of sets. We now compute π_0 of both sides.

For the classifying space of the bordism category, the left hand side of (20.43), we compute π_0 directly from the category. Namely, the morphisms define an equivalence relation on objects: two objects are equivalent if they are connected by a morphism. The construction of the classifying space shows that π_0 is the set of isomorphisms classes. Applied to the bordism category we obtain the bordism group

eq:463
$$\pi_0 B\bigl({}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}\bigr) \cong \Omega^{\mathcal{X}(n)}_{n-1}. \tag{20.45}$$

Assume that $\mathcal{X}(n)$ is induced from a stable tangential structure $\mathcal{X}$. Then for the right hand side of (20.43), we have for q large

eq:464
$$\begin{aligned}\pi_0\bigl(\Sigma MT\mathcal{X}(n)\bigr)_0 &= \pi_{n+q-1}\mathcal{X}(n,n+q)^{Q(q)}\\ &\cong \pi_{n+q-1}\mathcal{X}^{\perp}(q,n+q)^{S(q)}\\ &\cong \pi_{n-1}M\mathcal{X}^{\perp}.\end{aligned} \tag{20.46}$$

Therefore, on the level of π_0 the weak homotopy equivalence (20.43) is an isomorphism

eq:465
$$\Omega^{\mathcal{X}(n)}_{n-1} \xrightarrow{\cong} \pi_{n-1}M\mathcal{X}^{\perp}. \tag{20.47}$$

Recall the general Pontrjagin-Thom Theorem 10.33 which is precisely such an isomorphism. So we expect that the weak homotopy equivalence induces the Pontrjagin-Thom collapse map on the level of π_0, and that the GMTW theorem is a generalization of the classical Pontrjagin-Thom theorem.

Lecture 21: Sheaves on Man

sec:21

In this lecture and the next we sketch some of the basic ideas which go into the proof of Theorem 20.42. The main references for the proof are the original papers [GMTW] and [MW]. These lectures are an introduction to those papers.

The statement to be proved is a weak homotopy equivalence of two spaces. The main idea is to realize each space as a *moduli space* in C^∞ geometry. Moduli spaces are fundamental throughout geometry. A simple example is to fix a vector space V and construct the parameter space of lines in V: the projective space $\mathbb{P}V$. We could then omit the fixed ambient space V and ask for the moduli space of all lines. To formulate that precisely, consider arbitrary smooth families of lines, parametrized by a "test" manifold M. The first step is to define a 'smooth family of lines' as a smooth line bundle $L \to M$. The collection $\mathcal{F}(M)$ of line bundles is then a contravariant function of M: given a smooth map $f\colon M \to M'$ there is a pullback $f^*\colon \mathcal{F}(M') \to \mathcal{F}(M)$ of line bundles. We seek a universal line bundle $\mathcal{L} \to |\mathcal{F}|$ over a topological space so that any line bundle is pulled back from this family. Of course, we have arrived back at the idea of a classifying space, as discussed in Lecture 6. In this lecture we take up nonlinear versions—families of curved manifolds—and construct the universal space $|\mathcal{F}|$ directly from the map $\mathcal{F}$. For this we isolate certain properties of $\mathcal{F}$: it is a *sheaf*.

We introduce sheaves of *sets* and sheaves of *categories*. The sheaves are functions of an arbitrary smooth manifold, not of open sets on a fixed manifold. The map $\mathcal{F}$ in the previous paragraph is not a sheaf as stated, but is a sheaf of sets if we define $\mathcal{F}(M)$ as the set of line bundles $L \to M$ equipped with an embedding into a vector bundle $M \times \mathcal{H} \to M$ with constant fiber $\mathcal{H}$. In our nonlinear examples we consider fiber bundles of manifolds equipped with an embedding into affine space, as in the definition of the topological bordism category (Definition 20.19). In this lecture we discuss some basics and the construction of a topological space $|\mathcal{F}|$ from a sheaf of sets $\mathcal{F}$. We also introduce sheaves of categories and their classifying spaces.

Presheaves and sheaves

subsec:21.1

(21.1) *Sheaves on a fixed manifold.* Let X be a smooth manifold. A *presheaf* on X assigns a set $\mathcal{F}(U)$ to each open set $U \subset X$ and a restriction map $i^*\colon \mathcal{F}(U') \to \mathcal{F}(U)$ to each inclusion $i\colon U \hookrightarrow U'$. Formally, then, define the category $\mathrm{Open}(X)$ whose objects are open subsets of X and whose morphisms are inclusions. A *presheaf* is a functor

eq:468

$$\mathcal{F}\colon \mathrm{Open}(X)^{\mathrm{op}} \longrightarrow \mathrm{Set}\,. \tag{21.2}$$

It is a sheaf if it satisfies a gluing condition, which we specify below in the context we need. A typical example is the structure sheaf $F(U) = C^\infty(U)$ of smooth functions. Other sorts of "functions"—differential forms, sections of a fixed vector bundle—also form sheaves over a fixed manifold.

subsec:21.2

(21.3) *The category of smooth manifolds.* The sheaves we introduce are defined on all manifolds, not just on open submanifolds of a fixed manifold.

thm:371 **Definition 21.4.** The category Man has as objects smooth finite dimensional manifolds without boundary and as morphisms smooth maps of manifolds.

This category is quite general, and there are examples of sheaves defined only on interesting subcategories.[43]

subsec:21.3

(21.5) *Presheaves on* Man*.* Any contravariant function of manifolds is a presheaf.

thm:372 **Definition 21.6.** A *presheaf on* Man is a functor

eq:469
$$\mathcal{F}\colon \mathrm{Man}^{\mathrm{op}} \longrightarrow \mathrm{Set}\,. \tag{21.7}$$

We give several examples.

thm:373 **Example 21.8.** Let $\mathcal{F}(M) = C^\infty(M)$ be the set of smooth functions. A smooth map $(f\colon M \to M')$ of manifolds induces a pullback $\mathcal{F}(f) = f^*\colon \mathcal{F}(M') \to \mathcal{F}(M)$ on functions.

thm:374 **Example 21.9.** For any $X \in \mathrm{Man}$ define $\mathcal{F}_X(M) = \mathrm{Man}(M, X)$. (Recall that $\mathrm{Man}(M, X)$ is the hom-set in the category Man, so here the set of smooth maps $M \to X$.) The sheaf $\mathcal{F}_X$ is the *representable sheaf* associated to the manifold X. Intuitively, we "test" $\mathcal{F}_X$ with the probe M.

There is a category of presheaves.

thm:375 **Definition 21.10.** Let $\mathcal{F}, \mathcal{F}'\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$ be presheaves. A *map*, or morphism, $\varphi\colon \mathcal{F} \to \mathcal{F}'$ of presheaves is a natural transformation of functors.

The construction in Example 21.9 embeds the category of manifolds in the category of presheaves, as expressed by the following general and simple result.

thm:376 **Lemma 21.11** (Yoneda)**.** *Let* $X \in \mathrm{Man}$ *and* $\mathcal{F}\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$*. Then there is a bijection*

eq:470
$$\mathrm{Map}(\mathcal{F}_X, \mathcal{F}) \xrightarrow{\;\cong\;} \mathcal{F}(X). \tag{21.12}$$

Proof. We construct maps in each direction and leave the reader to prove they are inverse. First, a natural transformation $\varphi \in \mathrm{Map}(\mathcal{F}_X, \mathcal{F})$ determines $\varphi(X)(\mathrm{id}_X) \in \mathcal{F}(X)$. (Note $\varphi(X)\colon \mathcal{F}_X(X) \to \mathcal{F}(X)$ and $\mathcal{F}_X(X) = \mathrm{Man}(X, X)$.) In the other direction, if $s \in \mathcal{F}(X)$ then define $\varphi \in \mathrm{Map}(\mathcal{F}_X, \mathcal{F})$ by $\varphi(M)(f) = \mathcal{F}(f)(s)$, where $(f\colon M \to X) \in \mathrm{Man}(M, X)$. ☐

We give two examples of non-representable presheaves.

---

[43] For example, the sheaf $\mathcal{F}(M) = \{\text{orientations of } M\}$ is defined on the subcategory where maps are required to be local diffeomorphisms. One can further restrict the manifolds to be of a fixed dimension, as for example required by the notion of a 'local field' in theoretical physics [F2].

thm:378 **Example 21.13.** Let $q \in \mathbb{Z}^{\geq 0}$. Then $\mathcal{F}^q(M) = \Omega^q(M)$ is a presheaf. It is not representable: there is no finite dimensional (or infinite dimensional) smooth manifold Ω^q such that differential q-forms on M correspond to maps $M \to \Omega^q$. But the presheaf $\mathcal{F}^q$ is a stand-in for such a mythical manifold. In that sense presheaves on Man are generalized manifolds. In that regard, an immediate consequence of Lemma 21.11 is $\mathrm{Map}(\mathcal{F}_X, \mathcal{F}^q) = \Omega^q(X)$.

thm:379 **Example 21.14.** Define $\mathcal{F}(M)$ to be the set of commutative diagrams

eq:471 (21.15)
$$\begin{tikzcd} Y \arrow[rr, hook] \arrow[dr, "\pi"'] & & M \times \mathbb{A}^\infty \arrow[dl, "\pi_1"] \\ & M & \end{tikzcd}$$

in which π is a proper submersion and the top arrow is the Cartesian product of π and an embedding.[44] So $\mathcal{F}(M)$ is the set of submanifolds of $M \times \mathbb{A}^\infty$ whose projection onto M is a proper submersion. These form a presheaf: morphisms map to pullbacks of subsets and compositions of morphisms map strictly to compositions. (The reader should contemplate what goes wrong with compositions without the embedding.)

thm:382 *Remark* 21.16. An important theorem of Charles Ehresmann asserts that a proper submersion is a fiber bundle.

subsec:21.4

(21.17) *The sheaf condition.* A sheaf is a presheaf which satisfies a gluing condition; there is no extra data.

thm:380 **Definition 21.18.** Let $\mathcal{F}\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$ be a presheaf. Then $\mathcal{F}$ is a *sheaf* if for every open cover $\{U_\alpha\}$ of a manifold M, the diagram

eq:472 (21.19)
$$\mathcal{F}(M) \longrightarrow \prod_{\alpha_0} \mathcal{F}(U_{\alpha_0}) \rightrightarrows \prod_{\alpha_0,\alpha_1} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1})$$

is an equalizer.

This means that if $s_{\alpha_0} \in \mathcal{F}(U_{\alpha_0})$ is a family of elements such that the two compositions in (21.19) agree, then there is a unique $s \in \mathcal{F}(M)$ which maps to $\{s_{\alpha_0}\}$. If we view $\mathcal{F}(U)$ as the space of "sections" of the presheaf $\mathcal{F}$ on the open set U, then the condition is that local coherent "sections" of the presheaf glue uniquely to a global section.

thm:381 *Remark* 21.20. An open cover expresses M as a colimit of the diagram

eq:473 (21.21)
$$\coprod_{\alpha_0,\alpha_1} U_{\alpha_0} \cap U_{\alpha_1} \rightrightarrows \coprod_{\alpha_0} U_{\alpha_0}.$$

The sheaf condition asserts that $\mathcal{F}(M)$ is the limit of $\mathcal{F}$ applied to (21.21).

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[44] We also ~~want to~~ add the condition that for any compact $K \subset M$ the embedding $\pi^{-1}(K) \hookrightarrow \mathbb{A}^\infty$ factors through a smooth embedding $\pi^{-1}(K) \hookrightarrow \mathbb{A}^m$ for some finite m. This condition applies to all similar subsequent examples.

subsec:21.5

(21.22) *Intuition.* We can often regard $\mathcal{F}(M)$ as a smooth family of elements of $\mathcal{F}(\mathrm{pt})$ parametrized by M. So for a representable sheaf $\mathcal{F}_X$ we have $\mathcal{F}(\mathrm{pt}) = X$ and $\mathcal{F}_X(M)$ is a smooth family of points of X parametrized by M. Similarly, for the sheaf in Example 21.14, $\mathcal{F}(\mathrm{pt})$ is the set of submanifolds of $\mathbb{A}^\infty$ and $\mathcal{F}(M)$ is a smooth family of such submanifolds. In other examples, e.g. Example 21.13, the intuition must be refined: for $q > 0$ there are no nonzero q-forms on a point. In this case an element of $\mathcal{F}(M)$ is a smooth coherent family of elements of $\mathcal{F}(U)$ for arbitrarily small open sets U. That is exactly what the sheaf condition asserts.

The representing space of a sheaf

subsec:21.6

(21.23) *Extended simplices.* Recall (18.4) that a nonempty finite ordered set S determines a simplex $\Sigma(S)$ whose vertex set is S. The simplex $\Sigma(S)$ is a subspace of the abstract affine space $\Sigma_e(S)$ spanned by S. Whereas $\Sigma(S)$ is not a smooth manifold—it is a manifold with corners—the affine space $\Sigma_e(S)$ is. So

eq:474

$$\Sigma_e\colon \Delta \longrightarrow \mathrm{Man} \tag{21.24}$$

is a functor whose image consists of affine spaces and (very special) affine maps.

subsec:21.7

(21.25) *The space attached to a sheaf on* Man. The following definition allows us to represent topological spaces by sheaves.

thm:383 **Definition 21.26.** Let $\mathcal{F}\,\mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$ be a sheaf. The *representing space* $|\mathcal{F}|$ is the geometric realization of the simplicial set

eq:475

$$\Delta^{\mathrm{op}} \xrightarrow{\Sigma_e^{\mathrm{op}}} \mathrm{Man}^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathrm{Set}\,. \tag{21.27}$$

For example, if $\mathcal{F} = \mathcal{F}_X$ is the representable sheaf attached to a smooth manifold X, then $S \mapsto \mathcal{F}\bigl(\Sigma_e(S)\bigr)$ is the (extended, smooth) singular simplicial set associated to X, a manifold analog of Example 18.17. The Milnor theorem quoted after (18.20) holds for extended smooth simplices.

thm:384 **Theorem 21.28** (Milnor)**.** *The canonical map $|\mathcal{F}_X| \to X$ is a weak homotopy equivalence.*

The canonical map is induced from the evaluation

eq:476

$$\mathcal{F}\bigl((\Sigma_e(S))\bigr) \times \Sigma(S) = \mathrm{Man}\bigl(\Sigma_e(S), X\bigr) \times \Sigma(S) \xrightarrow{\ \mathrm{ev}\ } X. \tag{21.29}$$

thm:385 **Example 21.30.** Fix a (separable) complex Hilbert space $\mathcal{H}$. Define a sheaf $\mathcal{F}$ by letting $\mathcal{F}(M)$ be the set of commutative diagrams

eq:477

$$\begin{tikzcd} L \arrow[rr, hook] \arrow[dr, "\pi"'] & & M \times \mathcal{H} \arrow[dl, "\pi_1"] \\ & M & \end{tikzcd} \tag{21.31}$$

in which π is a complex line bundle and the horizontal embedding composed with projection onto $\mathcal{H}$ is linear on each fiber of π. (So it is an embedding of the line bundle $L \to M$ into the bundle with constant fiber $\mathcal{H}$.) In this case we claim there is a natural map $|\mathcal{F}| \to \mathbb{P}\mathcal{H}$ which is a weak homotopy equivalence. In essence $\mathcal{F}(M)$ is the space of *smooth* maps $M \to \mathbb{P}\mathcal{H}$, where we introduce an appropriate infinite dimensional smooth structure on $\mathbb{P}\mathcal{H}$. (As a simple special case, for which we do not need the smooth structure, consider $\mathcal{F}(\mathrm{pt})$.) So while $\mathcal{F}$ is not representable in Man, it is in a larger category which includes infinite dimensional smooth manifolds, and then the proof of Theorem 21.28 applies. We do not attempt details here.

