Seiberg-Witten Gauge Theory

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Contents

1 Introduction

Where the fire is churned, where the wind wafts, where the Soma juice flows over —there the mind is born.

Śvetāśvatara Upaniṣad 2.6[1]

In the fall of 1994 E. Witten announced a "new gauge theory of 4-manifolds", capable of giving results analogous to the earlier theory of Donaldson, but where the computations involved are "at least a thousand times easier" (Taubes). The dawn of the new theory began with the introduction of the monopole equations, whose roots lie in the depths of the still rather mysterious notion of S-duality in $N = 2$ supersymmetric Yang-Mills theory (IX: [59], [60])[2].

The equations are in terms of a section of a spinor bundle and a $U(1)$ connection on a line bundle L. The first equation just says that the spinor section ψ has to be in the kernel of the Dirac operator twisted with a connection A. The second equation describes a relation between the self-dual part of the curvature of the connection A and the section ψ in terms of the Clifford action.

The mathematical setting for Witten's gauge theory is considerably simpler than Donaldson's analogue: first of all it deals with $U(1)$-principal bundles (hermitian line bundles) rather than with $SU(2)$-bundles, and the abelian structure group allows simpler calculations; moreover the equation, which plays a role somehow analogous to the previous anti-self-dual equation for $SU(2)$-instantons, involves Dirac operators and $Spin_c$-structures, which are well known and long developed mathematical tools.

The main differences between the two theories arise when it comes to the properties of the moduli space of solutions of the monopole equation up to gauge transformations. As was immediately observed, the moduli space of Seiberg–Witten equations turns out to be always compact, essentially due to the Weitzenböck formula for the Dirac operator: a fact that avoids the complicated analytic techniques that were needed for the compactification of the moduli space of $SU(2)$-instantons.

In Witten's seminal paper (I:[19]) the monopole equation is introduced, and the main properties of the moduli space of solutions are deduced. The dimension of the moduli space is computed by an index theory technique, following an analogous proof for Donaldson's theory; and the Seiberg–Witten invariant (which depends on the Chern class of the line bundle L) is defined as the number of points, counted with orientation, in a zero-dimensional moduli space. The circumstances under which the Seiberg–Witten invariants provide a topological invariant of the four–manifold are illustrated in a similar way to the analo-

[1] The quotes from the Upaniṣads are from the Oxford University Press edition, in the English translation of P. Olivelle.

[2] References listed in the Introduction refer to the bibliographical appendix.

gous result regarding the Donaldson polynomials. Again as a consequence of the Weitzenböck formula, the moduli space turns out to be empty for all but finitely many choices of the line bundle L. Another advantage of this theory is that the singularities of the moduli space only appear at the trivial section $\psi \equiv 0$, since elsewhere the action of the gauge group is free. Thus, by perturbing the equation, it is possible to get a smooth moduli space.

Shortly after the new equations were introduced and their properties analysed, an impressive series of new results were proven in the span of a few months. As pointed out many times, the experts in the field had all the right questions to ask, questions that were too difficult to attack with the tools provided by Donaldson theory, and suddenly the new tool of Seiberg–Witten theory became available. A very rapid advancement in the field was thus possible.

The invariants were first computed for Kähler manifolds, and then, in extending the technique to the world of symplectic manifolds, Taubes uncovered a deep relation between Seiberg–Witten invariant and pseudo-holomorphic curves. The non-vanishing results for Seiberg–Witten invariants of symplectic manifolds led to conjecture that these constitute the most "basic" (indecomposable) kind of four-manifolds. The conjecture was later disproved, again by means of Seiberg–Witten theory, and there is at present no reasonable conjecture that would identify the "building blocks" of smooth four-manifolds. Thus, although in a way Seiberg–Witten theory helped proving several new results in four-manifold topology, it also showed that our understanding of the structure of four-manifolds is still very limited.

Among the very first applications of the new gauge theory was the proof of the Thom conjecture for embedded surfaces in $\mathbb{C}P^2$, in (II: [21]), later generalised to obtain various estimates on the minimal genus of embedded surfaces realising a given homology class in a four or three-dimensional manifold. This application led directly to the development of the three dimensional Seiberg–Witten theory and of an associated Floer homology which is in many ways analogous to the instanton homology constructed by Floer within the context of Donaldson theory. The Seiberg–Witten Floer homology, however, presents some interesting phenomena of metric dependence, due to non-trivial spectral flows of the Dirac operator.

In the original string-theoretic context, in which Seiberg–Witten theory arose, this can be seen as equivalent to Donaldson theory, the two being two different asymptotic limits of one common theory, which get interchanged under the symmetry of S-duality. Unfortunately, at present there is no clear mathematical understanding of this picture, but the equivalence of Donaldson and Seiberg–Witten theory is the object of current studies that are leading towards a proof, based on a slightly different perspective. The main idea is to realise both the Donaldson and the Seiberg–Witten moduli spaces as singular strata within a larger moduli space of solutions of twisted $PU(2)$-Seiberg–Witten equations. The corresponding relation between the Donaldson polynomial and the Seiberg–Witten invariants explained and justified (I: [19]) only at the level of physical intuition, is that the Seiberg–Witten invariants should coincide with the invariants derived by Kronheimer and Mrowka in the structure theorem for Donaldson polynomials.

After this brief recollection of the exciting days that saw the "change of paradigm" in gauge theory, we can discuss briefly how the material in this book is organised.

The first part of these notes, *Seiberg–Witten on four-manifolds*, collects some preliminary notions that are needed through the rest of the book, and then proceeds to give an introduction to Seiberg–Witten gauge theory on four-manifolds. It follows closely the original paper by Witten (I: [19]). We introduce the Seiberg–Witten equations as a variational principle, and discuss the analytic properties of the corresponding functional. Then, we proceed to introduce the moduli space of solutions. We describe reducibles and how to avoid them with a suitable perturbation. We discuss the compactness of the moduli space, the transversality result, and the orientation. Then we introduce the Seiberg–Witten invariants of a smooth four manifold X. We give a finiteness result, and discuss the independence of the metric and perturbation for the case $b_2^+(X) > 1$, and the wall crossing formula for $b_2^+(X) = 1$.

The second part, *Seiberg–Witten on three-manifolds*, deals with the dimensional reduction of the gauge theory. We first describe how to derive Seiberg–Witten equations on a three-manifold Y and construct a moduli space and a corresponding invariant, as in the four-dimensional case. We discuss the metric dependence of the invariant in the case of rational homology spheres, and the dependence on the cohomology class of the perturbation in the case of manifolds with $b_1(Y) = 1$. We then illustrate the relation between the invariant and classical invariants such as the Casson invariant, Milnor torsion, and the Alexander polynomial. We proceed to reinterpret the moduli space as the set of critical points of the Chern-Simons-Dirac functional, and the four-dimensional equations as its downward gradient flow. We describe the Hessian and the relative index of critical points, and give a gluing theorem for the moduli spaces of flow lines. This allows us to define compactification by lower dimensional strata of the spaces of flow lines. This is a crucial technical step in the construction of a boundary operator that connects critical points of relative index one. The corresponding homology is the Seiberg–Witten Floer homology, and the numerical invariant previously introduced can be reinterpreted as its Euler characteristic. We illustrate the construction of the equivariant Floer homology that bypasses the problem of metric dependence and reproves the wall crossing formula for the invariant. We finally discuss the problem of the exact triangles and surgery formulae.

The third part, *Topology and Geometry*, is an attempt to summarise in a somewhat coherent picture the current literature on the subject of topological and geometric results obtained via Seiberg–Witten theory. The results covered in these chapters span a wide range staring from Kähler manifolds and the Van de Ven conjectures for algebraic surfaces, to Taubes results on symplectic manifolds and the relation to pseudo-holomorphic curves and Gromov invariants, continuing with the estimates on the genus of embedded surfaces in four and three-manifolds, with related results on three-manifolds with contact structures. With the purpose of presenting the largest possible spectrum of results, in this part we often give only a brief sketch of the proofs, and we refer the reader to the appropriate sources in the literature, where she/he can find the detailed arguments. Given that this is still a field in ever increasing and rapid expansion, it is impossible to present all the contributions with a uniform level of attention and detail. Therefore a choice has to be made. This is forced upon by the specific research interests and taste of the author.

Through all these parts of the book, Seiberg–Witten gauge theory is considered as a completely self–contained subject and no a priori knowledge of Donaldson theory is assumed. In fact, all the sections that refer to Donaldson theory can be skipped by the reader who is not familiar with the non–abelian gauge theory, and this will not affect the comprehension of the remaining sections.

It is the author's belief, supported in this by more authoritative sources like (I: [19]) or (VIII: [2]), that it is inappropriate and in some sense misleading to present Seiberg–Witten theory without mentioning the context of physical theories that are responsible for the very existence of this new piece of mathematics. It is a hard and ambitious task to present quantum field theoretic results in a form that may be comprehensible and useful to a mathematician.

Thus, we have decided to enclose a fourth and last part in the book, dedicated to *Seiberg–Witten and Physics*. In this a fairly detailed description of the Mathai-Quillen formalism and the mathematically rigorous formulation of the regularised Euler class of Fredholm bundles have been included precisely to the purpose of making the reader familiar with the mathematical meaning of certain quantum field theoretic statements. This construction behaves particularly nicely when applied to Seiberg–Witten theory, since the compactness of the moduli space reduces some of the technicalities involved in the rigorous definition of the regularised Euler class.

This first step does not lead us close to the real heart of the matter. Therefore, we patiently proceed to attempt a brief exposition of some of the ideas of the theory of S-duality. This is especially difficult, since at the moment no clear mathematical understanding of the concept of S-duality is available and similarly for many of the physical concepts involved. However, this is by itself a good reason to include a brief discussion of what seems to be a promising direction for future studies and research.

We have also included a chapter where the mathematical approach to proving the equivalence of Seiberg–Witten and Donaldson theory is outlined. Some knowledge of $SU(2)$-gauge theory is needed in order to approach the content of this section.

The reader that approaches this fourth part of the book must keep in mind that the author is not a physicist, therefore the content of these chapters will inevitably appear to the eyes of a Physics-oriented reader as overly simplified.

In particular, due to the fact that the field is expanding so fast, the author should apologise to all the colleagues whose work might not have received the necessary attention in these notes. This especially applies to the Physics literature, where it is well known that the expansion rate of the set of new contributions to the subject is some orders of magnitude faster.

Because of all these and other limitations, these notes do not intend to be a comprehensive introduction to the Seiberg–Witten gauge theory. An initial version of these notes was prepared for a series of seminars that the author gave at the University of Milano in the summer 1995, intended mainly for an audience of first or second year doctorate students, and of advanced undergraduate students in Mathematics or Physics who were preparing their *laurea* dissertation in topology, differential geometry, or quantum field theory. The purpose of the set of notes was to help the students in reading the references available on the subject: strictly speaking, the purpose of the notes was to provide a guide on "how to approach the study of Seiberg–Witten gauge theory". As prerequisites, in fact, we assumed only some knowledge of the topology of fibre bundles, and notions like that of connection and curvature, that are usually covered in a European undergraduate course of algebraic topology or differential geometry; as well as of theoretical or mathematical physics. Some other concepts are briefly introduced, and the reader who is not familiar with the subject is addressed to references, where she/he can find more detailed information.

This initial version of the notes, roughly covering part one and part three, with a draft of part four, is still available on the electronic preprints as dg-ga/9509005, and it had the short-lived privilege of being the first available expository work that collected a good part of the existent literature on the subject. Many valuable references had since then appeared, which fulfill the same purpose, with particular attention to different mathematical orientations of the reader. Among these we recommend (I: [5]), but the reader should check the appropriate section of the bibliographical guide at the end of this volume for more references. Thus, it became in the least advisable to revise and update these notes. A first revision was made in the spring 1996, when the International Press Lecture Series took place at Irvine and many experts presented new results (among these Taubes' results on holomorphic curves, a first outline of Kronheimer and Mrowka's work on the interplay between three-dimensional Seiberg–Witten theory and contact geometry, and Furuta's progress in the direction of the 11/8 conjecture: the latter is certainly a sin of omission from this book). Shamelessly, I let other two years pass before attempting a final revision of the book.

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The first version and subsequent first revision of this book were written in the course of my short staying at the University of Chicago as a graduate student. I benefited in those days of conversations with Kevin Corlette, Victor Ginzburg, Jeff Harvey, J.Peter May, Dan Pollack, and, of course, with my advisor Mel Rothenberg, who helpfully and patiently supervised my study of gauge theory.

I worked on the final revision of the notes during my staying at the Max Planck Institut für Mathematik in Bonn, in the summer 1998, and during a subsequent visit to the Tata Institute of Fundamental Research in Mumbai in January 1999. I thank these institutions for the kind hospitality and for support. I also acknowledge support from two research grants of the CNR (Italian National Council of Research), during the academic years 1995-96 and 1996-97, as well as from the NSF (National Science Foundation, USA), grant DMS-9802480.

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After I completed the first revision of these notes, a draft of the manuscript was submitted to the Springer Lecture Notes for consideration. After a first refereeing, I was expected to revise the draft and resubmit it for a second refereeing and a final decision of the editor. Some textbooks on Seiberg–Witten theory had already appeared in the meanwhile, so I felt this volume would have a better use if published elsewhere, in such a way as to serve a different audience. Nonetheless, many thanks are due to Albrecht Dold for considering the previous draft, for his encouraging comments, and for submitting the draft to a refereeing process. Thanks are due to the referee as well. Her/his valuable comments and suggestions have been taken into consideration in preparing the

final version of the manuscript.

Last but not least, I should thank Renzo Piccinini who invited me to give the series of seminars at the University of Milano out of which the first draft of the book originated. I also thank all those who attended those lectures, for their intelligent comments and questions and for their appreciation.

This book is dedicated to my friend Niranjan, for help and encouragement through some difficult moments of life, and for always making mathematical conversation a most enjoyable experience.

It is also dedicated to the memory of my mother, a successful architect and fashion designer, who first taught me the love of mathematics as a form of art.

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Part I
Seiberg–Witten on four-manifolds

You have the glow of a man who knows brahman! Tell me –who taught you? 'Other than human beings' he acknowledged. 'But if it pleases you, sir, you should teach it to me yourself, for I have heard from people of your eminence that knowledge leads one most securely to the goal only when it is learnt from a teacher'.

Chāndogya Upaniṣad, 4.9.2-3

2 Preliminary Notions

We introduce here the basic concepts that are needed in order to define the Seiberg–Witten equations and invariants. This introduction will be rather sketchy: occasionally we will refer to more detailed references listed in the bibliography.

2.1 Clifford Algebras and Dirac Operators

Definition 2.1 *The* Clifford algebra $C(V)$ *of a real vector space* V *with an inner product* $(,)$ *is the algebra generated by the elements of* V*, subject to the relations*

$$e \cdot e' + e' \cdot e = -2(e, e').$$

The multiplication of elements of V *in the Clifford algebra is called Clifford multiplication.*

Given an orthonormal basis $\{e_i\}$ of V,

$$e_1^{\epsilon_1} \cdots e_n^{\epsilon_n},$$

where $\epsilon_i = 0$ or 1, is a vector space basis of $C(V)$.

If $\dim V = 2m$, there is a unique irreducible representation of $C(V)$ on a complex inner product space S, such that the elements of V act by skew-hermitian operators. This representation (which is usually called the *complex spinor representation*) has dimension 2^m. The representation S has a spectral decomposition $S = S^+ \oplus S^-$ with respect to the action of the volume form of $C(V)$.

In particular, given a Riemannian manifold X we shall consider the Clifford algebra associated to the tangent space at each point.

Definition 2.2 *The Clifford algebra of the tangent bundle of* X *is the bundle that has fibre over each point* $x \in X$ *the Clifford algebra* $C(T_xX)$*. We shall denote this bundle* $C(TX)$*.*

For the rest of this section we assume that the dimension of X is even.

Definition 2.3 *A* spinor bundle *over a Riemannian manifold* X *is a hermitian vector bundle* W *of complex rank* 2^m*, with a map* $\Gamma : TX \to End\ W$ *satisfying* $\Gamma(v) + \Gamma^*(v) = 0$ *and* $\Gamma(v)\Gamma^*(v) = -|v|^2 Id$*, for all* $v \in TX$*. A* spin connection *on* W *is a connection compatible with the Levi–Civita connection* ∇^{LC}*, that is, a connection*

$$\nabla : \mathcal{C}^\infty(W) \to \mathcal{C}^\infty(T^*X \otimes W)$$

such that, given any two vector fields u *and* v *on* X*, and any section* s *of* W *we have*

$$\nabla_u(v \cdot s) = \nabla_u^{LC}(v) \cdot s + v \cdot \nabla_u(s).$$

Not all manifolds admit a spinor bundle: the existence of such a bundle is equivalent [21] to the existence of a $Spin_c$–structure on the manifold X: we shall discuss $Spin_c$–structures in the next paragraph. If a spinor bundle exists, it splits as a direct sum of two vector bundles,

$$W = W^+ \oplus W^-, \tag{1}$$

where the splitting is given by the internal grading of the Clifford algebra. The determinant line bundles of W^+ and W^- are canonically isomorphic. We denote this bundle by L. A spin connection leaves W^+ and W^- invariant, and it induces the "same" connection on L. Clifford multiplication by elements of TX takes W^+ to W^-.

The following lemma is an easy consequence of Definition 2.3.

Lemma 2.4 *Let X be a manifold that admits a spinor bundle W. Let $\{e_i\}$ be a local orthonormal basis of sections of the tangent bundle TX and $<,>$ be the Hermitian structure on W. Then, for any section $\psi \in \Gamma(X, W)$, and for $i \neq j$, the expression*

$$< e_i e_j \psi, \psi >$$

is purely imaginary at each point $x \in X$.

Proof: In fact, by skew–adjointness of Clifford multiplication and the fact that the basis is orthonormal,

$$< e_i e_j \psi, \psi > = - < e_j \psi, e_i \psi > = < \psi, e_j e_i \psi > =$$

$$- < \psi, e_i e_j \psi > = -\overline{< e_i e_j \psi, \psi >}.$$

QED

Definition 2.5 *Given a spinor bundle W over X, endowed with a spin connection, the* Dirac operator *on W is the first order differential operator on the smooth sections*

$$D : \Gamma(X, W^+) \to \Gamma(X, W^-)$$

defined as the composition

$$D : \Gamma(X, W^+) \xrightarrow{\nabla} \Gamma(X, W^+ \otimes T^*X) \xrightarrow{g} \Gamma(X, W^+ \otimes TX) \xrightarrow{\bullet} \Gamma(X, W^-), \tag{2}$$

where the first map is the covariant derivative with respect to the spin connection, the second is the Legendre transform, that is, the isomorphism given by the Riemannian metric, and the third is Clifford multiplication.

It is easy to check (for more details see [21]) that this corresponds to the following expression in a local orthonormal system of coordinates:

$$Ds = \sum_k e_k \cdot \nabla_k s.$$

Considered as an operator on the space of sections of the full spinor bundle $W = W^+ \oplus W^-$, the Dirac operator of the form

$$\begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}$$

is formally self adjoint.

An essential tool in Spin geometry, which is very useful in Seiberg–Witten gauge theory, is the *Weitzenböck formula.*

Theorem 2.6 *The Dirac operator D satisfies the Weitzenböck formula:*

$$D^2 s = (\nabla^* \nabla + \frac{\kappa}{4} + \frac{-i}{4} F) s,$$

where ∇^* is the formal adjoint of the covariant derivative κ is the scalar curvature on X, F is the curvature of the induced connection on L, and $s \in \Gamma(X, W^+)$
.

Proof: In terms of local normal coordinates

$$D^2 s = \sum_{i,j} e_i \nabla_i (e_j \nabla_j s) = \sum_{i,j} e_i e_j \nabla_i \nabla_j s =$$

$$= -\sum_i \nabla_i^2 s + \sum_{i<j} e_i e_j (\nabla_i \nabla_j - \nabla_j \nabla_i) s.$$

The first summand is $\nabla^* \nabla$ ([21] pg. 28), and the second splits into a term which corresponds to the scalar curvature on X ([21] pg. 126) and the curvature $-iF$ of the induced connection on L (we identify the Lie algebra of $U(1)$ with $i\mathbb{R}$). QED

2.2 *Spin* and *Spin$_c$* Structures

The group $Spin(n)$ is the universal covering of $SO(n)$.

The group $Spin_c(n) = (Spin(n) \times U(1))/\mathbb{Z}_2$ is an extension

$$1 \to \mathbb{Z}_2 \to Spin_c(n) \to SO(n) \times U(1) \to 1. \tag{3}$$

This yields the exact sheaf–cohomology sequence:

$$\cdots \to H^1(X; Spin_c(n)) \to H^1(X; SO(n)) \oplus H^1(X; U(1)) \xrightarrow{\delta} H^2(X; \mathbb{Z}_2) \tag{4}$$

Recall that $H^1(X; G)$ represents the equivalence classes of principal G–bundles over X. It can be shown [14] that the connecting homomorphism of the sequence (4) is given by

$$\delta : (P_{SO(n)}, P_{U(1)}) \mapsto w_2(P_{SO(n)}) + \bar{c}_1(P_{U(1)}),$$

where $\bar{c}_1(P_{U(1)})$ is the reduction mod 2 of the first Chern class the principal bundle $P_{U(1)}$ and w_2 is the second Stiefel–Whitney class.

Definition 2.7 *A $Spin_c$–structure on an oriented n-dimensional Riemannian manifold X is a lift of the bundle Fr of oriented orthonormal frames to a principal $Spin_c(n)$ bundle. A Spin–structure is a lift of the bundle Fr to a principal $Spin(n)$ bundle.*

We shall refer to the set of $Spin_c$ structures on a Riemannian manifold X as $\mathcal{S}(X)$, or simply $\mathcal{S}$ when the manifold X is understood.

We have the following criteria for the existence of $Spin_c$ and $Spin$-structures.

Lemma 2.8 *A manifold X admits a $Spin_c$–structure iff $w_2(X)$ is the reduction mod 2 of an integral class. It has a Spin–structure iff $w_2(X) = 0$.*

We have the following theorem, which was proven originally in [10].

Theorem 2.9 *Every oriented 4–manifold admits a $Spin_c$–structure.*

2.3 Spinor Bundles

Two equivalent descriptions of spin bundles are possible. One is in terms of principal bundles, and the other in terms of vector bundles as in Definition 2.3. We briefly recall the principal bundle description following [14]. The equivalence of these two descriptions in the case of a four-manifold is dealt with in exercises at the end of the chapter.

Both $Spin(n)$ and $Spin_c(n)$ can be thought of as lying inside the Clifford algebra $C(\mathbb{R}^n)$. Therefore to a principal $Spin(n)$ or $Spin_c(n)$ bundle we can associate a vector bundle via the unique irreducible (complex, hermitian) representation of the Clifford algebra. This will be the bundle of Spinors over X associated to the $Spin_c$ or $Spin$ structure, as defined in 2.3.

It is instructive to think in terms of transition functions. Let $g_{\alpha\beta}$ be the transition functions of the frame bundle over X, which take values in $SO(n)$. Then locally they can be lifted to functions $\tilde{g}_{\alpha\beta}$ which take values in $Spin(n)$, since on a differentiable manifold it is always possible to choose open sets with contractible intersections that trivialise the bundle.

However, if the manifold is not $Spin$, then the $\tilde{g}_{\alpha\beta}$ will not form a cocycle, since $\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} = 1$ would imply that the second Stiefel–Whitney class vanishes.

If a $Spin_c$ structure exists we know that w_2 is the reduction of an integral class $c \in H^2(X, \mathbb{Z})$, which represents a complex line bundle, say with transition functions $\lambda_{\alpha\beta}$ with values in $U(1)$. Locally such functions will have a square root $\lambda_{\alpha\beta}^{1/2}$. However, the line bundle will not have a square root globally (in which case $\lambda_{\alpha\beta}^{1/2}$ will not form a cocycle).

However, the relation $w_2(X) + c = 0 \bmod 2$ that comes from (4) says that the product

$$\tilde{g}_{\alpha\beta}\lambda_{\alpha\beta}^{1/2} \tag{5}$$

is a cocycle. These are the transition functions of $S \otimes \sqrt{L}$, where S would be the spinor bundle of a $Spin$ structure and $\sqrt{L}$ would be the square root of a line bundle: neither of these objects is defined globally, but the tensor product is. Thus, we can represent the spinor bundle W as $W^{\pm} = S^{\pm} \otimes \sqrt{L}$.

We should point out that a $Spin_c$ structure can also be defined as a lift of $Fr \times P_{U(1)}$, where Fr is the oriented frame bundle, as before, and $P_{U(1)}$ is a principal $U(1)$ bundle satisfying $c_1(P_{U(1)}) = w_2(X)$ (mod 2). We can refer to the set of $Spin_c$ structures defined this way as $\mathcal{S}'$. Then there is a surjective map $\mathcal{S}' \to \mathcal{S}$. The fibre of this map is a homogeneous space for $H^1(X,\mathbb{Z})/2H^1(X,\mathbb{Z})$. Notice, moreover, that there is an injection of the set of $Spin$ structures into $\mathcal{S}'$, but the composite map to $\mathcal{S}$ is not an injection in general. There is an action of the group $H^2(X,\mathbb{Z})$ of equivalence classes of line bundles on X on the set $\mathcal{S}$, given by $W \to W \otimes H$, for $H \in H^2(X,\mathbb{Z})$. The set $\mathcal{S}$ is in fact a principal homogeneous space for $H^2(X,\mathbb{Z})$. The map $\mathcal{S} \to H^2(X,\mathbb{Z})$ given by $(W^{\pm},\Gamma) \to L = det(W^+)$ is equivariant with respect to the action of $H^2(X,\mathbb{Z})$, where the action on the right hand side is given by $L \to H^2 \otimes L$, for $H \in H^2(X,\mathbb{Z})$.

