Linear Cellular Automata and the Garden-of-Eden

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1. The All-Ones Problem

Suppose each of the squares of an $n \times n$ chessboard is equipped with an indicator light and a button. If the button of a square is pressed, the light of that square will change from off to on and vice versa; the same happens to the lights of all the edge-adjacent squares. Initially all lights are off. Now, consider the following question: is it possible to press a sequence of buttons in such a way that in the end all lights are on? We will refer to this problem as the *All-Ones Problem.*

A moment's reflection will show that pressing a button twice has the same effect as not pressing it at all. Thus a solution to our problem can be described by a subset of all squares (namely a set of squares whose buttons when pressed in an arbitrary order will render all lights on) rather than a sequence. In fact a set X of squares is a solution to the All-Ones Problem if and only if for every square s the number of squares in X adjacent to or equal to s is odd. Consequently, we will call such a set an odd-parity cover.

Trial and error in conjunction with a pad of graph paper will readily produce solutions for $n \leq 4$. A little more experimentation shows that an odd-parity cover —should one exist—is difficult to construct even for $n = 5$ or 6.

The brute-force approach to the problem, namely exhaustive search over all subsets of $\{1, \ldots, n\} \times \{1, \ldots, n\}$, presents 2^{n^2} candidates, and the search becomes infeasible for moderate values of n even with the help of a computer. A less brute-force method would be to try to solve the system

$$(A + I) \cdot X = \mathbf{1}$$

of linear equations over the field $GF(2) = \{0,1\}$, where A is the adjacency matrix of the $n \times n$ grid graph (interpreted as a matrix over $GF(2)$) and $\mathbf{1}$ is the vector with all components equal to 1. This method, which involves n^2 equations, again becomes unwieldy for small values of n. For a similar approach to a game related to the All-Ones Problem, see [3]. In any case, Figure 1 shows odd-parity covers for $n = 4, 5, 8$.

Several questions come to mind. For which n does a solution to the All-Ones Problem exist? More generally, how many odd-parity covers are there for an $n \times n$ board? What happens if the adjacency condition is changed—say, to an octal array (where a cell in the center has eight neighbors)? Can one replace an $n \times n$ rectangular grid by some other arrangement of sites and still obtain a solution? To answer some of these questions, we first rephrase the problem in terms of cellular automata.

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2. Cellular Automata on Graphs

A *cellular automaton* is a discrete dynamical system that consists of an arrangement of basic components called *cells* together with a *transition rule.* Every cell can assume a finite number of possible *states;* the collection of possible states is called the *alphabet* of the automaton. We will here restrict our attention to the case where the set of possible states is {off,on}, which we represent by {0,1}. The transition rule of a cellular automaton is local in the sense that the state of cell x at time $t + 1$ depends only on the states of the neighboring cells (including cell x itself) at time t. Traditionally the cells are arranged on a finite or infinite, one- or two-dimensional grid. With a view toward the All-Ones Problem, however, we will allow arbitrary adjacencies between the sites of our cellular automata. More precisely, let G be a locally finite graph, i.e., a graph such that every vertex v in G is adjacent to only finitely many vertices in G. Let V denote the set of vertices of the graph G, and for any vertex v define the closed neighborhood N_v of v by

$$N_v := \{ u \in V \mid u \text{ adjacent to } v \} \cup \{v\}.$$

A *pattern* of G is a function

$$X : V \to \{0,1\}$$

from the collection of all vertices V to the alphabet {0,1}. We let C_G denote the collection of all patterns of G and identify a pattern $X : V \to \{0,1\}$ with a subset of the vertex set V, namely the collection of all cells v with $X(v) = 1$.

Now let v be a vertex of G. A local pattern at v on G is a function

$$X_v : N_v \to \{0,1\}.$$

Clearly, any pattern $X : V \to \{0,1\}$ determines a local pattern X_v at v, for each vertex v, by setting

$$X_v(u) := X(u) \qquad (1)$$

for all u in N_v. A local rule ρ_v associates every local pattern at v with a state: the state of cell v in the next generation. Given a collection of local rules $\langle \rho_v : v \in V \rangle$ the corresponding global rule ρ is obtained by applying the local rules simultaneously—or in parallel, to use a modern term—to all the local patterns:

$$\rho : C_G \to C_G$$
$$\rho(X)(v) := \rho_v(X_v)$$

where X_v is defined by (1).