⇐

[**example of closed q-forms: degenerate simplices except in degree q labeled by $\mathbb{R}$, the integral over the usual q-simplex**]

`subsec:21.8`

(21.32) *Concordance.* We introduce an equivalence relation on sections of a sheaf. It is an adaptation of homotopy equivalence of functions to the sheaf world.

`thm:386` **Definition 21.33.** Let $\mathcal{F}\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$ be a sheaf, $M \in \mathrm{Man}$, and $s_0, s_1 \in \mathcal{F}(M)$. Then s_0 and s_1 are *concordant* if there exists $s \in \mathcal{F}(\mathbb{R} \times M)$ such that

`eq:478`
$$\begin{aligned} i_-^* s &= \pi_2^* s_0 \qquad \text{on } (-\infty, \epsilon) \times M, \\ i_+^* s &= \pi_2^* s_1 \qquad \text{on } (1-\epsilon, \infty) \times M \end{aligned} \tag{21.34}$$

for some $\epsilon > 0$.

The maps in (21.34) are the inclusions and projections

`eq:479`
$$\begin{tikzcd} M & (-\infty,\epsilon)\times M \arrow[l, "\pi_2"'] \arrow[r, "i_-"] & \mathbb{R}\times M & (1-\epsilon,\infty)\times M \arrow[l, "i_+"'] \arrow[r, "\pi_2"] & M \end{tikzcd} \tag{21.35}$$

This is just a smooth version of a homotopy, which would normally be expressed on the manifold-with-boundary $[0,1] \times M$, which is not in the category Man. See Figure 39.

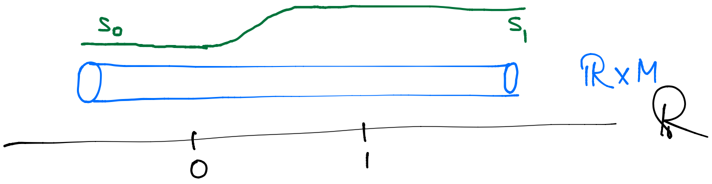

FIGURE 39. A concordance `fig:39`

Concordance is an equivalence relation. We denote the set of concordance classes of elements of $\mathcal{F}(M)$ as $\mathcal{F}[M]$. The map $M \mapsto \mathcal{F}[M]$ is not usually a sheaf: equivalence classes do not glue.

`thm:387` **Example 21.36.** For the sheaf $\mathcal{F}$ of Example 21.30 the set $\mathcal{F}[M]$ is the set of equivalence classes of complex line bundles $L \to M$. For the standard cover $\{S^2 \setminus \{p_1\}, S^2 \setminus \{p_2\}\}$ of $M = S^2$ by two open sets, the diagram (21.19) fails to be an equalizer.

subsec:21.9 **(21.37)** *The meaning of the representing space.* The representation space represents concordance classes.

thm:388 **Theorem 21.38** ([MW]). *There is a bijection*

eq:480
$$\mathcal{F}[M] \longrightarrow \bigl[M, |\mathcal{F}|\bigr], \tag{21.39}$$

where the codomain is the set of homotopy classes of continuous maps $M \to |\mathcal{F}|$.

Sketch of proof. A detailed proof may be found in [MW, Appendix]. We content ourselves here with describing the map (21.39), which is formal, and its inverse, which is less formal. We use the Yoneda Lemma 21.11, the Milnor Theorem 21.28, and the fact that the representing space $|-|$ is a functor to construct (21.39) as the composition

eq:481
$$\mathcal{F}(M) \cong \mathrm{Map}(\mathcal{F}_M, \mathcal{F}) \xrightarrow{\;|-|\;} \bigl[|\mathcal{F}_M|, \mathcal{F}|\bigr] \xleftarrow{\;\cong\;} \bigl[M, |\mathcal{F}|\bigr]. \tag{21.40}$$

To see that this passes to concordance classes, note that a concordance is a map $\mathcal{F}_{\mathbb{R}\times M} \to \mathcal{F}$, by Yoneda, and so induces[45]

eq:482
$$|\mathcal{F}_{\mathbb{R}\times M}| \simeq |\mathcal{F}_{\mathbb{R}}| \times |\mathcal{F}_M| \simeq \mathbb{R} \times |\mathcal{F}_M| \longrightarrow |\mathcal{F}|, \tag{21.41}$$

a homotopy of maps $M \to |\mathcal{F}|$.

The inverse construction is a bit more intricate. One begins with a map $g\colon M \to |\mathcal{F}|$, a representative of a homotopy class, and then must construct an element of $\mathcal{F}(M)$. This is accomplished using the sheaf property, which allows to construct a coherent family of elements of $\mathcal{F}(U)$ for a covering of M by open sets U. The first step is a simplicial approximation theorem, which realizes g up to homotopy as the geometric realization of a map $g'\colon sC \to s\mathcal{F}$ of simplicial sets, where sC is the simplicial set associated to an ordered simplicial complex C together with a homeomorphism $|C| \to M$—in fact, a smooth triangulation of M—and $s\mathcal{F}$ is the simplicial set (21.27). The second step is to construct a vector field on M from the triangulation C, a vector field which pushes towards lower dimensional simplices. This induces a map $h\colon M \to M$ homotopic to the identity such that each simplex Δ in C has an open neighborhood U_Δ which retracts onto Δ under h. Then $h^*g'(\Delta) \in \mathcal{F}(U_\Delta)$ is a coherent family of elements, so glues to the desired element of $\mathcal{F}(M)$, whose concordance class is independent of the choices. We refer to [MW, Appendix] for details. ☐

thm:389 **Example 21.42.** The application of Theorem 21.38 to Example 21.30 produces the theorem that $\mathbb{P}\mathcal{H}$ classifies equivalence classes of line bundles over a smooth manifold M, something we discussed in Lecture 6.

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[45] We assume the geometric realization commutes with products; see Remark 19.31.

Sheaves of categories

Let Cat be the category whose objects are categories $C_\bullet = (C_0, C_1)$ and whose morphisms are functors. We use the formulation **(13.7)** of categories as pairs of sets with various structure maps. A functor is a pair of maps (one on objects, one on morphisms), and composition of functors is associative on the nose.[46]

thm:390 **Definition 21.43.** A *sheaf of categories* $\mathcal{F}_\bullet\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Cat}$ is a pair of set-valued functors $(\mathcal{F}_0, \mathcal{F}_1)\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$ togther with structure maps (13.8) which satisfy the defining relations of a category.

So $\mathcal{F}_0$ and $\mathcal{F}_1$ separately satisfy the sheaf condition. For any test manifold M the category $\mathcal{F}_\bullet(M)$ is discrete: $\mathcal{F}_0(M)$ and $\mathcal{F}_1(M)$ are sets.

thm:391 **Definition 21.44.** Let $\mathcal{F}_\bullet\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Cat}$ be a sheaf of categories. The *representing category* is the topological category

eq:483
$$|\mathcal{F}_\bullet| = \bigl(|\mathcal{F}_0|\,,\,|\mathcal{F}_1|\bigr). \tag{21.45}$$

⇐

[**example of sheaf of double covers–including embeddings**]

A topological category has a classifying space, so there is a space $B|\mathcal{F}_\bullet|$ associated to a sheaf $\mathcal{F}_\bullet$ of categories. One of the constructions used in the proof, which we will not recount here, is a sheaf $\beta(\mathcal{F}_\bullet)$ of sets associated to a sheaf $\mathcal{F}_\bullet$ of categories with the property

eq:484
$$|\beta(\mathcal{F}_\bullet)| \simeq B|\mathcal{F}_\bullet|. \tag{21.46}$$

See [GMTW, §2.4], [MW, §4.1] for the construction of the *cocycle sheaf.*

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[46] There is a "weaker" notion involving natural transformations on functors: categories are objects of a 2-category. We will discuss higher categories, at least heuristically, in the last two lectures.

Lecture 22: Remarks on the proof of GMTW

sec:22

Recall that the GMTW Theorem 20.42 asserts the existence of a weak homotopy equivalence

eq:485

$$B\bigl({}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}\bigr) \simeq \bigl(\Sigma MT\mathcal{X}(n)\bigr)_0. \tag{22.1}$$

The left hand side is the classifying space of the topological bordism category whose morphisms are compact n-manifolds with $\mathcal{X}(n)$-structure. The right hand side is the 0-space of the suspension of the Madsen-Tillmann spectrum. Both of these pointed spaces were defined in Lecture 20, where we showed that the classical Pontrjagin-Thom theorem is the weak homotopy equivalence (22.1) composed with π_0. Indeed, the ideas of classical Pontrjagin-Thom theory are integral to the proof.

Rather than attempt a direct map between the spaces (22.1), the proof proceeds by constructing sheaves which represent these spaces. More precisely, there is a sheaf of sets $D = D_n^{\mathcal{X}(n)}$ on Man whose representing space is $\bigl(\Sigma MT\mathcal{X}(n)\bigr)_0$. The Pontrjagin-Thom theory, as well as Phillips' Submersion Theorem [Ph] is used to prove this representing statement. The value $D(M)$ of the sheaf on a test manifold M is a set of submersions over M. Intuitively, it is a set of fiber bundles of compact $(n-1)$-manifolds, but because the Phillips theorem only applies to noncompact manifolds there is a necessary modification. We explain the heuristic idea in the first section below, and then give the technically correct rendition, though not a complete proof. The space on the left hand side of (22.1) is the classifying space of a topological category, and it is fairly easy to construct a sheaf of categories $C = C_n^{\mathcal{X}(n)}$ on Man which represents this topological category (in the sense of Definition 21.44). Its value on a test manifold M is a category whose objects are fiber bundles over M with fibers closed $(n-1)$-manifolds. The remainder of the proof goes through auxiliary sheaves (of categories) which mediate between C and D. We content ourselves with an overview and refer to the reader to the original papers [GMTW, MW] for a full account.

The main construction: heuristic version

As mentioned in the introduction, this section is a useful false start.

subsec:22.1

(22.2) *A sheaf of $(n-1)$-manifolds.* Fix a positive integer n and an $\mathcal{X}(n)$-structure $\mathcal{X}(n) \to Gr_n(\mathbb{R}^\infty)$. We elaborate on Example 21.14. Let $E\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$ be the sheaf whose value on a test manifold M is a pair of maps

eq:486

$$\begin{tikzcd} Y \arrow[rr, hook] \arrow[dr, "\pi"'] & & M \times \mathbb{A}^\infty \arrow[dl, "\pi_1"] \\ & M & \end{tikzcd} \tag{22.3}$$

in which π is a fiber bundle with fibers closed $(n-1)$-manifolds and the top arrow is an embedding. For simplicity we do not include a tangential structure. Assume for simplicity that M is compact. Then for some $m > 0$ the embedding factors through an embedding into $\mathbb{A}^m$:

eq:488

$$\begin{tikzcd} Y \arrow[rr, hook] \arrow[dr, "\pi"'] & & M \times \mathbb{A}^m \arrow[dl, "\pi_1"] \\ & M & \end{tikzcd} \tag{22.4}$$

subsec:22.2

(22.5) *Relative Gauss map.* The *relative tangent bundle* $T(Y/M) \to Y$ is the kernel of the differential of π, the set of tangent vectors which point along the fibers of π. The embedding gives a Gauss map (see (**20.23**))

eq:487

$$\begin{tikzcd} T(Y/M) \arrow[r] \arrow[d] & S(n-1) \arrow[d] \\ Y \arrow[r] & Gr_{n-1}(\mathbb{R}^m) \end{tikzcd} \tag{22.6}$$

We emphasize: A fiber bundle, or proper submersion, has a tangent bundle along the fibers, which is identified with the pullback of the universal subbundle $S(n-1) \to Gr_{n-1}(\mathbb{R}^m)$.

The normal bundle $\nu \to Y$ to the embedding in (22.4) is also the normal bundle to the embedding of each fiber of π in $\mathbb{A}^m$, since π is a submersion, and the embedding induces a classifying map

eq:489

$$\begin{tikzcd} \nu \arrow[r] \arrow[d] & Q(m-n+1) \arrow[d] \\ Y \arrow[r] & Gr_{n-1}(\mathbb{R}^m) \end{tikzcd} \tag{22.7}$$

subsec:22.3

(22.8) *Pontrjagin-Thom collapse.* As in Lecture 2 and Lecture 10, choose a tubular neighborhood of $Y \subset M \times \mathbb{A}^m$. Then the Pontrjagin-Thom collapse induced by the embedding, followed by the map on Thom spaces induced from (22.7), is

eq:490

$$M_+ \wedge S^m \longrightarrow Y^\nu \longrightarrow Gr_{n-1}(\mathbb{R}^m)^{Q(m-n+1)}. \tag{22.9}$$

Here M_+ is the union of M and a disjoint basepoint, and the domain is the one-point compactification of $M \times \mathbb{A}^m$. According to Definition 20.31 the last space in (22.9) is the m^{th} space of the Madsen-Tillmann spectrum $MTO(n-1)$. So (22.9) is a pointed map of the m^{h} suspension of M_+ into the m^{th} space of the prespectrum which completes to the spectrum $MTO(n-1)$. Therefore, it represents a map of M into $MTO(n-1)_0$.

In summary, from a fiber bundle (22.4) of $(n-1)$-manifolds with embedding we have produced a map of the base into the 0-space of the spectrum $MTO(n-1)$.

subsec:22.4

(22.10) *An attempted inverse.* Conversely, a map $M \to MTO(n-1)_0$ is represented, for sufficiently large m, by a pointed map

eq:491

$$g\colon M_+ \wedge S^m \longrightarrow Gr_{n-1}(\mathbb{R}^m)^{Q(m-n+1)}. \tag{22.11}$$

After a homotopy we can arrange that g be transverse to the zero section of the bundle $Q(m-n+1) \to Gr_{n-1}(\mathbb{R}^m)$. Then the inverse image of the zero section is a submanifold $Y \subset M \times \mathbb{A}^m$ with $\dim Y - \dim M = n-1$. If we assume that M is compact, which we do, then Y is also compact. There is also a classifying map (22.7) of the normal bundle, the restriction of (22.11) to a map $Y \to Gr_{n-1}$. Let $V \to Y$ be the pullback of $S(n-1) \to Gr_{n-1}(\mathbb{R}^m)$.

