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Intersecting semi-disks and the synergy of three quadratic forms

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Abstract

In this paper, we study the Diophantine equation $x^2 = n^2 + mn + np + 2mp$ with m, n, p, and x being natural numbers. This equation arises from a geometry problem and it leads to representations of primes by each of the three quadratic forms: $a^2 + b^2$, $a^2 + 2b^2$, and $2a^2 - b^2$. We show that there are infinitely many solutions and conjecture that there are always solutions if $x \geq 5$ and $x \neq 7$; and, we find a parametrization of the solutions in terms of four integer variables.

1 Introduction

The following geometry problem recently appeared in various mathematics groups on social media ([3]). Referring to Figure 1, we have two semi-disks with centers at O and O' intersecting as shown.

Problem: *Given that $|DE| = m$, $|EF| = n$, and $|FC| = p$, find the diameter of the smaller semi-disk, $|AB| = 2x$, as a function of m, n, and p.*

The original problem was formulated with $m = 3$, $n = 7$, and $p = 2$, which gave the answer: $|AB| = 4\sqrt{6}$. Naturally, one may ask to solve this in general and perhaps some other related questions such as:
Q1) When is the value of $|AB|$ an integer given m, n, and p are non-negative

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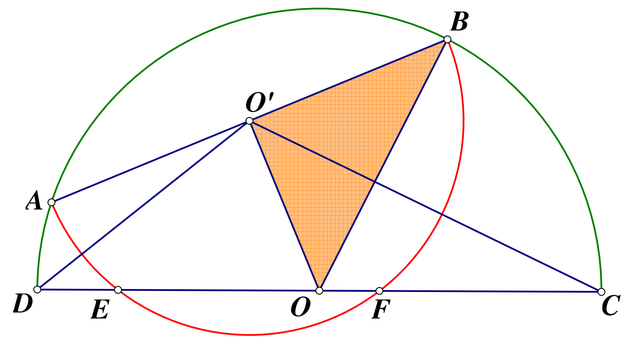

Figure 1: Intersecting semi-disks

integers?
Q2) What is the smallest integer value of $|AB|$, given m, n, and p are positive integers?
Q3) Is the set of integer values for $|AB|$ infinite?
Q4) What is the set of the integer values of $|AB|$ if we assume m, n, and p are natural numbers?
We are going to address all of these questions and relate this problem to the famous Fermat's characterizations of prime representations:

Theorem A (Fermat,[2]): *A prime number p can be written as $p = x^2 + y^2$ for some x and y integers if and only if $p = 2$ or $p \equiv 1$ (mod 4).*
This theorem is proved in [2] using Euler's ideas in Chapter I pages 7-12.

Theorem B (Euler,[1],[2],[8]): *A prime number p can be written as $p = x^2 + 2y^2$ for some x and y integers if and only if $p = 3$ or $p \equiv 1$ or 3 (mod 8).*

Our techniques involve the use of integer quaternions and they can be used in similar number theoretic problems of the same nature (see [7]). As a byproduct, we discovered a similar result to Theorem 1.6 from [8], which is stated in Theorem 4.1 in Section 4.

2 Solution of the geometry problem

The power of the point, D with respect to the small circle with center O', gives

$$|DE| \cdot |DF| = |DO'|^2 - x^2,$$

where x is the radius of the small circle. Similarly, the power of the point C is

$$|CF| \cdot |CE| = |CO'|^2 - x^2.$$

By the Apollonius median of a triangle formula, we have

$$4|OO'|^2 = 2(|DO'|^2 + |CO'|^2) - |DC|^2.$$

Now, we substitute to get

$$4|OO'|^2 = 2(|DE| \cdot |DF| + x^2 + |CF| \cdot |CE| + x^2) - |DC|^2.$$

Using the original data, we are going to substitute everything in terms of m, n, and p and simplify:

$$4|OO'|^2 = 2[m(m+n) + p(n+p) + 2x^2] - (m+n+p)^2.$$

On the other hand, in the right triangle $OO'B$, $|OO'|^2 + x^2 = |OB|^2$. Since $|OB| = \frac{DC}{2}$, then the equation above turns into

$$4(|OB|^2 - x^2) = 2[m(m+n) + p(n+p) + 2x^2] - (m+n+p)^2.$$

Solving this for x^2, we get:

$$8x^2 = 2(m+n+p)^2 - 2[m(m+n) + p(n+p)].$$

Simplifying and solving for $|AB|$, gives:

$$|AB| = 2x = \sqrt{n^2 + mn + np + 2mp}. \tag{1}$$

This formula (1) solves our geometry question in general. We observed that if $m = 3$, $n = 7$, and $p = 2$, we obtain $|AB| = 4\sqrt{6}$. We will later discuss the smallest integer value for $|AB|$, with m, n, and p being positive integers, in the last section.