Remark 2.10 *If we choose a $U(1)$-connection A on L, then we can write the Dirac operator on W^+ as a twisted Dirac operator on $S^+ \otimes \sqrt{L}$. This is well defined, since it is defined in terms of local quantities. The Weitzenböck formula 2.6 holds true for the twisted Dirac operator on $S^+ \otimes \sqrt{L}$.*

2.4 Topology of the gauge group

Recall that the gauge group of a G–bundle is defined as the group of self equivalences of the bundle, namely the group of smooth maps

$$\lambda_\alpha : U_\alpha \to G$$

$$\lambda_\beta = g_{\beta\alpha}\lambda_\alpha g_{\alpha\beta},$$

where the bundle is trivial over U_α and has transition functions $g_{\alpha\beta}$.

The gauge group is an infinite dimensional manifold. It is made into a Banach manifold with the choice of some fixed Sobolev norm. The definition of the L^2_k-Sobolev norms will be recalled later in this section. In the following we always assume to work with L^2_k-gauge transformations, with $k \geq 3$, since we think of the configuration space of pairs (A,ψ) endowed with the L^2_{k-1}-norm. If the structure group is abelian, or if the G-bundle is topologically trivial, then the gauge group has a simpler description as $\mathcal{G} = \mathcal{M}(X,G)$, the space of maps from X to G. We have the following easy lemma.

Lemma 2.11 *In the case $G = U(1)$, the set of connected components of the gauge group $\mathcal{G}$ is $H^1(X,\mathbb{Z})$.*

2.5 Symplectic and Kähler Manifolds

Recall that a manifold X is endowed with a symplectic structure if a closed 2–form ω is given on X such that it is non-degenerate, that is, its highest exterior power is nowhere vanishing.

An almost complex structure J on a manifold X is a complex structure on the tangent spaces $J_x : T_xX \to T_xX$, $J_x^2 = -1$. Given such a J and a non-degenerate 2-form ω on X, we say that the two are *compatible* if the expression $g(v, w) = \omega(v, Jw)$, with v and w tangent vectors, defines a Riemannian metric g on X, and J is orthogonal with respect to g. A symplectic form ω is said to tame an almost-complex structure J if it satisfies $\omega(v, Jv) > 0$ for all non zero vectors v.

Notice that the condition that ω is non-degenerate is necessary for g to be a Riemannian metric, but ω does not need to be closed, hence it may not give rise to a symplectic structure. A non-degenerate form ω supports a family of compatible almost-complex structures J, and corresponding metrics. The set $\mathcal{J}$ of almost-complex structures J that are ω–compatible is contractible.

When J is defined by an actual complex structure on X we say that J is integrable. If J is integrable, and the 2-form ω is closed, that is, if X is both symplectic and complex, then X is a Kähler manifold, ω is the Kähler form and $g(v, w) = \omega(v, Jw)$ a compatible Kähler metric.

A condition equivalent to the integrability of J can be used to characterise complex manifolds. In fact, on an almost complex manifold, the cotangent bundle can be written as

$$T^*(X)_{\mathbb{C}} = T^*(X)' \oplus T^*(X)'',$$

with the splitting given by the almost complex structure, and the complex of forms splits correspondingly as

$$\Lambda^{(p,q)}(X) = \{\alpha \in \Lambda^{p+q}(X) | \alpha \in \Lambda^p(T^*X)' \otimes \Lambda^q(T^*X)''\}.$$

On a complex manifold, the differential takes the form

$$d = \partial + \bar{\partial} : \Lambda^{(p,q)} \to \Lambda^{(p+1,q)} \oplus \Lambda^{(p,q+1)}.$$

If J is not integrable, the differential on $\Lambda^{(p,q)}$ also has components with values in $\Lambda^{(p-1,q+2)} \oplus \Lambda^{(p+2,q-1)}$. These components can be expressed in terms of the Nijenhuis tensor

$$N_J(v, w) = [v, w] + J[v, Jw] + J[Jv, w] - [Jv, Jw],$$

and a theorem of Newlander and Nirenberg asserts that the vanishing of the Nijenhuis tensor is in fact equivalent to the integrability of J. This property will be useful in discussing some aspects of Seiberg–Witten equations on Kähler and on symplectic manifolds.

The picture can be summarised as follows. We are considering three possible kinds of data on an even dimensional manifold X: a Riemannian metric g, an almost complex structure J and a 2-form ω. There is a compatibility condition expressed by the relation $g(v,w) = \omega(v, Jw)$: this implies that the 2-form is non-degenerate, but it is not sufficient to guarantee that it is closed.

If, in addition, the 2-form ω is closed, we obtain a symplectic manifold. Regardless of whether ω is closed, we can instead require another additional condition, which is the vanishing of the Nijenhuis tensor: this implies that the almost complex structure J is integrable, hence the manifold is complex.

If we require both conditions to hold simultaneously, $d\omega = 0$ and $N_J = 0$, this is equivalent to the condition $\nabla J = 0$ with respect to the covariant derivative induced by the Levi–Civita connection of g, and the manifold X is Kähler.

It is known that all symplectic structures are locally the same. (In fact by the Darboux theorem it is always possible to find a set of coordinates in which the symplectic form reduces to the "standard one": $\omega = \sum_i dx_i \wedge dy_i$.) In this sense symplectic geometry can be thought of as something more rigid than $\mathcal{C}^\infty$–geometry but less rigid than Riemannian geometry.

Note that, unlike symplectic geometry, which is less rigid than Riemannian geometry, Kähler geometry is much more rigid, as expected in passing from smooth to analytic geometry. In this sense, the condition of being Kähler is rather exceptional, although it may be non-trivial to provide examples of compact symplectic manifolds that are not Kähler, particularly in the simply connected case; see e.g. [17] and [9].

We recall some useful notions about group actions on symplectic manifolds. The action of a group on a symplectic manifold is said to be symplectic if it preserves the form ω.

Definition 2.12 *A symplectic action of a Lie group G on (X,ω) is* Hamiltonian *if each vector field v on X, given by the infinitesimal action of the Lie algebra $\mathcal{L} \to Vect(X)$, lifts to a map $H_v \in \mathcal{C}^\infty(X)$, via the relation*

$$\omega(v,\cdot) = dH_v(\cdot).$$

A Hamiltonian action is called Poisson *if the map $\mathcal{L} \to \mathcal{C}^\infty(X)$ given by $v \mapsto H_v$ is a Lie algebra homomorphism with respect to the Poisson bracket $\{,\}$ on $\mathcal{C}^\infty(X)$.*

The difference between Hamiltonian and Poisson actions is measured by a Lie algebra cocycle as explained in [18]. Notice that often a different terminology is encountered where "weakly Hamiltonian" is used instead of "Hamiltonian" and "Hamiltonian" is used instead of "Poisson".

Definition 2.13 *The* moment map *of a Poisson symplectic action of G on X is the map*

$$\mu : X \to \mathcal{L}^*$$

$$x \mapsto (\mu(x) : \mathcal{L} \to \mathbb{R})$$
$$\mu(x)(v) = H_v(x).$$

If a Riemannian manifold X is endowed with an orthogonal almost complex structure J, then there exists a canonical $Spin_c$−structure, namely, the spinor bundle S given by

$$S = \Lambda_J^{(0,*)}(X)$$

The Clifford multiplication on S is given by

$$(\alpha^{(0,1)} + \alpha^{(1,0)})\beta = \sqrt{2}(\alpha^{(0,1)} \wedge \beta - \alpha^{(1,0)} \rfloor \beta),$$

where $\alpha^{(1,0)} \rfloor \beta = \sum_{k,\ell} g_{k\ell} {\alpha^{(1,0)}}^k \beta^\ell$. Notice that, in the principal bundle description, the existence of a canonical $Spin_c$−structure associated to an almost complex structure depends on the embedding

$$U(n) \hookrightarrow Spin_c(2n)$$

.

One last observation that will be useful in discussing Seiberg–Witten theory on Kähler and symplectic manifolds is the following simple remark.

Remark 2.14 *let V be a 4−dimensional real vector space endowed with a positive definite scalar product $<,>$. Let J be an orthogonal complex structure on V, and let $\omega(v,w) = <v, Jw>$. Then we have a decomposition*

$$\Lambda^2_+ \otimes \mathbb{C} = \mathbb{C}\omega \oplus \Lambda^{(2,0)} \oplus \Lambda^{(0,2)}.$$

Symplectic geometry plays a prominent role in Seiberg–Witten gauge theory. Computation of the invariants is made easier in the presence of a symplectic structure and it is a deep result of Taubes that the Seiberg–Witten invariants are connected to other invariants of symplectic manifolds introduced by Gromov. The fact that the Seiberg–Witten invariants of symplectic manifolds have a "more basic" structure also led to a conjecture, suggested by Taubes, that symplectic manifolds may be among the most basic building blocks of the whole geometry of 4–manifolds. The conjecture was disproved by Z.Szabó: in fact, there seems to be currently no reasonable guess as to what the "basic building blocks" of smooth four-manifolds could be.

A good reference for the notions introduced in this section is [18].

2.6 The index theorem

We recall here very briefly some essential results of index theory. A very readable reference is the book by R. Boos and D.D. Bleecker, [2].

Recall that a bounded linear operator acting between Banach spaces is Fredholm if it has finite dimensional kernel and cokernel.

A differential operator of order m, mapping the smooth sections of a vector bundle E over a compact manifold Y to those of another such bundle F, can be described in local coordinates and local trivialisations of the bundles as

$$D = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha,$$

with $\alpha = (\alpha_1, \dots, \alpha_n)$. The coefficients $a_\alpha(x)$ are matrices of smooth functions and $D^\alpha = \frac{\partial}{\partial x_1^{\alpha_1}} \cdots \frac{\partial}{\partial x_n^{\alpha_n}}$.

Definition 2.15 *The* principal symbol *associated to the operator D is the expression*

$$\sigma_m(D)(x,p) = \sum_{|\alpha|=m} a_\alpha(x) p^\alpha.$$

Given the differential operator $D : \Gamma(Y, E) \to \Gamma(Y, F)$, the principal symbol with the local expression above defines a global map

$$\sigma_m : \pi^*(E) \to \pi^*(F),$$

where $T^*Y \stackrel{\pi}{\to} Y$ is the cotangent bundle, that is, the variables (x, p) are local coordinates on T^*Y.

Consider bundles E_i, $i = 1 \dots k$, over a compact n-dimensional manifold Y. Suppose there is a complex $\Gamma(E)$ formed by the spaces of (local) sections $\Gamma(E_i)$ and differential operators d_i of order m_i:

$$0 \to \Gamma(E_1) \stackrel{d_1}{\to} \cdots \stackrel{d_{k-1}}{\to} \Gamma(E_k) \to 0.$$

Construct the principal symbols $\sigma_m(d_i)$; these determine an associated symbol complex

$$0 \to \pi^*(E_1) \stackrel{\sigma_{m_1}(d_1)}{\to} \cdots \stackrel{\sigma_{m_{d-1}}(d_{k-1})}{\to} \pi^*(E_k) \to 0.$$

Definition 2.16 *The complex $\Gamma(E)$ is* elliptic *iff the associated symbol complex is exact off the zero section.*

In the case of just one operator D of order m, this means that $\sigma_m(D)$ is an isomorphism off the zero section. On a compact manifold Y the condition of ellipticity ensures that the differential operator D is Fredholm [2]. The Hodge theorem states that in this case the cohomology of the complex $\Gamma(Y, E)$ coincides with the harmonic forms; that is,

$$H^i(E) = \frac{Ker(d_i)}{Im(d_{i-1})} = Ker(\Delta_i),$$

where $\Delta_i = d_i^* d_i + d_{i-1} d_{i-1}^*$.

Without loss of generality, by passing to the assembled complex

$$E^+ = E^1 \oplus E^3 \oplus \cdots$$

$$E^- = E^2 \oplus E^4 \oplus \cdots,$$

we can always think of one elliptic operator $D : \Gamma(E^+) \to \Gamma(E^-)$, $D = \sum_i (d_{2i-1} + d_{2i}^*)$.

The index theorem can be stated as follows.

Theorem 2.17 *Consider an elliptic complex over a compact, orientable, even dimensional manifold Y without boundary. The index of D, which is given by*

$$Ind(D) = \dim Ker(D) - \dim Coker(D) = \sum_i (-1)^i \dim Ker \Delta_i = -\chi(E),$$

$\chi(E)$ being the Euler characteristic of the complex, can be expressed in terms of characteristic classes as

$$Ind(D) = (-1)^{n/2} < \frac{ch(\sum_i (-1)^i [E_i])}{e(Y)} td(TY_{\mathbb{C}}), [Y] > .$$

In the above ch is the Chern character, e is the Euler class of the tangent bundle of Y, and $td(TY_{\mathbb{C}})$ is the Todd class of the complexified tangent bundle.

2.7 Equivariant cohomology

We recall some basic notions of equivariant cohomology with real coefficients. These will be useful in discussing the approach to Seiberg–Witten gauge theory through Quantum Field Theory and in the construction of the Seiberg–Witten-Floer homology. The brief exposition presented here partly follows [4], which we recommend as a very good reference for the role of equivariant cohomology in Quantum Field Theory.

The basic idea is the following: when a group G (which we assume in the finite-dimensional case is a compact connected Lie group) acts freely on a manifold X, the quotient is a manifold. Thus we can compute the ordinary cohomology $H^*(M/G, \mathbb{R})$. However, if the action is not free, i.e. there are fixed points, the quotient fails to be a manifold. Equivariant cohomology is the algebraic object that plays the role of the ordinary cohomology of the quotient in this case.

In the case of ordinary cohomology, many different definitions can be given. All of them are proved to be equivalent since they satisfy the Steenrod axioms. Something similar happens with the equivariant cohomology. Since we are only interested in the case with real coefficients, we shall concentrate on a de Rham version of equivariant cohomology.

Let us recall the notions of a classifying space and the universal bundle. The latter is a principal G-bundle EG over a space BG which is defined uniquely up to homotopy, such that, given any principal G-bundle P over X, this is obtained as a pullback of the universal bundle via a classifying map $f : X \to BG$. The total space EG is contractible.

Definition 2.18 *Let X be a manifold with an action of a compact connected Lie group G. Consider the space $X \times EG$. This has a free action of G induced by the action on X and the free action on EG. Thus we can compute the ordinary cohomology of the quotient $XG = (X \times EG)/G$. This is the* equivariant cohomology *of X:*

$$H^*_G(X, \mathbb{R}) \equiv H^*(XG, \mathbb{R}).$$

The space XG is called the homotopy quotient of X.

Let $p : XG \to X/G$ denote the map induced by the projection $X \times EG \to X$. Then the fibre is

$$p^{-1}([x]) = EG/G_x,$$

where G_x is the stabiliser of the point $x \in X$. Thus, in general the map p is not a fibration. On the other hand, if the action of G is free, then the map p is a fibration with fibre EG. Since EG is contractible, the map p gives a homotopy equivalence.

Thus, in the case of a free action, the equivariant cohomology is just the ordinary cohomology of the quotient,

$$H^*_G(X, \mathbb{R}) \cong H^*(X/G, \mathbb{R}).$$

In other words, the idea underlying the above definition is that the product with EG makes the action free without changing the topology of X, since EG is a contractible space. If $X = pt$ is a point the equivariant cohomology is the cohomology of the classifying space BG,

$$H^*_G(pt, \mathbb{R}) \cong H^*(BG, \mathbb{R}).$$

This example shows that the equivariant cohomology is in general highly non-trivial.

There is an axiomatic version which is analogous to the Steenrod axioms for ordinary cohomology. Namely, equivariant cohomology is uniquely specified by the requirements that it coincides with the cohomology of the quotient in the case of a free action, that an equivariant homotopy equivalence induces an isomorphism, and that there is a Mayer-Vietoris sequence with open sets that are G-invariant.

The de Rham model that we are going to present is based on the construction of an "algebraic analogue" of the classifying space.

Definition 2.19 *The* Weil algebra *of a compact connected Lie group G is the differential graded algebra*

$$\mathcal{W}(\mathbf{g}) = S^*(\hat{\mathbf{g}}) \otimes \Lambda^*(\hat{\mathbf{g}}),$$

where $\mathbf{g}$ is the Lie algebra of G and $\hat{\mathbf{g}}$ is the dual.

$\mathcal{W}(\mathbf{g})$ is graded by assigning degree one to every element

$$\phi \in \hat{\mathbf{g}} \subset \Lambda^*(\hat{\mathbf{g}}),$$

and degree two to the corresponding element

$$u \in \hat{\mathbf{g}} \subset S^*(\hat{\mathbf{g}}).$$

Thus, if $\{X_1, ..., X_n\}$ is a basis of $\mathbf{g}$ and $\{\phi^1, ..., \phi^n\}$ is the dual basis of $\hat{\mathbf{g}}$, the algebra $\mathcal{W}(\mathbf{g})$ is freely generated as supercommutative graded algebra by $\phi^1, ..., \phi^n, u^1, ..., u^n$.

Let $\theta \in \hat{\mathbf{g}} \otimes \mathbf{g} \subset \Lambda(\hat{\mathbf{g}}) \otimes \mathbf{g}$ be defined as $\theta(X) = X$. Let Ω be the corresponging element in $S(\hat{\mathbf{g}}) \otimes \mathbf{g}$. We can write $\theta = \phi^h \otimes X_h$ and $\Omega = u^h \otimes X_h$, with an implicit sum over repeated indices.

The differential on $\mathcal{W}(\mathbf{g})$ is defined as follows. For every $\phi \in \hat{\mathbf{g}} \subset \Lambda(\hat{\mathbf{g}})$ and every $u \in \hat{\mathbf{g}} \subset S(\hat{\mathbf{g}})$ set

$$d_W(\phi) \;=\; \phi(\Omega - \frac{1}{2}[\theta, \theta]),$$

and

$$d_W(u) \;=\; u([\Omega, \theta]).$$

Extend d_W as degree one derivation. In particular, we have

$$d_W \phi^\alpha \;=\; u^\alpha - C^\alpha_{\beta\gamma}\phi^\beta\phi^\gamma$$

$$d_W u^\alpha \;=\; C^\alpha_{\beta\gamma} u^\beta \phi^\gamma,$$

where sums over repeated indices are understood. The $C^\alpha_{\beta\gamma}$'s are the structure constants of the Lie algebra $\mathbf{g}$ with respect to the given basis. It can be checked that $d_W^2 = 0$, hence $(\mathcal{W}(\mathbf{g}), d_W)$ is a differential graded algebra.

It is not hard to see that the cohomology of $\mathcal{W}(\mathbf{g})$ with respect to the differential d_W is trivial in degree higher than zero and is equal to $\mathbb{R}$ in degree zero. We think of $\mathcal{W}(\mathbf{g})$ as an algebraic analogue of the contractible space EG.

We introduce contractions and Lie derivatives (with respect to elements of $\mathbf{g}$) in $\mathcal{W}(\mathbf{g})$. Contractions with respect to elements of the dual basis of $\mathbf{g}$ are defined as $I_i(\theta^j) = \delta_i^j$ and $I(\Omega) = 0$ and the Lie derivative is the graded commutator of I_i and d_W, $L_i = [I_i, d_W]$.

The subcomplex $\mathcal{B}(\mathbf{g})$ of $\mathcal{W}(\mathbf{g})$ of elements that are killed by such contractions and Lie derivatives is the analogue of the space BG. In fact we have that

$$H^*(\mathcal{B}(\mathbf{g}), \mathbb{R}) \cong S^*(\hat{\mathbf{g}})^G,$$

the G-invariant polynomials on $\hat{\mathbf{g}}$. It is a well known theorem that for a connected compact Lie group G this is in fact the cohomology of the classifying space BG (with real coefficients), i.e.,

$$H^*(BG, \mathbb{R}) \cong S^*(\hat{\mathbf{g}})^G.$$

Given any principal G bundle P over X, a connection on P yields an algebra homomorphism (the Weil homomorphism) from $\mathcal{W}(\mathbf{g})$ to the de Rham complex $\Lambda^*(P)$. The subcomplex $\mathcal{B}(\mathbf{g})$ is mapped to the basic forms, i.e. the forms on P that are obtained as pullback of forms on X. These are killed by the contractions and the Lie derivatives, since they have no vertical component and are G-invariant. Let $\Lambda^*_G(X)$ denote the subcomplex of basic forms in $\mathcal{W}(\mathbf{g}) \otimes \Lambda^*(X)$.

We can construct a de Rham complex that computes equivariant cohomology of a manifold X with a G-action.

Proposition 2.20 *The subcomplex*

$$\Lambda^*_G(X) \subset \mathcal{W}(\mathbf{g}) \otimes \Lambda^*(X) \tag{6}$$

of basic forms with differential $d_W \otimes 1 + 1 \otimes d$ computes the equivariant cohomology of X,

$$H^*(\Lambda^*_G(X), d_W \otimes 1 + 1 \otimes d) \cong H^*_G(X, \mathbb{R}).$$

2.8 Sobolev norms

It is well know since [20] and the subsequent [23] (see also [8]) that in gauge theories the appropriate way of endowing the infinite dimensional spaces of connections and sections with a manifold structure is by appropriate Sobolev norms. We recall here a few basic notions.

Let X be a compact oriented n-dimensional manifold. On the space of $\mathcal{C}^\infty$ (real) functions on X we define

$$|\nabla^\ell f|^2 = \nabla^{\mu_1} \cdots \nabla^{\mu_\ell} f \nabla_{\mu_1} \cdots \nabla_{\mu_\ell} f,$$

where a sum over repeated indices is understood.

With this notation, we define the space $L^p_k(X)$, with $k \geq 0$ an integer and $p \geq 1$ a real number, to be the completion of $\mathcal{C}^\infty(X)$ in the norm

$$\|f\|_{L^p_k} = \left(\sum_{0 \leq \ell \leq k} \|\nabla^\ell f\|^p_{L^p(X)} \right)^{1/p} . \tag{7}$$

We shall occasionally consider non-compact complete Riemannian manifolds. In this case, we can choose whether to complete in the norm (7) the space of compactly supported smooth functions $\mathcal{C}^\infty_c(X)$ or the space $\{f \in \mathcal{C}^\infty(X) | |\nabla^\ell f| \in L^p(X), 0 \leq \ell \leq k\}$. If the Riemannian manifold is complete, the two spaces thus obtained coincide, but this is not the case on a non-complete manifold [1]. Similarly, the two completions can be considered for the case of smooth manifolds with boundary, and again (in cases where the boundary is a smooth compact $(n-1)$-dimensional manifold without boundary) the two spaces coincide [1].

The definition of L^p_k spaces of functions extends to L^p_k spaces of forms or sections of vector bundles over X, by patching together norms of the form (7) with a partition of unity. It is easy to show that different choices of the cutoff functions lead to equivalent norms. When $p = 2$ we have Hilbert spaces $L^2_k(X)$.

It is very useful, for many of the results discussed in the following chapters, to recall the Sobolev embedding theorems.

Proposition 2.21 *Let k and ℓ be positive integers with $k > \ell \geq 0$, and p and q real numbers with $1 \leq p < q$. Let r be a positive integer. Let X be an n-dimensional compact manifold. Then we have the following results:*

(1) If $\frac{1}{q} \geq \frac{1}{p} - \frac{k-\ell}{n}$ there is an inclusion $L^p_k(X) \hookrightarrow L^q_\ell(X)$ that is a bounded map; the inclusion is compact if $\frac{1}{q} > \frac{1}{p} - \frac{k-\ell}{n}$.

(2) If $\frac{k-r}{n} > \frac{1}{p}$ there is a continuous inclusion $L^p_k(X) \hookrightarrow \mathcal{C}^r(X)$.

Again the embedding theorem can be reformulated for the case of non-compact manifolds. The same result holds for complete Riemannian manifolds with bounded curvature and injectivity radius $\delta > 0$ ([1] pg.45) or for smooth manifolds with boundary ([1] pg.51). In these cases the space $\mathcal{C}^r(X)$ has to be replaced by the space $\mathcal{C}^r_B(X)$ of $\mathcal{C}^r$ functions that are bounded together with all derivatives up to order r on X. Proposition 2.21 then holds in this case as well. However, the Sobolev embedding has to be modified in a more essential way whenever the domain is such that the space of compactly supported functions is not dense in the Sobolev space. A good reference for Sobolev spaces on manifolds is [1].

As an example, consider the case of a compact 4-manifold X. If we want to guarantee that we are dealing with continuous functions we can choose to work in the space $L^p_1(X)$ with $p > 4$. In the case of the Hilbert spaces $L^2_k(X)$, it is sufficient to choose $k > 2$.

