Jaap Spies
Landschaplaan 88
7824 BM Emmen
jaapspies@gmail.com

Problemenrubriek Permanent solutions of Problem 29

Dancing School problems

The Dancing School Problems originated from Problem 29 of the March 2002 issue of the Nieuw Archief voor Wiskunde. The author of this problem, Lute Kamstra, found his inspiration in juggling. A flawed solution was published and after that the problem was declared open again. The present author, challenged by the editor, found a solution in January 2003. At the time, the Problem Section of the NAW was merged into the UWC, and there were still a few solutions to problems left unpublished, such as this solution. The methods used to solve this problem are connected to Graph Theory.

The following Dancing School Problem is equivalent to Problem 29: Imagine a group of n ($n > 0$) girls ranging in integer length from m to $m + n - 1$ cm and a corresponding group of $n + h$ boys ($h \geq 0$) with length ranging from m to $m + n + h - 1$ cm. Clearly, m is the minimal length of both boys and girls.

The location is a dancing school. The teacher selects a group of n out of $n + h$ boys. A girl of length l can now choose a dancing partner out of this group of n boys, someone either of her own length or taller up to a maximum of $l + h$. How many 'matchings' are possible?

A Solution

Let us return to the original problem of Lute Kamstra. Let $n > 0$ and $h \geq 0$ be given and let $A = \{a_1, a_2, \ldots, a_n\}$ be a subset of $\{1, 2, \ldots, n + h\}$, where we assume that $a_i < a_j$ whenever $i < j$. The problem asks us to count the number of bijections $\pi : \{1, 2 \ldots, n\} \to A$ that satisfy $k \leq \pi(k) \leq k + h$ for all k. We can code the restrictions in an n by n $(0,1)$-matrix $B = [b_{ij}]$ by setting $b_{ij} = 1$ if and only if $i \leq a_j \leq i + h$. The set of S_B of permitted bijections can then be characterized as follows:

$$S_B = \{\pi \mid \prod_{i=1}^{n} b_{i\,\pi(i)} = 1\}.$$

The number of elements of S_B is given by the following formula

$$|S_B| = \sum_{\pi} \prod_{i=1}^{n} b_{i\,\pi(i)},$$

the latter expression is the *permanent* of the matrix B, which we denote Per (B).

The matrix B can be interpreted as an incidence matrix of a bipartite graph G with vertices in $X = \{1, 2, ..., n\}$ and $Y = A = \{a_1, a_2, ..., a_n\}$. An edge of G is a pair (i, a_j) with $b_{ij} = 1$.

The Permanent of a Matrix

In 1812 permanents were introduced independently by Binet and Cauchy. They are defined as follows.
Let $A = (a_{ij})$ be an $m \times n$ matrix over any commutative ring, $m \leq n$. The *permanent of A* is given by

$$\text{Per}\,(A) = \sum_{\pi} a_{1\pi(1)} a_{2\pi(2)} \cdots a_{m\pi(m)} \qquad (1)$$

where the summation extends over all one-to-one functions π from $\{1, ..., m\}$ to $\{1, ..., n\}$.
The product $a_{1\pi(1)} a_{2\pi(2)} \cdots a_{m\pi(m)}$ is called a *diagonal product*. So the permanent of an $m \times n$ matrix A is the sum of all the diagonal products of A.
Binet gave algorithms for $m = 2, 3$ and 4. There are general algorithms by Binet/Minc and Ryser. See [3] and [4]. For an implementation of Ryser's algorithm in SAGE (Software for Algebra and Geometry Experimentation) see [8]. As far as we know SAGE [5] is the only Computer Algebra System (CAS) with a permanent function for rectangular matrices.

The Permanent of a Matrix of order n

The *permanent of a square matrix A of order n* is defined as Per $(A) = \sum_{\pi} a_{1\pi(1)} a_{2\pi(2)} ... a_{n\pi(n)}$ where we sum over all $n!$ possible permutations π of $1, 2, \ldots, n$.
What is this rather strange cousin of the determinant? Let us try to make this permanent more visible. We define a vector $\bar{x} = (x_1, x_2, \ldots, x_n)^T$ and a vector $\bar{y} = (y_1, y_2, \ldots, y_n)^T$ with $\bar{y} = A\bar{x}$. We define a multivariate polynomial

$$P(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} y_i = (a_{11}x_1 + \cdots + a_{1n}x_n) \cdot$$
$$(a_{21}x_1 + \cdots + a_{2n}x_n) \cdots (a_{n1}x_1 + \cdots + a_{nn}x_n)$$

Expanding this polynomial and summing the terms with $x_1 x_2 \cdots x_n$, we get

$$\sum_{\pi} a_{1\pi(1)} x_{\pi(1)} \cdot a_{2\pi(2)} x_{\pi(2)} \cdots a_{n\pi(n)} x_{\pi(n)}$$
$$= \left(\sum_{\pi} a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)} \right) \cdot x_1 x_2 \cdots x_n$$

So Per (A) is the coefficient of the term with $x_1 x_2 \cdots x_n$.