3 Reduction to intersection of quadratic forms

At this point, we are interested in integer solutions of equation (1); and, we continue by equivalently writing equation (1) in the following form:

$$4x^2 = n^2 + n(m+p) + 2mp$$

To proceed, we complete the square:

$$4x^2 + \frac{(m+p)^2}{4} = n^2 + n(m+p) + 2mp + \frac{(m+p)^2}{4} \implies$$

$$4x^2 + \frac{m^2+2mp+p^2}{4} - 2mp = (n + \frac{m+p}{2})^2.$$

We multiply everything by 4, simplify, and then complete the square a second time to obtain our final equation:

$$16x^2 + m^2 - 6mp + 9p^2 = (2n+m+p)^2 + 8p^2 \implies$$

$$(4x)^2 + (m-3p)^2 = (2n+m+p)^2 + 2(2p)^2. \tag{2}$$

Now, we need to look for the numbers N which are the sum of two squares and, also at the same time, the sum of three squares in which two are equal to each other. Such numbers are the numbers in the sequence: 1, 2, 4, 8, 9, 16, 17, 18, 25, 32, 34, ... etc, which is the sequence A155562 in the OEIS. For example,

$$17 = 4^2 + 1^2 = 3^2 + 2(2^2), \text{or}$$

$$34 = 5^2 + 3^2 = 4^2 + 2(3^2).$$

These numbers are basically at the intersection of two sets of numbers which are represented by two quadratic forms, i.e., $a^2 + b^2$ and $a^2 + 2b^2$. If these numbers are prime numbers, they are given by Theorem 1 and Theorem 2, stated in the Introduction.

Let us work out the case $N = 34$ and solve for m, n, p, and $2x$. Clearly, from equation (2), we get $p = \frac{3}{2}$. We can choose $m - 3p = -3$, which gives $m = \frac{3}{2}$. Then, $2n + m + p = 4$ implies $n = \frac{1}{2}$. Finally, $2x = \frac{5}{2}$. Scaling with a factor of 2, we obtain the integer solution of $m = p = 3$, $n = 1$, and $2x = 5$. We will show that this is the smallest integer solution for $|AB|$ given m, n, and p are natural numbers.

Obviously, not every writing of $N = U^2+V^2 = K^2+2L^2$ gives a meaningful solution for m, n, and p. For example, if $N = 73 = 8^2 + 3^2 = 1^2 + 2(6^2)$, we need to have $p = 3$ and $m - 3p = \pm 8$ or $m - 3p = \pm 3$. So, m can be 17, 1, 12, or 6. Then, $2n + m + p = 1$. Therefore, n cannot be positive for any of the options of m.

In general, we need to have the following conditions satisfied: $p = \frac{L}{2}$, $m = \frac{3L}{2} \pm U \geq 0$ and $K \geq 2L \pm U$, or $m = \frac{3L}{2} \pm V \geq 0$ and $K \geq 2L \pm V$ (the $\pm$ go the same way in the inequalities). To answer question Q1 in a way that is somewhat complete, we have the following theorem.

Theorem 3.1. *The value of $|AB|$ is an integer if and only if there exists a natural number N that can be written as $N = U^2 + V^2 = K^2 + 2L^2$, where U and V are integers and K and L are positive integers, with $|V| = 2AB$, $3L + 2U \geq 0$, $K \geq U + 2L$, and U and L are even.*

4 From Fermat to quaternions

Putting Theorem A and Theorem B together implies that a prime number p can be written as

$$p = x^2 + y^2 = u^2 + 2v^2$$

if and only if p is of the form $p = 8k + 1$, where k is an integer. Examples of such primes are:

$$17 = 4^2 + 1^2 = 3^2 + 2(2)^2,$$
$$41 = 5^2 + 4^2 = 3^2 + 2(4)^2,$$
$$73 = 8^2 + 3^2 = 1^2 + 2(6)^2, \ldots \text{ etc.}$$

Theorem 1.2 (II) in [8] states that a prime can be written as $2E^2 - F^2$ if and only if $p = 2$ or $p \equiv \pm 1 \pmod 8$. This implies that all the primes we have written above can be also written as $p = U^2 + V^2 = 2E^2 - F^2$ or

$$U^2 + V^2 + F^2 = 2E^2. \tag{3}$$

In [6], a similar equation was parameterized and the idea of integer quaternions was used. The quaternions are just the set of 4-dimensional vectors $q = a + bi + cj + dk$ where a, b, c, and d are real numbers, which can be added on components and multiplied by using the rules for i, j, and k which are given by:

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i \text{ and } ki = -ik = j.$$

If $q = a+bi+cj+dk$, the conjugate of q is $\overline{q} = a-bi-cj-dk$ and the norm of q is $N(q) = a^2 + b^2 + c^2 + d^2$. Quaternions whose components are all integers, are referred here as integer quaternions. It is well known that the norm N is multiplicative. In other words, $N(q_1q_2) = N(q_1)N(q_2)$, and $\overline{q_1q_2} = \overline{q_2}\ \overline{q_1}$.