The reason why we choose to work with Sobolev norms, is that they give a good control over the regularity of functions via the embedding theorems, but they also give rise to a Banach or Hilbert structure, thus enabling us to use an infinite dimensional analogue of the implicit function theorem which would not be available in the smooth topology. The implicit function theorem is a crucial tool in order to show that, after a generic perturbation, the moduli spaces of solutions of elliptic equations modulo gauge symmetries are smooth manifolds.

In a later chapter, in order to construct a gauge theory of three-manifolds, we shall consider the Seiberg–Witten equations on a tube $X = Y \times \mathbb{R}$, where Y is a compact oriented three-manifold without boundary. In this case it is possible to introduce weighted Sobolev norm, with a choice of a weight function that forces a certain rate of decay at $\pm\infty$. A reference for weighted Sobolev spaces is [16]. We shall consider $X = Y \times \mathbb{R}$ endowed with the cylindrical metric $dt^2 + g$.

We choose a weight function $e_\delta(t) = e^{\tilde{\delta}t}$, where $\tilde{\delta}$ is a smooth function with bounded derivatives, $\tilde{\delta} : \mathbb{R} \to [-\delta, \delta]$ for some fixed real number $\delta > 0$, such that $\tilde{\delta}(t) \equiv -\delta$ for $t \leq -1$ and $\tilde{\delta}(t) \equiv \delta$ for $t \geq 1$. The $L^2_{k,\delta}$ norm is defined as $\|f\|_{2,k,\delta} = \|e_\delta f\|_{2,k}$. The weight e_δ imposes an exponential decay as asymptotic condition along the cylinder.

Again, we have a Sobolev embedding theorem.

Proposition 2.22 *Let X be a cylinder $X = Y \times \mathbb{R}$, with Y a compact three-dimensional manifold. We have*

(i) The embedding $L^2_{k,\delta} \hookrightarrow L^2_{k-1,\delta}$ is compact for all $k \geq 1$.

(ii) If $k > m + 2$ we have a continuous embedding $L^2_{k,\delta} \hookrightarrow \mathcal{C}^m$.

(iii) If $k > m + 3$ the embedding $L^2_{k,\delta} \hookrightarrow \mathcal{C}^m$ is compact.

(iv) If $2 < k'$ and $k \leq k'$ the multiplication map $L^2_{k,\delta} \otimes L^2_{k',\delta} \stackrel{m}{\to} L^2_{k,2\delta}$ is continuous.

2.9 Fredholm properties

Suppose given a differential operator

$$D = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha,$$

of order m, acting on the smooth sections of a vector bundle E over a compact manifold X. Then D is bounded in the Sobolev norms

$$D : L^2_{k+m}(X, E) \to L^2_k(X, E).$$

As we recalled in the brief introduction to the index theorem, on a compact manifold if D is elliptic, this is a condition sufficient to ensure that it is Fredholm. However, if the manifold X is non-compact this is no longer true. Again, we are interested in the particular case where $X = Y \times \mathbb{R}$, with Y a compact three-manifold. We are also interested in one particular type of operators, namely D

a first order elliptic operator, such that, for $|t| \geq T_0$ on the cylinder, D is of the form

$$D = \partial_t + T_{\pm\infty,\delta} + \epsilon,$$

where the $T_{\pm\infty,\delta}$ are first order operators on Y, the "asymptotic" operators, and ϵ is a correction term that is small for $|t| \geq T_0$. We consider D acting on the Sobolev spaces

$$D : L^2_{1,\delta}(E) \to L^2_\delta(E),$$

where E is the pullback of a bundle on Y.

We have the following result [16], [19] pg.135, that imposes a constraint on the choice of the weight δ in the weighted Sobolev norms we want to consider on $X = Y \times \mathbb{R}$.

Proposition 2.23 *An elliptic operator D on $X = Y \times \mathbb{R}$ that is of the form*

$$D = \partial_t + T_{\pm\infty,\delta} + \epsilon,$$

for $|t| \geq T_0$, is Fredholm if and only if the operator $T_{\pm\infty,\delta} - \frac{\delta}{2} Id$ acting on L^2-sections is invertible, i.e. if $\frac{\delta}{2}$ is not in the spectrum of $T_{\pm\infty,\delta}$.

2.10 Exercises

- Let λ be an element of the gauge group $\mathcal{G}$. Prove that the connected component of $\mathcal{G}$ to which λ belongs is identified with the class $[\frac{-i}{\pi}\lambda^{-1}d\lambda]$ in $H^1(X,\mathbb{Z})$, under the isomorphism of lemma 2.11.

- Prove that, if $W^+ \oplus W^-$ is a $U(2) \times U(2)$-spinor bundle over the four-dimensional manifold X, in the sense of definition 2.3, then the structure group admits a reduction to $Spin_c(4) = \{(g,h) \in U(2) \times U(2) | \det(g) = \det(h)\}$.

- Let X be a four-manifold endowed with a $Spin_c(4)$-principal bundle that lifts the frame bundle. Let $W^\pm$ be the irreducible Clifford modules for $C(\mathbb{R}^4)$, and consider the bundle $P_{Spin_c(4)} \times_{Spin_c(4)} W^\pm$. For simplicity we also denote this bundle $W^\pm$. Then, $W^\pm$ is a spinor bundle in the sense of definition 2.3.

- Given a smooth vector bundle E over a $Spin_c$–manifold X and a connection A on E, the twisted Dirac operator $D_A : \Gamma(X, W^+ \otimes E) \to \Gamma(X, W^- \otimes E)$ is the operator acting on a section $s \otimes e$ as the Dirac operator on s and the composite of the covariant derivative $\tilde{\nabla}_A$ and the Clifford multiplication on e:

$$D_A : \Gamma(X, W^+ \otimes E) \stackrel{\nabla\otimes 1 + 1\otimes\tilde{\nabla}_A}{\to} \Gamma(X, W^+ \otimes E \otimes T^*X)$$

$$\stackrel{g}{\to} \Gamma(X, W^+ \otimes E \otimes TX) \stackrel{\bullet}{\to} \Gamma(X, W^- \otimes E).$$

The twisted Dirac operator satisfies the Weitzenböck formula

$$D_A^2 s = (\nabla_A^* \nabla_A + \frac{\kappa}{4} + \frac{-i}{4} F_A) s,$$

where ∇_A^* is the formal adjoint of the covariant derivative with respect to the *Spin*–connection on the Spinor bundle, and with respect to the connection A on E, $\nabla_A = \nabla \otimes 1 + 1 \otimes \tilde{\nabla}_A$; κ is the scalar curvature on X, F_A is the curvature of the connection A, and $s \in \Gamma(X, W^+ \otimes E)$.

- The choice of a connection $\hat{A}$ on the determinant line bundle L determines a unique spin connection. This shows existence of spin connections.

- Suppose given (X, g, J) a Riemannian manifold with metric g and an orthogonal almost complex structure J. Show that, with respect to the canonical $Spin_c$−structure with spinor bundle $S = \Lambda_J^{(0,*)}(X)$, the Dirac operator is of the form

$$D = \sqrt{2}(\overline{\partial} + \overline{\partial}^*) + Q,$$

with Q an endomorphism of S. Show that Q vanishes if X is Kähler. We shall return to this in discussing Seiberg–Witten theory on Kähler and symplectic manifolds.

3 The Functional and the Equations

In all the following we consider X to be a compact, connected, orientable, differentiable 4–manifold without boundary.

3.1 The Equations

The Seiberg–Witten equations are given in terms of a pair (A, ψ), where A is a spin connection and ψ is a section of W^+. For the present we assume that A and ψ are smooth. To stress the dependence of the Dirac operator on A, we denote it by D_A, we let $\hat{A}$ denote the induced connection on L and we similarly denote by $F_{\hat{A}}$ the curvature of the induced connection on L.

The equations are

$$D_A \psi = 0 \tag{8}$$

$$F^+_{\hat{A}} = \frac{1}{4} < e_i e_j \psi, \psi > e^i \wedge e^j, \tag{9}$$

Here $\{e_i\}$ is a local basis of TX that acts on ψ by Clifford multiplication (see the exercises), $\{e^i\}$ is the dual basis of T^*X, and $<,>$ is the inner product on the fibres of W^+.

The reader should be warned that the use of notation is not uniform in the literature. Here we chose to follow [12], but other references [13] [24] use a different notation. The main difference corresponds to interpreting the terms in the curvature equation (9) as purely imaginary self dual 2-forms, as we do, or as traceless hermitian endomorphisms of the positive spinor bundle. The equivalence of these notations is left as an exercise at the end of this chapter.

3.2 The Gauge Group

The gauge group $\mathcal{G}$ of $\sqrt{L}$ is well defined although $\sqrt{L}$ is not globally defined as a line bundle, since the definition of the gauge group is given just in terms of the transition functions. In particular, as in the case of a line bundle, $\mathcal{G} = \mathcal{M}(X, U(1))$.

There is an action of the gauge group on the space of pairs (A, ψ), where A is the spin connection and ψ a section of W^+, given by

$$\lambda : (A, \psi) \mapsto (A - \lambda^{-1} d\lambda, \lambda\psi). \tag{10}$$

Notice that the induced action on $\hat{A}$ will give $\hat{A} - 2i\lambda^{-1}d\lambda$.

Lemma 3.1 *The action defined in (10) induces an action of $\mathcal{G}$ on the space of solutions to the Seiberg–Witten equations.*

<u>Proof:</u> It is enough to check that

$$D_{A-\lambda^{-1}d\lambda}(\lambda\psi) = \lambda D_A \psi + d\lambda \cdot \psi - d\lambda \cdot \psi.$$

In the second equation

$$F^+_{\hat{A}-2\lambda^{-1}d\lambda} = F^+_{\hat{A}} - 2d^+(\lambda^{-1}d\lambda) = F^+_{\hat{A}},$$

and $< e_ie_j\lambda\psi, \lambda\psi > = | \lambda |^2 < e_ie_j\psi, \psi > = < e_ie_j\psi, \psi >$.
QED

It is clear from (10) that the action of $\mathcal{G}$ on the space of solutions is free iff ψ is not identically zero; while for $\psi \equiv 0$ the stabiliser of the action is $U(1)$, the group of constant gauge transformations.

3.3 The Seiberg–Witten Functional and the Variational Problem

Given a moduli problem formulated in terms of differential equations, one may consider a functional of which the solutions represent the absolute minima: an example is the case of the Yang–Mills functional, and the anti–self–dual equation for $SU(2)$ Donaldson gauge theory.

In the case of the Seiberg–Witten equations, it is not hard to figure out what such a functional could be:

Definition 3.2 *The* Seiberg–Witten functional *of a pair* (A, ψ) *is given by*

$$S(A,\psi) = \int_X (| D_A\psi |^2 + | F^+_{\hat{A}} - \frac{1}{4} < e_ie_j\psi, \psi > e^i \wedge e^j |^2) dv. \qquad (11)$$

Lemma 3.3 *Via the Weitzenböck formula (theorem 2.6) the Seiberg–Witten functional (11) can be rewritten as*

$$S(A,\psi) = \int_X (| \nabla_A\psi |^2 + | F^+_{\hat{A}} |^2 + \frac{\kappa}{4} | \psi |^2 + \frac{1}{8} | \psi |^4) dv. \qquad (12)$$

<u>Proof:</u> In fact we have that

$$< D_A^2\psi, \psi > = < \nabla_A^*\nabla_A\psi, \psi > + \frac{\kappa}{4} | \psi |^2 + \frac{1}{4} < F_{\hat{A}}\psi, \psi >$$

and

$$| F^+_{\hat{A}} - \frac{1}{4} < e_ie_j\psi, \psi > e^i \wedge e^j |^2 =$$

$$= | F^+_{\hat{A}} |^2 - \frac{1}{4}(F^+_{\hat{A}}, < e_ie_j\psi, \psi > e^i \wedge e^j) + \frac{1}{16} |< e_ie_j\psi, \psi >|^2 | e^i \wedge e^j |^2,$$

where $(,)$ denotes the pointwise inner product of two–forms: $(\alpha, \beta)dv = \alpha \wedge *\beta$. But $(F^+_{\hat{A}}, e^i \wedge e^j) = F^+_{\hat{A}\,ij}$ and $\frac{1}{4}(F^+_{\hat{A}}, < e_ie_j\psi, \psi > e^i \wedge e^j) = < \frac{1}{4} F^+_{\hat{A}\,ij} e_ie_j\psi, \psi >$, which is the expression of the action of $F^+_{\hat{A}}$ on $\Gamma(X, W^+)$, via Clifford multiplication. Thus two terms cancel out in the sum, as in the first summand of (11) only the self–dual part of the curvature acts non–trivially on the section

ψ (see the exercise at the end of the section). Moreover $| e^i \wedge e^j |^2 = 1$ and $|< e_i e_j \psi, \psi >|^2 = 2 | \psi |^4$.
QED

Some properties that follow from introducing the Seiberg–Witten functional are summarised in the following lemma.

Lemma 3.4 *If the scalar curvature of X is non-negative, all solutions of the Seiberg–Witten equations have $\psi \equiv 0$. If, further, X has $b_2^+ > 0$ then for a generic choice of the metric, the only solutions will be $\psi = 0$ and $\hat{A}$ flat.*

Proof: Under the assumption about the scalar curvature, (12), or equivalently the Weitzenböck formula, clearly implies $\psi \equiv 0$. Thus, the curvature equation becomes simply $F^+_{\hat{A}} = 0$. The first Chern class of the line bundle L, which is given by $c_1(L) = \frac{1}{2\pi}[F_{\hat{A}}]$, is an integral class modulo torsion. The equation implies that we have

$$\frac{1}{2\pi}[F_{\hat{A}}] \in H^{2-}(X; \mathbb{R}) \cap H^2(X; \mathbb{Z})/T,$$

with T the torsion subgroup of $H^2(X; \mathbb{Z})$. The set $H^2(X; \mathbb{Z})/T$ is a lattice in $H^2(X; \mathbb{R})$, and, under the assumption that we have $b_2^+ > 0$, the space $H^{2-}(X; \mathbb{R})$ is a proper subspace. For a generic choice of the metric, this subspace is in general position with respect to the lattice $H^2(X; \mathbb{Z})/T$ and does not intersect it outside the origin $F = 0$. This shows that A is a flat connection.
QED

The above argument fails in the case $b_2^+ = 0$. Clearly the following holds true as well.

Corollary 3.5 *If, moreover, the first Chern class of L in $H^2(X; \mathbb{Z})$ is not a torsion element, under the assumptions of the above lemma, we obtain that the only possible solution is the trivial one $\hat{A} = 0$, $\psi = 0$.*

In the case where $c_1(L)$ is torsion, we can still get rid of the flat connections by suitably perturbing the equations.

Definition 3.6 *The perturbed Seiberg–Witten equations are obtained by introducing a real self–dual two–form η as a perturbation parameter:*

$$\begin{aligned} D_A \psi &= 0, \\ F^+_{\hat{A}} + i\eta &= \tfrac{1}{4} < e_i e_j \psi, \psi > e^i \wedge e^j. \end{aligned} \tag{13}$$

The corresponding perturbed functional will be

$$S_\eta(A, \psi) = \int_X (| F^+_{\hat{A}} |^2 + | \nabla_A \psi |^2 + \frac{\kappa}{4} | \psi |^2) dv +$$

$$\int_X F_{\hat{A}}^+ \wedge i\eta + \int_X \mid \frac{1}{4} < e_i e_j \psi, \psi > e^i \wedge e^j - i\eta \mid^2 dv.$$

Notice that the corresponding equations, given in definition 3.6, no longer have solutions with $\psi \equiv 0$ and A a flat connection, since that would correspond to

$$\mid \frac{1}{4} < e_i e_j \psi, \psi > e^i \wedge e^j - i\eta \mid = 0,$$

which is not compatible with $\psi = 0$ and $\eta \neq 0$. The reducible solutions for the perturbed equations are of the form $\psi \equiv 0$ and A satisfying $F_{\hat{A}}^+ + i\eta = 0$.

Corollary 3.7 *Consider the perturbed Seiberg–Witten equations on a manifold X with $b_2^+ > 0$. Then, for any choice of a non-trivial self-dual 2-form η, the equations (13) admit no reducible solutions (A, ψ) with $\psi \equiv 0$.*

Proof: Under the assumption that $b_2^+ > 0$, we can guarantee that the equation $F_{\hat{A}}^+ + i\eta = 0$ has no solutions. In fact, there is a generic set of forms η such that the elements $\frac{1}{2\pi}[\eta] \in H^{2+}(X; \mathbb{R})$ avoid the the projection on $H^{2+}(X; \mathbb{R})$ of the elements $\frac{1}{2\pi}[F_{\hat{A}}]$ in the lattice $H^2(X; \mathbb{Z})/T$.
QED

It is a general fact in gauge theory that, given a functional like (11), one may look at the absolute minima, or just at the extremals, i.e. at solutions of the Euler–Lagrange equations. The equation for minima is in general a first order problem, while the Euler–Lagrange equations will give a second order problem. For instance, the functional considered in Donaldson theory is the Yang–Mills functional. The anti–self–dual connections are the absolute minima; while the corresponding variational problem gives rise to the Yang–Mills equation (see [22]).

The variational problem for the Seiberg–Witten functional (11) was analysed in [12], where it is proven that the Euler–Lagrange equations are of the form

$$D_A^2 \psi - \frac{i}{2} F_{\hat{A}}^+ \cdot \psi - \frac{1}{8} < e_i e_j \psi, \psi > e_i e_j \psi = 0 \quad (14)$$

and

$$d^*(F_{\hat{A}}^+ - \frac{1}{4} < e_i e_j \psi, \psi > e^i \wedge e^j) + \frac{i}{2} Im < D_A \psi, e_i \psi > e^i = 0. \quad (15)$$

Moreover, using the Weitzenböck formula, these equations can be rewritten as

$$\nabla_A^* \nabla_A \psi + \frac{\kappa}{4} \psi + \frac{1}{4} \mid \psi \mid^2 \psi = 0 \quad (16)$$

and

$$d^* F_{\hat{A}}^+ + \frac{i}{2} Im < \nabla_i \psi, \psi > e^i = 0.$$

It is known that in some gauge theoretic problems there is an equivalence of the first and second order equations, namely there are no critical points that are non-minimising [22]. One can formulate a similar question in the case of Seiberg–Witten theory. In our case, one would expect to find extra critical points that satisfy the Euler-Lagrange equations but do not minimise the energy. In fact, the intuitive reason is that, otherwise, one would have a contraction of the configuration space along the gradient flow lines of the functional down to the absolute minima. Since it is known that, in our case, the configuration space has non-trivial topology (in fact it has the homotopy type of $\mathbb{C}P^\infty \times K(H^1(X;\mathbb{Z}),1)$), there must be other critical points. However, in order to make this argument precise, one has to show that the functional (11) has a gradient flow that extends for all times, hence it is at least necessary to show that the functional has the Palais–Smale condition: this is proven in [12]. If indeed there are non-minimising critical points of the Seiberg–Witten functional, a natural question to ask is what is the geometric meaning of these solutions, and, in particular, whether they contain any more topological information on the differentiable manifold X. Usually, in a variational problem, coercivity ensures the existence of minima, whereas Palais–Smale gives some control over the existence of other critical points. In fact, Palais–Smale implies a deformation result, which is the analogue of the Morse lemma on the homotopy deformation of sub-level sets. This can be regarded as an existence result for a local gradient flow. In turn, this deformation lemma implies the Ambrosetti–Rabinowitz Mountain Pass theorem which is an existence result for critical points. In the context of Seiberg–Witten theory, an infinite dimensional Morse theory plays an essential role in the three-dimensional context, where it appears under the shape of Floer theory, which we shall discuss in a different part of the book. However, in the four-dimensional case, even if we have the right analytic properties for the functional, it is hard to use them to obtain significant geometric results. The reason for this difference between the three-dimensional and the four-dimensional case can be ascribed to the peculiar nature of Floer theory, where the gradient flow exists with respect to a weak L^2 inner product, and is given by the four-dimensional Seiberg–Witten equations. The Palais–Smale condition that we are going to discuss here in the four-dimensional context only ensures the existence of a local gradient flow with respect to a strong Sobolev metric. This makes the information more difficult to interpret geometrically. For existence and extension of local gradient flows for a Palais–Smale functional we refer to [3], and for a brief general overview of coercivity and Palais–Smale the reader can consult a textbook like [11].

Some analytic properties of the Seiberg–Witten functional have been analysed in [12] and will be described briefly in the next section.

3.4 Analytic properties of the Seiberg–Witten functional

Here we introduce some analytic properties of the Seiberg–Witten functional, following [12].

We first discuss the coercivity of the Seiberg–Witten functional (11), and then the Palais-Smale condition and the related property of compactness of the moduli space. Notice that exactly this property of compactness is the major difference between Seiberg–Witten and Donaldson gauge theory.

Let $\mathcal{A}$ be the space of connections and spinor sections,

$$\mathcal{A} = \mathcal{C} \times \Gamma(X, W^+).$$

Here we assume as analytic data $L^2_1(W^+)$ and $L^2_1(\mathcal{C})$, where the first denotes the completion in the L^2_1 Sobolev norm of the space of smooth sections $\Gamma(X, W^+)$ and the second is the L^2_1–completion of the space of 1–forms associated to the affine space of connections, given a fixed smooth connection A_0. We shall also consider, as gauge transformations, the space $\mathcal{G}^2_2$, which is locally the completion of $\mathcal{G}$ in the L^2_2–norm. These are all infinite dimensional Hilbert manifolds.

There is an action of $\mathcal{G}^2_2$ on $L^2_1(W^+) \times L^2_1(\mathcal{C})$, which is differentiable and is given by (10).

Consider the Seiberg–Witten functional on the space

$$L^2_1(\mathcal{A}) = L^2_1(W^+) \times L^2_1(\mathcal{A}).$$

It is proven in [12] that this functional is coercive, i.e. that the following holds.

Lemma 3.8 *There is a constant c such that, for some $\lambda \in \mathcal{G}^2_2$,*

$$S(A, \psi) \geq c^{-1}(\|\lambda\psi\|_{L^2_1} + \|A - \lambda^{-1} d\lambda\|_{L^2_1}) - c.$$

Given a weakly lower semicontinuous functional S defined on a Hilbert space, the property of coercivity, namely the fact that $S(x_k) \to \infty$ whenever $\|x_k\| \to \infty$, is enough to guarantee that the functional attains a minimum.

In our case this guarantees that there is an element (A, ψ) that satisfies minimises the Seiberg–Witten functional. Notice that this is not enough to guarantee the existence of solutions of the Seiberg–Witten equations, since, on a given manifold with a given choice of $Spin_c$ structure, the minima of $S(A, \psi)$ need not be zeroes or absolute minima. However, if on X with a given $s \in \mathcal{S}(X)$ we have

$$\inf S(A, \psi) = 0,$$

the coercivity properties ensures the existence of solutions.

3.4.1 The Palais–Smale condition: sequential compactness

In order to introduce the Palais–Smale condition, we need to recall the following definition:

Definition 3.9 *Suppose given a Hilbert manifold Y with the action of a (possibly infinite dimensional) Lie group G and a smooth functional $f : Y \to \mathbb{R}$, which is G invariant. Then f satisfies the Palais–Smale condition if, for any sequence $\{x_k\} \subset Y$ such that*

(i) $f(x_k)$ is bounded and

(ii) $df(x_k) \to 0$, as $k \to \infty$,

there is a subsequence $\{x'_k\}$ and a sequence $\{g_k\} \subset G$ such that $g_k x'_k \to x$, with $df(x) = 0$ and $f(x) = \lim_k f(x_k)$.

The Palais–Smale condition for a $\mathcal{C}^1$ functional on a Hilbert (Banach) space is stronger than coercivity. The most important use of the Palais Smale condition in four-dimensional Seiberg–Witten theory is the convergence result, which proves the sequential compactness of the moduli space.

In order to prove that the Seiberg–Witten functional over the space

$$L^2_1(W^+) \times L^2_1(\mathcal{C})$$

has the Palais–Smale condition, we need some preliminary results.

Lemma 3.10 *Let (A, ψ) be a solution of the Seiberg–Witten equations. Then the following estimate holds:*

$$\|\psi\|^2_{L^\infty} \leq \max(0, -\min_{x \in X} \kappa(x)).$$

<u>Proof:</u> Note that, since we are considering the equations in L^2_1–spaces, we are not assuming any regularity.

We can assume that the scalar curvature $\kappa \geq -1$. We want to show that the set S of points where $| \psi |> 1$ is of measure zero.

Define

$$\phi = \begin{cases} (|\psi| - 1)\frac{\psi}{|\psi|} & \text{if } |\psi| \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

As we computed in (16), using the Weitzenböck formula together with the Euler–Lagrange equations we obtain

$$\nabla^*_A \nabla_A \psi + \frac{\kappa}{4}\psi + \frac{1}{4}|\psi|^2\psi = 0.$$

This gives

$$\int_S (< \nabla_A \psi, \nabla_A \phi > + \frac{\kappa}{4} < \psi, \phi > + \frac{1}{4}|\psi|^2 < \psi, \phi >) dv =$$

$$\int_S (< \nabla_A \psi, \nabla_A \phi > + \frac{1}{4}(|\psi|^2 + \kappa)(|\psi| - 1)|\psi|)) dv$$

$$\geq \int_S (< \nabla_A \psi, \nabla_A \phi > + \frac{1}{4}(|\psi|^2 - 1)(|\psi| - 1)|\psi|) dv$$

now a straightforward computation shows that $< \nabla_A \psi, \nabla_A \phi > \geq 0$, hence S is of measure zero.
QED

Notice that the proof of [12] that we presented here does not assume any regularity of the solution ψ. A simpler proof of the bound 3.10 can be given as in [13], if we assume more regularity to start with: enough to allow us to differentiate $|\psi|^2$ twice and use $0 \geq \Delta|\psi|^2$ at a point where $|\psi|^2$ achieves a local maximum. The Laplacian is then estimated using the Weitzenböck formula and this results in a bound in terms of the scalar curvature as in 3.10. The initial assumption of regularity, which amounts to a choice of one particular Sobolev space to work with, is somewhat arbitrary. In fact, as we are going to see in theorem 3.12, we are really working with smooth solutions of the Seiberg–Witten equations.