How can we compute a permanent efficiently?
We know now where to find the permanent, but how can we compute it? There is the famous Ryser's Algoritm (see [3], p. 199-200). Here we give an alternative.
We define $Q(\bar{x}) = (\prod_{i=1}^{n} x_i) \cdot P(x_1, x_2, \ldots, x_n)$, and sum $Q(\bar{x})$ over all possible $\bar{x}$ with $x_i = \pm 1$, for i=1, 2, . . ., n.
As we can easily see only the term $\mathrm{Per}\,(A) \cdot x_1 x_2 ... x_n$ of $P(x_1, x_2, ..., x_n)$ always contributes to this sum ($x_i^2 = 1$ for $i = 1, 2, ..., n$). A term t of $Q(\bar{x})$ with factor x_k missing in $P(x_1, ..., x_n)$ is counted once as t and once as $-t$, so the overall result is 0. There are 2^n possible vectors $\bar{x}$ with $x_i = \pm 1$ for all i, so we have proved that the permanent of A is

$$\mathrm{Per}\,(A) = 2^{-n} \cdot \sum_{x_i = \pm 1} Q(\bar{x}).$$

A matching in G is a set of disjoint edges. A perfect matching is a matching containing n edges. The number of perfect matchings is $\mathrm{Per}\,(B)$. See [3] p. 44.

Rook Theory
There is a more algebraic approach to the computation of $|S_B|$. We interpret the matrix B as an $n \times n$ chess board. On squares with $b_{ij} = 1$ we may place a rook. Let $r_k(B)$ be the number of ways we can place k non-attacking rooks on the board (that is, choose k squares on B of which no two are on the same line). This again corresponds to a bipartite graph G, hence $r_k(B)$ is the number of matchings with k edges.

The *rook polynomial* $r(B, x)$ is defined as $r(B, x) = \sum_{k=0}^{n} r_k(B) x^k$ and $(r_0(B), r_1(B), \ldots, r_n(B))$ is called the *rook vector of* B. The number of perfect matchings is $r_n(B) = \mathrm{Per}\,(B)$.

In theory it is possible to determine the rook polynomial of arbitrary chessboards using the so-called expansion theorem. We mark a square on board B as special and let B_s denote the chessboard obtained from B by deleting the corresponding row and column. B_d is the board obtained from B by deleting the special square, meaning that we change the 1 into a 0. The ways of placing k non-attacking rooks can now be divided in two cases, those that have the rook in the special square and those that do not. In the first case we have $r_{k-1}(B_s)$ possibilities and in the second $r_k(B_d)$. We consequently do have the relation $r_k(B) = r_{k-1}(B_s) + r_k(B_d)$. This corresponds to $r(B, x) = x\, r(B_s, x) + r(B_d, x)$. We can now find the rook polynomial of arbitrary boards by repeatedly applying the expansion formula.

All Solutions
Let $m = \binom{n+h}{n} = \binom{n+h}{h}$ be the number of different subsets X_i of order n of the set $X = \{1, 2, ..., n+h\}$. We define a so called (0,1) configuration matrix $C = [c_{ij}]$ with $i = 1, ..., m$, $j = 1, ..., n+h$ and $c_{ij} = 1$ if and only if $x_j \in X_i$.

Let A_k, $k = 1, 2, ..., m$ be a possible subset of X. In the row $[c_{kj}]$ let h be the number of entries with $c_{kj} = 0$, n the number of entries with $c_{kj} = 1$ and $A_k = \{j | c_{kj} = 1\} = \{a_{k,1}, a_{k,2}, ..., a_{k,n}\}$ We define the matrix $B_k = [b_{ij}]$ of order n with $b_{ij} = 1$ if and only if $i \leq a_{k,j} \leq i + h$. In terms of these notions the answer of Problem 29 for A_k is $\mathrm{Per}\,(B_k)$ for $k = 1, 2, ..., m$.

In [6] a C-program can be found which for given n and h calculates the answers for all possible subsets A_k.

Related Problems
What if the girls assume power, depose the teacher and choose directly out of the set of $n+h$ boys (still observing the length restrictions)? Formulated in the spirit of Problem 29: we can simply count the injective maps $\pi : \{1, 2 \ldots, n\} \to \{1, 2 \ldots, n+h\}$ subject to the original restrictions: $k \leq \pi(k) \leq k+h$ for all k.