Algebraically, the collection of all patterns forms a vector space over {0,1} construed as the two-element field $GF(2)$. This vector space will be called the pattern space. The vector sum of X and Y in C_G is the symmetric difference of X and Y. The singletons $\{v\}$, $v \in V$, provide a standard basis for C_G in the case of a finite graph G. To keep notation simple we will write v instead of $\{v\}$, so that in particular if $u \neq v$, then $v + u$ stands for the pattern $\{u,v\}$. We will write **1** for the "All-Ones" pattern: $\mathbf{1}(v) = 1$ for all v in V. Similarly, **0** denotes the empty set as an element of C_G. Any pattern Z such that $\rho(Z) = X$ is called a predecessor of X under the global rule ρ. A pattern that fails to have any predecessors is frequently called a Garden-of-Eden: once "lost" it remains inaccessible forever; see [1], [2].

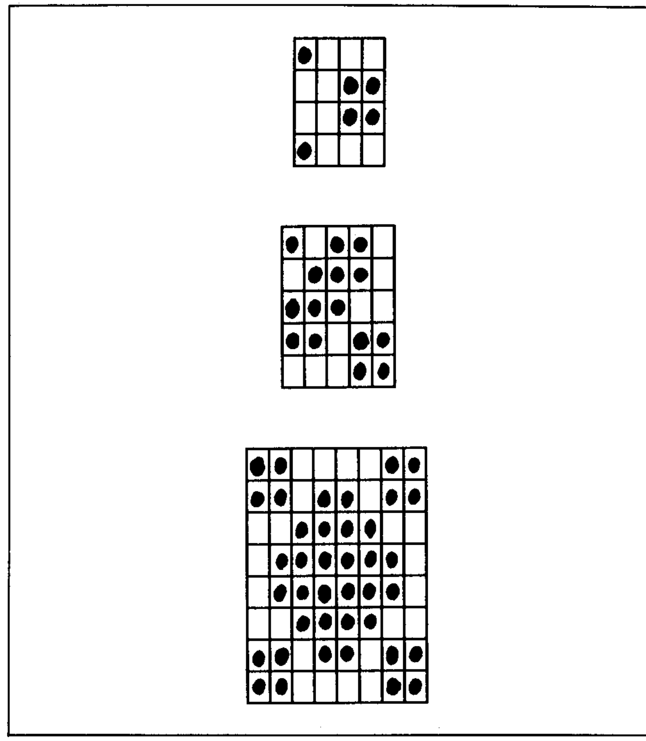

Figure 1. Solutions to the All-Ones Problem for square grids of size 4 × 4, 5 × 5, and 8 × 8. Note that a solution for the 6 × 6 grid is contained in the center part of the 8 × 8 solution.

To tackle our chessboard problem, define a global rule σ by defining a collection σ_v of local rules as follows:

$$\sigma_v(X_v) := \text{card}(X_v) \bmod 2.$$

In other words, $\sigma(X)(v) = 1$ iff $\text{card}(X_v)$ is odd. A graph together with rule σ will be called a σ-automaton. The predecessors of **1** in a σ-automaton are exactly the solutions of the All-Ones Problem, which can now be restated as follows:

Given an $n \times n$ grid graph G, is **1** *a Garden-of-Eden under rule σ in* G?

Before we answer the above question, let us digress briefly. The rule σ is a typical member of the class of "linear rules," which means that σ is linear as a map from the pattern space C_G to itself. In fact, $\sigma(X) = (A+I) \cdot X$, where pattern X is construed as a column vector and the adjacency matrix A of G is construed as