Lemma 3.11 *Fix a smooth connection A_0. Given a connection $A \in L^2_1(\mathcal{C})$, there exists a gauge transformation λ in the identity component of $\mathcal{G}^2_2$ such that the 1–form*

$$A - \lambda d\lambda - A_0$$

is co–closed.

Proof: Consider an irreducible element (A, ψ) in the configuration space $\mathcal{A}$. The directions in the tangent space $\mathcal{T}_{(A,\psi)}\mathcal{A}$ spanned by the action of the gauge group $\mathcal{G}$ are given by the image of the map

$$G_{(A,\psi)} : \Lambda^0(X) \to i\Lambda^1(X) \oplus \Gamma(X, W^+)$$

given by

$$f \mapsto (-df, if\psi).$$

The tangent space $\mathcal{T}_{[A,\psi]}\mathcal{B}$ to the quotient $\mathcal{B} = \mathcal{A}/\mathcal{G}$ is then identified with

$$(i\Lambda^1(X) \oplus \Gamma(X, W^+))/Im(G_{(A,\psi)}).$$

In particular, this means that we can gauge transform (A, ψ) by an element λ in the identity component of the gauge group to obtain that $\lambda(A, \psi)$ is in the orthogonal complement to the image of $G_{\lambda(A,\psi)}$, with respect to the L^2-inner product, that is, it is in the kernel of d^*.
QED

Notice that, fixing a background connection A_0, one can work in the Coulomb gauge. This is a global gauge fixing, and leaves only the ambiguity of constant gauge transformations and of large gauge transformations λ such that $\lambda^{-1}d\lambda$ is an integral harmonic 1-form.

Theorem 3.12 *Given a sequence $\{(A_k, \psi_k)\}$ of solutions of the Seiberg–Witten equations in $L^2_1(W^+) \times L^2_1(\mathcal{C})$, there exist a subsequence $\{(A_{k'}, \psi_{k'})\}$ and a sequence of gauge transformations $\{\lambda_{k'}\}$ in $\mathcal{G}^2_2$, such that the sequence*

$$\{(A_{k'} - \lambda_{k'} d\lambda_{k'}, \lambda_{k'}\psi_{k'})\}$$

converges with all derivatives to a smooth solution (A, ψ) of the Seiberg–Witten equations.

Proof: Using lemma 3.11 we can assume that $\{\hat{A}_k - \hat{A}_0\}$ is a sequence of co–closed 1–forms. Notice that we have $\hat{A}_k - \hat{A}_0 = 2(A_k - A_0)$. Moreover, since $F_{\hat{A}_k} = d\hat{A}_k$, we have $dF_{\hat{A}_k} = 0$. Thus, $d^* F_{\hat{A}_k} = (d + d^*)F_{\hat{A}_k}$ and $d^* d\hat{A}_k = (d^*d + dd^*)\hat{A}_k$, where both $d + d^*$ and $d^*d + dd^*$ are elliptic operators.

Hypothesis (i) of definition 3.9 holds, since $S(A_k, \psi_k) \equiv 0$, and, since the Seiberg–Witten functional is coercive, this means that we have

$$\|\lambda_k \psi\|_{L^2_1}, \text{ and} \|A_k - \lambda_k d\lambda_k - A_0\|_{L^2_1}$$

are bounded (possibly after composing with another family of gauge transformations).

The bound on the L^2_1–norm of the connections gives an L^2 bound:

$$\|A_k - A_0\|_{L^2} \leq \|A_k - A_0\|_{L^2_1}.$$

In order to show that there is a subsequence that converges with all derivatives, it is sufficient to show that all Sobolev norms are bounded and use the Sobolev embedding theorem.

To bound the higher Sobolev norms observe that we have

$$d^* F_{\hat{A}_k} = 2d^* F^+_{\hat{A}_k} = -i \sum_j Im(< \nabla_j \psi_k, \psi_k >) e^j.$$

This follows from the variational equation (15) and the identities $*d * F = *dF^+ - *dF^-$ and $0 = *dF = *dF^+ + *dF^-$. This gives a bound on $\|d^* F_{\hat{A}}\|_{L^2}$, and therefore on $\|d^* d\hat{A}_k\|_{L^2}$.

Now the elliptic estimate applied to the operator $dd^* + d^*d$ gives

$$\|\hat{A}_k\|_{L^2_2} \leq c(\|d^* d\hat{A}_k\|_{L^2} + \|\hat{A}_k\|_{L^2}).$$

It is clear that this procedure can be carried over for all higher Sobolev norms.

The result for the sections ψ_k follows from lemma 3.10, which gives an L^2 bound, and the coercivity property of the Seiberg–Witten functional, which gives the L^2_1 bound. The bounds on the higher norms are obtained by applying the elliptic estimate to the Dirac operator. The transformed sequence is smooth provided A_0 is chosen smooth.

QED

The result of theorem 3.12 holds true also for solutions of the variational problem (14), (15). Moreover, the argument given here can be slightly modified in order to show the following result (whose proof is given in [12]).

Theorem 3.13 *The Seiberg–Witten functional (11) satisfies the Palais–Smale condition of definition 3.9.*

3.5 Exercises

- The equation (9) is often written in terms of endomorphisms of W^+. Following [6] we can consider the map $\gamma : TX \to Hom(S^+, S^-)$ given in local coordinates by $\gamma(e_i) = \sigma_i$, where the σ_i are the Pauli matrices; S^+ and S^- are complex 2-plane bundles. This induces an action of the 2-forms Λ^2 on S^+ given by the expression $e \wedge e'(s) = -\gamma^*(e)\gamma(e')s$, where $*$ denotes the adjoint ([6] pg. 76). Check that Λ^{2-} acts trivially, and therefore this can be considered as an action of the self-dual 2-forms. Hence we get a map $\rho : \Lambda^{2+} \to End(S^+)$. Check that (9) can be written as

  $$\rho(F^+_{\hat{A}}) = \sigma(\psi \otimes \bar{\psi}),$$

  where σ is the projection on the traceless part of $End(S^+ \otimes \sqrt{L}) \cong S^+ \otimes S^+$.

- Check that the map ρ in the above problem changes norms by a factor 2.

- The equations (8), (9) can be written as

  $$D_A \psi = 0$$

  and

  $$(F^+_{\hat{A}})_{ij} = -\frac{i}{2}\bar{\psi}\Gamma_{ij}\psi,$$

  where the $\Gamma_{ij} = [\sigma_i, \sigma_j]$ are defined in terms of the Clifford matrices (see [24]). Since $\psi \in \Gamma(X, S^+ \otimes \sqrt{L})$, $\bar{\psi} \in \Gamma(X, S^+ \otimes (\sqrt{L})^{-1})$ and thus the right hand side can also be read as

  $$\bar{\psi} \cdot \psi \in \Gamma(X, S^+ \otimes S^+) \cong Hom(S^+, S^+).$$

  Check that this sheaf splits as

  $$Hom(S^+, S^+) \cong \Lambda^0(TX) \oplus \Lambda^{2+}(TX)$$

  due to the Clifford action, and that, in the second equation, the self-dual part of the curvature has to coincide with the projection of $\bar{\psi} \cdot \psi$ on $\Lambda^{2+}(TX)$.

- Complete the details of the proof of lemma 3.10.

- Prove that $| < e_i e_j \psi, \psi > |^2 = 2 \mid \psi \mid^4$.

4 Seiberg–Witten invariants of 4-manifolds

4.1 The Moduli Space

We can consider the moduli space of solutions of (8) and (9) modulo the action of the gauge group. The purpose of studying the topology of the moduli space is to have a somehow "simpler" model of the manifold X by means of which to compute invariants associated to the differentiable structure of X.

Definition 4.1 *The moduli space M is the set of solutions of the Seiberg–Witten equations, modulo the action of the gauge group. M depends on the choice of the $Spin_c$ structure $s \in \mathcal{S}$.*

In principle, by the above definition, it seems that we are considering a whole, possibly infinite, family of moduli spaces, according to the choice of the $Spin_c$ structure $s \in \mathcal{S}$; however in the following we shall see that only finitely many choices will give rise to a nontrivial moduli space.

Some of the most important geometric properties of the moduli space are finite dimensionality, compactness, orientability, and the fact that it has at most very "nice" singularities. We are going to present these results in the rest of the section. We follow mainly the original paper of Witten [24], and occasionally [13].

4.1.1 Computation of the Dimension

Here we are going to see an interesting application of the Atiyah–Singer index theorem, namely, how the Fredholm property of the linearisation of the equations, together with the index theorem, allows us to estimate the dimension of the moduli space of solutions. This is a procedure that is well known from many different contexts: roughly, whenever one wishes to compute the dimension of the moduli space of solutions of certain differential equations modulo the action of some large symmetry group, one tries to construct a local model of the moduli space by linearising the equations to some Fredholm operator and then fit the linearisation into a short chain complex (the deformation complex) such that its Euler characteristic, computed via the index theorem, gives the dimension of the moduli space.

This procedure works in some generic case (e.g. for an open dense set of metrics). For metrics that are "non–generic" in that sense, the moduli space may have a dimension that is not the expected one. This phenomenon corresponds to an obstruction that lies in the second cohomology of the deformation complex. A case in Seiberg–Witten theory where the obstruction is non-vanishing, that is, where the actual dimension of the moduli spaces exceeds the expected one, occurs when one considers Kähler metrics. We shall discuss this case in Part III.

In order to compute the dimension of M, we shall linearise (8) and (9) in a neighbourhood of a solution (A_0, ψ_0). It should be pointed out that the computation of the virtual dimension that we are going to present relies on the a priori assumption that the moduli space is non-empty. If the virtual dimension thus obtained is negative, this is enough to guarantee that, under a generic perturbation, the moduli space M is empty. However, generic moduli spaces can be empty even though the virtual dimension computed by the index theorem is positive.

The linearisation of (8) and (9) is easily obtained as follows.

Lemma 4.2 *The linearised Seiberg–Witten equations at a pair (A_0, ψ_0) are*

$$D_{A_0}\phi + i\alpha \cdot \psi_0 = 0$$

and

$$d^+\alpha - \frac{1}{2}Im(< e_i e_j \psi_0, \phi >)e^i \wedge e^j = 0.$$

where α is a real 1–form, and ϕ a section of W^+.

Now consider the infinitesimal action of the gauge group on the solution (A_0, ψ_0). If we write an element of the gauge group as a map $\lambda = e^{if}$ for some $f : X \to \mathbb{R}$, this means that the infinitesimal action is given by the map

$$(A_0, \psi_0) \mapsto (A_0 - idf, if\psi_0),$$

according to the definition (10) of the action of $\mathcal{G}$.

Now consider the following short sequence of spaces and maps:

$$0 \to \Lambda^0 \xrightarrow{G} i\Lambda^1 \oplus \Gamma(X, W^+) \xrightarrow{T} i\Lambda^{2+} \oplus \Gamma(X, W^+) \to 0. \tag{17}$$

Here G is the map given by the infinitesimal action of $\mathcal{G}$

$$G_{(A,\psi)}(f) = (-df, if\psi),$$

T is the operator defined by the linearisation of the Seiberg–Witten equations, i.e. the left hand side of the equations in lemma 4.2, Λ^q is the space of real q–forms on X, and $\Gamma(X, W^+)$ is the space of smooth sections of the spinor bundle. We define the operator G^* as the adjoint of G with respect to the L^2-inner product.

Lemma 4.3 *The sequence (17) is a chain complex. We shall denote this complex by C^*. The operator $T \oplus G^*$ on the assembled complex*

$$0 \to i\Lambda^1 \oplus \Gamma(X, W^+) \xrightarrow{T \oplus G^*} i\Lambda^{2+} \oplus \Gamma(X, W^+) \oplus \Lambda^0 \to 0$$

is Fredholm.

Proof: We need to check that $T \circ G = 0$. But in fact

$$D_{A_0}(if\psi_0) - idf \cdot \psi_0 = 0,$$

because of (8), and

$$d^+(df) + \frac{1}{2} Im(< e_i e_j \psi_0, if\psi_0 >) e^i \wedge e^j =$$

$$= Im(\frac{if}{2} < e_i e_j \psi_0, \psi_0 >) e^i \wedge e^j = 0.$$

Here we used the facts that $d^+ d = p_{\Lambda^{2+}} \circ d^2 = 0$ and that $< e_i e_j \psi_0, \psi_0 >$ is purely imaginary (as proved in lemma 2.4).

The complex (17) is elliptic and the assembled operator $T \oplus G^*$ is Fredholm since, up to zero–order terms, it is given by the elliptic differential operators $d^+ + d^*$ and D_A, as we discuss in the proof of theorem 4.4.

QED

By definition, the tangent space of M at the point (A_0, ψ_0) is the quotient $Ker(T)/Im(G)$: in fact we consider the linear approximation to the Seiberg–Witten equations modulo those directions that are spanned by the action of the gauge group. Thus we need to compute $H^1(C^*)$ to get the virtual tangent space. The index theorem provides a way to compute the Euler characteristic of C^* in terms of some characteristic classes.

Theorem 4.4 *The Euler characteristic of C^* is*

$$-\chi(C^*) = Ind(D_A + d^+ + d^*),$$

where d^* is the adjoint of the exterior derivative.

Proof: Up to zero–order operators G can be deformed to the exterior derivative d; and T can be deformed to the pair of operators

$$D_A : \Gamma(X, W^+) \to \Gamma(X, W^-)$$

and

$$d^+ : \Lambda^1 \to \Lambda^{2+}.$$

Hence the assembled complex becomes

$$0 \to \Lambda^1 \oplus \Gamma(X, W^+) \stackrel{D_A + d^+ + d^*}{\to} \Lambda^0 \oplus \Lambda^{2+} \oplus \Gamma(X, W^-) \to 0.$$

The Euler characteristic of the original complex is not affected by this change, hence:

$$-\chi(C^*) = Ind(D_A + d^+ + d^*).$$

QED

Note that, since $\sqrt{L}$ is not really a line bundle, its first Chern class is defined to be $c_1(\sqrt{L}) = \frac{c_1(L)}{2}$, and it makes sense in the coefficient ring $\mathbb{Z}[\frac{1}{2}]$.

Corollary 4.5 *The index of the complex C^* is equal to*

$$c_1(\sqrt{L})^2 - \frac{2\chi + 3\sigma}{4},$$

where χ is the Euler characteristic of X, σ is the signature of X, and $c_1(\sqrt{L})^2$ is the cup product with itself of the first Chern class of $\sqrt{L}$ integrated over X (with a standard abuse of notation we write $c_1(\sqrt{L})^2$ instead of $< c_1(\sqrt{L})^2, [X] >$).

Proof: Use the additivity of the index. By the index theorem for the twisted Dirac operator it is known that

$$Ind(D_A) = -\int_X ch(\sqrt{L})\hat{A}(X).$$

The Chern character is $ch(\sqrt{L}) = 2(1 + c_1(\sqrt{L}) + \frac{1}{2}c_1(\hat{L})^2 + \cdots)$ (the rank of L over the reals is two). The $\hat{A}$ class is $\hat{A}(X) = 1 - \frac{1}{24}p_1(X) + \cdots$, where $p_1(X)$ is the first Pontrjagin class of the tangent bundle. Thus the top degree term of $ch(\sqrt{L})\hat{A}(X)$ will be $\frac{1}{12}p_1(X) + c_1(L)^2$. On the other hand, by the index theorem for the signature operator it is known that $\frac{1}{3}\int_X p_1(X) = \sigma$, hence we get

$$Ind(D_A) = c_1(\sqrt{L})^2 - \frac{\sigma}{4}.$$

The index of $d^+ + d^*$ can be read off from the chain complex

$$0 \to \Lambda^0 \stackrel{d}{\to} \Lambda^1 \stackrel{d^+}{\to} \Lambda^{2+} \to 0.$$

The Euler characteristic of this complex turns out to be

$$-Ind(d^* + d^+) = \frac{1}{2}(\chi + \sigma),$$

by another application of the index theorem.

Summing up together it follows that

$$\chi(C^*) = c_1(\sqrt{L})^2 - \frac{2\chi + 3\sigma}{4}.$$

QED

Now, in order to obtain from this index computation the dimension of the moduli space, we need the following lemmata:

Lemma 4.6 *The tangent space $\mathcal{T}_{(A,\psi)}M$ at a regular point $\psi \neq 0$ can be identified with $H^1(C^*)$.*

Proof: $H^1(C^*)$ describes exactly those directions that are spanned infinitesimally at the point (A, ψ) by the solutions of the Seiberg–Witten equations, modulo those directions that are spanned by the action of the gauge group.
QED

Notice that in the following, unless otherwise specified, we use the notation M to denote the moduli space corresponding to a fixed choice of the $Spin_c$ structure in $\mathcal{S}$, and for a generic choice of the metric and of the perturbation. Sometimes the dependence on the $Spin_c$ structure is stressed by adding a subscript M_s, with $s \in \mathcal{S}$.

Lemma 4.7 *In the above complex $H^0(C^*) = 0$ and, under a suitable perturbation of the Seiberg–Witten equations, also $H^2(C^*) = 0$.*

Proof: $H^0(C^*) = 0$ since the map G, which describes the infinitesimal action of the gauge group, as in (17), is injective. In order to show that $H^2(C^*) = 0$, we use the following strategy. We allow the perturbation to vary in Λ^{2+}, and consider at a given point (A, ψ, η) the operator

$$\tilde{T}_{A,\psi,\eta}(\alpha, \phi, \epsilon) = (D_{A_0}\phi + i\alpha \cdot \psi_0, d^+\alpha + \epsilon - \frac{1}{2} Im < e_i e_j \psi_0, \phi > e^i \wedge e^j),$$

that maps the L^2_k-tangent space to the L^2_{k-1} one. Here (A, ψ) is a solution of the perturbed equations as in definition 3.6, with perturbation η. We prove that the operator $\tilde{T} + G^*$, with

$$0 \to \Lambda^1 \oplus \Lambda^{2+} \oplus \Gamma(X, W^+) \stackrel{\tilde{T}+G^*}{\to} \Lambda^{2+} \oplus \Gamma(X, W^+) \oplus \Lambda^0 \to 0,$$

is surjective. This implies, by the infinite dimensional Sard theorem, that, for a generic choice of the perturbation η, the original operator $T + G^*$ is surjective, hence $H^2(C^*) = 0$.

The operator $\tilde{T}$ has closed range. Suppose there is a section in $\Lambda^{2+} \oplus \Gamma(S^+ \otimes L)$ that is orthogonal to the image of $\tilde{T}$ in the space of L^2_{k-1} sections. We want to show that such an element must be identically zero. By an argument of [21], this is enough to prove that $\tilde{T}$ is surjective on smooth sections as well. So assume that (χ, β) is orthogonal to any section of the form

$$(D_{A_0}\phi + i\alpha \cdot \psi_0, d^+\alpha + \eta - \frac{1}{2} Im < e_i e_j \psi_0, \phi > e^i \wedge e^j).$$

This means that

$$< D_{A_0}\phi + i\alpha \cdot \psi_0, \chi >= 0,$$

with the inner product of sections of W^-, and

$$(d^+\alpha + \eta - \frac{1}{2} Im < e_i e_j \psi_0, \phi > e^i \wedge e^j, \beta) = 0$$

as 2–forms. Using the fact that (χ, β) is in the kernel of $\tilde{T}^*$, which is an elliptic operator with L^2_k coefficients, we can assume that (χ, β) is in fact in L^2_k. By the arbitrariness of α and ϵ, both χ and β must be zero.
QED

The above lemmata and computations yield the following result.

Theorem 4.8 *The dimension of the moduli space M at a regular point is given by*

$$dim(M) = c_1(\sqrt{L})^2 - \frac{2\chi + 3\sigma}{4}.$$

The result proven here is an infinitesimal result; the technique for passing to a local result is the same used in the $SU(2)$ gauge theory: as a reference we can point to [6].

Theorem 4.8 and the previous lemmata 4.7 and 4.6 can be read as follows. There are two sources of obstruction that make moduli space non-regular at a given point (A, ψ). One is, in fact, an obstruction to the smoothness of the moduli space at (A, ψ), and is represented by the $H^0(C^*)$ of the deformation complex: if the infinitesimal action of the gauge group at (A, ψ) is not injective, then the gauge class $[A, \psi]$ is not a smooth point in the quotient $\mathcal{B}$. Results like corollary 3.7 guarantee the vanishing of this obstruction. The other obstruction, which is given by the $H^2(C^*)$ of the deformation complex measures the lack of transversality, that is, the excess intersection, as we shall explain in Part IV.

4.1.2 Compactness

Even though everything needed in order to prove compactness of the moduli space is contained in theorem 3.12, we recall it here, since this property of compactness is precisely the main feature that distinguishes Seiberg–Witten theory from other gauge theories. In particular it makes it more tractable than Donaldson theory where the analytically complicated Uhlenbeck compactification is needed.

We proved in theorem 3.12 that every sequence of solutions of the Seiberg–Witten equations has a convergent subsequence, up to gauge transformations. This implies the following.

Theorem 4.9 *Any sequence of points in the moduli space of connections and sections (A, ψ) that satisfies the Seiberg–Witten equations, modulo the action of the gauge group $\mathcal{G}$, has a convergent subsequence. Therefore, the moduli space is compact.*

Notice that we want to consider the perturbed Seiberg–Witten equations as in definition 3.6. The compactness argument carries over to this case upon modifying the estimate of lemma 3.10. We obtain

$$\|\psi\|^2_{L^\infty} \leq \max(0, -\min_{x\in X}(\kappa(x) - 4\sqrt{2}|\eta|^2).$$

The rest of the argument follows analogously.

4.1.3 Orientation

The proof of the orientability of the moduli space just mimics an analogous argument in Donaldson theory: the orientation is given by a trivialisation of the determinant line bundle associated to the linearised Seiberg–Witten equations.

There is another proof of orientability given for Donaldson theory [8], which does not go through to the Seiberg–Witten case unless some stronger assumptions are made on the manifold X, for example that X be simply connected. Though this proof is not as good, it is a sufficient simplification for our purposes, therefore we shall present it briefly.

We have seen that, under a generic choice of the perturbation, we have no reducible solutions, moreover, as seen in lemma 4.7, the moduli space M is a smooth manifold that is cut out transversely by the equations. Following [8], we can now describe a way to give an orientation of M. There is an embedding of $\hat{M}$ in the space $\hat{\mathcal{A}}/\mathcal{G}$ of gauge equivalence classes of pairs (A, ψ) with $\psi \neq 0$,

$$\hat{M} \hookrightarrow \hat{\mathcal{A}}/\mathcal{G}.$$

The action of $\mathcal{G}$ on $\hat{\mathcal{A}}$ is free.

It is known (see e.g. [2]) that a family of Fredholm operators on a space Y determines an *index bundle* $[Ind(D)] \in KO(Y)$, in K–theoretic language. By definition $[Ind(D)]$ is orientable iff the characteristic class $w_1([Ind(D)]) = 0$. Recall that, for an element in $KO(Y)$, the Stiefel–Whitney class is given by $w_1(\xi - \zeta) := w_1(\xi) - w_1(\zeta)$ in $\mathbb{Z}_2$.

Moreover, if the space Y is simply connected, then every bundle over Y is orientable.

Thus, the strategy to prove the orientability of the moduli space is to realise the tangent bundle TM as a subbundle of the index bundle of a family of Fredholm operators over a simply connected space.

According to lemma 4.6, the fibre of TM over (A, ψ) is given by $H^1(C^*)$. In other words, the complex C^* corresponds to an assignment of a Fredholm operator to the point (A, ψ). Hence we get a family of Fredholm operators parametrised by all possible choices of (A, ψ), $\psi \neq 0$, namely the linearisation of the Seiberg–Witten equations at the chosen connection and section modulo the action of the gauge group. By theorem 4.4 this family of Fredholm operators may be thought of as the following:

$$T_{(A,\psi)} = D_A + d^+ + d^*.$$

To make this argument precise we should consider the appropriate Sobolev norms on $\mathcal{G}$ and on the space of sections where the family T acts. We address the reader to [8], where the analysis for $SU(2)$ gauge theory is developed.

Consider the index bundle of the above family T. By construction TM is the pullback of $[Ind(T)]$ via the embedding of M in $\hat{\mathcal{A}}/\mathcal{G}$.

Thus, we have the following:

Theorem 4.10 *Suppose that the manifold X has $H^1(X; \mathbb{Z}) = 0$. Then the moduli space M is orientable.*

Proof: We want to show that the bundle TM is orientable. By the above argument, it is enough to show that $\hat{\mathcal{A}}/\mathcal{G}$ is simply connected.