Clearly we can once again associate a bipartite graph G to this problem. The n-set X of girls and the $(n+h)$-set Y of boys provide the vertices. If a girl x can choose a boy y of appropriate length we have an edge $\{x, y\}$ of G.

The adjacency matrix A has the special form

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}.$$

The matrix B that codes the restrictions now is an n by $n+h$ matrix of zeros and ones, defined by $b_{ij} = 1$ if and only if $i \leq j \leq i+h$.

In the matrix B a matching M with cardinality n corresponds to a set of n 1's with no two of the 1's on the same row. The total number of matchings with $|M| = n$ is $\mathrm{Per}\,(B)$.

It is clear that this problem can be translated into a Rook Problem: find the number of all possible non-attacking placings of n rooks on an $n \times (n+h)$-chessboard, when placing a rook on the i-th row and the j-th column is subject to the condition $i \leq j \leq i+h$.

As stated above we already have a solution to this problem: $\mathrm{Per}\,(B)$. However, the special form of the matrix B may allow for a more explicit formula. There is one in Ryser's Algorithm: We translate Theorem 7.1.1 of [3] to our situation.

Let $B = [b_{ij}]$ the $n \times (n+h)$ (0,1)-matrix with $b_{ij} = 1$ if and only if $i \leq j \leq i+h$. For any submatrix C of B we let $\prod(C)$ denote the product of its row sums and for a number r with $h \leq r \leq n+h-1$ the expression $\sum \prod(B_r)$ denote the sum of all $\prod(B_r)$, where B_r runs over all $n \times (n+h-r)$-submatrices of B. Then

$$\mathrm{Per}\,(B) = \sum_{k=0}^{n-1} (-1)^k \binom{h+k}{k} \sum \prod (B_{h+k}).$$

Some Concrete Results
Let A be a (0,1)-matrix with m rows and n columns ($m \leq n$), let α be a k-subset of the m-set $\{1, 2, ..., m\}$ and let β be a k-subset of $\{1, 2, ..., n\}$. $A[\alpha, \beta]$ is the $k \times k$ submatrix of A determined by rows i with $i \in \alpha$ and columns j with $j \in \beta$.

The permanent $\mathrm{Per}\,(A[\alpha, \beta])$ is called a *permanental k-minor of A*. We define the sum over all possible α an β

$$p_k(A) = \sum_{\beta} \sum_{\alpha} \mathrm{Per}\,(A[\alpha, \beta]).$$

We define $p_0(A) = 1$ and note that $p_m(A) = \mathrm{Per}\,(A)$. The number $p_k(A)$ counts the number of sets of k ones in the matrix A with no two ones in the same row, so $(p_1(A), \ldots, p_m(A))$ is the rook vector of A. According to Theorem 7.2.1 of [3] we can evaluate the permanent of a (0,1)-matrix in terms of the permanental minors of the complementary matrix $J_{m,n} - A$, where $J_{m,n}$ is the m by n matrix with all entries 1.

Translated to our matrix B of the previous section, we get

$$\mathrm{Per}\,(B) = \sum_{k=0}^{n} (-1)^k p_k(J_{n,n+h} - B) \frac{(n+h-k)!}{h!}.$$

The Van der Waerden Conjecture: a story told by J.H. van Lint

Few mathematicians have ever heard of permanents and if so, it is the Permanent Theorem, better known as the Van der Waerden conjecture. Cited from [1]: "Van Lint, a well-known Dutch mathematician, wrote (see [2]) about the previous history of the Van der Waerden conjecture:

'Much of the work on permanents is in some way connected to this conjecture and about 75% of the work on permanents is less than 20 years old! ... In 1926 B.L. van der Waerden proposed as a problem (!) to determine the minimal permanent among all doubly stochastic matrices. It was natural to assume that this minimum is Per $J_n = n!n^{-n}$. Let us denote by Ω_n the set of all doubly stochastic matrices. The assertion

$$A \in \Omega_n \cap A \neq J_n \Rightarrow (\mathrm{Per}\,(A) > per(J_n))$$

became known as the Van der Waerden conjecture. Sometimes just showing that $n!n^{-n}$ is the minimal value is referred to as the conjecture.

This note allows me to save for posterity a humorous experience of the late sixties. Van der Waerden, retired by then, attended a meeting on combinatorics, a field he had never worked in seriously. A young mathematician was desperate to present his thesis in 20 minutes. I was sitting in the front row next to Van der Waerden when the famous conjecture was mentioned by the speaker and the alleged author inquired what this famous conjecture stated!! The exasperated speaker spent a few seconds of his precious time to explain and at the end of his talk wandered over to us to read the badge of the person who had asked this inexcusable question. I could foresee what was to happen and yet, I remember how he recoiled. You needn't worry — he recovered and now is a famous combinatorialist. The lesson for the reader is the following. If you did not know of the 'conjecture' then it is comforting to realize that it was 40 years old before Van der Waerden heard that it had this name.