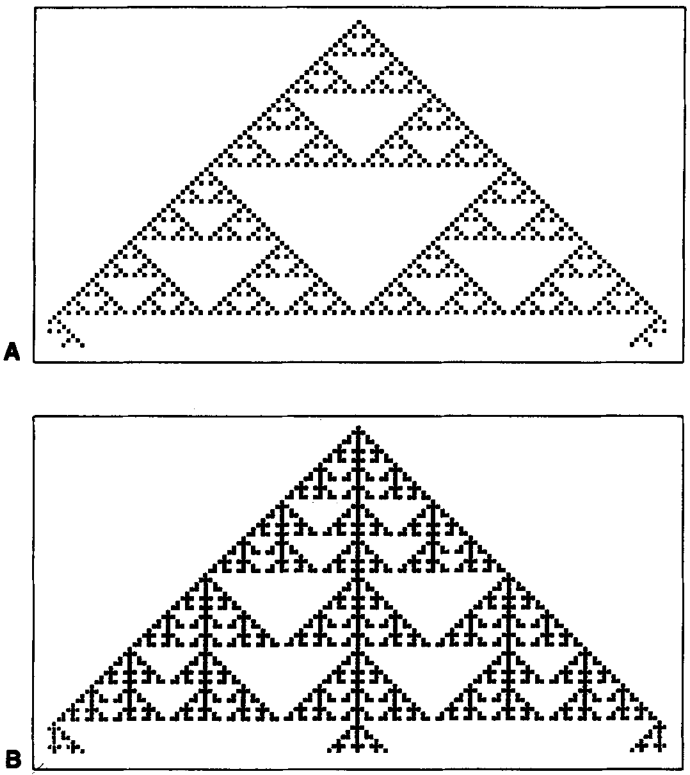

Figure 2A. The first 70 steps in the evolution of pattern ⟨0⟩ in P_∞ with rule σ^-. **B.** The first 70 steps in the evolution of pattern ⟨0⟩ in P_∞ with rule σ.

Table 1. Irreversible $n \times n$ grid automata, $n \leqslant 100$. Here d_n denotes the dimension of $K_{P_{n,n},\sigma}$.

n	d_n	n	d_n
4	4	53	2
5	2	54	4
9	8	59	22
11	6	61	40
14	4	62	24
16	8	64	28
17	2	65	42
19	16	67	32
23	14	69	8
24	4	71	14
29	10	74	4
30	20	77	2
32	20	79	64
33	16	83	6
34	4	84	12
35	6	89	10
39	32	92	20
41	2	94	4
44	4	95	62
47	30	98	20
49	8	99	16
50	8		

a matrix over the field {0,1}. Another rule very closely related to σ is obtained by excluding the vertex v from its own neighborhood: $\sigma^-{}_v(X_v) := \text{card}(X_v - v) \bmod 2$. Thus $\sigma^-(X) = A \cdot X$. Any rule ρ that is composed of linear local rules ρ_v is called a linear rule.

Despite the simplicity of linear rules like σ and σ^-, the cellular automata obtained from them show quite complicated behavior. A classic example is the two-sided infinite path graph P_∞ (the vertices of P_∞ are the integers, and v is adjacent to u if and only if $|u - v| = 1$). Figure 2 shows the evolution of the seed pattern {0} on P_∞ using various linear rules. The two-dimensional patterns generated in this fashion have non-integer fractal dimension and display such complicated geometric properties as self-similarity. The fractal dimension of the pattern asymptotically generated by rule σ is $\log_2(1 + \sqrt{5})$, whereas rule σ^- generates a pattern of dimension $\log_2 3$.

A wealth of information about these and other cellular automata rules as well as many fascinating pictures can be found in [5] through [8] (rules σ and σ^- are called rule 150 and 90, respectively, in [5]).

3. Finite σ-Automata

One of the basic questions of the theory of cellular automata concerns the global reversibility of the transition rule: can pattern X be reconstructed from $\rho(X)$? Locally irreversible systems are interesting from a thermodynamic point of view: unlike locally reversible systems, they may evolve from unordered to ordered states. Linear rules are locally irreversible in the sense that different patterns can lead to the same state in one particular cell in the next generation. Globally linear rules may well be reversible, however.