If—and this is not generally true—the composition $Y \hookrightarrow M \times \mathbb{A}^m \to M$ is a submersion, then since Y is compact it is a fiber bundle. We would deduce that maps into the Madsen-Tillmann spectrum give fiber bundles. But that is not true. Nor is it true, even if the composition is a submersion, that $V \to Y$ can be identified with the relative tangent bundle.

The main construction: real version

The main tool to obtain a submersion is the Phillips Submersion Theorem. It is part of a circle of ideas in differential topology called *immersion theory* [Sp], and one of the main tools used in the proofs is Gromov's *h-principle* [ElMi]. We simply quote the result here.

thm:392

Theorem 22.12 (Phillips [Ph])**.** *Let X be a smooth manifold with no closed components and M a smooth manifold with $\dim M \le \dim X$. Then the differential*

eq:492

$$\mathrm{Submersion}(X,M) \xrightarrow{\ d\ } \mathrm{Epi}(TX,TM) \tag{22.13}$$

is a weak homotopy equivalence.

Here $\mathrm{Epi}(TX,TM)$ is the space of smooth maps $L\colon TX \to TM$ which sends fibers to fibers and restricts on each fiber to a surjective linear map (epimorphism). The domain of (22.13) is the space of submersions $X \to M$, and the differential maps a submersion to an epimorphism on tangent bundles. Note that a manifold with no closed components is often called an *open manifold*.

Theorem 22.12 is precisely the tool needed to deform the map $Y \to M$ in **(22.10)** to a submersion. But to do so we must replace Y be a noncompact manifold. The simplest choice is $X = \mathbb{R} \times Y$. We indicate the modifications to the previous heuristic section which incorporate this change.

subsec:22.5

(22.14) *The sheaf D.* We introduce a sheaf $D = D_n^{\mathcal{X}(n)}\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$ which represents $\bigl(\Sigma MT\mathcal{X}(n)\bigr)_0$.

thm:393

Definition 22.15. Fix $M \in \mathrm{Man}$. An element of $D(M)$ is a pair (X,θ) consisting of a submanifold $X \subset M \times \mathbb{R} \times \mathbb{A}^\infty$ and an $\mathcal{X}(n)$-structure θ. The submanifold must satisfy

(i) $\pi_1\colon X \to M$ is a submersion with fibers of dimension n, and
(ii) $\pi_1 \times \pi_2\colon X \to M \times \mathbb{R}$ is proper.

The relative tangent bundle $T(X/M) \to X$ has rank n and, because of the embedding, comes equipped with a Gauss map

eq:493 (22.16)
$$\begin{tikzcd} T(X/M) \arrow[r] \arrow[d] & S(n) \arrow[d] \\ X \arrow[r] & Gr_n(\mathbb{R}\times\mathbb{R}^\infty) \end{tikzcd}$$

The tangential structure is a fibration $\mathcal{X}(n) \to Gr_n(\mathbb{R}\times\mathbb{R}^\infty)$, fixed once and for all. The relative $\mathcal{X}(n)$-structure θ is a lift

eq:494 (22.17)
$$\begin{tikzcd} T(X/M) \arrow[r, "\theta"] \arrow[d] & S(n) \arrow[d] \\ X \arrow[r] & \mathcal{X}(n) \end{tikzcd}$$

of the Gauss map, as in (20.25).

We remark that the embedding $X \hookrightarrow M\times\mathbb{R}\times\mathbb{A}^\infty$ is required to satisfy the technical condition in the footnote of Example 21.14. For this exposition we restrict to M compact.

The first condition in Definition 22.15 implies that π_1 is a family of n-manifolds, but it is not a fiber bundle as the fibers may be noncompact, so as we move over the base M the topology of the fibers can change. The second condition implies that each fiber comes with a real-valued function π_2 with compact fibers. The inverse image of a regular value $a \in \mathbb{R}$ is a closed $(n-1)$-manifold, but the topology depends on the regular value. The inverse images of two regular values $a_0 < a_1$ comes with a bordism: the inverse image of $[a_0, a_1]$. See Figure 40.

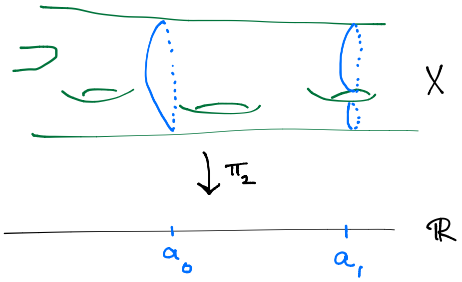

FIGURE 40. A fiber of $X \to M$ fig:40

subsec:22.6

(22.18) *Statement of theorems.* Recall the notion of concordance **(21.32)**.

thm:394 **Theorem 22.19.** *For any $M \in$ Man there is a bijection*

eq:495 (22.20)
$$D[M] \cong \bigl[M\,,\,(\Sigma MT\mathcal{X}(n))_0\bigr]$$

between concordance classes of elements of $D(M)$ and homotopy classes of maps into the 0-space of the suspended Madsen-Tillmann spectrum.

We sketch the construction of the bijection (22.20) in the remainder of this section.

Recall from Theorem 21.38 that the representing space $|D|$ also satisfies

eq:496

$$D[M] \cong \bigl[M\,,\,|D|\bigr], \tag{22.21}$$

and so the following is not surprising.

thm:395 **Corollary 22.22.** *There is a weak homotopy equivalence*

eq:497

$$|D| \simeq \bigl(\Sigma MT\mathcal{X}(n)\bigr)_0. \tag{22.23}$$

The proof uses an auxiliary sheaves which keep track of the contractible choices (of a tubular neighborhood, of a regular value) which are used below. We refer to [Po, §4.3] for a sketch of how that argument goes.

subsec:22.7

(22.24) *Sketch of* (22.20) $\longrightarrow$. Given an element $X \subset M \times \mathbb{R} \times \mathbb{A}^\infty$ of $D(M)$, choose $a \in \mathbb{R}$ a regular value of π_2, m a positive integer such that $X \subset M \times \mathbb{R} \times \mathbb{A}^m$. Let $Y \subset M \times \mathbb{A}^m$ be the intersection of X and $M \times \{a\} \times \mathbb{A}^m$. The normal bundle to $Y \subset M \times \mathbb{A}^m$ is the restriction of the normal bundle to $X \subset M \times \mathbb{R} \times \mathbb{A}^m$. Therefore, as in (22.7) but now using the lift (22.17) from the $\mathcal{X}(n)$-structure, we obtain a classifying map

eq:498

$$\begin{tikzcd} \nu \arrow[r] \arrow[d] & Q(m-n+1) \arrow[d] \\ Y \arrow[r] & \mathcal{X}(n,m+1) \end{tikzcd} \tag{22.25}$$

where the southeast space is the pullback

eq:499

$$\begin{tikzcd} \mathcal{X}(n,m+1) \arrow[r] \arrow[d] & \mathcal{X}(n) \arrow[d] \\ Gr_n(\mathbb{R} \times \mathbb{R}^m) \arrow[r] & Gr_n(\mathbb{R} \times \mathbb{R}^\infty) \end{tikzcd} \tag{22.26}$$

Choose a tubular neighborhood of $Y \subset M \times \mathbb{A}^m$. The Pontrjagin-Thom collapse, as in (22.9), is a map

eq:500

$$M_+ \wedge S^m \longrightarrow \mathcal{X}(n,m+1)^{Q(m-n+1)}. \tag{22.27}$$

The last space is the $(m+1)$-space of the Madsen-Tillmann spectrum $MT\mathcal{X}(n)$, so that (22.27) represents a (non-pointed) map of M into the 0-space of the shifted spectrum $\Sigma MT\mathcal{X}(n)$.

subsec:22.8 **(22.28)** *Sketch of* (22.20) $\longleftarrow$. We proceed as in **(22.10)**, but now mapping into the last space in (22.27), to obtain as there

(i) a submanifold $Y \subset M \times \mathbb{A}^m$ with $\dim Y - \dim M = n - 1$,
(ii) a classifying map of the normal bundle

eq:501 (22.29)
$$\begin{tikzcd} \nu \arrow[r] \arrow[d] & Q(m-n+1) \arrow[d] \\ Y \arrow[r, "g"] & \mathcal{X}(n, m+1) \end{tikzcd}$$

and

(iii) a rank n vector bundle $W \to Y$, defined as $g^*\bigl(S(n) \to \mathcal{X}(n, m+1)\bigr)$.

These bundles over Y come equipped with isomorphisms

eq:502 (22.30)
$$\begin{aligned} \nu \oplus W &\xrightarrow{\cong} \underline{\mathbb{R}^{m+1}} \\ \nu \oplus TY &\xrightarrow{\cong} TM \oplus \underline{\mathbb{R}^m} \end{aligned}$$

The first is induced from the tautological exact sequence (6.9) after choosing once and for all a splitting over $\mathcal{X}(n, m+1)$. The second comes from splitting the usual exact sequence for a submanifold. Combining these isomorphisms we obtain an isomorphism

eq:503 (22.31)
$$\underline{\mathbb{R}^{m+1}} \oplus TY \xrightarrow{\cong} W \oplus \nu \oplus TY \xrightarrow{\cong} W \oplus TM \oplus \underline{\mathbb{R}^m}.$$

The next step is to "strip off" the trivial bundle of rank m in the isomorphism (22.31). This is possible for m sufficiently large. The proof is an application of the following general principle, which can be proved by obstruction theory. Recall that for $k \in \mathbb{Z}^{\geq 0}$ a space is *k-connected* if it is connected and all homotopy groups π_q, $q \leq k$ vanish. A map is k-connected if its mapping cylinder is k-connected.

thm:396 **Proposition 22.32.**

(i) *Let $E \to Y$ be a (continuous) fiber bundle with k-connected fiber and base Y a CW complex of dimension ℓ. Then the space $\Gamma(Y; E)$ of sections is $(k - \ell)$-connected.*
(ii) *Let $(E_1 \to E_2) \longrightarrow Y$ be a map of fiber bundles. Assume the map on each fiber is k-connected and $\dim Y = \ell$. Then the induced map of sections $\Gamma(Y; E_1) \to \Gamma(Y; E_2)$ is $(k - \ell)$-connected.*

Our application is to the map

eq:504 (22.33)
$$\mathrm{Iso}\bigl(\underline{\mathbb{R}} \oplus TY\, ,\, W \oplus TM\bigr) \longrightarrow \mathrm{Iso}\bigl(\underline{\mathbb{R}^{m+1}} \oplus TY\, ,\, W \oplus TM \oplus \underline{\mathbb{R}^m})\bigr)$$

of fiber bundles of isomorphisms of vector bundles over Y. On fibers this is the standard embedding of general linear groups $GL_{n+d}(\mathbb{R}) \hookrightarrow GL_{n+d+m}$, where $d = \dim M$. This map is $(n + d - 1)$-connected, so the induced map on sections is $(n - 1)$-connected. Since $n - 1 > 0$, this implies that the isomorphism (22.31) is isotopic to an isomorphism which is the stabilization of an isomorphism

eq:505 (22.34)
$$\underline{\mathbb{R}}\oplus TY\xrightarrow{\;\cong\;}W\oplus TM.$$

Now compose the isomorphism (22.34) with projection onto TM to obtain an epimorphism

eq:506 (22.35)
$$T(\mathbb{R}\times Y)\cong\underline{\mathbb{R}}\oplus TY\xrightarrow{\;\cong\;}TM.$$

The Phillips Submersion Theorem 22.12 implies that there is a submersion $\pi_1\colon X=\mathbb{R}\times Y\to M$ whose differential is isotopic to (22.35). The isomorphism (22.34) induces an isomorphism

eq:507 (22.36)
$$W\xrightarrow{\;\cong\;}T(X/M)=\ker d\pi_1.$$

Projection gives a function $\pi_2\colon X\to\mathbb{R}$, and we can use the Whitney embedding theorem to construct $\pi_3\colon X\hookrightarrow\mathbb{A}^{m'}$ for m' sufficiently large. The product $\pi_1\times\pi_2\times\pi_3\colon X\hookrightarrow M\times\mathbb{R}\times\mathbb{A}^{m'}$ is the desired element of $D(M)$. (A more delicate argument produces the $\mathcal{X}(n)$-structure.)

thm:397 *Remark* 22.37. This completes the sketch construction of the two maps in (22.20). The proof that they are inverse is based on [MW, Lemma 2.5.2].

A sheaf model of the topological bordism category

It is fairly straightforward to construct a sheaf of (discrete) categories whose representing space is the topological bordism category ${}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$. This is a more elaborate version of Example 21.30, where there is a "representing category in the category of smooth infinite dimensional manifolds".

thm:398 **Definition 22.38.** The sheaf of categories $C=C_n^{\mathcal{X}(n)}\colon\mathrm{Man}^{\mathrm{op}}\to\mathrm{Cat}$ is defined on a test manifold $M\in\mathrm{Man}$ as follows. The objects of $C(M)$ are triples (a,Y,θ) consisting of a smooth function $a\colon M\to\mathbb{R}$, an embedding $Y\hookrightarrow M\times\mathbb{A}^\infty$ such that $\pi_1\colon Y\to M$ is a proper submersion, and an $\mathcal{X}(n)$-structure θ on the relative tangent bundle. A morphism $(a_0,Y_0,\theta_0)\to(a_1,Y_1,\theta_1)$ is a pair (X,Θ) consisting of a neat submanifold $X\hookrightarrow M\times[a_0,a_1]\times\mathbb{A}^\infty$ with $\mathcal{X}(n)$-structure Θ such that $\pi_1\colon X\to M$ is a proper submersion and, for some $\delta\colon M\to\mathbb{R}^{>0}$

eq:508 (22.39)
$$\begin{aligned}X\cap\bigl(M\times[a_0,a_0+\delta)\times\mathbb{A}^\infty\bigr)&=Y_0\times[a_0,a_0+\delta)\\X\cap\bigl(M\times(a_1-\delta,a_1]\times\mathbb{A}^\infty\bigr)&=Y_1\times(a_1-\delta,a_1]\end{aligned}$$

as $\mathcal{X}(n)$-manifolds.

Here $M\times[a_0,a_1]\subset M\times\mathbb{R}$ is the subset of pairs (m,t) such that $a_0(m)\le t\le a_1(t)$. Composition is by union, as usual when we have embeddings.