What is the origin of the problem? Upon my request Van der Waerden went far back in his memory and came up with the following. One day in 1926 during the discussion that took place daily in Hamburg O. Schreier mentioned that G.A. Miller had proved that there is a mutual system of representatives for the right and left cosets of a subgroup H of a finite group G. At this moment Van der Waerden observed that this was a property of any two partitions of a set of size μn into μ subsets of size n. This theorem was published in "Hamburger Abhandlungen" in 1927. In the note, added in the proof, Van der Waerden acknowledged that he had rediscovered the theorem which is now known as the Konig-Hall theorem ...

In the terminology of permanents we can formulate the problem Schreier and Van der Waerden were considering as follows. Let $A_i (1 \le i \le \mu)$ and $B_k (1 \le k \le \mu)$ be the subsets in two partitions and let $a_{ik} := |A \cap B_k|$. Then $A = (a_{ik})$ is a matrix with constant line sums (= n). The assertion that there is a mutual system of representatives of the sets A_i respectively of the sets B_k is the same as to say that Per $A > 0$. At this point Van der Waerden wondered what the minimal permanent, under the side condition that all line sums are 1, is? He posed this as a problem in Jber. d. D.M.V. 35 and thus the Van der Waerden conjecture was born.' "

Remarks

The matrix J_n is the $n \times n$ matrix with all entries equal to $1/n$. In a double stochastic matrix all rows and all columns sum up to 1. G.P. Egorychev provided one of the proofs in 1981. The other independent proof is from D.I. Falikman.

This is in particular interesting for $h \ge n-2$, in which case we can easily see that $p_k(J_{n,n+h} - B)$ is independent of h, meaning that Per $(B) = f(n,h)$ is polynomial in h. For example, we have:

$$f(5,h) = h^5 - 5h^4 + 25h^3 - 55h^2 + 80h - 4 \ (h \ge 3).$$

We have polynomials up to $f(10,h)$. See the sequences A079908-A079914 in [7]. In [6] a small SAGE-program can be found that generates these polynomials.

The Free Dancing School and Juggling

What if the girls choose directly out of the set of $n+h$ boys, and take a boy either of their own length or taller as partner.

Again B is a $(0,1)$-matrix of size $n \times (n+h)$ that specifies the possible dancing pairs. We have $b_{ij} = 1$ if and only if $i \le j \le n+h$. The number of matchings with cardinality n is Per (B). Let $b_1, b_2, ..., b_m$ be integers with $0 \le b_1 \le b_2 \le ... \le b_m$. The $m \times b_m$ $(0,1)$-matrix $A = [a_{ij}]$ defined by $a_{ij} = 1$ if and only if $1 \le j \le b_i$, $(i = 1,2,...,m)$ is called a *Ferrers matrix*, and is denoted by $F(b_1, b_2, ..., b_m)$. According to [3], Corollary 7.2.6, we can calculate the permanent with Per $(F(b_1, b_2, ..., b_m)) = \prod_{i=1}^{m}(b_i - i + 1)$. We can associate a matrix B to a Ferrers matrix $F(b_1, b_2, ..., b_n)$ by setting $b_i = h + i$. So Per $(B) = \prod_{i=1}^{n}(h + i - i + 1) = (h+1)^n$; a result we could also have found by direct counting.

We note that the counting problem in Proposition 3.1 in *Juggling polynomials* by Kamstra can be solved by calculating the permanent of the corresponding Ferrers matrix. See [9] page 5.

References

1 G.P. Egorychev, 'Van der Waerden Conjecture and Applications', in *Handbook of Algebra* **1**, edited by M. Hazewinkel, Elsevier, 1996.

2 J.H. van Lint, 'The van der Waerden Conjecture: two proofs in one year', *Math. Intelligencer* **4** (1982), 72–77.

3 Richard A. Brualdi, Herbert J. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, Cambridge, 1991.

4 Henryk Minc, *Permanents*, Addison-Wesley, Reading, 1978.

5 William Stein, David Joyner, *SAGE: Software for Algebra and Geometry Experimentation*, Comm. Computer Algebra **39** (2005), pp. 61–64.

6 www.jaapspies.nl/mathfiles

7 *The Online Encyclopedia of Integer Sequences*, www.research.att.com/~njas/sequences

8 http://sage.math.washington.edu/sage/

9 Lute Kamstra *Juggling polynomials*, CWI Report PNA-R0113, 2001.