Let us assume from now on that G is a finite graph. For linear rules the situation is simple: rule ρ is injective if and only if it is surjective. Let $K_{G,\rho} \subseteq C_G$ be the kernel of rule ρ on G and set $d(G,\rho) := dim(K_{G,\rho}) = \log_2(\text{card}(K_{G,\rho}))$. Then the automaton G with rule ρ is reversible if and only if it has no Garden-of-Eden if and only if $d(G,\rho) = 0$. In particular, the All-Ones Problem can be solved in any finite reversible automaton. For rule σ the 3×3 grid is an example for a reversible automaton; indeed, all $(2^i - 1) \times (2^i - 1)$ square grids are reversible. A polyhedron gives rise to a graph that has the corners of the polyhedron as vertices and an edge joining two vertices if and only if an edge of the polyhedron joins the two corresponding corners. Of the five graphs obtained in this fashion from the Platonic solids, only the octahedron is reversible; whereas the tetrahedron, the cube, the dodecahedron, and the icosahedron are irreversible. The d-dimensional hypercube $\mathbf{2}^d$ is the graph with vertex set $\{0,1\}^d$ and there is an edge between v and u if and only if v and u have Hamming distance one (i.e., they disagree in exactly one component). The hypercube $\mathbf{2}^d$ is

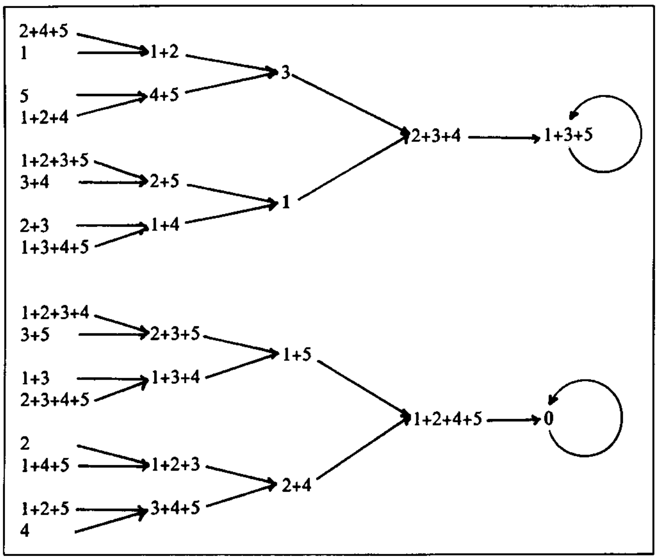

Figure 3. The transition diagram $\mathfrak{E}_{P_5}$.

reversible if and only if d is even; for a proof of these results see [4].

Let us agree on some notation for graphs: P_m (C_m) will denote the path (cycle) graph on m points $\{1, \ldots, m\}$ and $P_{m,n}$ the rectangular $m \times n$ grid graph. Table 1 shows the dimension d_n of $K_{P_{n,n},\sigma}$ for all irreversible $n \times n$ grid automata, $n \leq 100$. Note that dimension appears to be even for any square grid (it is not even in general for rectangular grids). Observe that $d_{2n+1} = 2 \cdot d_n + \delta_n$, where $\delta_n \in \{0,2\}$. Indeed, the table suggests $\delta_{2n+1} = \delta_n$. We are not aware of a proof for any of these conjectures.

Now suppose pattern X has predecessor Y. By linear algebra the collection of all predecessors of X is the affine subspace $\rho^{-1}(X) = Y + K_{G,\rho}$; thus the number of predecessors of X is either 0 or $2^{d(G,\rho)}$. The table provides an explanation for the extra difficulty of constructing an odd-parity cover for the 5×5 or 6×6 grid compared to the 4×4 grid: there are 16 solutions for 4×4 grid, 4 for the 5×5 grid, but only one for the 6×6 grid. One can show that in the 4×4 board one can pick an *arbitrary* pattern in the first row and always expand it to an odd-parity cover of the whole grid.

The predecessor relation is best expressed graphically by means of the transition diagram $\mathfrak{E}_{G,\rho}$ of G. Formally $\mathfrak{E}_{G,\rho}$ is a directed graph that has as vertices the patterns of G and an edge from X to Y if and only if $\rho(X) = Y$. For a linear rule ρ the in-degree of every point in $\mathfrak{E}_{G,\rho}$ is either 0 or $2^{d(G,\rho)}$. The out-degree is 1, of course, so the connected components of $\mathfrak{E}_{G,\rho}$ are all unicyclic. Indeed, they consist of one cycle and a number of $2^{d(G,\rho)}$-ary trees anchored on that cycle. Clearly, rule ρ is reversible on G if and only if the connected components of $\mathfrak{E}_{G,\rho}$ are cycles. The orbit $\{\sigma^i(X) \mid i \geq 0\}$ of any pattern X forms a one-generated monoid.