As indicated, $C(M)$ is the space of smooth maps $M\to{}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$, with the appropriate smooth structure on ${}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$, and this gives a map of topological categories

eq:509 (22.40)
$$\eta\colon|C|\longrightarrow{}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}\,.$$

We have not defined a weak equivalence of topological categories, which is what we'd like to say (22.40) is, but in any case the definition would amount to the following.

thm:399 **Theorem 22.41.** *η induces a weak homotopy equivalence of classifying spaces*

eq:510 (22.42)
$$B|C| \longrightarrow B\bigl({}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}\bigr).$$

Sketch of proof. The space of q-simplices $N_q|C|$ in the nerve of $|C|$ is the geometric realization of the extended, smooth singular simplices on the space $N_q{}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$, so by Theorem 21.28 the map

eq:511 (22.43)
$$N_q|C| \longrightarrow N_q{}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}$$

induced by η is a weak homotopy equivalence. Thus $B\eta$ is also a weak homotopy equivalence. ☐

Comments on the rest

We hope to have given a "reader's guide" to much of the proof in [GMTW]. At this point we have a sheaf of categories C which represents the space $B\bigl({}^t\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}\bigr)$ and a sheaf of spaces D which represents the space $\bigl(\Sigma MT\mathcal{X}(n)\bigr)_0$. To put them on equal footing we regard D as a sheaf of categories with only identity morphisms. The goal now is to construct a weak homotopy equivalence of these sheaves. This is not done directly, but by means of two intermediate sheaves of categories. There are two main discrepancies between $C(\mathrm{pt})$ and $D(\mathrm{pt})$, and so between $C(M)$ and $D(M)$ which are parametrized families. First, objects in D are to be thought of as fibers at an unspecified regular value $a \in \mathbb{R}$ of a proper map $X \to \mathbb{R}$, whereas objects in C have a specified value of a. Second, morphisms in D are manifolds without boundary ($X \to \mathbb{R}$) whereas morphisms in C are manifolds with boundary. The intermediate sheaves $D^{\pitchfork}$, $C^{\pitchfork}$ mediate these discrepancies. We sketch the definitions below. The main work is in proving that straightforwardly defined maps

eq:512 (22.44)
$$D \xleftarrow{\ \alpha\ } D^{\pitchfork} \xrightarrow{\ \gamma\ } C^{\pitchfork} \xleftarrow{\ \delta\ } C$$

are weak homotopy equivalences.

subsec:22.9

(22.45) *The sheaf $D^{\pitchfork}$ and the map α.* For convenience we omit the $\mathcal{X}(n)$-structures from the notation: they are just carried along.

The objects are a subsheaf of $\mathcal{F}_{\mathbb{R}} \times D$ where $\mathcal{F}_{\mathbb{R}}$ is the representable sheaf of real-valued functions (see Example 21.9). An object in $D^{\pitchfork}(M)$ is a pair (a, X) where $X \subset M \times \mathbb{R} \times \mathbb{A}^\infty$ is an object of $D(M)$ and $a\colon M \to \mathbb{R}$ has the property that $a(m)$ is a regular value of $\pi_2\big|_{\pi_1^{-1}(M)}$. It is a category of partially ordered sets: there is a unique morphisms $(a_0, X_0) \to (a_1, X_1)$ if $(a_0, X_0) \le (a_1, X_1)$, and the latter is true if and only if $X_0 = X_1$, the functions satisfy $a_0 \le a_1$ and $a_0 = a_1$ on a union of components of M.

The map α is the forgetful map which forgets a.

subsec:22.10

(22.46) *The sheaf $C^{\pitchfork}$ and the maps δ, γ.* This is very similar to C, but the objects and morphisms are not "sharply cut off" at points $a \in \mathbb{R}$. So objects have a bicollaring and morphisms are open with collars at a_0, a_1. We refer to [GMTW, §2] for details.

The map δ puts product bicollars and collars on the objects and morphisms of C.

To define $\gamma(a, X)$ we use the fact that a consists of regular values to find a function $\epsilon\colon M \to \mathbb{R}^{>0}$ so that $(a-\epsilon, a+\epsilon)$ also consists of regular values. (The notation is as in Definition 22.38.) Then $Y = (\pi_1 \times \pi_2)^{-1}\bigl(M \times (a-\epsilon, a+\epsilon)\bigr)$ is an object of $C^{\pitchfork}$. There is a similar construction on morphisms.

subsec:22.11

(22.47) *Proofs of equivalences.* The techniques to prove that the maps α, γ, δ are weak equivalences are presented in [GMTW] with technical details in [MW].

Lecture 23: An application of Morse-Cerf theory

sec:23

We review quickly the idea of a Morse function and recall the basic theorems of Morse theory. Passing through a single critical point gives an *elementary bordism*; a very nice Morse function—an *excellent* function—decomposes an arbitrary bordism as a sequence of elementary bordisms. The space of excellent functions is not connected, but is if we relax the excellence standard slightly. This basic idea of Cerf theory relates different decompositions. We use it to classify 2-dimensional oriented TQFTs with values in the category of vector spaces. This is one of the earliest theorems in the subject, dating at least from Dijkgraaf's thesis [Dij].

Morse functions

subsec:23.1

(23.1) *Critical points and the hessian.* Let M be a smooth manifold and $f\colon M \to \mathbb{R}$ a smooth function. Recall that $p \in M$ is a *critical point* if $df_p = 0$. A number $c \in \mathbb{R}$ is a *critical value* if $f^{-1}(c)$ contains a critical point. At a critical point p the second differential, or Hessian,

eq:513

$$d^2f_p\colon T_pM \times T_pM \longrightarrow \mathbb{R} \tag{23.2}$$

is a well-defined symmetric bilinear form. To evaluate it on $\xi_1, \xi_2 \in T_pM$ extend ξ_2 to a vector field to near p, and set $d^2f_p(\xi_1,\xi_2) = \xi_1\xi_2 f(p)$, the iterated directional derivative. We say p is a *nondegenerate critical point* if the Hessian (23.2) is a nondegenerate symmetric bilinear form.

thm:400

Lemma 23.3 (Morse)**.** *If p is a nondegenerate critical point of the function $f\colon M \to \mathbb{R}$, then there exists a local coordinate system $x^1, \dots, x^n$ about p such that*

eq:514

$$f = (x^1)^2 + \cdots + (x^r)^2 - (x^{r+1})^2 - \cdots - (x^n)^2 + c \tag{23.4}$$

for some p.

The number $n - r$ of minus signs in (23.4) is the *index* of the critical point p.

An application of Sard's theorem proves that Morse functions exist, and in fact are open and dense in the space of C^∞ functions (in the Whitney topology **(20.1)**).

subsec:23.2

(23.5) *Morse functions on bordisms.* If X is a manifold with boundary we consider smooth functions which are constant on ∂X and have no critical points on ∂X. The following terminology is apparently due to Thom.

thm:401

Definition 23.6. Let $X\colon Y_0 \to Y_1$ be a bordism. An *excellent* function $f\colon X \to \mathbb{R}$ satisfies

(i) $f(Y_0) = a_0$ is constant;

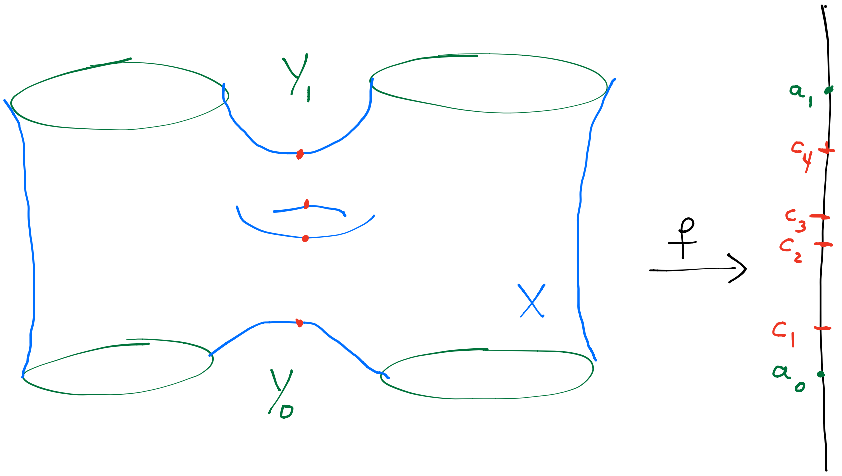

FIGURE 41. An excellent function on a bordism fig:41

(ii) $f(Y_1) = a_1$ is constant; and

(iii) The critical points $x_1, \dots, x_N$ have distinct critical values $c_1, \dots, c_N$ which satisfy

eq:515 (23.7)
$$a_0 < c_1 < \cdots < c_N < a_1.$$

We depict an excellent function on a bordism in Figure 41.

thm:402 **Proposition 23.8.** *Let $X\colon Y_0 \to Y_1$ be a bordism. Then the space of excellent functions on X is open and dense.*

subsec:23.3

(23.9) *Passing a critical level.* The basic theorems of Morse theory tell the structure of $X_{a',a''} = f^{-1}([a',a''])$ if a', a'' are regular values. If there are no critical values in $[a',a'']$, then $X_{a',a''}$ is diffeomorphic to the Cartesian product of $[a',a'']$ and $Y = f^{-1}(a)$ for any $a \in [a',a'']$. If there is a single critical value $c \in [a',a'']$ and $f^{-1}(c)$ contains a single critical point of index q, then $X_{a',a''}$ is obtained from $X_{a',c-\epsilon}$ by attaching an *n-dimensional q-handle*. We defer to standard books [M4, PT] for a detailed treatment of Morse theory.

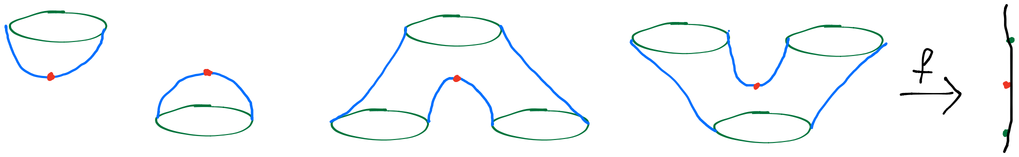

FIGURE 42. Some elementary 2-dimensional bordisms fig:42

thm:403 **Definition 23.10.** A bordism $X\colon Y_0 \to Y_1$ is an *elementary bordism* if it admits an excellent function with a single critical point.

The elementary 2-dimensional bordisms are depicted in Figure 42.

subsec:23.4

(23.11) *Decomposition into elementary bordisms.* An excellent function on *any* bordism $X\colon Y_0 \to Y_1$ expresses it as a composition of elementary bordisms

eq:516

$$X = X_N \circ \cdots \circ X_1 \tag{23.12}$$

where $X_1 = f^{-1}([a_0, c_1 + \epsilon])$, $X_2 = f^{-1}([c_1 + \epsilon, c_2 + \epsilon])$, …, $X_N = f^{-1}([c_{N-1} + \epsilon, a_1])$. Excellent functions connected by a path of excellent functions lead to an equivalent decomposition: corresponding elementary bordisms are diffeomorphic. We can track the equivalence class by a *Kirby graphic* (Figure 43) which indicates the distribution of critical points and their indices. The space of excellent functions is not connected; a bordism has (infinitely) many decompositions with different Kirby graphic.

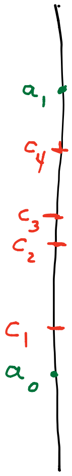

FIGURE 43. The Kirby graphic of Figure 41

fig:43

Elementary Cerf theory

Jean Cerf [C] studied a filtration on the space of smooth functions. The subleading part of the filtration connects different components of excellent functions.

thm:404

Definition 23.13. A smooth function $f\colon M \to \mathbb{R}$ on an n-manifold M has a *birth-death singularity* at $p \in M$ if there exist local coordinates $x^1, \dots, x^n$ in which

eq:517

$$f = (x^1)^3 + (x^1)^2 + \cdots + (x^r)^2 - (x^{r+1})^2 - \cdots - (x^n)^2 + c \tag{23.14}$$

We say the index of p is $n - r$.

There is an intrinsic definition: p is a degenerate critical point, the null space $N_p \subset T_pM$ of d^2f_p has dimension one, and the third differential d^3f_p is nonzero on N_p.

thm:405

Definition 23.15. Let $X\colon Y_0 \to Y_1$ be a bordism and $f\colon X \to \mathbb{R}$ a smooth function.

(i) f is *good of Type* α if f is excellent except at a single point at which f has a birth-death singularity.
(ii) f is *good of Type* β if f is excellent except that there exist exactly two critical points x_i, x_{i+1} with the same critical value $f(x_i) = f(x_{i+1})$.

We say f is *good* if it is either excellent or good of Type α or good of Type β.

thm:407 **Theorem 23.16** (Cerf [C]). *Let $X\colon Y_0 \to Y_1$ be a bordism. Then the space of good functions is connected. More precisely, if f_0, f_1 are excellent, then there exists a path f_t of good functions such that f_t is excellent except at finitely many values of t.*

There is an even more precise statement. The space of good functions is an infinite dimensional manifold, the space of good functions which are not excellent is a codimension one submanifold, and the path $t \mapsto f_t$ crosses this submanifold transversely at finitely many values of t.

A path of good functions has an associated Kirby graphic which encodes the excellent chambers and wall crossings of the path. The horizontal variable it t and the vertical is the critical value. The curves in the graphic are labeled by the index of the critical point in the preimage. Birth-death singularities occur with critical points of neighboring indices. Kirby uses these graphics in his calculus [Ki]. Figure 44 shows some simple Kirby graphics.

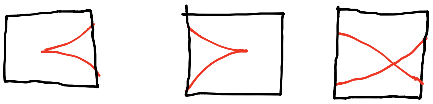

FIGURE 44. Kirby graphics of a birth, death, and exchange fig:44

thm:408 **Example 23.17.** The prototype for crossing a wall of Type α is the path of functions

eq:518 (23.18)
$$f_t(x) = \frac{x^3}{3} - tx$$

defined for $x \in \mathbb{R}$. Then f_t is Morse for $t \neq 0$, has no critical points if $t < 0$, and has two critical points $x = \pm\sqrt{t}$ for $t > 0$. As t increases through $t = 0$ the two critical points are born; as t decreases through $t = 0$ they die. The critical values are $\pm t^{3/2}$, up to a multiplicative constant, which explains the shape of the Kirby graphic.

These Cerf wall crossings relate different decompositions (23.12) of a bordism into elementary bordisms. In the next section we apply this to construct a 2-dimensional TQFT by "generators and relations": we define it on elementary bordisms and use the Cerf moves to check consistency.