Consider the fibration

$$\mathcal{G} \to \hat{\mathcal{A}} \to \hat{\mathcal{A}}/\mathcal{G}.$$

The space of all pairs of a connection A and a section ψ is contractible as product of an affine and a vector space. The subspace $\hat{\mathcal{A}}$ is obtained by imposing the condition that $\psi \neq 0$. Thus it satisfies the condition $\pi_1(\hat{\mathcal{A}}) = 0$.

Thus, in the long homotopy sequence of the fibration, we have

$$\cdots \to \pi_1(\mathcal{G}) \to \pi_1(\hat{\mathcal{A}}) \to \pi_1(\hat{\mathcal{A}}/\mathcal{G}) \to \pi_0(\mathcal{G}) \to \pi_0(\hat{\mathcal{A}});$$

where we have that $\pi_1(\hat{\mathcal{A}}) = 0$. Moreover, by lemma 2.11, $\pi_0(\mathcal{G}) = H^1(X; \mathbb{Z})$, which is trivial by hypothesis. Hence $\pi_1(\hat{\mathcal{A}}/\mathcal{G}) = 0$.

QED

As already pointed out in the beginning of this paragraph, a more general proof can be given of the orientability, which does not assume any hypothesis on the cohomology of X. Note, moreover, that a technical difficulty we omitted to mention in the above argument arises in considering the K–ring $KO(\hat{\mathcal{A}}/\mathcal{G})$, since $\hat{\mathcal{A}}/\mathcal{G}$ is not a compact space. We refer the reader to [2] or [8] for a more detailed treatment of this problem.

4.2 The Invariants

Throughout this section we shall assume that the result of lemma 4.7 holds, namely, that the moduli space M_s contains no reducibles and is cut out transversely by the equations.

When the $Spin_c$ structure $s \in \mathcal{S}$ is such that the dimension of M_s satisfies

$$c_1(\sqrt{L})^2 - \frac{2\chi + 3\sigma}{4} < 0,$$

there are generically no solutions to the Seiberg–Witten equations, hence we define the Seiberg–Witten invariant N_s to be zero.

If the $Spin_c$ structure $s \in \mathcal{S}$ is such that the dimension of M_s is zero, then we have

$$c_1(\sqrt{L})^2 = \frac{2\chi + 3\sigma}{4},$$

hence generically the moduli space M consists of a finite number of points, due to the compactness property. Since we also have an orientation of the moduli space, we can associate to M a number, which is obtained by counting the point with a sign given by the orientation.

Definition 4.11 *The Seiberg–Witten invariant, relative to a choice of a* $Spin_c$ *structure* $s \in \mathcal{S}$ *with*

$$c_1(\sqrt{L})^2 - \frac{2\chi + 3\sigma}{4} = 0,$$

is given by

$$N_s \equiv \sum_{p \in M} \epsilon_p,$$

with $\epsilon_p = \pm 1$ *according to the orientation of* M *at the point* p.

It is more difficult to define the invariant when the dimension of the moduli space is positive. In fact a priori there are many possible ways to get a number by evaluating some cohomology class over the cycle M, so as to generalise the counting of points.

We shall adopt the following definition. Consider the group of all gauge transformations that fix a base point, $\mathcal{G}_b \subset \mathcal{G}$. Take the moduli space M^b of solutions of the Seiberg–Witten equations modulo the action of $\mathcal{G}_b$. Since we are considering perturbed equations and M contains no reducibles, we have the following.

Lemma 4.12 *This space* M^b *fibres as a principal* $U(1)$ *bundle over the moduli space* M.

Let $\mathcal{L}$ denote the line bundle over M associated to this principal $U(1)$ bundle via the standard representation. Then we introduce the following invariant.

Definition 4.13 *The Seiberg–Witten invariant, relative to a choice of* $s \in \mathcal{S}$ *such that the dimension of* M *is positive and even,*

$$d = c_1(\sqrt{L})^2 - \frac{2\chi + 3\sigma}{4} > 0,$$

is given by the pairing of the $(d/2)$*th-power of the Chern class of the line bundle* $\mathcal{L}$ *with the moduli space* M,

$$N_s \equiv \int_M c_1(\mathcal{L})^{d/2}.$$

If the dimension of M *is odd, the invariant is set to be zero.*

In the section regarding the quantum field theoretic approach to Seiberg–Witten theory we spend some more words on how to define the invariants. In fact, when the moduli space has positive dimension, it seems that there is a certain arbitrariness in the choice of the cohomology class to integrate in order to obtain the invariant. The choice of Definition 4.13 is in a sense canonical, in fact the space $\hat{\mathcal{A}}/\mathcal{G}$ is a model for the classifying space of $\mathcal{G}$. Thus we have a homotopy equivalence

$$\hat{\mathcal{A}}/\mathcal{G} \sim \mathbb{C}P^\infty \times K(H^1(X,\mathbb{Z}),1).$$

There is a canonical line bundle that has Chern class given by the generator of the $\mathbb{C}P^\infty$ factor in $H^2(\hat{\mathcal{A}}/\mathcal{G},\mathbb{Z})$. The corresponding principal $U(1)$-bundle is the one considered in lemma 4.12.

We show in the following section that only finitely many choices of L determine a nontrivial invariant.

4.3 Finiteness

Again this is going to be a consequence of the Weitzenböck formula, proved in theorem 2.6. Consider the Seiberg–Witten functional (11), rewritten in the form (12).

Lemma 4.14 *Solutions to the Seiberg–Witten equations have a uniform bound on* $\int_X \mid F^+_{\hat{A}} \mid^2 dv$.

Proof:Complete the term $\frac{\kappa}{4} \mid \psi \mid^2 + \frac{1}{8} \mid \psi \mid^4$ to a square: this gives the estimate

$$0 \leq \int_X (\frac{\kappa}{4} \mid \psi \mid^2 + \frac{1}{8} \mid \psi \mid^4 + \frac{1}{8}\kappa^2) dv.$$

At a solution we have $S(A,\psi)=0$, hence

$$\int_X \mid F^+_{\hat{A}} \mid^2 dv = -\int_X (\mid \nabla_A \psi \mid^2 + \frac{\kappa}{4} \mid \psi \mid^2 + \frac{1}{8} \mid \psi \mid^4) dv.$$

Thus we get that

$$\int_X \mid F^+_{\hat{A}} \mid^2 dv \leq \int_X (\mid F^+_{\hat{A}} \mid^2 + \mid \nabla_A \psi \mid^2) dv =$$

$$-\int_X (\frac{\kappa}{4} \mid \psi \mid^2 + \frac{1}{8} \mid \psi \mid^4) dv \leq \frac{1}{8} \int_X \kappa^2 dv.$$

QED

lemma 4.14 gives the following result.

Lemma 4.15 *Solutions to Seiberg–Witten equations have both*

$$I^+ = \int_X \mid F^+_{\hat{A}} \mid^2 dv \quad and \quad I^- = \int_X \mid F^-_A \mid^2 dv$$

uniformly bounded, i.e. bounded by geometric quantities that do not depend on the line bundle L.

Proof: The result follows from lemma 4.14 and the fact that

$$c_1(\sqrt{L})^2 = \int_X c_1(L) \wedge c_1(L) = \frac{1}{(2\pi)^2} \int_X (| F_{\hat{A}}^+ |^2 - | F_{\hat{A}}^- |^2) dv.$$

QED

Notice that there may be only finitely many choices of L, hence of $s \in \mathcal{S}$, such that both I^+ and I^- are bounded [6], [8], [24]. Thus we have one important result of Seiberg–Witten gauge theory:

Theorem 4.16 *Only finitely many choices of the $Spin_c$ structure give rise to non–trivial invariants.*

This is enough for us to introduce the Seiberg–Witten basic classes and the notion of simple type. We shall return on this in discussing the relation between Seiberg–Witten and Donaldson theory.

Definition 4.17 *We call* "Seiberg–Witten basic classes" *those classes* $c_1(L) \in H^2(X, \mathbb{Z})$ *that give rise to a non-trivial Seiberg–Witten invariant.*

Definition 4.18 *A closed connected 4-manifold X is of* "Seiberg–Witten simple type" *if all the basic classes correspond to zero-dimensional moduli spaces. That is $N_s \neq 0$ iff* $\dim M_s = c_1(\sqrt{L})^2 - \frac{2\chi+3\sigma}{4} = 0$.

4.4 A Cobordism Argument

The reason we are considering the Seiberg–Witten invariants is to introduce diffeomorphism invariants of a 4-manifold. Hence we need to show that the construction above leads to a value of N_s that is independent of the metric on X. This turns out to be the case, at least under the assumption that $b_2^+(X) > 1$. In the case $b_2^+(X) = 1$ the space of metrics breaks into *chambers*: inside each chamber the choice of the metric doesn't affect the invariant; while when a path of metrics crosses a *wall* between two chambers, the invariant jumps by a certain amount. An accurate study of the structure of chambers in the Kähler case can be found in [7].

Here we shall present briefly an argument that shows the invariance with respect to the metric for manifolds with $b_2^+ > 1$. The proof of the following results carries over to the present case from [6] chapters 4 and 9.

Theorem 4.19 *The invariants N_s of a manifold X with $b_2^+(X) > 1$ are diffeomorphism invariants, independent of the metric.*

Proof:

Let us consider two Riemannian metrics g_0 and g_1 on the manifold X. Let η_0 and η_1 be two corresponding small perturbations. We can choose a generic smooth path of metrics g_t that connects g_0 and g_1 and a corresponding path of forms η_t. If we assume that $b_2^+(X) > 1$, then we can choose g_t and η_t in such a way as to avoid singular solutions with $\psi \equiv 0$ at any fixed t.

Thus, for a fixed $Spin_c$ structure on X, we consider a trivial infinite dimensional bundle over $\hat{\mathcal{A}}/\mathcal{G} \times \{g_t, \eta_t\}$, with fibre $\Lambda^{2+} \oplus \Gamma(X, W^-)$, and the section

$$\sigma(A, \psi) = (F^+_{\hat{A}} + i\eta_t - \frac{1}{4} < e_i e_j \psi, \psi > e^i \wedge e^j, D_A \psi).$$

We define the *universal moduli space* to be the zero set of σ:

$$\mathcal{M}_s = \sigma^{-1}(0).$$

We have chosen $\{g_t, \eta_t\}$ so that $\mathcal{M}_s$ avoids the reducible locus. Moreover, we can ensure that $\mathcal{M}_s$ is a smooth manifold, of the dimension prescribed by the index theorem, cut out transversely by σ. The argument depends on an infinite dimensional generalisation of the transversality theorem [6], pg.145. In fact, if $f : \mathcal{X} \to \mathcal{Y}$ is a Fredholm map between infinite dimensional Banach manifolds and $h : W \to \mathcal{Y}$ is a smooth map from a finite dimensional manifold W, then h can be arbitrarily approximated in the $\mathcal{C}^\infty$ topology by a map which is transverse to f.

If we apply this argument to the Fredholm projection $\pi : \mathcal{M}_s \to \{g_t, \eta_t\}$ and to the map (g, η) of the interval I in the space of metrics and perturbations, then we get the transversality result: we can find a path $\{g_t, \eta_t\}$ such that the moduli space $\pi^{-1}(g_t, \eta_t) = M_{s, g_t, \eta_t}$ is a smooth manifold of codimension one in $\mathcal{M}_s$.

Suppose for simplicity that the moduli spaces M_s for (g_0, η_0) and (g_1, η_1) are zero dimensional. Then the set $\mathcal{M}_s$ is a smooth 1-dimensional manifold with boundary. From theorem 4.9, we deduce that $\mathcal{M}_s$ is compact. But the total oriented boundary of a compact 1-dimensional manifold is zero.

The case when M_s has positive dimension is analogous. In fact the universal moduli space is of dimension $\dim \mathcal{M}_s = \dim M_s + 1$ with oriented boundary $M_s(g_1, \eta_1) - M_s(g_0, \eta_0)$. Therefore the invariant

$$\int_{M_s(g_1, \eta_1)} c_1(\mathcal{L})^{d/2} = \int_{M_s(g_0, \eta_0)} c_1(\mathcal{L})^{d/2}.$$

In the case $b_2^+ = 1$, we expect to have a wall crossing formula that describes how the invariant jumps in correspondence to a metric and perturbation (g, η) where reducible solutions $\psi \equiv 0$ occur.

We assume that we have two pairs of metrics and perturbations (g_0, η_0) and (g_1, η_1) where there are no reducibles in the moduli space, and a path (g_t, η_t) that crosses the wall

$$\{(g, \eta) | H^{2-}(X, \mathbb{R}) \cap (\eta + H^2(X, \mathbb{Z})/T) \neq \emptyset\}$$

only once at (g, η) and transversely.

The following case of wall crossing was proven in [13].

Proposition 4.20 *Let X be a compact four-manifold with $b_2^+ = 1$ and $b_1 = 0$. Then we have*

$$N_s(X, (g_0, \eta_0)) - N_s(X, (g_1, \eta_1)) = \pm 1.$$

This formula was generalised in [15] as follows.

Proposition 4.21 *Let X be a compact four-manifold with $b_2^+ = 1$ and b_1 even. Then there is a torus T^{b_1} of reducible solutions in correspondence to the "bad element" (g, η). The corresponding wall crossing formula is*

$$N_s(X, (g_0, \eta_0)) - N_s(X, (g_1, \eta_1)) = \frac{\pm 1}{(b_1/2)!}(\frac{1}{2}(c_1(\mathcal{E})^2 c_1(L)[X])^{b_1/2}[T^{b_1}]),$$

where $\mathcal{E}$ is the tautological line bundle over $T^{b_1} \times X$.

References

[1] T. Aubin, *Non-linear analysis on manifolds: Monge–Ampére equations*, Springer 1982.

[2] B. Boos, D.D. Bleecker, *Topology and analysis: the Atiyah–Singer index formula and gauge–theoretic Physics*, Springer 1985.

[3] K.C. Chang, *Infinite dimensional Morse theory and multiple solution problems*, Birkhäuser 1993.

[4] S. Cordes, G. Moore, S. Ramgoolam, *Lectures on 2D Yang-Mills theory, equivariant cohomology, and topological field theories*, Nuclear Phys. B Proc. Suppl. 41 (1995), 184-244.

[5] S. Cordes, G. Moore, S. Ramgoolam, *Large N 2D Yang-Mills theory and topological string theory*, Comm. Math. Phys. 185 (1997), no. 3, 543–619.

[6] S. Donaldson, P. Kronheimer, *The geometry of four–manifolds*, Oxford 1990.

[7] R. Friedman, J.W. Morgan, *Algebraic Surfaces and Seiberg–Witten Invariants*, J. Alg. Geom. 6 (1997) N.3 445-479.

[8] D. Freed, K.K. Uhlenbeck, *Instantons and four–manifolds*, Springer 1984. Second edition. MSRI Publications, 1. Springer, 1991.

[9] R. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. (2) 142 (1995), N.3, 527-595.

[10] F. Hirzebruch, H. Hopf, *Felder von Flachenelementen in 4–dimensionalen Mannigfaltigkeiten*, Math. Annalen 136 (1958) 156-172.

[11] S. Kesavan, *Topics in functional analysis and applications*, Wiley 1989.

[12] J. Jost, X. Peng, G. Wang, *Variational aspects of the Seiberg–Witten functional*, Calc. Var. Partial Differential Equations 4, N.3 (1996) 205-218.

[13] P. Kronheimer, T. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Research Letters 1 (1994) 797-808.

[14] H.B. Lawson, M.L. Michelsohn, *Spin geometry*, Princeton University Press, 1989.

[15] T.J. Li, A. Liu, *General wall crossing formula*, Math. Res. Lett. (1995) N.2, 797-810.

[16] R.B. Lockhart, R.C. McOwen, *Elliptic operators on non-compact manifolds*, Ann. Sci. Norm. Sup. Pisa, IV-12 (1985), 409-446.

[17] D. McDuff, *Examples of simply–connected symplectic manifolds*, J. Diff. Geom. 20 (1984), 267-278.

[18] D. McDuff, D. Salamon, *Introduction to symplectic topology*, Oxford 1995.

[19] J.W. Morgan, T.S. Mrowka, D. Ruberman, *The L^2-moduli space and a vanishing theorem for Donaldson polynomial invariants*, International Press 1994.

[20] M.S. Narasimhan, T.R. Ramadas, *Geometry of* $\mathrm{SU}(2)$ *gauge fields*, Comm. Math. Phys. 67 (1979), no. 2, 121–136.

[21] J. Roe, *Elliptic operators, topology, and asymptotic methods*, Longman Sc. 1988.

[22] C.H. Taubes, *On the equivalence of first and second order equations for gauge theories*, Commun. Math. Phys. 75 (1980), 207-227.

[23] K.K. Uhlenbeck, *Connections with L^p bounds on curvature*, Comm. Math. Phys. 83 (1982), no. 1, 31–42.

[24] E. Witten, *Monopoles and Four–Manifolds*, Math. Research Letters 1 (1994) 769-796.

Part II
Seiberg–Witten on three-manifolds

I do not think
that I know it well;
but I know not
that I do not know.
Who of us knows that,
he does know that;
but he does not know
that he does not know.

Kena Upaniṣad, 2.2

5 Three-manifolds

We briefly review some basic notions of 3-manifold topology. We refer the reader to [31] for a brief but informative overview.

Consider a Riemann surface Σ_g of genus g, embedded in $\mathbb{R}^3$. The region of $\mathbb{R}^3$ bounded by Σ_g is a *handlebody* N_g of genus g. The complement of N_g in the one point compactification S^3 is also homeomorphic to N_g, hence this determines a decomposition of $S^3 = N_g \cup_{\Sigma_g} N_g$.

In general, a *Heegaard splitting* of a compact oriented three-manifold Y is a decomposition

$$Y = N_g \cup_{\phi:\Sigma_g \to \Sigma_g} N_g,$$

where the two handlebodies are glued along the homeomorphism $\phi : \Sigma_g \to \Sigma_g$ of the Riemann surface. Two isotopic homeomorphisms produce topologically equivalent manifolds (hence smoothly equivalent, since in dimension three we do not distinguish between the smooth and the topological category).

The first useful result is an existence result.

Proposition 5.1 *Every compact oriented three-manifold Y admits a Heegaard splitting,*

$$Y = N_g \cup_{\phi:\Sigma_g \to \Sigma_g} N_g.$$

This result is far from providing a classificatory scheme. In fact, as the trivial example of S^3 shows, Heegaard splittings are not unique, and a more serious problem consists of the fact that surface homeomorphisms are difficult to classify. However, a surface diffeomorphism can be reduced, up to isotopy, to a sequence of simpler operations, known as *Dehn twists* along some homologically non-trivial curve on the Riemann surface. The Dehn twist consists of cutting along the curve and gluing back after a full twist.

Thus, the information encoded in the Heegaard splitting can be rephrased in terms of the Dehn twists as follows.

Corollary 5.2 *Every compact oriented three-manifold Y is obtained from S^3 by Dehn twists on homologically non-trivial curves on a Riemann surface Σ_g.*

If we isolate the set of such curves, we obtain a link in S^3. Every component of the link is a knot $K \subset S^3$. Thus, we can define in general the *Dehn surgery* on a knot. An $r = p/q$ surgery on a knot is the operation of removing from S^3 a tubular neighbourhood $\nu(K)$ of the knot, and gluing it back via a homeomorphism $\phi : T^2 \to T^2$ of the boundary T^2. The number $r = p/q$ is determined by the class of the image of the meridian γ_1 of the knot under the map ϕ,

$$\phi[\gamma_1] = p[\gamma_1] + q[\gamma_2],$$

in $H^1(T^2, \mathbb{Z})$. The parameter r is the *framing* of the surgery. A similar operation can be performed starting with a link in a three-manifold Y. A surgery on a link is obtained by performing the surgery on every component with the corresponding assigned framing. In the following we will be especially interested in the case of 0-surgery, which exchanges parallel and meridian, and 1-surgery on a knot in a homology sphere, which produces a different homology sphere. In general, it is hard to determine when two different surgeries produce equivalent manifolds. There are moves of the framed link , known as Kirby moves, that do not affect the resulting topology, see [31].

Thus, the previous statement can be formulated as follows.

Proposition 5.3 *Every compact oriented three-manifold Y is obtained from S^3 by Dehn surgery on the components of a link $L \subset S^3$.*

Again, this is not a classification, because of the difficulty of determining the topology given the surgery presentation and identifying equivalent presentations. In particular, the problem of the possible existence of homotopy 3-spheres is still, at this time, open. Recently, Kronheimer and Mrowka have outlined a program that combines the information of instanton and Seiberg–Witten Floer homology and may lead to substantial results in this direction [27].

An invariant associated to the fundamental group of a three-manifold is the Casson invariant [1]. This was originally defined for homology spheres, later extended to rational homology spheres [41], and more recently to more general three-manifolds [17]. The Casson invariant of homology spheres counts the conjugacy classes of representations of the fundamental group in $SU(2)$. The counting is achieved by considering a Heegaard splitting of Y. On each handlebody N_g, the variety of conjugacy classes of irreducible representations of $\pi_1(N_g)$ in $SU(2)$ is a smooth $3g-3$ dimensional manifold. The pullback by the inclusion map of the boundary $\Sigma_g \subset N_g$ give embeddings of these varieties in the $6g-6$ dimensional variety of conjugacy classes of irreducible representations of $\pi_1(\Sigma_g)$ in $SU(2)$. If Y is a homology sphere, the two subvarieties determined by the two handlebodies can be made to intersect transversely in the representations of $\pi_1(\Sigma_g)$. The intersection avoids the reducible locus, and consists of a finite set of points with an orientation. This counts precisely the conjugacy classes of irreducible representations of $\pi_1(Y)$.

Thus, the representations are counted with signs determined by the relative orientations of the intersecting character varieties of the two handlebodies of a Heegaard splitting of Y. It can be shown that the resulting invariant is independent of the choice of the Heegaard splitting.

6 A three-manifold invariant

So far we have always considered Seiberg–Witten gauge theory as a tool for studying the differentiable structure of four-manifolds. In the present section we describe the dimensional reduction that gives rise to a three-dimensional gauge theory. In particular, this gives rise to an analogue of the four-dimensional invariant. This is an invariant of compact connected oriented three-manifolds without boundary, which is closely related to other classical invariants, such as the Casson invariant, Milnor torsion, and the Alexander polynomial of knots.

6.1 Dimensional reduction

Let Y be a compact connected oriented three-manifold without boundary. Consider a manifold $X = Y \times \mathbb{R}$ endowed with a cylindrical metric.

Although so far we have been considering the Seiberg–Witten equations on compact four-manifolds, we may as well define the same equations on X. Oriented three-manifolds are parallelisable, hence, upon fixing a trivialisation of the tangent bundle, the set of $Spin_c$ structures $\mathcal{S}(Y)$ can be identified (non-canonically) with the set of line bundles $H^2(Y, \mathbb{Z})$. Namely, a fixed lifting of the frame bundle to a $Spin$-bundle defines a $Spin$-structure with spinor bundle $\tilde{S}$, where we have $Spin(3) = SU(2)$. All the $Spin_c$-structures are then obtained by twisting $\tilde{S}$ with a line bundle H, this gives the $U(2)$-bundle $\tilde{W} = \tilde{S} \otimes H$. Again we introduce the notation $L = \det(\tilde{W})$. As in the four-dimensional case, the fibre of the map $\mathcal{S}(Y) \to H^2(Y, \mathbb{Z})$ given by $\tilde{W} \mapsto \det(\tilde{W})$ is a principal homogeneous space for the 2-torsion in $H^2(Y, \mathbb{Z})$, hence finite. We define a $Spin_c$-structure on X by fixing isomorphisms of the positive and negative spinor bundles $W^{\pm}$ with the pullback of $\tilde{W}$ via the projection $\pi : Y \times \mathbb{R} \to Y$, so that the composite isomorphism $W^+ \to W^-$ is given by Clifford multiplication by dt, where t is the coordinate in the $\mathbb{R}$-direction.

Definition 6.1 *The choice of isomorphisms $W^{\pm} \cong \pi^* \tilde{W}$ gives a reference connection in the t-direction. Therefore it makes sense to say that a pair (A, ψ) on X is* in a temporal gauge *if the dt component of the connection A is identically zero.*

A pair (A, ψ) on X that is in a temporal gauge induces a path of connections and sections of the spinor bundle over Y by defining $A(t)$ and $\psi(t)$ to be the restrictions of A and ψ to $Y \times \{t\}$. Notice that each gauge equivalence class of pairs (A, ψ) on X contains a representative which is in a temporal gauge. Once the temporal gauge is fixed, the remaining gauge degrees of freedom are time independent transformations in the gauge group $\mathcal{G} = \mathcal{M}(Y, U(1))$ of the line bundle L on Y.

Thus, on the manifold Y we can introduce the configuration space $\mathcal{A} = \mathcal{C} \times \Gamma(Y, \tilde{W})$ on which the group $\mathcal{G}$ acts. The action is free away from the reducible points, namely the points with $\psi \equiv 0$. We can form the quotient $\mathcal{B} = \mathcal{A}/\mathcal{G}$, and use the notation $\mathcal{B}^*$ for the irreducible part.

We perform a dimensional reduction as follows.