Figure 3 shows the transition diagram for the σ-automaton P_5. Notice that P_5 is irreversible but nonetheless pattern **1** has a predecessor. For arbitrary n, it is easy to determine all odd-parity covers for P_n:

n	odd-parity covers
$n \equiv 0 \pmod 3$	$2 + 5 + \ldots + (n-4) + (n-1)$
$n \equiv 1 \pmod 3$	$1 + 4 + 7 + \ldots + (n-3) + n$
$n \equiv 2 \pmod 3$	$1 + 4 + 7 + \ldots + (n-4) + (n-1)$ and $2 + 5 + 8 + \ldots + (n-3) + n$.

The corresponding pictures may be more convincing than algebra; see Figure 4.

So P_n is reversible if and only if $n \not\equiv 2 \pmod 3$. For $n \equiv 2 \pmod 3$ there exists exactly one predecessor of 0 (other than 0 itself), namely $1 + 2 + 4 + 5 + \ldots + (n-1) + n$; thus $d(P_n,\sigma) = 1$. Similarly, for the cycle C_n one has the following situation:

n	odd-parity covers
$n \not\equiv 0 \pmod 3$	**1**
$n \equiv 0 \pmod 3$	**1**, $1 + 4 + \ldots + (n-2)$, $2 + 5 + \ldots (n-1)$, $3 + 6 + \ldots + n$.

Thus **1** is a fixed point under σ in C_n and $d(C_n,\sigma) = 2$ for $n \equiv 0 \pmod 3$.

Finding odd-parity covers for each of the ladder graphs $P_{n,2}$, $n \geq 2$ is still straightforward. For even larger grids the search becomes hopeless, as patient readers may readily convince themselves. It is not clear that an odd-parity cover should exist for arbitrary grids.

As we have seen, reversibility is a sufficient but by no means necessary condition for **1** to have a predecessor. Indeed, we will show that an arbitrary finite graph G possesses an odd-parity cover, or, equivalently, the All-Ones Problem has a solution. We do not know of any purely geometric (read: graph theoretic) argument to prove this. We will present an algebraic proof.* To this end let $v_1, \ldots, v_n$ be the standard base of the pattern space C_G and let C^*_G be the dual space equipped with the dual base $v_1^*, \ldots, v_n^*$. Every vector $X = \Sigma\xi_i v_i$ in C_G gives rise to a linear functional $X^* := \Sigma\xi_i v_i^*$ in C^*_G; define

$$\langle Z,X\rangle := X^*(Z).$$

C_G is finite dimensional and therefore isomorphic to C^*_G under the map $X \mapsto X^*$; we will identify both spaces to keep notation manageable. Z is said to be perpendicular to X, in symbols $Z \perp X$, if $\langle Z,X\rangle = 0$. Hence Z and X are perpendicular if and only if their intersection has even cardinality. Z is perpendicular to a subspace W of C_G if and only if $Z \perp X$ for all $X \in W$. Define the orthogonal complement $W^\perp$ of the subspace W by $W^\perp := \{Z \in C_G \mid Z \perp W\}$. Using the notion of orthogonal complement one can now characterize

* R. Tindell pointed out that there is a purely graph theoretic proof for trees (connected acyclic graphs). See pages 31–32.

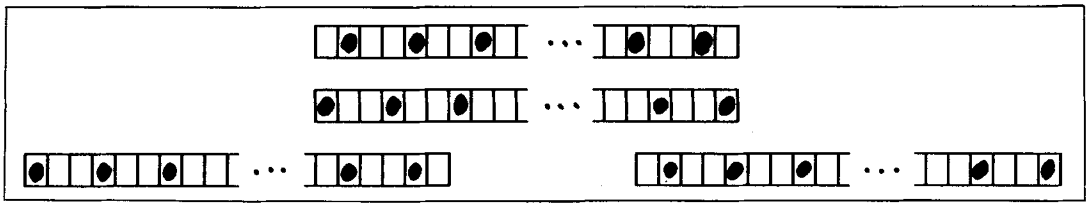

Figure 4. Odd-parity covers for P_n.