Application to TQFT

subsec:23.5

(23.19) *Frobenius algebras.* Before proceeding to 2-dimensional field theories, we need some algebra.

thm:409 **Definition 23.20.** Let k be a field. A *commutative Frobenius algebra* (A, τ) over k is a finite dimensional unital commutative associative algebra A over k an a linear map $\tau \colon A \to k$ such that

eq:519

$$\begin{aligned} A \times A &\longrightarrow k \\ x, y &\longmapsto \tau(xy) \end{aligned} \tag{23.21}$$

is a nondegenerate pairing.

thm:410 **Example 23.22** (Frobenius)**.** Let G be a finite group. Let A be the vector space of functions $f \colon G \to \mathbb{C}$ which are *central*: $f(gxg^{-1}) = f(x)$ for all $x, g \in G$. Define multiplication as convolution:

eq:520

$$f_1 * f_2(x) = \sum_{x_1 x_2 = x} f_1(x_1) f_2(x_2). \tag{23.23}$$

A straightforward check shows $*$ is commutative and associative and the unit is the "δ-function", which is 1 at the identity $e \in G$ and 0 elsewhere. The trace is

eq:521

$$\tau(f) = \frac{f(e)}{\#G}. \tag{23.24}$$

If we remove the central condition, then we obtain the noncommutative Frobenius algebra of all complex-valued functions on G.

thm:411 **Example 23.25.** Let M be a closed oriented n-manifold. Then $H^\bullet(M; \mathbb{C})$ is a super commutative Frobenius algebra. Multiplication is by cup product and the trace is evaluation on the fundamental class. The 'super' reflects the sign in the cup product. For $M = S^2$ we obtain an ordinary commutative Frobenius algebra since there is no odd cohomology. This is a key ingredient in the original construction of Khovanov homology [Kh].

subsec:23.6

(23.26) *2-dimensional oriented TQFT.* The following basic result was well-known by the late 1980s. It appears in Dijkgraaf's thesis [Dij]. More mathematical treatments can be found in [Ab, Ko]. The Morse theory proof we give below is taken from [MoSe, Appendix].

thm:412 **Theorem 23.27.** *Let* $F \colon \mathrm{Bord}^{SO}_{\langle 1,2 \rangle} \to \mathrm{Vect}_k$ *be a TQFT. Then* $F(S^1)$ *is a commutative Frobenius algebra. Conversely, if* A *is a commutative Frobenius algebra, then there exists a TQFT* $F_A \colon \mathrm{Bord}^{SO}_{\langle 1,2 \rangle} \to \mathrm{Vect}_k$ *such that* $F_A(S^1) = A$.

thm:413 *Remark* 23.28. The 2-dimensional field theory constructed from the Frobenius algebra in Example 23.22 has a "classical" description: it counts principal G-bundles, which for a finite group G are regular covering spaces with Galois group G. The invariant $F(X)$ of a closed surface of genus g is given by a classical formula of Frobenius. The TQFT provides a proof of that formula by cutting a surface of genus g into elementary pieces.

We give the proof of Theorem 23.27 which is in [MS].

Proof. Given $F\colon \mathrm{Bord}^{SO}_{\langle 1,2\rangle} \to \mathrm{Vect}_k$ define the vector space $A = F(S^1)$. The elementary bordisms in Figure 45 define a unit $u\colon k \to A$, a trace $\tau\colon A \to k$, and a multiplication $m\colon A\otimes A \to A$. (We read "time" as flowing up in these bordisms; the bottom boundaries are incoming and the top boundaries are outgoing.) The bilinear form (23.21) is the composition in Figure 46, and it has an inverse given by the cylinder with both boundary components outgoing, as is proved by the S-diagram argument. Therefore, it is nondegenerate. This proves that (A, u, m, τ) is a commutative Frobenius algebra.

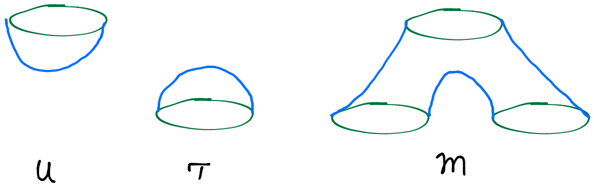

FIGURE 45. Elementary bordisms which define the Frobenius structure fig:45

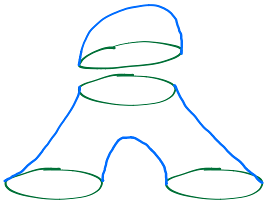

FIGURE 46. The bilinear form fig:46

Next we compute the map defined by the time-reversal of the multiplication (Figure 47). Let $x_1, \dots, x_n$ and $x^1, \dots, x^n$ be dual bases of A relative to (22.42): $\tau(x^i x_j) = \delta^i_j$. Then

eq:522
$$\begin{aligned} m^*\colon A &\longrightarrow A\otimes A \\ x &\longmapsto xx_i\otimes x^i \end{aligned} \tag{23.29}$$

This is the adjoint of multiplication relative to the pairing (23.21). Similarly, note that the unit $u = \tau^*$ is adjoint to the trace. In fact, these adjunctions follow from general duality in symmetric monoidal categories. The time-reversal is the dual in the bordism category (Definition 1.22, (**2.20**), Theorem 15.29),[47] and the dual in the category of vector spaces is the usual dual. A symmetric monoidal functor, such as F_A, maps duals to duals (Proposition 15.34).

For the converse, suppose A is a commutative Frobenius algebra. We construct a 2-dimensional TQFT F_A.

It is easy to prove that the topological group $\mathrm{Diff}^{SO}(S^1)$ of orientation-preserving diffeomorphisms retracts onto the group of rotations, which is connected. Since diffeomorphisms act on A through their isotopy class, the action is trivial. Thus is Y is any oriented manifold diffeomorphic to a circle, there is up to isotopy a unique orientation-preserving diffeomorphism $Y \to S^1$. For any

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[47] We also note that an oriented surface admits an orientation-reversing involution, so is diffeomorphic to the same underlying manifold with the opposite orientation.

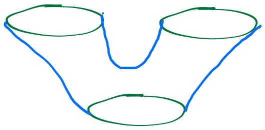

FIGURE 47. The adjoint m^*

fig:47

closed oriented 1-manifold Y define $F_A(Y) = A^{\otimes(\#\pi_0 Y)}$; orientation-preserving diffeomorphisms of closed 1-manifolds act as the identity.

The value of F_A on elementary 2-dimensional bordisms (Figure 42) are given by the structure maps $u = \tau^*, \tau, m, m^*$ of the Frobenius algebra. An arbitrary bordism is a composition of elementary bordisms (tensor identity maps) via an excellent Morse function, and we use such a decomposition to define F_A. However we must check that the value is independent of the excellent Morse function. For that we use Cerf's Theorem 23.16. It suffices to check what happens when we cross a wall of Type α or of Type β.

First, a simplification. Since time-reversal implements duality, if an equality of maps holds for a wall-crossing it also holds for its time-reversal. This cuts down the number of diagrams one needs to consider.

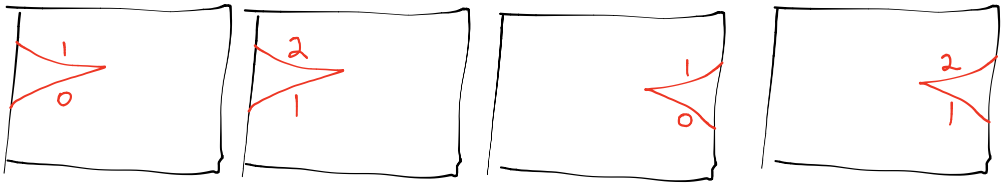

FIGURE 48. The four Type α wall-crossings

fig:49

There are four Type α wall-crossings, as indicated by their Kirby graphics in Figure 48. The numbers indicate the index of the critical point. If f_t is a path of Morse functions with the first Kirby graphic, then the three subsequent ones may be realized by $-f_t$, f_{1-t}, and $-f_{1-t}$, respectively. (Here $0 \leq t \leq 1$.) It follows that we need only check the first. The corresponding transition of bordisms is indicated in Figure 49. These bordisms both map to $\mathrm{id}_A \colon A \to A$: for the first this expresses that u is an identity for the multiplication m.

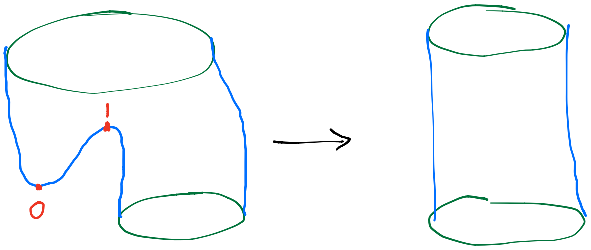

FIGURE 49. Crossing a birth-death singularity

fig:48

In a Type β wall-crossing there are two critical points and the critical levels cross. So on either side of the wall the bordism X is a composition of two elementary bordisms. We assume

X is connected or there is nothing to prove. Furthermore, if the indices of the critical points are q_1, q_2, then the Euler characteristic of the bordism is $(-1)^{q_1} + (-1)^{q_2}$, by elementary Morse theory. Let C denote the critical contour at the critical time t_{crit}, when the two critical levels cross. Since the bordism is connected there are two possibilities: either C is connected or it consists of two components, each with a single critical point. In the latter case there would have to be another critical point in the bordism to connect the two components, else the bordism would not be connected. Therefore, C is connected and it follows easily that both critical points have index 1, whence X has Euler characteristic -2.

Now in each elementary bordism (Figure 42) the number of incoming and outgoing circles differs by one, so in a composition of two elementary bordisms the number of circles changes by two or does not change at all. This leads to four possibilities for the number of circles: 1 → 1, 2 → 2, 3 → 1, or 1 → 3. The last is the time-reversal of the penultimate, so we have three cases to consider.

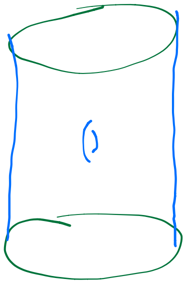

FIGURE 50. $1 \to 1$

fig:50

The first, $1 \to 1$, is a torus with two disks removed. Figure 50 is not at the critical time—the two critical levels are distinct. Note that at a regular value between the two critical values, the level curve has two components, by the classification of elementary bordisms (Figure 42). So the composition is

eq:523

$$A \xrightarrow{m^*} A \otimes A \xrightarrow{m} A \tag{23.30}$$

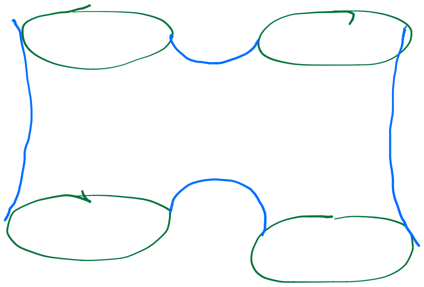

FIGURE 51. $2 \to 2$

fig:51

The second case, $2 \to 2$, is somewhat more complicated than the others. The number of circles in the composition is either $2 \to 1 \to 2$ or $2 \to 3 \to 2$. The $2 \to 1 \to 2$ composition, depicted in

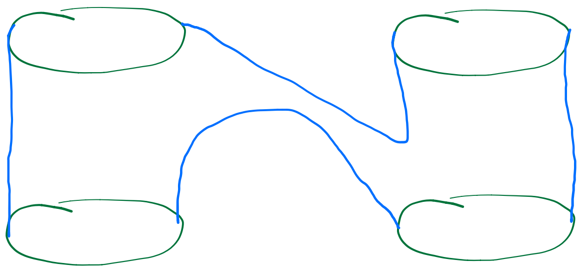

FIGURE 52. 2 → 2 fig:53

Figure 51, is $m^* \circ m$, which is the map

eq:524
$$x \otimes y \longmapsto xy \longmapsto xyx_i \otimes x^i, \tag{23.31}$$

using the dual bases introduced above. The 2 → 3 → 2 composition, depicted in Figure 52, is either $(m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes m^*)$ or $(\mathrm{id} \otimes m) \circ (m^* \otimes \mathrm{id})$, so either

eq:525
$$x \otimes y \longmapsto x \otimes yx_i \otimes x^i \longmapsto xyx_i \otimes x^i \tag{23.32}$$

or

eq:526
$$x \otimes y \longmapsto xx_i \otimes x^i \otimes y \longmapsto xx_i \otimes x^i y. \tag{23.33}$$

To see that these are equal, use the identity $z = \tau(zx_j)x^j$ for all $z \in A$. Thus

eq:527
$$xx_i \otimes x^i y = \tau(x^i y x_j) x x_i \otimes x^j = y x_j x \otimes x^j = xyx_i \otimes x^i. \tag{23.34}$$

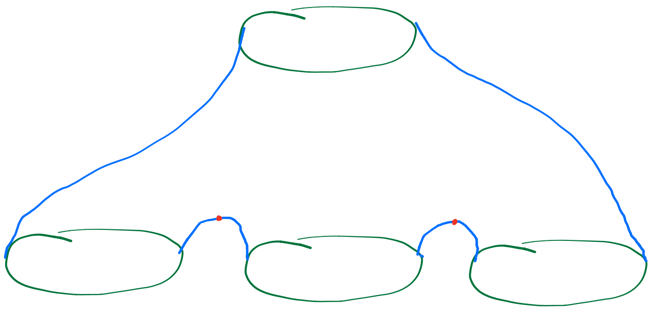

FIGURE 53. 3 → 1 fig:52

The last case, 3 → 1, is depicted in Figure 53 at the critical time. On either side of the wall we have a composition 3 → 2 → 1, and the two different compositions $A^{\otimes 3} \longrightarrow A^{\otimes 2} \longrightarrow A$ are equal by the associative law for m. ☐

Lecture 24: The cobordism hypothesis

sec:24

In this last lecture we introduce the Baez-Dolan cobordism hypothesis [BD], which has been proved by Hopkins-Lurie in dimension 2 and by Lurie [L1] in all dimensions. We begin by motivating the notion of an *extended* topological quantum field theory. This leads to the idea of higher categories, which are also natural for bordisms. We then state the cobordism hypothesis for framed manifolds. We refer to [F1, Te] for more thorough introductions to the cobordism hypothesis.

In this lecture we extract from the geometry of bordisms an even more elaborate algebraic gadget than before: an *(∞, n)-category*.

We have no pretense of precision, and indeed to define an (∞, n)-category, much less a symmetric monoidal (∞, n)-category, is a nontrivial undertaking. At the same time we discuss some motivating examples which we do not explain in complete detail. The circle of ideas around the cobordism hypothesis is under rapid development as we write. We hope the reader is motivated to explore the references, the references in the references, and the many forthcoming references.