Theorem 6.2 *For a pair (A, ψ) in a temporal gauge, the Seiberg–Witten equa-*

tions (8), (9) on X can be rewritten as the following equations on Y:

$$\frac{d}{dt}\psi = -\partial_A \psi \tag{18}$$

and

$$\frac{d}{dt}A = - * F_A + \tau(\psi, \psi), \tag{19}$$

where $\tau(\psi, \psi)$ is the 1-form written in local coordinates as

$$\tau(\psi, \psi) = \sum_i < e_i \psi, \psi > e^i.$$

The equations represent the flow of the vector field

$$(-\partial_A \psi, - * F_A + \tau(\psi, \psi)) \tag{20}$$

on the configuration space $\mathcal{A}$.

Proof: The Dirac operator on X twisted with the connection A has the form

$$D_A = \begin{pmatrix} 0 & D_A^+ \\ D_A^- & 0 \end{pmatrix},$$

$$D_A^+ = \partial_t + \partial_{A(t)},$$

where ∂_A is the self-adjoint Dirac operator on Y twisted with the time dependent connection $A(t)$.

For the curvature equation (19), observe that 1-forms on Y give endomorphisms of the spinor bundle $\tilde{W}$. Via pullback and the isomorphism of W^+ and W^-, a 1-form α on Y acts on $\tilde{W}$ as the 2-form $\alpha \wedge dt$ acts on W^+. Write $F_A^+ = \frac{1}{2}(F_A + *F_A)$, and $F_A = dA$ in coordinates. Since F_A^- acts trivially on W^+, it is not hard to see that the action of F_A^+ on W^+ corresponds exactly to the action of $\frac{dA}{dt} + *F_A$ on $\tilde{W}$ (here $*$ is the Hodge operator on Y). An analogous argument explains the presence of the term $< e_i\psi, \psi > e^i$.
QED

In this way, every solution of the four-dimensional Seiberg–Witten equations gives rise to a flow line of the vector field (20) that extends backward and forward for all time. It may be useful to point out that, though we are describing the solutions of (18) and (19) as the gradient flow of a functional, these are not evolution equations. In fact, as we are going to discuss in the following, the linearisation of the equations is a first order elliptic operator. This implies that such flow exists only at solutions of the 4-dimensional Seiberg–Witten equations. At a generic point (A, ψ) of the configuration space on Y the gradient vector field will not in general give rise to a flow: not even local existence is guaranteed. This is one aspect that makes Floer theory different from infinite dimensional Morse theory, even though it is modeled on the Morse–Smale complex.

The stationary points of the flow, satisfying $\frac{d}{dt}(A(t),\psi(t)) = 0$ correspond to the solutions of the stationary equations

$$\partial_A \psi = 0 \tag{21}$$

$$*F_A = \tau(\psi,\psi). \tag{22}$$

These are elliptic equations on Y, and the goal is to use a suitable counting of the solutions modulo gauge to construct an invariant of three-manifolds. We proceed to analyse equations (21) and (22) more closely.

Notice that the vector field (20) is gauge invariant, hence it descends to a vector field on $\mathcal{B}^*$.

6.2 The moduli space and the invariant

We can topologise the configuration space $\mathcal{A} = \mathcal{C}\times\Gamma(Y,\tilde{W})$ with an L^2_k Sobolev norm, with $k \geq 2$, and consider the gauge group of L^2_{k+1} gauge transformations on Y. With this choice the irreducible part $\mathcal{B}^*$ of the quotient is a Hilbert manifold. We define the moduli space $M_c(Y,s)$ of solutions of (21) and (22) in $\mathcal{B}$, where L determines the choice of the $Spin_c$-structure.

The linearisation of (21) and (22) at a solution (A,ψ) gives the deformation complex

$$\Lambda^0(Y)\oplus\Lambda^1(Y)\oplus\Gamma(Y,\tilde{W}) \stackrel{\Theta}{\to} \Lambda^0(Y)\oplus\Lambda^1(Y)\oplus\Gamma(Y,\tilde{W}). \tag{23}$$

Here Θ is the Fredholm operator

$$\Theta_{(A,\psi)}(f,\alpha,\phi) = \left\{ \begin{array}{l} G^*_{(A,\psi)}(\alpha,\phi) \\ T_{(A,\psi)}(\alpha,\phi) + G_{(A,\psi)}(f), \end{array}\right. \tag{24}$$

where the operator T is the linearisation of the equations

$$T_{(A,\psi)}(\alpha,\phi) = \left(\begin{array}{c} -*d\alpha + 2Im(< e_i\psi,\phi >)e^i \\ -\partial_A\phi - i\alpha\cdot\psi \end{array}\right),$$

the operator G is the infinitesimal action of the gauge group, and G^* is the adjoint of G with respect to the L^2-inner product,

$$G^*_{(A,\psi)}(\alpha,\phi) = -d^*\alpha + iIm < \psi,\phi > .$$

It is not hard to see that the operator Θ has index zero. Thus, the virtual dimension of the moduli space $M_c(Y,s)$ is zero. However, if the operator is not surjective, the transversality theorem does not hold: even if $M_c(Y,s)$ does not meet the reducible locus of $\mathcal{B}$ and is a smooth manifold in $\mathcal{B}^*$, it may not be cut out transversely; hence the dimension may be larger than the expected one. Transversality, like in the four-dimensional case, can be achieved by a suitable choice of a perturbation, but whether $M_c(Y,s)$ can be made to avoid the reducible locus in $\mathcal{B}$ depends on the homology of the underlying manifold Y, as we are going to see.

The natural choice of perturbations for the three-dimensional equations is obtained by dimensional reduction of the perturbed four-dimensional case, under the identification $\Lambda^{2+}(Y \times \mathbb{R}) \cong \pi^*(\Lambda^1(Y))$, given by $\pi^*(\rho(t)) = *\rho(t) + \rho(t) \wedge dt$, with respect to the projection $\pi : Y \times \mathbb{R} \to Y$. We obtain the perturbed equations

$$\partial_A \psi = 0 \tag{25}$$

$$* F_A = \tau(\psi, \psi) + 2i\rho, \tag{26}$$

with $\rho \in \Lambda^1(Y)$. It is easy to verify that, for the equations (25) and (26) to have solutions, we need to choose ρ co-closed. In fact, the equation (25) implies that the 1-form $\tau(\psi,\psi)$ is co-closed.

We have the following result about the moduli space $M_c(Y,s)$.

Theorem 6.3 *If Y has $b_2(Y) > 0$ and ρ is a co-closed 1-form on Y satisfying $[*\rho] \neq \pi c_1(L)$ in $H^2(Y,\mathbb{R})$, then the moduli space $M_c(Y,s)$ is contained in the irreducible part $\mathcal{B}^*$.*

If Y is a rational homology sphere, then, for every choice of the line bundle L and of the co-closed 1-form $\rho = *d\nu$, there is exactly one reducible point θ in $M_c(Y,s)$. If Y is an integral homology sphere, the class θ is the gauge equivalence class of $(\nu, 0)$, with ν as above.

In both cases, for a generic choice of the perturbation ρ, the linearisation $\Theta_{A,\psi}$ at an irreducible solution $\psi \neq 0$ is surjective, hence the irreducible part of the moduli space $M^*_c(Y,s) = M_c(Y,s) \cap \mathcal{B}^*$ is cut out transversely by the equations.

Proof: With an argument analogous to the four-dimensional case it is possible to show that, under a generic choice of the perturbation ρ, the transversality theorem holds. Thus the linearisation at the solutions of the perturbed equations is surjective [20]. Let Y be a rational homology sphere. The unperturbed Seiberg–Witten equations (21) and (22) on Y have a reducible gauge class of solutions $[A_0, 0]$, with A_0 the abelian flat connection determined by the $U(1)$-representation of $\pi_1(Y)$. In the case of an integral homology sphere such connection A_0 lies in the gauge orbit of the trivial connection.

Thus, when $b_1(Y) = 0$, there is a unique reducible solution $\theta = [A_0, 0]$ up to gauge transformations. With the perturbation $\rho = *d\nu$ the reducible solutions are flat connections shifted by ν, that is, they are in the gauge orbit of the element $(A_0 + \nu, 0)$.
QED

Theorem 6.4 *On any three-manifold Y, and for every choice of the $Spin_c$-structure $s \in \mathcal{S}(Y)$, the moduli space $M_c(Y,s)$ is compact. Moreover, $M_c(Y,s)$*

is empty for all but finitely many choices of the class $c_1(L) \in H^2(Y, \mathbb{Z})$. The determinant line bundle of the linearisation $\Theta_{A,\psi}$ determines an orientation of the moduli space $M_c(Y, s)$.

The compactness follows from the Weitzenböck formula, with an argument similar to the four-dimensional case. However, notice that the compactness of the moduli space in three dimensions does not follow immediately from the compactness of the one in four dimensions. In fact, in some situations, the dimensional reduction may cause a loss of compactness.

The finiteness result again follows from the Weitzenböck formula, since one can obtain a rough but sufficient bound on the Chern class [16] in terms of the scalar curvature, namely we have $|c_1(L)| \leq \frac{|\kappa|}{4\pi}$.
QED

Theorem 6.5 *If Y is a rational homology sphere, and $\rho = *d\nu$ is a choice of perturbation, let g be a metric on Y such that the twisted Dirac operator ∂_ν satisfies $Ker(\partial_\nu) = 0$. Then the reducible point $\theta = [A_0 + \nu, 0]$ in $M_c(Y, s)$ is isolated.*

Thus, on any three-manifold Y, with any choice of the $Spin_c$-structure $s \in \mathcal{S}(Y)$, under a generic choice of the perturbation ρ as in theorem 6.3, the moduli space $M_c(Y, s)$ consists of a finite set of points with an attached $\pm$ sign determined by the orientation.

Proof: Suppose Y is a rational homology sphere. We want to show that the point $[\nu, 0]$ is isolated in $M_c(Y, s)$. This is shown in [8] [42] using the local Kuranishi model, namely by expanding a solution near $(A_0 + \nu, 0)$ as

$$A = \nu + \epsilon\alpha_1 + \epsilon^2\alpha_2 + \cdots$$

$$\psi = \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots$$

The condition that the pair (A, ψ) satisfies the equations implies that $\alpha_i \equiv 0$ and $\phi_i \equiv 0$, provided that for the chosen metric g on Y the Dirac operator has trivial kernel, $Ker(\partial^g_\nu) = 0$.
QED

We can introduce the Seiberg–Witten invariant of three-manifolds as follows.

Definition 6.6 *Let Y be any compact oriented three-manifold without boundary. For a fixed $Spin_c$-structure $s \in \mathcal{S}$ and for a generic choice of a co-closed 1-form ρ on Y, we define the Seiberg–Witten invariant $\chi(Y, s)$*

$$\chi(Y, s) = \#M^*_c(Y, s) \tag{27}$$

as the number of irreducible points, counted with the orientation, in the zero-dimensional moduli space $M_c(Y, s)$.

This counting of the critical points of the vector field (20) on $\mathcal{B}$ is reminiscent of one of the many possible definitions of the Euler characteristic on finite dimensional manifolds, although we still have not discussed the meaning of the orientation attached to the points of $M_c(Y,s)$ in terms of the vector field. In fact, $\chi(Y,s)$ can be thought of as a regularised Euler characteristic of $\mathcal{B}$, in the sense that will be discussed in the last part of the book.

The invariant $\chi(Y,s)$ is defined by counting only irreducible points and by a choice of a generic perturbation. Thus, the following two considerations are in order.

Remark 6.7 *(1) If Y is a rational homology sphere, some of the irreducible points counted in (27) may collide with the reducible θ if the metric and perturbation undergo a smooth deformation. One or more irreducibles can disappear into or arise from the reducible, thus changing the value of $\chi(Y,s)$.*

(2) If Y has $b_1(Y)=1$, the condition $[*\rho]=\pi c_1(L)$ has codimension one, hence it disconnects $H^2(Y,\mathbb{R})$ in two separate chambers. Thus, suppose that ρ_0 and ρ_1 are perturbations satisfying the condition $[*\rho]\neq\pi c_1(L)$, necessary for $M_c(Y,s)$ to avoid the reducible locus. A generic path of perturbations connecting ρ_0 to ρ_1 may cross the wall $[*\rho]=\pi c_1(L)$, and the resulting value of $\chi(Y,s)$ can jump, $\chi_{\rho_0}(Y,s)\neq\chi_{\rho_1}(Y,s)$.

As we are going to see, both bad cases discussed in remark 6.7 actually arise, and corresponding wall crossing formulae can be computed.

6.3 Cobordism and wall crossing formulae

We first prove the metric independence of $\chi(Y,s)$ in the case of manifolds Y with $b_1(Y)>1$. The cobordism argument is modeled on the analogous proof on four-manifolds.

Theorem 6.8 *Let Y be a three-manifold with $b_1(Y)>1$, and (g_0,ρ_0) and (g_1,ρ_1) two generic choices of metrics and co-closed perturbations. We have $\chi_{g_0,\rho_0}(Y,s)=\chi_{g_1,\rho_1}(Y,s)$.*

Proof: Suppose given a path (g_t,ρ_t) connecting (g_0,ρ_0) and (g_1,ρ_1). Consider the infinite dimensional manifold $\mathcal{B}^*\times[0,1]$. We want to construct a "universal moduli space" for the equations. Consider the product bundle over $\mathcal{B}^*\times[0,1]$ with fibre $\Lambda^1(Y)\oplus\Gamma(Y,\tilde{W})$. We have a section given by

$$s(A,\psi,t)=(\partial_A^t\psi, *_tF_A-2i\rho_t-\tau(\psi,\psi)). \qquad (28)$$

Note that in (28) both the Dirac operator and the Hodge $*$-operator depend on the metric, hence on the parameter t. Here $\{\rho_t\}$ is any family of 1-forms that are co-closed with respect to $*_t$ and away from the wall, i.e. $[*_t\rho_t]\neq\pi c_1(L)$.

The universal moduli space is the zero set of the section s, $M_U = s^{-1}(0)$. It is not hard to see that the section s has a Fredholm linearisation Ds of the form

$$Ds_{(A,\psi,\rho,t)}(\alpha,\phi,\eta,\epsilon) = \epsilon\frac{\partial}{\partial t}s(A,\psi,\rho,t) + \tilde{T}\mid_{(A,\psi,\rho,t)}(\alpha,\phi,\eta),$$

where (α,ϕ) are coordinates in the tangent space of $\mathcal{B}$, η is a 1-form, and $\epsilon \in \mathbb{R}$. If the path is generic, Ds is onto.

Now we can apply again the implicit function theorem on Banach manifolds, and we get that s is transverse to the zero section. Therefore the universal moduli space M_U is a smooth manifold. Notice that the condition $[*_t\rho_t] \neq \pi c_1(L)$ ensures that the moduli space corresponding to each value of t does not contain reducibles. The dimension of the universal moduli space is $Ind(T)+1 = 1$. The proofs of the compactness and orientability of the universal moduli space are analogous to the case of M_c.

The independence of the metric now follows from the fact that the moduli spaces corresponding to the metrics g_0 and g_1 form the boundary of a compact oriented 1-manifold, and the total oriented boundary of such a manifold is zero. QED

Now consider the case with $b_1(Y) = 1$. Here more care has to be taken, since in this case it is no longer possible to choose arbitrarily the metric and the perturbation. In fact we cannot ensure that a generic path of perturbations will avoid the wall $[*\rho] = \pi c_1(L)$. Nevertheless, for small enough perturbations, the previous argument ensures independence of the metric and of the perturbation.

In the following it is useful to restrict the perturbation ρ to a fixed cohomology class. In particular we shall often work with a cohomologically trivial perturbation $[*\rho] = 0$. Then an argument similar to the previous one proves transversality in this case. Notice, however, that the result does not immediately follow from the proof of transversality for unrestricted ρ, since in principle the subspace $[*\rho] = 0$ may miss the generic set given by the Sard–Smale theorem. It is necessary to check with a direct computation the vanishing of the Cokernel of the linearisation $Ds_{(A,\psi,\rho,t)}(\alpha,\phi,\eta,\epsilon)$, with $*\rho$ and $*\eta$ cohomologically trivial.

If the path (g_t,ρ_t) crosses the wall $[*\rho] = \pi c_1(L)$, there is a wall crossing formula [19], as follows.

Theorem 6.9 *Suppose Y has $b_1(Y) = 1$ and (g_t,ρ_t) is a path of metrics and perturbations that crosses the wall $[*\rho] = \pi c_1(L)$ once transversely. With the pair (g,ρ) where the path crosses the wall, the moduli space $M_c(Y,s,g,\rho)$ has a circle S^1 of reducible points. A local analysis of the universal moduli space near the pair (g,ρ) shows that there are some irreducibles that collide with the circle of reducibles. The invariant $\chi(Y,s)$ changes correspondingly by an amount equal to the spectral flow of the linearisation around the circle of reducibles.*

The proof can be found in [19]. It is based on the local model of the universal moduli space near the bad metric and perturbation.

Consider the case of homology spheres. Let ∂^g_ν be the Dirac operator for the choice of the metric g, twisted with the 1-form ν. We analyse how the condition $Ker(\partial^g_\nu) = 0$ creates a metric dependence for the invariant $\chi(Y, s)$. The following preliminary lemma, which does not simply follow from the fact that ∂^g_ν is of index zero, is needed in order to define the chamber structure in the space of metrics and perturbations. Let Met be the set of metrics on Y.

Lemma 6.10 *Given Y a homology sphere, consider the set of pairs (g, ν) in $Met \times Z^1(Y)$. The condition $Ker(\partial^g_\nu) = 0$ is satisfied by a generic set in $Met \times Z^1(Y)$. The wall*

$$\mathcal{W} = \{(g, \nu) | Ker(\partial^g_\nu) \neq 0\}$$

is a stratified set, with the top stratum of codimension one in $Met \times Z^1(Y)$.

For the proof we refer to [21]. For a more detailed analysis of the chambers, see also [30].

Theorem 6.11 *Given Y a homology sphere, we consider the Seiberg–Witten equations (25) and (26), with a perturbation $\rho = *d\nu$. In the space of metrics and perturbations $Met \times Z^1(Y)$ consider a path (g_t, ν_t). The wall crossing formula is*

$$\chi_{g_0,\nu_0}(Y) - \chi_{g_1,\nu_1}(Y) = SF(\partial^{g_t}_{\nu_t}),$$

where $SF(\partial^{g_t}_{\nu_t})$ is the spectral flow of the twisted Dirac operator along the path (g_t, ν_t) connecting (g_0, ν_0) to (g_1, ν_1).

Recall that the spectral flow is the total number of eigenvalues $\lambda(t)$ of the operator $\partial^{g_t}_{\nu_t}$ that cross zero from negative to positive minus the number of those that cross from positive to negative along the path $0 < t < 1$. There are various possible proofs of this wall crossing formula. One way is the local analysis of the universal moduli space near the bad points where a generic path (g_t, ν_t) crosses the wall, just like the proof of the $b_1(Y) = 1$ case [19], [21]. Another approach is via Floer homology [21], as we shall discuss later. Moreover, in the case of an integral homology sphere, the wall crossing formula will also follow from the results discussed in the following section, where $\chi(Y)$ (there is a unique choice of $Spin_c$-structure on a homology sphere) is decomposed into the sum of a topological invariant and a metric dependent term.

6.4 Casson invariant and Alexander polynomial

Unlike the Casson invariant of homology spheres that has a nice geometric definition in terms of representations of the fundamental group $\pi_1(Y)$, the Casson-type invariant $\chi(Y, s)$ obtained via Seiberg–Witten theory on three-manifolds does not have a simple geometric description, due to the presence of the spinor equation that is not easily interpreted in terms of topological quantities. However, the invariant $\chi(Y, s)$ satisfies a surgery formula under Heegaard splittings [20] that is reminiscent of the Casson invariant [1], [35]. As we are going to see in this section, the two invariants are closely related: an explicit formula that describes $\chi(Y, s)$ in terms of the Casson invariant and other correction terms can be derived from suitable gluing theorems.

The Casson invariant is defined only in the case when Y is a homology three-sphere [1] (with a generalisation to rational homology spheres [41]), whereas we have defined the invariant $\chi(Y, s)$ for all three-manifolds. Moreover, we should really think of $\chi(Y, s)$ as a family of invariants parametrised by the choice of the $Spin_c$-structure $s \in \mathcal{S}$. This has been done to the expense of introducing a chamber structure. As we are going to see, in the case of integral homology spheres there is a direct relation between $\chi(Y)$ (there is no non-trivial choice of $s \in \mathcal{S}$ in this case) and the Casson invariant, with a correction term that takes into account the metric dependence discussed in the previous section.

For some manifolds with $b_1(Y) > 0$ the relation to the Casson invariant is via the Alexander polynomial of a knot and a surgery formula. This relation to the Alexander polynomial generalises to other manifolds with $b_1(Y) > 0$. In fact, as proven by Meng and Taubes [22], a suitable combination of the Seiberg–Witten invariants $\chi(Y, s)$ for various choices of $s \in \mathcal{S}$ reproduces the Milnor torsion invariant. The Alexander polynomial of a link can be regarded as a particular case.

In this chapter we shall only state the result of Meng and Taubes in the more restrictive case where it gives rise to the Alexander polynomial, as stated in theorem 6.12 below. We refer the reader to [22] for the result in complete generality. The general statement, that we shall omit here, holds for all compact three-manifolds with $b_1(Y) > 0$ (and more generally for some cases of three-manifolds with boundary), and is formulated in terms of Milnor torsion.

Here we only consider manifolds Y_0 obtained by zero-surgey on a knot K in S^3. Such manifolds have the homology type of $S^1 \times S^2$, hence they have $b_1(Y) = 1$. We shall not review the definition of Milnor torsion, and its relation to the Alexander polynomial for our class of manifolds. The reader can find all this in [40].

Theorem 6.12 *Let Y_0 be obtained as zero-surgery on a knot K in an integral homology sphere. Let*

$$\Delta_K(t) = a_0 + a_1(t^{-1} + t) + \cdots + a_r(t^{-r} + t^r)$$

be the symmetrised Alexander polynomial of the knot K. The manifold Y_0 has the homology of $S^1 \times S^2$. Consider $Spin_c$-structures s_k on Y_0, defined by the condition $c_1(L_k) = 2k$. The Seiberg–Witten invariants, obtained under a generic choice of the perturbation, are of the form

$$\chi(Y_0, s_k) = \sum_{j>0} j a_{j+|k|}.$$

The theorem follows from the results of [22]. In the case of the $Spin_c$ structure s_0, with $c_1(L_0) = 0$, the invariant can be computed with respect to a cohomologically non-trivial perturbation, namely a 1-form ρ satisfying

$$0 \neq \epsilon = [*\rho] \in H^2(Y, \mathbb{R}).$$

We have the following useful corollary.

Corollary 6.13 *Let Y_0 and $\Delta_K(t)$ be as above. The invariants $\chi(Y_0, s_k)$ can be assembled in a unique invariant*

$$\chi(Y_0) = \sum_k \chi(Y_0, s_k) = \frac{1}{2}\Delta''_K(t)|_{t=1}.$$

Proof: This is just the simple computation

$$\chi(Y_0) = \sum_k \sum_{j>0} j a_{j+|k|} = \sum_{j>0} \frac{j(j-1)}{2} a_j.$$

QED

The expression $\lambda'(K) = \frac{1}{2}\Delta''_K(t)|_{t=1}$ is exactly the term that appears in the surgery formula that defines axiomatically the Casson invariant of integral homology spheres [1]. Namely, given an integral homology sphere Y and a knot $K \subset Y$, consider the manifold Y_1 obtained by 1-surgery on the knot. This is again an integral homology sphere. The Casson invariant $\lambda(Y)$ satisfies the relation

$$\lambda(Y) - \lambda(Y_1) = \lambda'(K). \tag{29}$$

This simple observation, together with the result of Meng and Taubes, naturally suggests to investigate the analogous of the surgery formula (29) for Seiberg–Witten invariants, namely to express $\chi(Y) - \chi(Y_1)$ in terms of some polynomial invariant of the knot K, and to identify explicitly the correction term that $\chi(Y) - \lambda(Y)$ that relates the Seiberg–Witten and the Casson invariants of integral homology spheres. We are going to see how these questions can be answered.

If Y is a homology sphere with an embedded knot K, and Y_1 and Y_0 are the manifolds obtained by 1-surgery and 0-surgery on K, as above, then Y, Y_1 and Y_0 split as $V \cup_{T^2} \nu(K)$, where V is the knot complement and $\nu(K)$ is a tubular neighbourhood of the knot. The manifolds with boundary V and $\nu(K)$ are glued along T^2 by the diffeomorphism of T^2 specified by the Dehn surgery. It is convenient to consider the manifolds V and $\nu(K)$ completed with an infinite cylinder $T^2 \times [0, \infty)$ and a flat cylindrical metric on the end. For simplicity, we shall use the notation V and $\nu(K)$ to indicate both the manifolds with boundary or these complete manifolds.

The following theorem [7] [10] [11] is an answer to the first question.