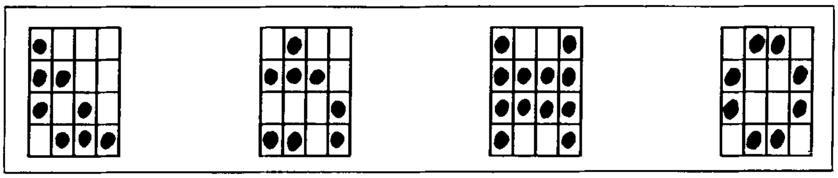

Figure 5. A basis for $K_{P_{4,4}}$.

the patterns that lie in the range of σ.

3.1 THEOREM: *Let G be an arbitrary finite graph and let ρ be one of the rules σ or σ^-. Then pattern X has a predecessor in G under rule ρ if and only if X is perpendicular to the kernel of ρ.*

Proof: The neighborhood (and adjacency) matrix of a graph is symmetric. Thus ρ is selfadjoint and

$$\langle \rho(X),Y\rangle = \langle X,\rho(Y)\rangle.$$

for any two patterns, X,Y. Also note that $\langle .\,,\,.\rangle$ is nondegenerate in the sense that

$$X = \mathbf{0} \text{ iff } \langle Z,X\rangle = 0 \text{ for all } Z \text{ in } C_G.$$

Now let W be the range of ρ. We get

$$\begin{aligned} Y \in W^\perp &\Leftrightarrow \text{for all } X \text{ in } C_G\ \langle \rho(X),Y\rangle = 0 \\ &\Leftrightarrow \text{for all } X \text{ in } C_G\ \langle X,\rho(Y)\rangle = 0 \\ &\Leftrightarrow \rho(Y) = 0, \end{aligned}$$

i.e., $W^\perp = K_{G,\rho}$. By a well-known theorem of algebra $W^{\perp\perp} = W$. Hence $W = K_{G,\rho}{}^\perp$ and the proof is complete. ■

3.2 THEOREM: *The All-Ones Problem has a solution in any finite graph.*

Proof: The key observation is that any pattern X such that $\sigma(X) = \mathbf{0}$ must have even cardinality. To see this, first note that every vertex x in X has odd degree in the subgraph G with vertex set X. Second, recall the handshaking theorem: the number of odd-degree points in any finite graph is even. Now, $\langle X,\mathbf{1}\rangle$ is the cardinality of X modulo 2, so $\langle X,\mathbf{1}\rangle = 0$. Hence $\mathbf{1}$ is perpendicular to $K_{G,\sigma}$, and has a predecessor under rule σ by Theorem 3.1. ■

Thus $\mathbf{1}$ fails to be a Garden-of-Eden under rule σ in any finite graph. We note in passing that Theorem 3.2 can be generalized to locally finite graphs. For finite graphs one can compute a basis for the affine subspace of all odd-parity covers in polynomial time, i.e., in a number of steps polynomial in the number of points of the graph. Surprisingly, the problem of determining an odd-parity cover of *minimal cardinality* turns out to be computationally difficult: the corresponding optimization problem is *NP*-hard; see [4]. Thus it is unlikely that any efficient (read: deterministic polynomial time) algorithm exists to construct an odd-parity cover of minimal size for a given graph.

For certain simple graphs like $P_{4,4}$, where the kernel of σ_G is known explicitly, Theorem 3.1 provides an easy test of whether a given pattern has a predecessor. Figure 5 shows a basis for the kernel of σ on $P_{4,4}$. It is easy to check that no singleton can have a predecessor. Or consider the path P_n, where $n \equiv 2 \pmod 3$ and $n > 2$. Then in P_n exactly the patterns of the form $X = X_0 \cup X_1$, where X_0 is a subset of $1 + 2 + 4 + 5 + \ldots + (n-1) + n$ of even cardinality and X_1 is a subset of $3 + 6 + \ldots + (n-2)$, have a predecessor under rule σ. Hence exactly half the patterns have a predecessor, and the other half are Gardens-of-Eden.

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