Extended TQFT

subsec:24.7

(24.1) *Factoring numerical invariants.* Let

eq:532
$$F\colon \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle} \to \mathrm{Vect}_k \tag{24.2}$$

be a topological field theory with values in the symmetric monoidal category of vector spaces over k. Thus the theory assigns a number in k to every closed n-manifold X (with $\mathcal{X}(n)$-structure, which we do not mention in the sequel). Suppose X is cut in two by a codimension one submanifold Y, as indicated in Figure 54. We view $X_1\colon \emptyset^{n-1} \to Y$ and $X_2\colon Y \to \emptyset^{n-1}$, so that $F(X_1)\colon k \to F(Y)$ and $F(X_2)\colon F(Y) \to k$. Let $\xi_1, \dots, \xi_k$ be a basis of $F(Y)$ and $\xi^1, \dots, \xi^k$ the dual basis of $F(Y)^\vee$. Write

eq:530
$$\begin{aligned} F(X_1) &= a^i \xi_i \\ F(X_2) &= b_i \xi^i \end{aligned} \tag{24.3}$$

for some $a^i, b_i \in k$. Then the fact that $F(X) = F(X_2) \circ F(X_1)$ means

eq:528
$$F(X) = a^i b_i. \tag{24.4}$$

In other words, the TQFT allows us to factorize the numerical invariant of a closed n-manifold into a sum of products of numbers. An n-manifold with boundary has an invariant which is not a single number, but rather a vector of numbers.

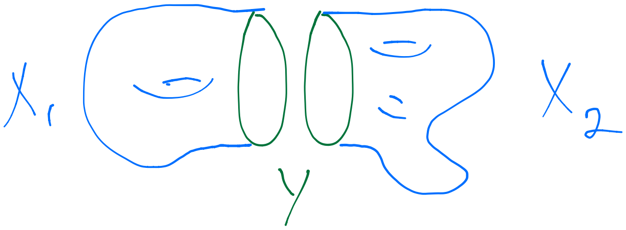

FIGURE 54. Factoring the numerical invariant $F(X)$ fig:54

subsec:24.8

(24.5) *Factoring the "quantum Hilbert space".* We ask: can we factor the vector space $F(Y)$? If so, what kind of equation replaces (24.4)? Well, it must be an equation of sets rather than numbers, and more precisely an equation for vector spaces. Our experience teaches us we should not write an *equality* but rather an *isomorphism*, and that isomorphism takes place in the *category* Vect_k. (Compare: the *equation* (24.4) takes place in the *set* k.) So given a decomposition of the closed $(n-1)$-manifold Y, as in Figure 55, we might by analogy with (24.3) write

eq:529

$$\begin{aligned} F(Y_1) &= V^i c_i \\ F(Y_2) &= W_i c^i \end{aligned} \tag{24.6}$$

for vector spaces $V^i, W_i \in \mathrm{Vect}_k$, and by analogy with (24.4) write

eq:531

$$F(Y) \cong \bigoplus_i V^i \otimes W_i \tag{24.7}$$

In these expressions $V^i, W_i \in \mathrm{Vect}_k$. But what are c_i, c^i? By analogy they should be dual bases of a Vect_k-*module* $F(Z)$ which is associated to the closed $(n-2)$-manifold Z. Of course, the TQFT (24.2) does not assign anything in[48] codimension 2, so we must *extend* our notion of TQFT to carry out this factorization.

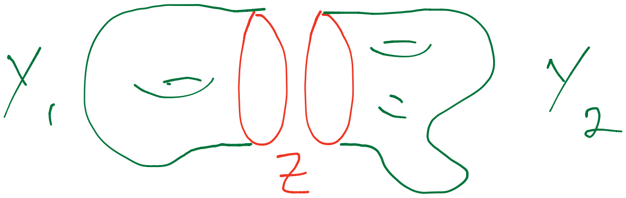

FIGURE 55. Factoring the vector space $F(Y)$ fig:55

Indeed, one of the main ideas of this lecture is to extend the notion of a TQFT to assign invariants to manifolds of arbitrary codimension—down to points—and thus allow gluing which is completely local.

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[48] By 'codimension 2' we mean $(n-2)$-manifolds.

thm:414 *Remark* 24.8. In realistic quantum field theories the vector space $F(Y)$ in codimension 1 is usually called the *quantum Hilbert space*. (It *is* a Hilbert space in unitary theories.) The idea that it should be local in the sense that it factors when Y—physically a *spacelike slice* in a Lorentz manifold—is split in two, is an idea which is present in physics. For systems with discrete space, such as statistical mechanical models in which space is a lattice, the quantum Hilbert space is a tensor product of Hilbert spaces attached to each lattice site and obviously obeys a gluing law. For continuous systems one sometimes attaches a *von Neumann algebra* to what corresponds to Z in Figure 55, and then the Hilbert spaces $F(Y_1), F(Y_2)$ are modules over that von Neumann algebra.

Example: $n = 3$ Chern-Simons theory

This topological quantum field theory was introduced[49] in [Wi1]. It was the key example for many of the early mathematical developments in topological quantum field theory; see [F3] for a recent survey. Here we just make some structural remarks which indicate the utility of viewing quantum Chern-Simons as an *extended* TQFT.

subsec:24.10

(24.9) *Definition using the functional integral.* The data which defines the theory is a compact Lie group G and a class in $H^4(BG;\mathbb{Z})$ called the *level* of the theory. For G a connected simple group, $H^4(BG;\mathbb{Z}) \cong \mathbb{Z}$ and the level can be identified with an integer (usually denoted 'k' in the literature). Let X be a closed oriented 3-manifold. The field in Chern-Simons theory is a connection A on a principal G-bundle over X. The Chern-Simons invariant is a number $\Gamma_X(A) \in \mathbb{C}^\times$, which in fact has unit norm.[50] Suppose $L \subset X$ is a link with components $L_1, \dots, L_\ell$. Let $\rho_1, \dots, \rho_\ell$ be finite dimensional unitary representations of G, which we use to label the components of the link. Then there is an invariant

eq:533 (24.10)
$$W_{L;\rho_1,\dots,\rho_\ell}(A) \in \mathbb{C}$$

defined as the product of the characters of the representations ρ_i applied to the holonomy of the connection A around the various components L_i of the link. Physicists call this the "Wilson line" operator. Formally, the *quantum* Chern-Simons invariant is a *functional integral*

eq:534 (24.11)
$$F(X, L; \rho_1, \dots, \rho_\ell) = \int DA\,\Gamma_X(A) W_L(A)$$

over the infinite dimensional space of G-connections. It is not well-defined mathematically—an appropriate measure $\Gamma_X(A)\,DA$ has not been rigorously constructed—but as a heuristic leads to many predictions which have been borne out, both theoretically and numerically.[51]

---

[49] We have been lax in not pointing out earlier that the whole notion of a topological quantum field theory was introduced by Witten in an earlier paper [Wi2]

[50] This numerical invariant extends to an invertible quantum field theory which is *not* topological: it is defined on the bordism category of oriented manifolds equipped with a G-connection. Similarly, there is an invertible theory which includes the Wilson line operators (24.10) described below.

[51] One subtlety: in the quantum theory the manifolds have an additional tangential structure—a trivialization of the first Pontrjagin class p_1—which is very close to a 3-framing (Example 9.51).

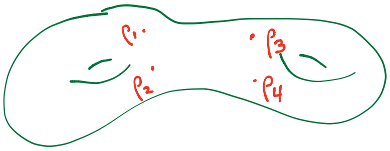

Figure 56. A surface with marked points

fig:56

subsec:24.11

(24.12) *Categorical interpretation.* It is natural to make a (topological) bordism category whose objects are oriented 2-manifolds with a p_1-structure and a finite set of marked points; see Figure 56. A bordism between two such surfaces is then a 3-manifold with boundary and a link; see Figure 57. The link is a neat compact 1-dimensional submanifold, and it hits the boundary in the marked points. Each component of the link is labeled by a representation of G. Composition and the symmetric monoidal product (disjoint union) are as usual. Then the Chern-Simons theory is a symmetric monoidal functor from this category to $\mathrm{Vect}_{\mathbb{C}}$.

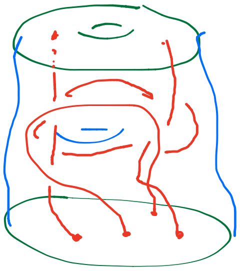

Figure 57. A bordism with a link/braid

fig:57

subsec:24.12

(24.13) *Cutting out the links.* The idea now is to convert to a standard bordism category by cutting out a tubular neighborhood of the marked points and links. Already from an object (Figure 56) we obtain a 2-manifold with boundary in a 3-dimensional theory. Thus codimension 2 manifolds (1-manifolds) are immediately in the game. If we cut out a tubular neighborhood of the link in Figure 57, then we obtain a 3-manifold with *corners*.

Consider a closed component of a link. A tubular neighborhood is a diffeomorphic to a solid torus, but not canonically so: the isotopy classes form a $\mathbb{Z}$-torsor where the generator of $\mathbb{Z}$ acts by a Dehn twist. To fix this indeterminacy the links are given a normal framing. Then, up to isotopy, there is a unique identification of a tubular neighborhood of each closed component with $D^2 \times S^1$, and in the 3-manifold with the tubular neighborhood removed there is a contribution of a standard $S^1 \times S^1$ to the boundary. Now the labels ρ_i can be interpreted as a basis for the vector space $F(S^1 \times S^1)$. In fact, there is a finite set of labels in the quantum theory.[52]

For a component of the link with boundary, the normal framing fixes up to isotopy a diffeomorphism of a tubular neighborhood with a solid cylinder, and the intersection with the incoming or outgoing 2-manifold is a disk, as in Figure 58. This can be re-drawn as in Figure 59, which suggests

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[52] For a connected and simply connected group G the vector space is a quotient of the representation ring of G; the story is more complicated for a general compact Lie group.

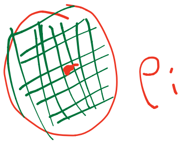

FIGURE 58. Tubular neighborhood of marked point

fig:58

that ρ_i be interpreted as an object in the linear category $F(S^1)$. This is indeed what happens in the extended TQFT.

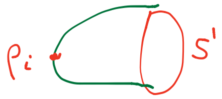

FIGURE 59. An object $\rho_i \in F(S^1)$

fig:59

Morse functions revisited

subsec:24.13

(24.14) *Multi-cuttings and locality.* In Lecture 23 we used a single Morse function—in fact, an excellent function—to decompose a bordism into a composition of elementary bordisms (Figure 41). But the elementary bordisms (e.g. Figure 42) are not completely local; they contain more than a local neighborhood of the critical point. To achieve something entirely local we must slice again in the other direction, say by a second Morse function. For a 2-dimensional manifold this is enough to achieve locality (Figure 60). For an n-dimensional manifold we need n functions.

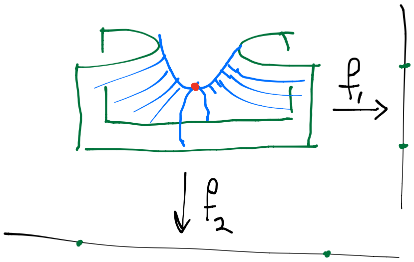

FIGURE 60. Cutting a surface with 2 Morse functions near a critical point

fig:60

The multi-categorical nature of multi-cuttings is already evident in Figure 60. Recall from Figure 38 that a single function on a manifold, thought of as "time", gives rise to a composition law for bordisms. Hence n time functions induce n composition laws. These should be thought of as "internal" to an n-category; there is still disjoint union which induces a symmetric monoidal structure.

subsec:24.14

(24.15) *Collapsing identity maps.* The standard Morse picture collapses the four vertical lines in Figure 60. The resulting manifold with corners is a (closed) square $D^1 \times D^1$. As time (f_1) flows from bottom to top two of the boundary edges ($S^0 \times D^1$) flow to the other two boundary edges ($D^1 \times S^0$) through the square. The four corner points ($S^0 \times S^0$) remain inert through the flow. In this interpretation the square is a map

eq:535 (24.16)
$$D^1 \times D^1 \colon S^0 \times D^1 \Longrightarrow D^1 \times S^0$$

and the two pairs of boundary edges are maps

eq:536 (24.17)
$$S^0 \times S^0 \to S^0 \times S^0.$$

We combine (24.16) and (24.17) into a single diagram:

eq:537 (24.18)
$$\begin{tikzcd} S^0 \times S^0 \arrow[r, bend left, "D^1\times S^0"] \arrow[r, bend right, "S^0\times D^1"'] \arrow[r, phantom, "\Uparrow D^1\times D^1"] & S^0 \times S^0 \end{tikzcd}$$

The general n-dimensional handle of index q is depicted as

eq:538 (24.19)
$$\begin{tikzcd} S^{p-1} \times S^{q-1} \arrow[r, bend left, "D^p\times S^{q-1}"] \arrow[r, bend right, "S^{p-1}\times D^q"'] \arrow[r, phantom, "\Uparrow D^p\times D^q"] & S^{p-1} \times S^{q-1} \end{tikzcd}$$

where $p = n - q$.

Higher categories

subsec:24.15

(24.20) *(m, n)-categories.* Intuitively, a higher category has objects, 1-morphisms which map between objects, 2-morphisms which map between 1-morphisms, etc. The diagrams (24.18) and (24.19) are 2-morphisms (double arrows) which map between 1-morphisms (single arrows). There are k composition laws for k-morphisms, and the composition laws are no longer required to be associative. We allow ∞-categories which have morphisms of all orders. An *(∞, n)-category* is an ∞-category in which all k-morphisms are invertible for $k > n$. In this notation a $(1, 1)$-category is an ordinary category and a $(1, 0)$-category is a groupoid.

What follows are two examples of 2-categories. Together with the multi-bordism category indicated in the previous section, these give some of the most important ways in which multi-categories arise.

thm:415 **Example 24.21** (Higher groupoids from a topological space)**.** This generalizes Example 13.14. Let Y be a topological space. The simplest invariant $\pi_0 Y$ is the *set* of path components. The next simplest is $\pi_{\leq 1} Y$, the fundamental *groupoid* of Y. Its objects are the points of Y and a morphism

$y_0 \to y_1$ is a *homotopy class* of continuous paths $\gamma\colon [0,1] \to Y$ with $\gamma(0) = y_0$ and $\gamma(1) = y_1$. It is clear how to go further. We construct a *2-groupoid* $\pi_{\le 2}Y$ as follows. (A 2-groupoid is a $(2,0)$-category, i.e., a 2-category in which all morphisms are invertible.) An object is a point of Y as before. A 1-morphism in $\pi_{\le 2}Y$ is a continuous path—there is no identification of homotopic paths. Let $y_0, y_1 \in Y$ and $\gamma, \gamma'\colon [0,1] \to Y$ two continuous paths from y_0 to y_1. A 2-morphism $\Gamma\colon \gamma \Rightarrow \gamma'$ is a *homotopy class* of continuous maps $\Gamma\colon [0,1] \times [0,1] \to Y$ such that

eq:539

$$\begin{aligned} \Gamma(t_1, 0) &= \gamma(t_1) \\ \Gamma(t_1, 1) &= \gamma'(t_1) \\ \Gamma(0, t_2) &= y_0 \\ \Gamma(1, t_2) &= y_1 \end{aligned} \tag{24.22}$$

for all $t_1, t_2 \in [0,1]$. The last two equations allow us to factor Γ through the lune obtained by collapsing the vertical boundary edges of the square $[0,1] \times [0,1]$. Thus the domain has the shape of the diagram (24.18), as befits a 2-morphism. We identify homotopic maps Γ, where the map on the boundary is static during the homotopy. Vertical composition of 2-morphisms is associative on the nose, but other compositions are only associative up to homotopy.