Theorem 6.14 *There exists a perturbation μ on $\nu(K)$ such that the following holds. There are chambers of metrics and perturbations for Y and Y_1 such that the following surgery formula for the Seiberg–Witten invariant holds:*

$$\chi(Y) - \chi(Y_1) = \chi(Y_0).$$

<u>Sketch of the Proof:</u> Notice that the two terms $\chi_{g,\nu}(Y)$ and $\chi_{g_1,\nu_1}(Y_1)$ on the left hand side depend on the choice of metric and perturbation (g, ν) according to the wall crossing described above. The perturbation ν is obtained by combining the perturbation μ on $\nu(K)$ that "simulates the effect of surgery" [6] with a perturbation on V that guarantees transversality. On Y_1 and on Y_0 with a non-trivial $Spin_c$ structure we consider perturbations obtained from a perturbation on V. In the case of Y_0 with the trivial $Spin_c$ structure s_0 we consider a perturbation ρ obtained by combining a perturbation on V with a perturbation μ_0 on $\nu(K)$ so that ρ satisfies

$$0 \neq \epsilon = [*\rho] \in H^2(Y, \mathbb{R}).$$

In order to fix a choice of the metric and perturbation on the homology spheres Y and Y_1, we assume that the manifold Y has a long cylinder $[-r, r] \times T^2$ around the boundary of the tubular neighbourhood of the knot where the surgery is performed. We choose the flat metric on T^2, the corresponding flat cylindrical metric on $[-r, r] \times T^2$ and we extend this inside the tubular neighbourhood of the knot $\nu(K) \cong D^2 \times S^1$ to a metric of positive scalar curvature, flat near the boundary. The manifolds Y_1 and Y_0 have the corresponding induced metrics g_1 and g_0.

The essential ingredient of the proof is a gluing theorem. The manifolds Y, Y_1, and Y_0 decompose as $V \cup_{T^2} [-r, r] \times T^2 \cup_{T^2} \nu(K)$, where V is the knot complement. The moduli spaces of solutions decompose accordingly as a fibre product

$$M_c^*(Y, g, \nu) = M_c^*(V, \tilde{\nu}) \times_{\chi_0(T^2,Y)} M_c(\nu(K), \mu),$$

$$M_c^*(Y_1, g_1, \nu_1) = M_c^*(V, \tilde{\nu}) \times_{\chi_0(T^2,Y_1)} M_c(\nu(K)),$$

and

$$M_c^*(Y_0, s_k, \rho) = M_c^*(V, \tilde{\nu}) \times_{\chi_0(T^2,Y_0,s_k)} M_c(\nu(K)),$$

for s_k non-trivial, with $[*\rho] = 0$, and

$$M_c^*(Y_0, s_k, \rho) = M_c^*(V, \tilde{\nu}) \times_{\chi_0(T^2,Y_0,s_k)} M_c(\nu(K), \mu_0),$$

with $[*\rho] \neq 0$. We need to explain the terms that appear in these fibre products. The moduli space $M_c^*(V, \tilde{\nu})$ counts irreducible solutions modulo gauge of the perturbed equations on the knot complement V endowed with an infinite cylindrical end $T^2 \times [0,\infty)$. The fibre product is defined with respect to boundary value maps. These are continuous maps

$$\partial : M_c^*(V,\tilde{\nu}) \to \chi_0(T^2, Y_i)$$

and

$$\partial : M_c^*(\nu(K)) \to \chi_0(T^2, Y_i).$$

The space $\chi_0(T^2, Y_i)$ is the covering $\chi_0(T^2, Y_i) \to \chi(T^2)$ of

$$\chi(T^2) = H^1(T^2, \mathbb{R})/H^1(T^2, \mathbb{Z})$$

with fibre $H^1(T^2,\mathbb{Z})/H(Y_i)$. The subgroup $H(Y_i)$ is the image of the map $J_i^* : H^1(V,\mathbb{Z}) \to H^1(T^2,\mathbb{Z})$ defined by the inclusion $T^2 \overset{J_i}{\hookrightarrow} V$ inside the manifold Y_i, which is either Y or Y_1 or Y_0. The boundary value map from $M_c^*(V,\tilde{\nu})$ is generically finite to one and the one from $M_c^*(\nu(K))$ is an embedding.

There are two main steps in the proof. The first one is to describe the geometric limits of solutions, namely to show that, upon stretching the long neck $T^2 \times [-r,r]$, solutions of the Seiberg–Witten equations on Y_i converge smoothly on compact sets and up to gauge transformations to a pair of a solution on V and a solution on $\nu(K)$. The second half of the proof involves establishing the existence of the asymptotic value maps and proving the fibre product structure. This implies showing that a pair of solutions on V and $\nu(K)$ with matching asymptotic values can be glued to form a solution on Y_i with a long neck. The way the gluing over T^2 is obtained depends on the construction of the perturbation μ on $\nu(K)$, which therefore encodes the information about the Dehn surgery. We can summarise briefly the steps of the proof as follows.

Geometric limits: Suppose given a sequence of solutions (A_r, ψ_r) of the Seiberg–Witten equations on the manifold $V \cup_{T^2} [-r,r] \times T^2 \cup_{T^2} \nu(K)$. By taking the limit smoothly on compact sets we obtain a finite energy solution (A', ψ') on $V \cup_{T^2} [0,\infty) \times T^2$, and a finite energy solution (A'', ψ'') on $V \cup_{T^2} (-\infty, 0] \times T^2$.

Asymptotic values: An irreducible finite energy solution (A', ψ') on $V \cup_{T^2} [0,\infty) \times T^2$ decays exponentially along the cylindrical end $T^2 \times [0,\infty)$ to an asymptotic value. Since the induced $Spin_c$-structure on the torus T^2 is trivial, the asymptotic value is given by a flat connection a_∞ on T^2 and vanishing spinor. The rate of decay is determined by the first non-trivial eigenvalue of the operator $\bar{\partial}_{a_\infty} + \bar{\partial}^*_{a_\infty}$ on T^2.

Under the assumption of non-negative scalar curvature on the tubular neighbourhood of the knot, and since we have the trivial $Spin_c$-structure on the torus T^2, the finite energy solution (A'', ψ'') has vanishing spinor $\psi'' \equiv 0$, hence it is just a flat connection on $\nu(K)$.

Given these results, it is possible to consider the space $\mathcal{A}_{k,\delta}(V)$ of extended $L^2_{k,\delta}$ solutions on V, namely solutions (A,ψ) that are locally in L^2_k and that satisfy

$$\|(A,\psi) - (a_\infty, 0)\|_{L^2_{k,\delta}(T^2 \times [0,\infty))} < \infty,$$

with a_∞ as defined above. On this configuration space we can consider the action of $L^2_{k+1,\delta}$ gauge transformations on V. This gives rise to a well defined boundary value map

$$\partial : M^*_c(V, \tilde{\nu}) \to \chi_0(T^2, Y_i).$$

A perturbation $\nu = \tilde{\nu} + \mu$ of the equations on Y is then chosen in such a way that the asymptotic values in $\chi_0(T^2, Y)$, lifted to the common covering $H^1(T^2, \mathbb{R})$ of $\chi_0(T^2, Y)$, $\chi_0(T^2, Y_1)$, and $\chi_0(T^2, Y_0, s_k)$ become close to the union of the asymptotic values of solutions of the equations on Y_1 and on Y_0 with the various s_k. A similar technique is known in Donaldson theory as the "geometric triangle" [6].

A careful analysis of the splitting of the spectral flow under surgery is the needed in order to guarantee that the counting of orientations remains unchanged.

This completes the sketch of the argument: we are not going to discuss any of the details here.
QED

The surgery formula is the same as the one satisfied by the Casson invariant. This indicates that the difference of the two invariants should consist precisely of a term that captures the metric dependence of $\chi(Y)$. The following correction term was conjectured by Kronheimer and Mrowka, and then proved independently by Chen and Lim [10], [11], [18].

Theorem 6.15 *Let Y be an integral homology sphere. There exists a Spin four-manifold X with boundary, such that $\partial X = Y$. The difference between the Seiberg–Witten and the Casson invariant of Y is given by*

$$\lambda(Y) - \chi(Y) = Ind(D_X) + \frac{1}{8}\sigma(X), \tag{30}$$

where D_X is the Dirac operator on X with Atiyah-Patodi-Singer boundary conditions, and $\sigma(X)$ is the signature.

The proof can be found in the references quoted above. Notice that the correction term can be rewritten as a sum of η-invariants [2] as

$$Ind(D_X) + \frac{1}{8}\sigma(X) = \eta_{\partial^g_\nu}(0) + \frac{1}{8}\eta_{*d-d*}(0).$$

Recall that the η-invariant $\eta_A(0)$ of a self-adjoint elliptic operator A on a compact manifold Y is defined by the value at zero of the analytic continuation of $\eta_A(s) = \sum sign(\lambda)|\lambda|^{-s}$. Here the sum is over all the non-zero eigenvalues of A. Upon deforming the operator A, in our case via a deformation of the metric and perturbation (g, ν), the η-invariant jumps as the spectral flow, [2]. So in particular one can obtain (see [7]) a proof of theorem 6.15 as a consequence of the surgery formula of theorem 6.14 and the wall crossing formula of theorem 6.11.

Instead of giving the details of this argument, we prefer to digress briefly and give some motivation for the initial conjecture that identified (30) as the right correction term. In fact, this may lead to other interesting conjectures.

Several classes of three-manifolds can be realised as the link of an isolated singularity in a complex surface. This is the case, for instance, for very simple manifolds like lens spaces, or for the more interesting class of Brieskorn spheres. The latter, usually denoted by $\Sigma(p,q,r)$, can be regarded as the link of the isolated singularity at zero of the complex hypersurface $x^p + y^q + z^r = 0$ in $\mathbb{C}^3$. A generalisation of Brieskorn spheres is the class of Seifert fibred homology spheres $\Sigma(a_1, \ldots, a_n)$. These are manifolds that can be constructed out of two-dimensional orbifolds [33], but they also have a description as the link of the singularity at zero of maps $f : \mathbb{C}^n \to \mathbb{C}^{n-2}$. Other interesting three-manifolds that are obtained as link of singularities are some torus bundles over S^1 that arise as link of some cusp singularities in Hirzebruch's Hilbert modular surfaces.

In these cases, one can consider the Milnor fibre $F = f^{-1}(\delta)$ of the singularity. We have to introduce some hypotheses on the surface in order to guarantee the existence of smoothings, that is, of deformations of the surface such that the general fibre is smooth. We can assume that the surface is a complete intersection, namely a surface obtained from $n-2$ non-singular hypersurfaces in $\mathbb{C}P^n$ intersecting transversely. Thus, the Milnor fibre F can be thought of as a smooth four-manifold with boundary $\partial F = Y$, where the homology sphere Y is the link of the singularity. It is a conjecture that, under this hypothesis, the Casson invariant of Y should satisfy the relation

$$\lambda(Y) = \frac{1}{8}\sigma(F). \tag{31}$$

The conjecture has been verified for Seifert fibred homology spheres [27]. Notice that the conjecture only concerns singularities in complete intersections such that the link of the singularity is a homology sphere. Links of singularities that are homology spheres are classified in [26], but it is not known which of them arise from complete intersections.

Computations of the Seiberg–Witten invariants $\chi(Y,s)$ were explicitly obtained [25] in the case of Brieskorn spheres and more generally Seifert fibred spaces. When Y is a Seifert-fibred rational homology sphere, the calculation of [25] shows that the invariant $\chi(Y)$ can be related to the η-invariant $\eta_{\partial^g_\nu}(0)$. For a direct calculation of these η-invariants see [29]. This, together with the relation (31) for Seifert fibred spaces, leads then naturally to identify the correction term as (30). For a computation in the case of Brieskorn spheres following this principle, see [28].

The result of theorem 6.15 implies that the conjectural relation (31) can be rephrased equivalently as follows.

Conjecture 6.16 *Let Y be a homology sphere that arises as the link of an isolated singularity in a complex surface X. Assume that X is a complete inter-*

section, so that we have a smoothing of X, and a Milnor fibre F with $\partial F = Y$. Then the Seiberg–Witten invariant satisfies the relation

$$\chi(Y) = -Ind(D_F),$$

where D_F is the Dirac operator on the smooth Spin four-manifold F.

It is known from the results of [35] that, in Donaldson theory, the Casson invariant can be regarded as the Euler characteristic of the instanton Floer homology [13]. The latter is obtained when the dimensional reduction of Donaldson's self dual equations is thought of as the gradient flow of the Chern-Simons functional. This behaves in some sense like a Morse function on the configuration space and the instanton homology is the associated Morse homology.

It is natural then to attempt the same construction in the case of Seiberg–Witten theory and reinterpret the invariants $\chi(Y, s)$ as Euler characteristics of suitable Floer homologies. The remainder of this chapter will be dedicated to defining and introducing the main properties of the Seiberg–Witten Floer homology. We shall see that the questions of wall crossing, surgery formulae, and the relation to known classical invariants that we discussed in this section have precise analogues in terms of Floer homologies.

The conjectural relation (31) for homology spheres Y that arise as links of singularities in complete intersections also admits generalisations. In fact [27], it is possible to ask what information about the Milnor fibre F is captured by the instanton Floer homology of Y. Similarly, in the case of the equivalent conjecture 6.16, we can ask what information on the Milnor fibre F is captured by the Seiberg–Witten Floer homology.

7 Seiberg–Witten Floer homology

The main idea in the construction of the Floer homology [13] is to interpret the flow equations (18) and (19) as the gradient flow of a functional that can be regarded as a Morse function on the configuration space $\mathcal{B}^*$. Mimicking the construction of the ordinary homology of a finite dimensional manifold out of a Morse function and its gradient flow [32], one can consider a complex generated by the critical points, with a boundary operator defined by a suitable counting of flow lines.

We are going to describe the various steps of the construction in the following sections. As a general remark, some care is necessary in adapting the intuition from the case of finite dimensional Morse theory. In fact, there are a few crucial differences that make Floer homology more subtle than just an infinite dimensional Morse theory. First of all, the equations (18) and (19) are not really flow equations, since the linearisation is an elliptic operator. Thus, the natural intuition that comes from parabolic problems (local existence of the solution forward in time, extension for all times) cannot be applied to this case.

The vector field (20) in general may not be integrable even for small times. However, we are really considering solutions of the four-dimensional Seiberg–Witten equations on $Y \times \mathbb{R}$. These correspond to particular choices of initial conditions for which the solution of (18) and (19) extends for all times. This is an essential difference with respect to the finite dimensional Morse theory, where every point of the underlying manifold corresponds to a unique flow line (up to reparametrisations) that extends for all times. Another important difference to keep in mind is that the configuration space $\mathcal{B}^*$ is topologised with L^2_k-Sobolev norms, but the gradient, as we are going to discuss in the next section, is taken with respect to the L^2-inner product hence it is not, strictly speaking, a Morse flow on $\mathcal{B}^*$. Nonetheless, we shall see that we can define an associated Floer homology which gives a refined invariant of the underlying three-manifold Y.

7.1 The Chern-Simons-Dirac functional

Let us introduce a functional defined on the space of connections on L and sections of $\tilde{W}$ over Y.

Definition 7.1 *The Chern-Simons-Dirac functional on the space* $\mathcal{A} = \mathcal{C} \times \Gamma(Y, \tilde{W})$ *is defined as*

$$C(A, \psi) = \frac{1}{2} \int_Y (A - A_0) \wedge (F_A + F_{A_0}) + \frac{1}{2} \int_Y < \psi, \partial_A \psi > dv, \qquad (32)$$

where A_0 *is a fixed smooth background connection.*

The first summand in (32) is just the abelian Chern–Simons functional , whereas the second term contains the information on the spinor and the interaction between the connection and the spinor. The gradient of the functional C is given by the vector field (20)

$$\nabla C(A, \psi) = (\partial_A \psi, *F_A - \tau(\psi, \psi)).$$

Thus, the critical points of the Chern-Simons-Dirac functional are exactly the solutions of the equations (21) and (22).

We want the functional (32) modulo the action of the gauge group. The functional is invariant under gauge transformations connected to the identity; however, in the case of maps that belong to other connected components in $\pi_0(\mathcal{G}) = H^1(Y, \mathbb{Z})$, an easy computation shows that the functional changes according to

$$C(A - \lambda^{-1} d\lambda, \lambda\psi) = C(A, \psi) + 4\pi^2 < c_1(L) \cup [\lambda], [Y] > . \qquad (33)$$

Here $[\lambda]$ is the cohomology class of the 1-form $-\frac{1}{\pi}\lambda^{-1} d\lambda$, representing the connected component of λ in the gauge group.

It is clear that the functional (32) is real valued on $\mathcal{B}$, whenever Y is a rational homology sphere. It is multiple valued, with values in the circle $\mathbb{R}/c\mathbb{Z}$ with $c = 4\pi^2\ell$ and

$$\ell = g.c.d.\{< c_1(L) \cup h, [Y] > | h \in H^1(Y, \mathbb{Z})\},$$

when Y has $b_1(Y) > 0$. In the case of a manifold Y with $b_1(Y) > 0$, we want to obtain a real valued functional (32), hence we have to pass to a covering of $\mathcal{B}$. There are two possible choices of the covering that make the functional real valued.

Let $\mathcal{B}_0$ be a covering of $\mathcal{B}$ with fibre $H^1(Y, \mathbb{Z})$,

$$\mathcal{B}_0 = \mathcal{A}/\mathcal{G}_0,$$

where $\mathcal{G}_0$ is the connected component that contains the identity in the gauge group

$$\mathcal{G} = \mathcal{M}(Y, U(1)).$$

The Chern-Simons-Dirac functional is certainly real valued on the space $\mathcal{B}_0$. However, there is a smaller covering which makes the functional real valued.

Consider the space $\mathcal{B}_L$ given by

$$\mathcal{B}_L = \mathcal{A}/\mathcal{G}_L,$$

where $\mathcal{G}_L$ is the subgroup of $\mathcal{G}$ of all gauge transformations of the form

$$\mathcal{G}_L = \{\lambda \in \mathcal{G} | \langle c_1(L) \cup [\lambda], [Y] \rangle = 0\}.$$

Let H_L denote the subgroup of $H^1(Y, \mathbb{Z})$ given by

$$H_L = \{h \in H^1(Y, \mathbb{Z}) | \langle c_1(L) \cup h, [Y] \rangle = 0\}.$$

The space $\mathcal{B}_L$ is a covering of $\mathcal{B}$ with fibre $H^1(Y, \mathbb{Z})/H_L = \mathbb{Z}$.

Thus, in the case of a rational homology sphere Y, the moduli space of critical points $\tilde{M}_c(Y, s)$ is just the space $M_c(Y, s)$ discussed previously. In the case of a manifold Y with $b_1(Y) > 0$, the moduli space $\tilde{M}_c(Y, s) \subset \mathcal{B}_L$ is a $\mathbb{Z}$-covering of $M_c(Y, s)$. Analogously, we can consider the set of critical points in $\mathcal{B}_0$ that covers $M_c(Y, s)$ with fibre $H^1(Y, \mathbb{Z})$.

When $b_1(Y) > 0$, in order to avoid reducible solutions and to achieve transversality of the moduli space $\tilde{M}^*_c(Y, s)$, we introduce a co-closed perturbation ρ as in the previous section. The perturbed Chern-Simons-Dirac functional is of the form

$$C_\rho(A, \psi) = C(A, \psi) - 2i \int_Y (A - A_0) \wedge *\rho, \tag{34}$$

with gradient flow

$$\frac{d}{dt}\psi = -\partial_A \psi \tag{35}$$

$$\frac{d}{dt}A = - * F_A + \tau(\psi,\psi) + 2i\rho. \quad (36)$$

If Y has $b_1(Y) > 0$, we choose ρ so that $*\rho \neq \pi c_1(L)$, as in the previous section. If Y is a rational homology sphere we write $\rho = *d\nu$.

When gauge transformations act on the perturbed Chern-Simons-Dirac functional, there is a further correction term which depends on the cohomology class of ρ,

$$C_\rho(\lambda(A,\psi)) = C_\rho(A,\psi) + 4\pi^2 < c_1(L) \cup [\lambda], [Y] > +\pi < [\rho] \cup [\lambda], [Y] > . \quad (37)$$

Thus, whenever we have a manifold Y with $b_1(Y) > 0$ and $c_1(L) \neq 0$, we shall restrict to the class of perturbations with $[\rho] = 0$, so that the Chern-Simons-Dirac functional still descends to a real valued functional on $\mathcal{B}_0$ or $\mathcal{B}_L$.

Notice that the case where $b_1(Y) > 0$ and $c_1(L) = 0$ is more subtle. In fact, if we choose a cohomologically trivial perturbation $\rho = *d\nu$, we have a torus T^{b_1} of reducibles in the moduli space $M_c(Y,s)$, given by solutions of $d(A-\nu) = 0$ modulo gauge. If we want to eliminate reducibles, we have to fix a non-trivial cohomology class for the perturbation ρ. We shall return to this special case in the following and discuss the implications at the level of the spaces of flow lines for the functional C_ρ.

7.2 Hessian and relative index

In order to proceed with the analogue of the finite dimensional Morse homology, an important step is the definition of the Morse index of a critical point. This is related to the Hessian of the functional (32).

Proposition 7.2 *Up to a zero-order operator, the Hessian $H_{(A,\psi)}$ of the functional (34) at a critical point (A,ψ) is the same as the operator Θ in the deformation complex.*

Proof: Consider a parametrised family

$$(A_s,\psi_s) = (A,\psi) + s(\alpha,\phi)$$

of connections and sections. The linear part of the increment, that is, the coefficient of s in

$$C_\rho(A_s,\psi_s) - C_\rho(A,\psi),$$

is a 1-form on the infinite dimensional space of connections and spinors which is identified via the metric with the gradient of (32). The coefficient of $s^2/2$ is a 2-form which, together with the gauge fixing condition $G^*_{A,\psi}(\alpha,\phi) = 0$, induces an operator $H_{(A,\psi)}$ on the tangent space of $\mathcal{B}$ at (A,ψ). The explicit form of the increment $C_\rho(A_s,\psi_s) - C_\rho(A,\psi)$ is

$$\frac{s}{2}(-\int_Y \alpha \wedge F_A + (A - A_0) \wedge d\alpha +$$

$$\int_Y (< \psi, \alpha\psi > + < \phi, \partial_A \psi > + < \psi, \partial_A \phi >) dv$$

$$+4 \int_Y \alpha \wedge * \rho) +$$

$$\frac{s^2}{2}(- \int_Y \alpha \wedge d\alpha + \int_Y (< \phi, \alpha\psi > + < \psi, \alpha\phi >) dv$$

$$+ \int_Y < \phi, \partial_A \phi > dv).$$

We can write the first order increment as

$$\begin{aligned} \mathcal{F} \mid_{(A,\psi,\rho)} (\alpha, \phi) = & - \textstyle\int_Y i\alpha \wedge (F_A - 2 * \rho - *\tau(\psi, \psi)) + \\ & + \tfrac{1}{2} \textstyle\int_Y (< \phi, \partial_A \psi > + < \partial_A \phi, \psi >) dv. \end{aligned} \tag{38}$$

This is the L^2-inner product of the tangent vector (α, ϕ) with the gradient flow (20). The Hessian is a quadratic form in the increment (α, ϕ), that can be written as

$$\frac{s^2}{2}(- \int_Y \alpha \wedge d\alpha + 2Re \int_Y < \phi, \alpha\psi > dv + \int_Y < \phi, \partial_A \phi > dv) =$$

$$\frac{s^2}{2}(- \int_Y \alpha \wedge d\alpha + \int_Y \alpha \wedge (*\tau(\phi, \psi) + *\tau(\psi, \phi) + Re \int_Y < \phi, \alpha\psi + \partial_A \phi > dv).$$

Thus we have the Hessian of the form

$$\begin{aligned} < H_{(A,\psi,\rho)}(\alpha, \phi), (\alpha, \phi) >= & < \alpha, *d\alpha - \tau(\psi, \phi) - \tau(\phi, \psi) > \\ & + Re < \phi, \partial_A \phi + \alpha\psi > . \end{aligned} \tag{39}$$

The first term is the L^2-inner product of forms and the second is the L^2-inner product of sections of $\tilde{W}$.

The quadratic form (39) induces an operator on the tangent bundle of $\mathcal{B}_L$, which is the same as the linearisation Θ of the equations on Y. At a critical point this is the Hessian of the Chern-Simons-Dirac functional.
QED

Thus, under a generic choice of the perturbation ρ, the Hessian at a critical point has trivial kernel. This means that we can state the following result.

Proposition 7.3 *Assume that Y has $b_1(Y) > 0$ and $c_1(L) \neq 0$, and that the perturbation ρ is cohomologically trivial. The functional (34) on the configuration space $\mathcal{B}_L$ has the Morse property, that is, the critical points are isolated and non-degenerate. If Y is a rational homology sphere, the functional (34), restricted to the irreducible component $\mathcal{B}^*$ has the Morse property.*

In the finite dimensional context, the index of a critical point of a Morse function is defined by counting the dimension of the eigenspaces of the Hessian with negative eigenvectors. Notice, however, that in this infinite dimensional case the Hessian has spectrum unbounded from below, hence the Morse index is not well defined. Nevertheless, it makes sense to define a relative Morse index

$$\mu(a) - \mu(b) \quad a, b \in \tilde{M}_c(Y, s)$$

as the spectral flow of the operator H along a path that connects the two critical points a and b. In principle, it is not clear that this notion is well defined. In fact, we need to check that the spectral flow does not depend on the path in $\mathcal{B}_L$ chosen to connect the points a and b.

