# Problem C NAW 5/6 nr. 2 June 2005

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## The problem

### Introduction

In what follows, $P$ stands for the set consisting of all odd prime numbers; $M$ is the set consisting of all natural 2-powers $1, 2, 4, 8, 16, 32, ...$; $T$ is the set consisting of all positive integers that can be written as a sum of at least three consecutive natural numbers.

1. Show that the set theoretic union of $P$, $M$ and $T$ coincides with the set consisting of all the natural numbers..
2. Show that the sets $P$, $M$ and $T$ are pairwise disjoint.
3. Given $b \in T$, determine $t(b)$ in terms of the prime decomposition of $b$, where by definition $t(b)$ stands for the minimum of all those numbers $t > 2$ for which $b$ admits an expression as sum of $t$ consecutive natural numbers.
4. Consider the cardinality $C(b)$ of the set of all odd positive divisors of some element $b$ of $T$. Now think of expressing this $b$ in all possible ways as a sum of at least three consecutive natural numbers. Suppose this can be done in $S(b)$ ways. Determine the numerical connection between the numbers $C(b)$ and $S(b)$.

Remark: In this problem we clearly follow the convention not to include zero in the natural numbers.

### Solution

First let

$$a = p_0^{e_0} \cdot p_1^{e_1} \cdots p_m^{e_m} \tag{1}$$

be the 'prime decomposition' of a positive integer $a$ with $p_0 = 2$, $e_0 \geq 0$ and $p_1, ..., p_m$ odd primes with $e_i > 0$ for $i = 1, ..., m$. We want to write $a$ as the sum of $k$ consecutive natural numbers starting with $n$.

$$a = n + (n+1) + \cdots (n+k-1) = k \cdot n + \frac{k(k-1)}{2} = k(2n+k-1)/2$$

So

$$2a = k \cdot (2n + k - 1) \tag{2}$$

We define $k$ to be the smallest factor, thus $k < \sqrt{2a}$. We observe that only one of the factors is odd.

**Part 1 and 2**

When $a$ is a power of 2 we can only have $k = 1$. A power of two is clearly not an odd prime and vice versa. An odd prime can only be written as a sum of 2 consecutive natural numbers ($k = 2$). For all other positive integers we have at least one odd prime divisor $p_i$. Let $k = p_i \geq 3$ and $n = (2a/k - k + 1)/2$. It follows that $a$ can be written as the sum of at least three consecutive positive integers starting with $n$. The rest is trivial.

**Part 3**

Let $b = a \in T$ and $p_1$ the smallest odd prime divisor of $b$. From (2) it follows that if $e_0 = 0$, meaning $b$ is odd, we have $t(b) = p_1$, else $t(b) = min(2^{e_0+1}, p_1)$.

**Part 4**

Let again be $b = a \in T$. We use the prime decomposition (1) to find the number of all odd divisors of $b$. We easily see that this number must be
$(e_1 + 1) \cdot (e_2 + 1) \cdots (e_m + 1)$. So $C(b) = (e_1 + 1) \cdot (e_2 + 1) \cdots (e_m + 1)$.
$S(b)$ is the number of ways $b$ can be expressed as sum of at least three positive integers.
From (2) it follows that for each odd divisor of $b$ we can find a $k < \sqrt{2a}$. We must exclude $k = 1$ and $k = 2$. Only in case of an odd $b$ we can have $k = 2$, so $S(b) = C(b) - 2$ if $b$ is odd and $S(b) = C(b) - 1$ if $b$ is even.