It should be clear how to define the *fundamental m-groupoid* $\pi_{\le m}Y$ of the topological space Y for any $m \in \mathbb{Z}^{\ge 0}$. There is an assertion (either a definition or theorem, depending on the approach, though I don't know a reference in which it is a theorem) in higher category theory that an $(\infty, 0)$-category *is* a topologial space.

thm:416

Example 24.23 (The Morita 2-category of algebras)**.** Let k be a field. We construct a 2-category $C = \mathrm{Alg}_k$ which is not a groupoid. (In the above nomenclature it is a $(2,2)$-category.) The objects are algebras over k. For algebras A_0, A_1 a morphism $B\colon A_0 \to A_1$ is an (A_1, A_0)-bimodule. That is, B is a k-vector space which is simultaneously a left module for A_1 and a right module for A_0. The actions commute, so equivalently B is a left $(A_1 \otimes A_0^{\mathrm{op}})$-module. The collection $C(A_0, A_1)$ of these bimodules is a 1-category: a morphism $f\colon B \Rightarrow B'$ is a linear map $f\colon B \to B'$ which intertwines the (A_1, A_0)-action. So f is a 2-morphism in C:

eq:540

$$\begin{tikzcd} A_0 \arrow[r, bend left, "B'"{name=U}] \arrow[r, bend right, "B"'{name=D}] & A_1 \arrow[from=D, to=U, phantom, "\Uparrow f"] \end{tikzcd} \tag{24.24}$$

Composition of bimodules (1-morphisms) is by tensor product over an algebra. Thus if A_0, A_1, A_2 are k-algebras, $B_1\colon A_0 \to A_1$ an (A_1, A_0)-bimodule, and $B_2\colon A_1 \to A_2$ an (A_2, A_1)-bimodule, then $B_2 \circ B_1\colon A_0 \to A_2$ is the (A_2, A_0)-bimodule $B_2 \otimes_{A_1} B_1$. This composition is only associative up to isomorphism.

The cobordism hypothesis

subsec:24.16

(24.25) *The (∞,n)-category of bordisms.* We motivated above the idea that using multiple Morse functions we can make out of n-manifolds an n-category: n-manifolds with corners of all codimensions form the n-morphisms in that category. This is an (n,n)-category in the nomenclature of **(24.20)**. This is already a huge step above what we had before, an n-categorical generalization of Definition 14.3. Now we want to generalize Definition 20.19 in the sense that we will consider a *topological space* of n-morphisms. Now an n-morphism is an n-manifold with corners, together with partitions of the various corners telling which are incoming and which are outgoing. (There are also collar neighborhoods.) The discussion in Lecture 20 indicates how that can be done. However, using the assertion at the end of Example 24.21 we can replace that space by its fundamental ∞-groupoid, which amounts to saying that an $(n+1)$-morphism is a diffeomorphism of n-dimensional bordisms, an $(n+2)$-morphism is an isotopy of such diffeomorphisms, etc. In this way we obtain an (∞,n)-bordism category which we denote Bord_n. Of course, we can include a tangential structure as well. The relevant example for us is n-framings (Example 9.51), which we denote as $\mathcal{X}(n) = EO(n)$, and thus denote the resulting bordism category $\mathrm{Bord}_n^{EO(n)}$.

subsec:24.17

(24.26) *Fully extended TQFT.* Following Definition 14.20 we define a (fully) extended topological quantum field theory to be a homomorphism of symmetric monoidal (∞,n)-categories

eq:541
$$F\colon\ \mathrm{Bord}_n^{EO(n)} \longrightarrow C \tag{24.27}$$

into an arbitrary symmetric monoidal (∞,n)-category C.

subsec:24.18

(24.28) *Finiteness.* Recall Theorem 15.36 which asserts that the objects which appear in the image of an ordinary TQFT are dualizable. The corresponding finiteness condition in an (∞,n)-category is *n-dualizability*, or *full dualizability*. We do not elaborate here, but defer to [L1, §2.3].

subsec:24.19

(24.29) *The cobordism hypothesis.* The cobordism hypothesis is the next in a sequence of theorems in the course. The first is stated in **(2.28)**: the oriented bordism group Ω_0^{SO} is the free abelian group on one generator. It may be accurate to attribute this to Brouwer as it is the basis of oriented intersection theory. This statement only uses 0- and 1-manifolds, and on such manifolds an orientation is equivalent to a 1-framing. This result was restated in Theorem 16.8. The second result in this line is Theorem 16.10. It roughly asserts that $\mathrm{Bord}_{\langle 0,1\rangle}^{SO} = \mathrm{Bord}_{\langle 0,1\rangle}^{EO(1)}$ is the free 1-category with duals[53] with a single generator pt_+. But it is much easier to formulate in terms of homomorphisms out of $\mathrm{Bord}_{\langle 0,1\rangle}$, and that is how Theorem 16.10 is stated. Still, it is a theorem about the structure of the bordism category, a statement about 0- and 1-manifolds. The cobordism hypothesis is a similar statement, but about the bordism (∞,n)-category.

thm:417 **Theorem 24.30** (cobordism hypothesis)**.** *Let C be a symmetric monoidal (∞,n)-category. Then the map*

eq:542
$$\begin{aligned} \Phi\colon\ \mathrm{TQFT}_n^{EO(n)}(C) &\longrightarrow (C^{\mathrm{fd}})^{\sim} \\ F &\longmapsto F(\mathrm{pt}_+) \end{aligned} \tag{24.31}$$

is an equivalence of ∞-groupoids.

---

[53] i.e., every object has a dual

The domain is the multi-category of homomorphisms $\mathrm{Bord}_n^{EO(n)} \to C$. The multi-category analog of Proposition 15.34(ii) implies that the domain is an ∞-groupoid: all morphisms are invertible. The notation in the codomain follows Definition 16.4 and Definition 16.5: it is the maximal ∞-groupoid underlying the subcategory of fully dualizable objects.

The cobordism hypothesis is a statement about the n-framed bordism category. There are many variations. We will stop here and not comment on the proof nor on the applications.

Appendix: Fiber bundles and vector bundles

sec:26

This appendix is provided for reference as these topics may not be covered in the prelim class.pp I begin with fiber bundles. Then I will discuss the particular case of vector bundles and the construction of the tangent bundle. Intuitively, the tangent bundle is the disjoint union of the tangent spaces (see (25.20)). What we must do is define a manifold structure on this disjoint union and then show that the projection of the base is *locally trivial*.

Fiber bundles

thm:a1 **Definition 25.1.** Let $\pi\colon E \to M$ be a map of sets. Then the *fiber* of π over $p \in M$ is the inverse image $\pi^{-1}(p) \in E$.

In some cases, as in the context of fiber bundles, it it convenient to denote the fiber $\pi^{-1}(p)$ as E_p. If π is surjective then each fiber is nonempty, and the map π partitions the domain E:

eq:a1

$$E = \coprod_{p \in M} E_p \tag{25.2}$$

Recall that '∐' is the notation for *disjoint union*; that is, an ordinary union in which the sets are disjoint. (So 'disjoint' functions as an adjective; 'disjoint union' is not a compound noun.)

thm:a2 **Definition 25.3.** Let $\pi\colon E \to M$ be a surjective map of manifolds. Then π is a *fiber bundle* if for every $p \in M$ there exists a neighborhood $U \subset M$ of p, a manifold F, and a diffeomorphism

eq:a2

$$\varphi\colon \pi^{-1}(U) \longrightarrow U \times F \tag{25.4}$$

such that the diagram

eq:a3

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd} \tag{25.5}$$

commutes. If $\pi'\colon E' \to M$ is also a fiber bundle, then a *fiber bundle map* $\varphi\colon E \to E'$ is a smooth map of manifolds such that the diagram

eq:a4

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd} \tag{25.6}$$

commutes. If φ has an inverse, then we say φ is an *isomorphism* of fiber bundles.

In the diagram $\pi_1\colon U\times F\to U$ is projection onto the first factor. (We will often use the notation $\pi_k\colon X_1\times X_2\times\cdots\times X_n\to X_k$ for projection onto the k^{th} factor of a Cartesian product.) The commutation of the diagram is the assertion that $\pi=\pi_1\circ\varphi$, which means that φ maps fibers of π diffeomorphically onto F. The manifold F may vary with the local trivialization.

thm:a3 **Definition 25.7.** The modification of Definition 25.3 in which F is fixed once and for all defines a *fiber bundle with fiber* F.

You should prove that we can always take F to be fixed on each component of M.

Terminology: E is called the *total space* of the bundle and M is called the *base*. As mentioned, F is called the *fiber*.

thm:a4 **Example 25.8.** The simplest example of a fiber bundle is $\pi=\pi_1\colon M\times F\to M$, where M and F are fixed manifolds. This is called *the trivial bundle with fiber* F. A fiber bundle is *trivializable* if it is isomorphic to the trivial bundle.

The characteristic property of a fiber bundle is that it is *locally trivializable*: compare (25.5) and (25.6).

thm:a5 **Exercise 25.9.** Prove that every fiber bundle $\pi\colon E\to M$ is a submersion.

thm:a6 *Remark* 25.10. We can also define a fiber bundle of topological spaces: in Definition 25.3 replace 'manifold' by 'topological space' and 'diffeomorphism' by 'homeomorphism'.

Transition functions

A local trivialization of a fiber bundle is analogous to a chart in a smooth manifold. Notice, though, that a topological manifold has no intrinsic notion of smoothness, so we must define smooth manifolds by comparing charts via transition functions and then specifying an atlas of C^∞ compatible charts. By contrast, when defining the notion of a fiber bundle we already know what a smooth manifold is and so only assert the existence of smooth local trivializations. But we can still construct fiber bundles by a procedure analogous to the construction of smooth manifolds when we don't have the total space as a manifold.

Let $\pi\colon E\to M$ be a fiber bundle and $\varphi_1\colon\pi^{-1}(U_1)\to U_1\times F$ and $\varphi_2\colon\pi^{-1}(U_2)\to U_2\times F$ two local trivializations with the same fiber F. Then the *transition function* from φ_1 to φ_2 is

eq:a5
$$g_{21}\colon U_1\cap U_2\longrightarrow \operatorname{Aut}(F) \tag{25.11}$$

defined by

eq:a6
$$(\varphi_2\circ\varphi_1^{-1})(p,f)=\bigl(p,g_{21}(p)(f)\bigr),\qquad p\in U_1\cap U_2,\quad f\in F. \tag{25.12}$$

Here $\operatorname{Aut}(F)$ is the group of diffeomorphisms of F. The map g_{21} is smooth in the sense that the associated map

eq:a11
$$\begin{aligned}\tilde g_{21}\colon (U_1\cap U_2)\times F&\longrightarrow F\\ (p\quad,\quad f)&\longmapsto g_{21}(p)(f)\end{aligned} \tag{25.13}$$

is smooth. When we come to vector bundles F is a vector space and the transition functions land in the finite dimensional Lie group of linear automorphisms; then the map (25.11) is *smooth* if and only if (25.13) is smooth. Note in the formulas that $g_{21}(p)(f)$ means the diffeomorphism $g_{21}(p)$ applied to f.

Just as the overlap, or transition, functions between coordinate charts encode the smooth structure of a manifold, the transition functions between local trivializations encode the global properties of a fiber bundle.

We can use transition functions to construct a fiber bundle when we are only given the base and fiber but not the total space. For that start with the base manifold M and the fiber manifold F and suppose $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M. Now suppose given transition functions

eq:a7

$$g_{\alpha_1\alpha_0}\colon U_{\alpha_0}\cap U_{\alpha_1} \longrightarrow \operatorname{Aut}(F) \tag{25.14}$$

for each pair $\alpha_0,\alpha_1\in F$, and assume these are smooth in the sense defined above using (25.13). We demand that $g_{\alpha\alpha}(p)$ be the identity map for all $p\in U_\alpha$, that $g_{\alpha_1\alpha_0}=g^{-1}_{\alpha_0\alpha_1}$ on $U_{\alpha_0}\cap U_{\alpha_1}$, and that

eq:a8

$$(g_{\alpha_0\alpha_2}\circ g_{\alpha_2\alpha_1}\circ g_{\alpha_1\alpha_0})(p)=\mathrm{id}_F,\qquad p\in U_{\alpha_0}\cap U_{\alpha_1}\cap U_{\alpha_2}. \tag{25.15}$$

Equation (25.15) is called the *cocycle condition*. We are going to use the transition functions (25.14) to construct E from the local trivial bundles $U_\alpha\times F\to U_\alpha$, and the cocycle condition (25.15) ensures that the gluing is consistent. So define

eq:a9

$$E=\coprod_{\alpha\in A}(U_\alpha\times F)\,\big/\!\sim \tag{25.16}$$

where

eq:a10

$$\bigl(p_{\alpha_0},f\bigr)\sim\bigl(p_{\alpha_1},g_{\alpha_1\alpha_0}(f)\bigr),\qquad p_{\alpha_0}=p_{\alpha_1}\in U_{\alpha_0}\cap U_{\alpha_1},\quad f\in F. \tag{25.17}$$

The projections $\pi_1\colon U_\alpha\times F\to U_\alpha$ fit together to define a surjective map $\pi\colon E\to M$. It is straightforward to verify that each fiber $\pi^{-1}(p)$ of π is diffeomorphic to F. Also, observe that the quotient map restricted to $U_\alpha\times F$ is injective.

thm:a7 **Proposition 25.18.** *The quotient* (25.16) *has the natural structure of a smooth manifold and* $\pi\colon E\to M$ *is a fiber bundle with fiber* F*.*

I will only sketch the proof, which I suggest you think through carefully. Once we prove E is a manifold then the fiber bundle property—the local triviality—is easy as the construction comes with local trivializations. Equip E with the quotient topology: a set $G\subset E$ is open if and only if its inverse image in $\coprod\limits_{\alpha\in A}(U_\alpha\times F)$ is open. This topology is Hausdorff: if $q_1,q_2\in E$ have different projections in M they can be separated by open subsets of M; if they lie in the same fiber, then we use the fact that F is Hausdorff to separate them in some $U_\alpha\times F$. The topology is also second countable: since M is second countable there is a countable subset of A for which E is the quotient (25.16), and then as F is second countable we can find a countable base for the topology. To construct an atlas, cover each U_α by coordinate charts of M and cover F by coordinate charts. Then the Cartesian product of these charts produces charts of $U_\alpha \times F$, and so charts of E. It remains to check that the overlap of these coordinate charts is C^∞.