Following the same technique as in [6], we can think of the equality (3) as the norm of $q(i+j)\overline{q}$, whose conjugate is $q(\overline{i}+\overline{j})\overline{q} = -q(i+j)\overline{q}$ which means the real part of $q(i+j)\overline{q}$ is zero. Therefore,

$$Ui + Vj + Fk = q(i+j)\overline{q}$$

where $q = t + ui + vj + wk$ is an integer quaternion. As a result we obtain

$$N(Ui + Vj + Fk) = N(q(i+j)\overline{q}) = N(q)N(i+j)N(\overline{q}) = 2N(q)^2.$$

This gives the parametrization of (3):

$$\begin{cases} U = u^2 + 2uv + t^2 - 2tw - w^2 - v^2, \\ V = 2uv + v^2 + 2tw - w^2 + t^2 - u^2, \\ F = 2uw + 2vw - 2tv + 2tu \text{ and} \\ E = u^2 + v^2 + w^2 + t^2. \end{cases} \tag{4}$$

From here, to deduce a parametrization of the equality $A^2 + B^2 = C^2 + 2D^2$, we use Lagrange's identity $(\alpha^2 + \beta^2)(\gamma^2 + \theta^2) = (\alpha\gamma + \beta\theta)^2 + (\alpha\theta - \beta\gamma)^2$:

$$U^2 + V^2 = 2E^2 - F^2 = 2(u^2 + v^2 + w^2 + t^2)^2 - (2uw + 2vw - 2tv + 2tu)^2 \implies$$

$$U^2+V^2 = 2[(u^2+v^2)-(w^2+t^2)]^2+8(u^2+v^2)(w^2+t^2)-(2uw+2vw-2tv+2tu)^2 \implies$$

$$U^2+V^2 = 2[(u^2+v^2)-(w^2+t^2)]^2+4[2(uw-tv)^2+2(vw+tu)^2-(uw+vw-tv+tu)^2]$$

or finally

$$U^2 + V^2 = 2[(u^2 + v^2) - (w^2 + t^2)]^2 + 4(uw - tv - vw - tu)^2 = K^2 + 2L^2.$$

Hence, we obtain

$$\begin{cases} A = u^2 + 2uv + t^2 - 2tw - w^2 - v^2 \\ B = 2uv + v^2 + 2tw - w^2 + t^2 - u^2 \\ C = 2(vw + tv - uw + tu) \\ D = u^2 + v^2 - t^2 - w^2 \end{cases} . \tag{5}$$

If we multiply the equation $U^2+V^2=K^2+2L^2$ by 2, it becomes $(U+V)^2+(U-V)^2=2K^2+4L^2$. Dividing this equation by 4, we get $(\frac{U+V}{2})^2+(\frac{U-V}{2})^2=L^2+\frac{K^2}{2}$. Since $(\frac{U+V}{2})=t^2-w^2+2uv$ and $(\frac{U-V}{2})=u^2-2tw-v^2$, solving for U and V and switching K and L roles, we obtain a new parametrization of the equality $U^2+V^2=K^2+2L^2$:

$$\begin{cases} U = t^2 - w^2 + 2uv \\ V = u^2 - v^2 - 2tw \\ K = u^2 + v^2 - t^2 - w^2 \\ L = vw + tv - uw + tu \end{cases} . \tag{6}$$

Hence, we recover the parametrization given in [5].

The beginning of this discussion was mostly about primes. However, one can prove that in general, a similar result to Theorem 1.6 in [8]:

Theorem 4.1. *Given the sets*

$$\mathcal{A} := \{t \in \mathbb{Z} | t = 2x^2 - y^2, x, y \in \mathbb{Z}\},$$

$$\mathcal{B} := \{t \in \mathbb{Z} | t = x^2 + y^2, x, y \in \mathbb{Z}\}, \textit{and}$$

$$\mathcal{C} := \{t \in \mathbb{Z} | t = 2x^2 + y^2, x, y \in \mathbb{Z}\}$$

then $\mathcal{A}\cap\mathcal{B}\subset\mathcal{C}$, $\mathcal{A}\cap\mathcal{C}\subset\mathcal{B}$, $\mathcal{B}\cap\mathcal{C}\subset\mathcal{A}$.

The proof to this theorem follows the same steps as Theorem 1.6 in [8] and the details of the argument will appear in subsequent work.

Definition 4.2. *Given three quadratic forms, we say that they form a trinity, if the sets of integers that they represent, say* $\mathcal{A}$, $\mathcal{B}$, *and* $\mathcal{C}$, *satisfy:* $\mathcal{A}\cap\mathcal{B}\subset\mathcal{C}$, $\mathcal{A}\cap\mathcal{C}\subset\mathcal{B}$, $\mathcal{B}\cap\mathcal{C}\subset\mathcal{A}$.

