Problem A NAW 5/5 nr. 4 december 2004

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December 2004

The problem

Introduction

1. Show that there exist infinitely many $n \in \mathbf{N}$, such that $S_n = 1+2+...+n$ is a square.
2. Let $a_1, a_2, a_3, ...$ be those squares. Calculate $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution

We know $S_n = \frac{1}{2}n(n+1)$ so we have to solve the diophantine equation

$$\frac{1}{2}n(n+1) = m^2 \tag{1}$$

Rewriting gives $4n^2 + 4n = 8m^2$ or $(2n+1)^2 - 1 = 2(2m)^2$. Substituting $x = 2n+1$ and $y = 2m$ we get the famous Pell equation

$$x^2 - 2y^2 = 1 \tag{2}$$

with an infinite number of solutions $(3,2), (17,12), (99,70)...$ with corresponding $n = 1, 8, 49, 288, ...$
A well known result gives solutions of (2)

$$x_k = \frac{(3+2\sqrt{2})^k + (3-2\sqrt{2})^k}{2}$$

and

$$y_k = \frac{(3+2\sqrt{2})^k - (3-2\sqrt{2})^k}{2\sqrt{2}}$$

The sequence $\{a_i\}_{i=1,2,3,...}$ starts with $1, 36, 1225, 41616, ...$ and can be calculated with

$$a_k = \frac{y_k^2}{4} = \frac{((3+2\sqrt{2})^k - (3-2\sqrt{2})^k)^2}{32} \tag{3}$$

which can be rewritten as

$$a_k = \frac{((17+12\sqrt{2})^k + (17-12\sqrt{2})^k) - 2}{32} \tag{4}$$

And so

$$\frac{a_{k+1}}{a_k} = \frac{((17+12\sqrt{2})^{k+1} + (17-12\sqrt{2})^{k+1}) - 2}{((17+12\sqrt{2})^k + (17-12\sqrt{2})^k) - 2} \tag{5}$$

We easily see that $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 17 + 12\sqrt{2}$.

Remark

Finding a triangular number S_n that is cubic, except the trivial 1, would be spectacular. As we try to solve

$$\frac{1}{2}n(n+1) = m^3$$

substituting $X = 2m$ and $Y = 2n+1$ we get the elliptic curve with equation

$$Y^2 = X^3 + 1 \tag{6}$$

We find this curve as A36 in the Cremona table. The torsion group is of order 6 with real members (-1,0), (0,-1), (0,1), (2,-3) and (2,3). This means the only cubic triangular number is 1.
Moreover the above equation is also known from the Catalan conjecture, or should we say Catalan theorem: the only non-trivial integer powers that differ 1 are 2^3 and 3^2.

References

[1] http://www.research.att.com/projects/OEIS?Anum=A001110