Vector bundles

The notion of fiber bundle is very general: the fiber is a general manifold. In many cases the fibers have extra structure. In lecture we met a fiber bundle of affine spaces. There are also fiber bundles of Lie groups. One important special type of fiber bundle is a *vector bundle*: the fibers are vector spaces.

`thm:a9` **Definition 25.19.** A *vector bundle* is a fiber bundle as in Definition 25.3 for which the fibers $\pi^{-1}(p)$, $p \in M$ are vector spaces, the manifolds F in the local trivialization are vector spaces, and for each $p \in U$ the local trivialization (25.4) restricts to a vector space isomorphism $\pi^{-1}(p) \to F$.

As mentioned earlier, the transition functions (25.14) take values in the Lie group of linear automorphisms of the vector space F. (For $F = \mathbb{R}^n$ we denote that group as $GL_n\mathbb{R}$.)

You should picture a vector bundle over M as a smoothly varying locally trivial family (25.2) of vector spaces parametrized by M. “Smoothly varying” means that the collection of vector spaces fit together into a smooth manifold.

The tangent bundle

Let M be a smooth manifold and assume $\dim M = n$. (If different components of M have different dimensions, then make this construction one component at a time.) One of the most important consequences of the smooth structure is the *tangent bundle*, the collection of tangent spaces

`eq:a14`
$$\pi\colon \coprod_{p\in M} T_pM \longrightarrow M \tag{25.20}$$

made into a vector bundle. We can construct it as a vector bundle using Proposition 25.18 as follows. Let $\{(U_\alpha, x_\alpha)\}_{\alpha\in A}$ be a countable covering of M by coordinate charts. (As remarked earlier countability is not an issue and we can use the entire atlas.) Then we obtain local trivializations (25.4) for each coordinate chart:

`eq:a15`
$$\begin{aligned} \varphi_\alpha\colon \pi^{-1}(U_\alpha) &\longrightarrow U \times \mathbb{R}^n \\ \xi = \sum_i \xi^i \frac{\partial}{\partial x^i} &\longmapsto (p; \xi^1, \xi^2, \dots, \xi^n), \end{aligned} \tag{25.21}$$

where $\xi \in T_pM$. This is well-defined, but so far only a map of sets as we have not even topologized the total space in (25.20). But we can still use (25.21) to compute the transition functions via (25.12). Namely, define $g_{\alpha_1\alpha_0}\colon U_{\alpha_0} \cap U_{\alpha_1} \to GL_n\mathbb{R}$ by

eq:a16

$$g_{\alpha_1\alpha_0}(p) = d(x_{\alpha_1} \circ x_{\alpha_0}^{-1})_p. \tag{25.22}$$

In other words, the transition functions for the tangent bundle are the differentials of the overlap functions for the charts.

Problems

sec:25

thm:418 **Exercise 26.1.** Derive the signature formula for a closed oriented 8-manifold. You may use the result that $\Omega_8^{SO} \otimes \mathbb{Q}$ is 2-dimensional with basis the classes of $\mathbb{CP}^2 \times \mathbb{CP}^2$ and $\mathbb{CP}^4$.

thm:419 **Exercise 26.2.** Check the signature formula in the previous problem for the quaternionic projective plane $\mathbb{HP}^2$.

thm:420 **Exercise 26.3.** Suppose $V_1 \to M_1$ and $V_2 \to M_2$ are real vector bundles. Find a relationship among the Thom complexes $M_1^{V_1}$, $M_2^{V_2}$, and $(M_1 \times M_2)^{V_1 \times V_2}$.

thm:421 **Exercise 26.4.** Prove that $\mathbb{CP}^4$ does not embed in $\mathbb{A}^{11}$. (Hint: Consider Pontrjagin classes.)

thm:422 **Exercise 26.5.** Construct a 20-dimensional closed oriented manifold with signature 2012.

thm:434 **Exercise 26.6.**

(i) Construct a double cover homomorphism $SU(2) \times U(1) \to U(2)$.
(ii) Compute the rational homotopy groups $\pi_i U(8) \otimes \mathbb{Q}$ for $i = 1, \dots, 4$.
(iii) Compute as much of $H_\bullet\bigl(BU(8);\mathbb{Q}\bigr)$ as you can.

thm:435 **Exercise 26.7.**

(i) Recall from (14.4) in the lecture notes that a diffeomorphism $f\colon Y \to Y$ of a closed manifold Y determines a bordism X_f. Let f_0, f_1 be diffeomorphisms. Prove that X_{f_0} is diffeomorphic to X_{f_1} (as bordisms) if and only of f_0 is pseudoisotopic to f_1.
(ii) Find a manifold Y and diffeomorphisms $f_0, f_1\colon Y \to Y$ which are pseudoisotopic but not isotopic.

thm:436 **Exercise 26.8.**

(i) Let S be a set with composition laws $\circ_1, \circ_2\colon S \times S \to S$ and distinguished element $1 \in S$. Assume (i) 1 is an identity for both $\circ_1$ and $\circ_2$; and (ii) for all $s_1, s_2, s_3, s_4 \in S$ we have

$$(s_1 \circ_1 s_2) \circ_2 (s_3 \circ_1 s_4) = (s_1 \circ_2 s_3) \circ_1 (s_2 \circ_2 s_4).$$

Prove that $\circ_1 = \circ_2$ and that this common operation is commutative and associative.
(ii) Let C be a symmetric monoidal category. Apply (a) to $C(1,1)$, where $1 \in C$ is the tensor unit.

thm:423 **Exercise 26.9.** Let $y \in C$ be a dualizable object in a symmetric monoidal category, and suppose $(y^\vee, c, e)$ and $(\widetilde{y^\vee}, \tilde{c}, \tilde{e})$ are two sets of duality data. Prove there is a unique map $(y^\vee, c, e) \to (\widetilde{y^\vee}, \tilde{c}, \tilde{e})$.

thm:437 **Exercise 26.10.** For each of the following symmetric monoidal categories determine all of the dualizable objects.

(i) $(\mathrm{Top}, \amalg)$, the category of topological spaces and continuous maps under disjoint union.
(ii) $(\mathrm{Ab}, \oplus)$, the category of abelian groups and homomorphisms under direct sum.
(iii) $(\mathrm{Mod}_R, \otimes)$, the category of R-modules and homomorphisms under tensor product, where R is a commutative ring.
(iv) $(\mathrm{Set}, \times)$, the category of sets and functions under Cartesian product.

thm:424 **Exercise 26.11.** Recall that every category C has an associated groupoid $|C|$ obtained from C by inverting all of the arrows. What is $|\mathrm{Bord}_{\langle 1,2\rangle}|$? $|\mathrm{Bord}^{\mathrm{Spin}}_{\langle 1,2\rangle}|$? What are all $\mathrm{Vect}_{\mathbb{C}}$-valued invertible topological quantum field theories with domain $\mathrm{Bord}_{\langle 1,2\rangle}$? $\mathrm{Bord}^{\mathrm{Spin}}_{\langle 1,2\rangle}$?

thm:425 **Exercise 26.12.** Fix a finite group G. Let C denote the groupoid $G/\!/G$ of G acting on itself by conjugation. Let D denote the groupoid of principal G-bundles over S^1. (A principal G-bundle is a regular, or Galois, cover with group G.) Prove that C and D are equivalent groupoids. You should spell out precisely what these groupoids are.

thm:438 **Exercise 26.13.** Explain why each of the following fails to be a natural map $\eta\colon F \to G$ of symmetric monoidal functors $F, G\colon C \to D$.

(i) F, G are the identity functor on Vect_k for some field k, and $\eta(V)\colon V \to V$ is multiplication by 2 for each vector space V.
(ii) C, D are the category of algebras over a field k, the functor F maps $A \mapsto A \otimes A$, the functor G is the identity, and $\eta(A)\colon A \otimes A \to A$ is multiplication.

thm:439 **Exercise 26.14.** In this problem you construct a simple TQFT $F\colon \mathrm{Bord}_{\langle 0,1\rangle} \to \mathrm{Vect}_{\mathbb{Q}}$. For any manifold M let $\mathcal{C}(M)$ denote the groupoid of principal G-bundles over M, as in Exercise 26.12.n

(i) For a compact 0-manifold Y, define $F(Y)$ as the vector space of functions $\mathcal{C}(Y) \to \mathbb{Q}$. Say what you mean by such functors on a groupoid.
(ii) For a *closed* 1-manifold X define

$$F(X) = \sum_{[P]\in\pi_0\mathcal{C}(X)} \frac{1}{\#\,\mathrm{Aut}(P)},$$

where the sum is over equivalence classes of principal G-bundles. Extend this to all bordisms $X\colon Y_0 \to Y_1$.
(iii) Check that F is a symmetric monoidal functor.
(iv) Calculate F on a set of duality data for the point $\mathrm{pt} \in \mathrm{Bord}_{\langle 0,1\rangle}$. Use it to compute $F(S^1)$.

thm:426 **Exercise 26.15.** Fix a nonzero number $\lambda \in \mathbb{C}$. Construct an invertible TQFT $F\colon \mathrm{Bord}^{SO}_{\langle 1,2\rangle} \to \mathrm{Vect}_{\mathbb{C}}$ such that for a closed 2-manifold X we have $F(X) = \lambda^{\chi(X)}$, where $\chi(X)$ is the Euler characteristic. Can you extend to the bordism category $\mathrm{Bord}_{\langle 1,2\rangle}$?

thm:440 **Exercise 26.16.** Here are some problems concerning invertibility in symmetric monoidal categories, as in Lecture 17.

(i) Construct a category of invertibility data (Definition 17.18), and prove that this category is a contractible groupoid.
(ii) Prove Lemma 17.21(i).

(iii) Let $\alpha\colon \mathrm{Bord}^{SO}_{\langle 0,1\rangle} \to C$ be a TQFT. Prove that if $\alpha(\mathrm{pt}_+)$ is invertible, then α is invertible.

`thm:427` **Exercise 26.17.** Compute the invariants of the Picard groupoid of superlines. (See (17.27) and (17.35) in the notes.)

`thm:428` **Exercise 26.18.** Show that a special Γ-set determines a commutative monoid. More strongly, construct a category of special Γ-sets, a category of commutative monoids, and an equivalence of these categories.

`thm:429` **Exercise 26.19.** Let $\mathbb{S}$ denote the Γ-set $\mathbb{S}(S) = \Gamma^{\mathrm{op}}(S^0, S)$, for $S \in \Gamma^{\mathrm{op}}$ a finite pointed set. Compute $\pi_1|\mathbb{S}|$.

`thm:441` **Exercise 26.20.** Let C be a category. An object $* \in C$ is *initial* if for every $y \in C$ there exists a unique morphism $* \to y$, and it is *terminal* if for every $y \in C$ there exists a unique morphism $y \to *$.

(i) Prove that an initial object is unique up to unique isomorphism, and similarly for a terminal object.

(ii) Examine the existence of initial and terminal objects for the following categories: Vect, Set, Space, Set_*, Space_*, the category of commutative monoids, a bordism category, a category of topological quantum field theories.

(iii) Prove that if C has either an initial or final object, then its classifying space is contractible.

`thm:430` **Exercise 26.21.** Let K denote the classifying spectrum of the category whose objects are finite dimensional complex vector spaces and whose morphisms are isomorphisms of vector spaces. Compute $\pi_0 K$. Compute $\pi_1 K$.

⇒

`thm:431` **Exercise 26.22.** [**rewrite as quotient of diagonal in** $M \times M$.] Let M be a *commutative* monoid. We described a general construction of the group completion of any monoid. Give a much simpler construction of the group completion $|M|$ by imposing an equivalence relation on $M \times M$. You may wish to think about the examples $M = (\mathbb{Z}^{\ge 0}, +)$ and $M = (\mathbb{Z}^{>0}, \times)$.

`thm:442` **Exercise 26.23.** Let G be a topological group, viewed as a category C with a single object. (Normally we use 'G' in place of 'C', but for clarity here we distinguish.)

(i) Describe the nerve NC of G explicitly.

(ii) Define a groupoid $\mathcal{G}$ whose set of objects is G and with a unique morphism between any two objects. Construct a free right action of G on $\mathcal{G}$ with quotient C. First, define carefully what that means.

(iii) Prove that the classifying space $B\mathcal{G}$ is contractible.

(iv) Show that G acts freely on $B\mathcal{G}$ with quotient BC.

So we would like to assert that $B\mathcal{G} \to BC$ is a principal G-bundle, and by Theorem 6.45 in the notes a universal bundle, which then makes BC a classifying space in the sense of Lecture 6. The only issue is local triviality; see Segal's paper.

`thm:432` **Exercise 26.24.** The embedding $U(m) \hookrightarrow O(2m)$ of the unitary group into the orthogonal group determines a $2m$-dimensional tangential structure $BU(m) \to BO(2m)$. Compute the integral homology $H_\bullet(MTU(m))$ of the associated Madsen-Tillmann spectrum.

thm:443 **Exercise 26.25.** For each of the following maps $\mathcal{F}\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{Set}$, answer : Is $\mathcal{F}$ a presheaf? Is $\mathcal{F}$ a sheaf?

(i) $\mathcal{F}(M) =$ the set of smooth vector fields on M
(ii) $\mathcal{F}(M) =$ the set of orientations of M
(iii) $\mathcal{F}(M) =$ the set of sections of $\mathrm{Sym}^2 T^*M$
(iv) $\mathcal{F}(M) =$ the set of Riemannian metrics on M
(v) $\mathcal{F}(M) =$ the set of isomorphism classes of double covers of M
(vi) $\mathcal{F}(M) = H^q(M; A)$ for some $q \geq 0$ and abelian group A

thm:433 **Exercise 26.26.** Define a sheaf $\mathcal{F}$ of categories on Man which assigns to each test manifold M a groupoid of double covers of M. Be sure to check that you obtain a presheaf—compositions map to compositions—which satisfies the sheaf condition. Describe $|\mathcal{F}|$ and $B|\mathcal{F}|$. Compute the set $\mathcal{F}[M]$ of concordance classes of double covers on M.

thm:444 **Exercise 26.27.**

(i) Fix $q \geq 0$. Define a sheaf $\mathcal{F}$ of sets on Man which assigns to each test manifold M the set of closed differential q-forms. Compute $\mathcal{F}[M]$. Identify $|\mathcal{F}|$.
(ii) Fix $k > 0$. Fix a complex Hilbert space $\mathcal{H}$. Define a sheaf $\mathcal{F}$ of sets on Man which assigns to each test manifold M the set of rank k vector bundles $\pi\colon E \to M$ together with an embedding $E \hookrightarrow M \times \mathcal{H}$ into the vector bundle with constant fiber $\mathcal{H}$ and a flat covariant derivative operator. (The flat structure and embedding are uncorrelated.) Discuss briefly why $\mathcal{F}$ is a sheaf. Compute $\mathcal{F}[M]$. Identify $|\mathcal{F}|$.

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