This brings the question of how many triples of quadratic forms form a trinity.

5 Answers to our questions

An answer to question Q1 in a way that is somewhat complete, is contained in Theorem 5.1. We can use the parametrization (5) and (6) to be more precise in this theorem but it is difficult to say when the inequalities involved are satisfied. However, a special case can be easily worked out: if $u=w=0$, we obtain $U=t^2-v^2$, $V=t^2+v^2$, $K=2tv$, and $L=v^2-t^2$. To make K and L positive, we can choose $v>t>0$. This gives $AB=\frac{v^2+t^2}{2}$. This shows that we have infinitely many solutions for $|AB|$ if we take v and t of the same parity and $2tv > v^2 - t^2$. This answers question Q3 and partially answers question Q4.

We notice that these examples have $m = p$. To answer question Q2, let us prove that 5 is the smallest integer value of AB, given m, n, and p are positive integers.

Proposition 5.1. *The smallest integer value of $2x$ where $4x^2 = n^2 + mn + np + 2mp$, with m, n, and p as natural numbers, is* 5.

Proof: By way of contradiction from equation (1), if we assume that $2x \leq 4$, we obtain $n^2 + mn + np + 2mp \leq 16$. Since $n \geq 1$, this implies $1 + m + n + p + 2mp \leq 16$ or equivalently $(2m+1)(2p+1) \leq 31$. Since the situation is perfectly symmetric in m, n, and p, we may assume that $m \leq p$. This implies $(2n+1)^2 \leq 31$ or $2n+1 \leq 5$ which means $m = 1$ or $m = 2$. If $m = 1$, then $2p + 1 \leq \frac{31}{3}$, which implies that p is in $1, 2, 3, 4$.

Case 1: $m = p = 1$
Now, we substitute in equation (1) and obtain $2x = \sqrt{n^2 + 2n + 2} = \sqrt{(n+1)^2 + 1}$, which is clearly not an integer for any value of $n \geq 1$.

Case 2: $m = 1$, $p = 2$
In this case, we have $2x = \sqrt{n^2 + 3n + 4}$, which is strictly between $n + 1$ and $n + 2$. Therefore, $2x$ cannot be an integer.

Case 3: $m = 1$. $p = 3$
Here, we have $2x = \sqrt{n^2 + 4n + 6}$ which is strictly between $n + 2$ and $n + 3$. Hence, $2x$ cannot be an integer.

Case 4: $m = 1$, $p = 4$
Now, $2x = \sqrt{n^2 + 5n + 8}$ which is strictly between $n + 2$ and $n + 3$. Thus, $2x$ cannot be an integer.

Case 5: $m = p = 2$
In this case, $2x = \sqrt{n^2 + 4n + 8}$ which is strictly between $n + 2$ and $n + 3$. $\square$

Let us denote the set of values $|AB|$ for which (1) has positive integer solutions by $\mathcal{S}$. One simple observation is that $\mathcal{S}$ has the property "$x \in \mathcal{S} \implies kx \in \mathcal{S}$ for every k positive integer". Computer searches show that $\mathcal{S} = \{k \in \mathbb{N} | k \geq 5\} \setminus \{7\}$. The reason why 7 is excluded is because the equation (1) is equivalent with

$$(2m + n)(2p + n) = 98 - n^2 \tag{7}$$

and this restricts n to be in the set 1 through 6. If n is odd, $98 - n^2$ is a prime and that makes equation (7) impossible. If n is even, the left hand side of (7) is divisible by 4 but the right hand side is not. We observed that

$$6 \in \mathcal{S} \text{ for } m = 2, n = 3, \text{and } p = 3$$

$$8 \in \mathcal{S} \text{ for } m = 2, n = 3, \text{and } p = 7$$

$$14 \in \mathcal{S} \text{ for } m = 8, n = 1, \text{and } p = 11.$$

This shows that we can concentrate on odd numbers in $\mathcal{S}$:

$$(2m+n)(2p+n) = 2(2k+1)^2 - n^2 \tag{8}$$

In order to get a solution from this equation, it is necessary to write the right hand side of (8) as a product of two factors which are at least n. Now, it turns out that for most values of k in equation (8), this can be accomplished with n is in the set $\{1, 2, 3, 4, 5, 6\}$. The first odd number for which $n \geq 7$ is $63091 = 7 \cdot 9013$. However,

$$9013 \in \mathcal{S} \text{ for } m = 8, n = 1, \text{and } p = 4778480.$$

We are going to close this paper with the conjecture that $\mathcal{S} = \{k \in \mathbb{N} | k \geq 5\} \setminus \{7\}$.

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