Proposition 7.4 *In the case of a manifold Y with $b_1(Y) > 0$, the relative Morse index in $\mathcal{B}_L$ or $\mathcal{B}_0$ is well defined, in fact the spectral flow of H around a loop is zero. Similarly, for a homology sphere, the spectral flow in $\mathcal{B}$ is well defined. However, for a manifold Y with $b_1(Y) > 0$, the spectral flow in the quotient $\mathcal{B}$ by the action of the full gauge group is only defined modulo a periodicity given by*

$$\ell = g.c.d.\{< c_1(L) \cup h, [Y] > | h \in H^1(Y, \mathbb{Z})\}.$$

Proof: Consider a loop $[A(t), \psi(t)]$ in $\mathcal{B}_L$, with $t \in [0, 1]$. If we lift it to a path $(A(t), \psi(t))$, the endpoints differ by a gauge transformation $\lambda \in \mathcal{G}_L$. This $\lambda : Y \to U(1)$ determines a $U(1)$-bundle over $Y \times S^1$ obtained by identifying the ends of the cylinder; the connection $A(t)$ gives rise to a connection on this line bundle $\hat{L}$ over $Y \times S^1$. The spectral flow along $[A(t), \psi(t)]$ can be computed [2] as the index of the operator $\frac{\partial}{\partial t} + H_{(A(t),\psi(t))}$. This is given by

$$Ind(\frac{\partial}{\partial t} + L_{(A(t),\psi(t))}) = -\frac{1}{16\pi^2} \int_{Y \times S^1} c_1(\hat{L})^2 - \frac{2\chi + 3\sigma}{4}.$$

The term $2\chi + 3\sigma = 0$ on a manifold of the form $Y \times S^1$. As for the first term, notice that we can write [44]

$$F_{A(t)} \wedge F_{A(t)} = F_{A(t)} \wedge \frac{dA(t)}{dt} \wedge dt,$$

and therefore we get

$$\frac{-1}{8\pi^2} \int_{Y \times S^1} F_{A(t)} \wedge \frac{dA(t)}{dt} \wedge dt = \frac{i}{2\pi} \int_Y c_1(L) \int_{S^1} \frac{dA}{dt} =$$

$$= \frac{i}{2\pi} \int_Y c_1(L) \wedge \lambda^{-1} d\lambda,$$

since the difference of $A(t)$ at the two endpoints is $\lambda^{-1} d\lambda$.

Thus we have obtained the following spectral sum formula [44]:

$$Ind(\frac{\partial}{\partial t} + L_{(A(t),\psi(t))}) = \frac{i}{2\pi}\int_Y c_1(L)\wedge \lambda^{-1}d\lambda.$$

This implies immediately that the relative index is well defined in $\mathcal{B}_L$ or $\mathcal{B}_0$, and in $\mathcal{B}$ for homology spheres. It also implies that, if $b_1(Y) > 0$, the spectral flow in $\mathcal{B}$ is only defined up to the periodicity

$$\ell = g.c.d.\{< c_1(L)\cup h, [Y] > \ |h \in H^1(Y, \mathbb{Z})\}.$$

Notice that such ℓ is an even number.
QED

By the additivity of the spectral flow, for any three elements a, b, and c in $\tilde{M}_c(Y,s)$, we have

$$\mu(a) - \mu(c) = \mu(a) - \mu(b) + \mu(b) - \mu(c).$$

This justifies the notation $\mu(a) - \mu(b)$ for the spectral flow that defines the relative index.

Notice that if the spectral flow is computed in $\mathcal{B}$, for a manifold with $b_1(Y) > 0$, we find that the relative index is only defined in $\mathbb{Z}_\ell$. This periodicity is reminiscent of the $\mathbb{Z}_8$ periodicity in the instanton Floer homology [13] of homology spheres. However, there is a conceptual difference between these two periodicity phenomena. In our case, the periodicity is simply related to the existence of the covering $\mathcal{B}_L$, whereas in Donaldson theory the periodicity has a more subtle interpretation connected to a bubbling phenomenon [13] that does not have a direct analogue in Seiberg–Witten theory.

7.3 Flow lines: asymptotics

Now we want to consider the gradient flow equations (18) and (19) on the cylinder $Y \times \mathbb{R}$. To build up the right intuition, it is perhaps better to think of a finite dimensional analogue given by gradient flow lines of a proper Morse function on a complete non-compact Riemannian manifold. Over a compact manifold gradient flow lines of a Morse function will approach asymptotic values as $t \to \pm\infty$ that are critical points, whereas, on a non-compact manifold there may be flow lines that have no asymptotic value and simply go off to infinity. This in general will be the case for solutions of our flow equations as well. Thus, we have to select a suitable class of solutions of (18) and (19) for which we can guarantee the existence of asymptotic values. The guiding principle goes under the name of Simon's principle "finite energy implies finite length". For the moment we shall not work modulo gauge: we shall return to discuss the effect of the gauge group action in the next section.

Notice that, as we pointed out already, the analogy with finite dimensional Morse theory holds with the *caveat* that the equations (18) and (19) are not evolution equations, hence they should not be thought of as defining a local flow on $\mathcal{B}^*$.

Definition 7.5 *Let $(A(t), \psi(t))$ be a solution of the equations (35) and (36) on the cylinder $Y \times \mathbb{R}$. We say that the path $(A(t), \psi(t))$ is of finite energy if the following condition is satisfied:*

$$\mathcal{E} = \int_{-\infty}^{\infty} \|\nabla C_\rho(A(t), \psi(t))\|^2_{L^2(Y\times\{t\})} dt < \infty. \tag{40}$$

Suppose we have a solution $(A(t), \psi(t))$ with asymptotic values

$$\lim_{t\to-\infty} (A(t), \psi(t)) = (A_a, \psi_a),$$

$$\lim_{t\to+\infty} (A(t), \psi(t)) = (A_b, \psi_b),$$

where limits are taken in the L^2_k-topology on the configuration space. Here $a = [A_a, \psi_a]$ and $b = [A_b, \psi_b]$ are critical points in $\tilde{M}_c(Y, s)$. Then the solution $(A(t), \psi(t))$ is of finite energy. In fact, the total variation of the Chern-Simons-Dirac functional is

$$C_\rho(A_a, \psi_a) - C_\rho(A_b, \psi_b) = -\int_{-\infty}^{\infty} \frac{d}{dt} C_\rho(A(t), \psi(t)) dt$$

$$= -\int_{-\infty}^{\infty} < \nabla C_\rho(A(t), \psi(t)), \frac{d}{dt}(A(t), \psi(t)) > dt = \int_{-\infty}^{\infty} |\nabla C_\rho(A(t), \psi(t))|^2_{L^2} dt,$$

by equations (18) and (19).

The interesting point is the fact that the finite energy condition (40) is also sufficient to provide the existence of asymptotic values. This implication is more subtle and it relies upon the proof of some estimates for solutions on cylinders [23] [24].

Suppose given a finite energy solution $(A(t), \psi(t))$ of (18) and (19) on $Y \times \mathbb{R}$. The solution $(A(t), \psi(t))$ is irreducible if it is an irreducible solution of the four-dimensional Seiberg–Witten equations, that is, if the spinor $\psi(t)$ is not identically vanishing.

Proposition 7.6 *Let $(A(t), \psi(t))$ be an irreducible finite energy solution of (18) and (19) on $Y \times \mathbb{R}$. Then there exist critical points $a = [A_a, \psi_a]$ and $b = [A_b, \psi_b]$ in $\tilde{M}_c(Y, s)$, such that we have*

$$\lim_{t\to-\infty} (A(t), \psi(t)) = (A_a, \psi_a),$$

$$\lim_{t\to+\infty} (A(t), \psi(t)) = (A_b, \psi_b),$$

in the L_1^2 topology. Moreover, if $[A_a, \psi_a]$ is an irreducible critical point, $\psi_a \neq 0$, then we have exponential decay

$$\|(A(t), \psi(t)) - (A_a, \psi_a)\|^2_{L_1^2(Y\times\{t\})} \leq Ce^{-\delta|t|},$$

for all $t \leq -T_0$. The rate of decay δ is controlled by the eigenvalues of the Hessian at the critical point: $\delta < \min\{|\lambda_a|\}$, where $\{\lambda_a\}$ is the set of non-zero eigenvalues of the Hessian $H_{(A_a,\psi_a)}$. We have a similar decay as $t \to +\infty$ if $[A_b, \psi_b]$ is an irreducible critical point.

A proof of the existence of asymptotic values and an estimate of the rate of decay can be derived from the Łojasiewicz inequalities [34]. These estimate the distance from a critical point in terms of the L^2-norm of the gradient ("finite energy implies finite length"). The argument depends on the fact that the Chern-Simons-Dirac functional C_ρ is a real analytic function on the configuration space of connections and sections.

Since the rate δ of the exponential decay is controlled by the smallest nontrivial absolute value of the eigenvalues of the Hessian H_{A_a,ψ_a}, we can choose a unique δ for all critical points.

Thus, given two irreducible critical points $a = [A_a, \psi_a]$ and $b = [A_b, \psi_b]$ in $\tilde{M}_c(Y, s)$, we consider all solutions $(A(t), \psi(t))$ of the flow equations with limits

$$\lim_{t\to-\infty}(A(t), \psi(t)) = (A_a, \psi_a) \quad \text{and} \quad \lim_{t\to+\infty}(A(t), \psi(t)) = (A_b, \psi_b),$$

and such that

$$\|(A(t), \psi(t)) - (A_a, \psi_a)\|_{L_1^2} \leq Ce^{\delta t}$$

for $t \in (-\infty, -T_0]$ and

$$\|(A(t), \psi(t)) - (A_b, \psi_b)\|_{L_1^2} \leq Ce^{-\delta t}$$

for $t \in [T_0, \infty)$.

In the next section we describe how to fit these solutions into a configuration space with a free action of a gauge group, so that we can finally introduce the moduli spaces of flow lines.

7.4 Flow lines: moduli spaces

There are other technical issues that need to be addressed in order to define moduli spaces of flow lines. The reader may refer to [21], and to [23] for the analogous problems treated within the context of Donaldson theory.

So far we have considered the flow equations (18) and (19) for $(A(t), \psi(t))$ in a temporal gauge. With this notation, we can consider the action of the group of time independent gauge transformations on the set of solutions. Notice, however, that, in order to define virtual tangent spaces to the moduli spaces of flow lines and set up the corresponding Fredholm analysis, we need to consider slices of the gauge action. This corresponds to the analysis worked out in [23] for the case of Donaldson Floer theory.

The slice at an element (A,ψ) is defined by the condition $G^*_{(A,\psi)}(\alpha,\phi)=0$. It is easy to verify that the flow $(A(t),\psi(t))$ is tangent to the slice at $(A(t),\psi(t))$ itself. However, if we need to consider positive dimensional moduli spaces (this will be the case especially in the context of the equivariant Floer theory discussed later in the chapter), we need to consider solutions in the slice at a fixed element (A_0,ψ_0). A solution $(A(t),\psi(t))$ of (18) and (19) is then no longer in the slice at (A_0,ψ_0), unless time dependent gauge transformations are allowed, but these will break the temporal gauge condition.

This problem can be overcome by replacing the temporal gauge condition with the condition of *standard form* introduced in [23]. This allows time–dependent gauge transformations. One can then add a correction term $\gamma(A,\psi)$ in the flow equations in order to make the flow tangent to a fixed slice of the gauge action at the point (A_0,ψ_0). The linearisation of the equations contains the extra term Θ_γ that linearises $\gamma(A,\psi)$,

$$\mathcal{D}_{(A(t),\psi(t))}(\alpha,\phi)=\frac{\partial}{\partial t}+\Theta_{A(t),\psi(t)}+\Theta_\gamma.$$

Anisotropic Sobolev norms $L^2_{(k,m)}$ can be introduced on the spaces of connections and sections and gauge transformations, as analysed in [23].The linearisation Θ_γ is a compact operator with respect to these norms.

However, it is perhaps more convenient to follow [13] and return to the original description of the solutions of (18) and (19) as solutions of the four-dimensional Seiberg–Witten equations on $Y\times\mathbb{R}$. This is the approach followed in [21]. In this case we define the moduli space of flow lines $\mathcal{M}(a,b)$. Consider the set of pairs (A,ψ) on $Y\times\mathbb{R}$ that solve the four-dimensional Seiberg–Witten equations (8) and (9), and such that there are elements (A_a,ψ_a) and (A_b,ψ_b) in the classes $[A_a,\psi_a]=a$ and $[A_b,\psi_b]=b$ in $M_c(Y,s)$ such that $(A,\psi)-(A_a,\psi_a)$ is in $L^2_{k,\delta}$ on $Y\times[T_0,\infty)$ and $(A,\psi)-(A_b,\psi_b)$ is in $L^2_{k,\delta}$ on $Y\times(-\infty,-T_0]$. The gauge group $\mathcal{G}(a,b)$ of gauge transformations on $Y\times\mathbb{R}$ that decay exponentially to elements in the stabilisers G_a and G_b as $t\to\pm\infty$ acts on the space of solutions. The quotient is the moduli space of flow lines $\mathcal{M}(a,b)$. We have the following index counting [8], [14], [44].

Proposition 7.7 *The linearisation*

$$\mathcal{D}_{(A(t),\psi(t))}$$

of the four-dimensional Seiberg–Witten equations on $Y\times\mathbb{R}$, at a solution with asymptotic values in the classes $[A_a,\psi_a]=a$ and $[A_b,\psi_b]=b$, is a Fredholm operator whose index is the spectral flow of the family of operators $\Theta_{(A(t),\psi(t))}$. Thus, the virtual dimension of the moduli space $\mathcal{M}(a,b)$ is given by

$$\dim\mathcal{M}(a,b)=\mu(a)-\mu(b)-\dim G_a.$$

Another subtle problem is the transversality result, which ensures that the moduli space $\mathcal{M}(a,b)$ is of the proper dimension and cut out transversely by the equations. The problem is finding a suitable perturbation theory for the flow equations. In order to define the boundary operator of the Floer homology, we want to consider spaces of unparametrised flow lines, that is, solutions of the flow equations modulo an action of $\mathbb{R}$ by reparametrisations. Thus, we want to introduce a perturbation that preserves the translation invariance of the flow equations. Moreover, the perturbation has to achieve transversality. It is not hard to check, for instance, that a time independent perturbation would satisfy the first requirement but not the second. A class $\mathcal{P}$ of perturbations satisfying both conditions was introduced by Froyshov [14], see also [21]. Another class of holonomy perturbations is described in [16], [7]. We obtain the following result [21].

Proposition 7.8 *Let $P_{A,\psi}$ be a generic choice of perturbation in the class $\mathcal{P}$ of [14]. Consider the space $\mathcal{M}(a,b)$ of solutions of the perturbed equations*

$$D_A\psi = 0 \tag{41}$$

$$F_A^+ = \frac{1}{4} < e_i e_j \psi, \psi > e^i \wedge e^j + iP_{A,\psi}, \tag{42}$$

with asymptotic values in the classes $[A_a, \psi_a] = a$ and $[A_b, \psi_b] = b$, modulo the action of $\mathcal{G}(a,b)$. The space $\mathcal{M}(a,b)$ is of the proper dimension and is cut out transversely by the equations.

Thus, by the translation invariance of the equations (41) and (42), we can form the quotient by the action of $\mathbb{R}$ by translations

$$\hat{\mathcal{M}}(a,b) = \mathcal{M}(a,b)/\mathbb{R}.$$

This gives the moduli spaces of unparametrised flow lines, of dimension

$$\dim \hat{\mathcal{M}}(a,b) = \mu(a) - \mu(b) - \dim G_a - 1.$$

Proposition 7.9 *The manifold $\mathcal{M}(a,b)$ is oriented by a trivialization of the determinant line bundle of the operator $\mathcal{D}$. This is obtained from an orientation of*

$$H^0_\delta(Y\times\mathbb{R}) \oplus H^{2+}_\delta(Y\times\mathbb{R}) \oplus H^1_\delta(Y\times\mathbb{R}),$$

the cohomology groups of δ-decaying forms.

<u>Sketch of the Proof:</u> The strategy for the proof is to deform the operator $\mathcal{D}$ to the operator $d^+ + d^*_\delta + D_A$, where D_A is the Dirac operator on the 4-manifold

$Y \times \mathbb{R}$. An orientation of the determinant line of this operator is determined by an orientation of

$$H^0_\delta(Y \times \mathbb{R}) \oplus H^{2+}_\delta(Y \times \mathbb{R}) \oplus H^1_\delta(Y \times \mathbb{R}),$$

since the Dirac operator is complex linear and it preserves the orientation induced by the complex structure on the spinor bundle. The subtle point is to guarantee that we can deform $\mathcal{D}$ through a family of Fredholm operators. This depends on the behaviour of the asymptotic operators at $t \to \pm\infty$. In particular, the condition is satisfied if the weight δ of the exponential decay is such that, along all the perturbation, $\delta/2$ is never in the spectrum of the asymptotic operators (see proposition 2.23). In our case, however, this condition might fail, hence a more subtle approach is needed. This was originally suggested by L. Nicolaescu and is based on the excision formulae for the index. This separates the deformation into a part where the asymptotic operator remains constant and a part where the change in the index is computed by the excision formula and explicitly accounts for the change of orientation produced along the original deformation. Notice that this contribution can be non-trivial, as computed explicitly in some examples in [29]. The details of this proof can be found in [21].
QED

Now we can address the question of compactness for the spaces of flow lines. The following proposition shows that there is a stratification of the moduli spaces $\hat{\mathcal{M}}(a,b)$ as manifolds with corners, via boundary strata given by lower dimensional moduli spaces.

Proposition 7.10 *There is a natural compactification of the manifold* $\hat{\mathcal{M}}(a,b)$ *obtained by adding boundary strata of codimension* k *of the form*

$$\hat{\mathcal{M}}(a,c_1) \times_{c_1} \hat{\mathcal{M}}(c_1,c_2) \times_{c_2} \cdots \times_{c_k} \hat{\mathcal{M}}(c_k,b), \tag{43}$$

that is, the union of gradient flow lines that break through other critical points with Morse indices $\mu(a) > \mu(c_1) > \cdots > \mu(b)$.

This compactification is the result of a gluing theorem. We shall not give here a complete argument, and we refer the reader to [21] for details. We just point out that the result 7.10 consists of two main steps. One is a convergence result, which proves the inclusion

$$\partial^{(k)}\hat{\mathcal{M}}(a,b) \subseteq \hat{\mathcal{M}}(a,c_1) \times_{c_1} \hat{\mathcal{M}}(c_1,c_2) \times_{c_2} \cdots \times_{c_k} \hat{\mathcal{M}}(c_k,b).$$

Here the notation $\partial^{(k)}$ means the stratum of codimension k in the boundary of $\hat{\mathcal{M}}(a,b)$. This is achieved by showing that a sequence of solutions in $\hat{\mathcal{M}}(a,b)$ has a subsequence that converges smoothly on compact sets to a finite energy solution on $Y \times \mathbb{R}$ which is in some $\hat{\mathcal{M}}(c,d)$ with $\mu(a) \geq \mu(c) > \mu(d) \geq \mu(b)$.

It is important to notice that the convergence that gives rise to the compactification 7.10 is not in the norm topology $L^2_{k,\delta}$ but only in the weak topology of $\mathcal{C}^r$ convergence on compact sets, then improved by elliptic regularity to $\mathcal{C}^\infty$ convergence on compact sets.

The second step is the proof of the reverse inclusion, namely the fact that indeed all the components $\hat{\mathcal{M}}(c,d)$ with $\mu(a) \geq \mu(c) > \mu(d) \geq \mu(b)$ appear as boundary components of $\hat{\mathcal{M}}(a,b)$,

$$\hat{\mathcal{M}}(a,c_1) \times_{c_1} \hat{\mathcal{M}}(c_1,c_2) \times_{c_2} \cdots \times_{c_k} \hat{\mathcal{M}}(c_k,b) \subseteq \partial\hat{\mathcal{M}}(a,b).$$

This follows from a gluing argument. We first construct approximate solutions by patching together solutions in the components $\hat{\mathcal{M}}(c_j,c_{j+1})$ with smooth cut-off functions. There is a gluing parameter $T \geq T_0$, that fixes the choice of a parametrised representative of an element in $\hat{\mathcal{M}}(c_j,c_{j+1})$, and substantially measures how close to the critical points the cutting and pasting of solutions happens. The purpose then is to show that for large enough $T \geq T_0$, the approximate solution can be perturbed to an actual solution. The latter is achieved by considering the linearisation at the approximate solution and showing that it is a contraction in a small ball. A fixed point argument then proves the existence of the solution, [13], [21].

7.5 Homology

The previous results make it possible to define the chain complex and the boundary operator.

As we have already seen in dealing with the invariant $\chi(Y,s)$, the case of a rational homology sphere requires a strategy in order to deal with the presence of the reducible point. In the case of the invariant we have seen that, under generic assumptions on the choice of the metric and perturbation, we can guarantee that the bad point θ is isolated, and this is enough to justify the definition of $\chi(Y,s)$ as the algebraic sum of the irreducible points in $M^*_c(Y,s)$. A similar strategy can be adopted in defining the Seiberg–Witten Floer homology for rational homology spheres [8], [14], [42].

Under a choice of the metric and perturbation (g,ν) such that $Ker(\partial^g_\nu) = 0$, we know that the bad point θ is isolated. Thus, we can consider only irreducibles among the generators. In other words, the complex $FC_*(Y,g)$ is generated in degree q by all the irreducible critical points of relative index q with respect to the reducible solution. In the case of manifolds with $b_1(Y) > 0$ all the generators are counted, and the index is with respect to an arbitrary solution, thus it is defined up to an integer shift.

Definition 7.11 *We can fix arbitrarily the Morse index of one of the points of $\tilde{M}_c(Y,s)$. If Y is a rational homology sphere, we assign index zero to the unique reducible, $\mu(\theta) = 0$. In order to construct the chain complex FC_* that computes* *the Seiberg–Witten Floer homology, we take as generators of FC_q all the critical points of relative index q with respect to the chosen one,*

$$FC_q = \{a \in \tilde{M}^*_c(Y,s) | \mu(a) - \mu(a_0) = q\}.$$

The boundary operator $\partial_q : FC_q \to FC_{q-1}$ has matrix elements

$$< \partial_q a, b >= n_{ab}, \tag{44}$$

with $a \in FC_q$, $b \in FC_{q-1}$. The coefficient n_{ab} is the number of points in $\hat{\mathcal{M}}(a,b)$ counted with the orientation. The Seiberg–Witten Floer homology is defined as

$$FH_*(Y) = H_*(FC_*, \partial_*).$$

The operator ∂ as defined in 7.11 is indeed a boundary operator, namely $\partial \circ \partial = 0$. The proof of this fact relies upon the gluing formula for moduli spaces of flow lines described in the previous section. This is the analogue in the infinite dimensional context of the transitivity of Morse-Smale flow, namely the fact that the closure of the space of flow lines that connect two critical points with relative Morse index 2 is the union of trajectories that break through critical points with intermediate Morse index.

The operator $\partial \circ \partial$ has matrix elements $< \partial^2 a, c >= \sum_b n_{ab} n_{bc}$. We know from the gluing formula 7.10 that the moduli space $\hat{\mathcal{M}}(a,c)$ is an oriented 1-dimensional manifold with boundary

$$\partial \hat{\mathcal{M}}(a,c) = \cup_b \hat{\mathcal{M}}(a,b) \times \hat{\mathcal{M}}(b,c).$$

The total oriented boundary of a one dimensional manifold is zero, and this proves that $\partial^2 = 0$.

In the case of a rational homology sphere we have removed the bad point θ from the set of generators. Thus, we have to check more carefully that the relation $\partial^2 = 0$ still holds. In other words, consider again the moduli space $\hat{\mathcal{M}}(a,c)$, with a and c in $\tilde{M}^*_c(Y,s)$ with $\mu(a) - \mu(c) = 2$. We need to ensure that the boundary of $\hat{\mathcal{M}}(a,c)$ does not contain flow lines in $\hat{\mathcal{M}}(a,\theta) \times \hat{\mathcal{M}}(\theta,c)$. This may occur when $\mu(a) = 1$ and $\mu(c) = -1$. By dimensional count,

$$\dim \hat{\mathcal{M}}(\theta,c) = \mu(\theta) - \mu(c) - \dim G_\theta - 1 = -1,$$

hence $\hat{\mathcal{M}}(\theta,c)$ is generically empty.

This shows that the Floer complex is well defined in all cases.

In the case $b_1(Y) > 0$, with the definition given above, the Floer homology is finitely generated in each degree, but it extends over infinitely many degrees. However, we can define a $\mathbb{Z}_\ell$-periodic complex [8] that is finitely generated, by reassembling the various $CF_q(Y,s)$. Consider two elements a and b in $M_c(Y,s)$, with relative index $\mu(a) - \mu(b)$ defined mod ℓ. Consider all the elements a_k and b_k, in $\tilde{M}_c(Y, s)$, with $k \in \mathbb{Z}$, that map to a and b under the projection. Consider all the moduli spaces of flow lines $\hat{\mathcal{M}}(a_k, b_{k'})$. There are two representatives a_{min} and b_{min} such that the flow lines in $\hat{\mathcal{M}}(a_{min}, b_{min})$ have the least energy among all the $\hat{\mathcal{M}}(a_k, b_{k'})$. The boundary operator in the Floer complex that defines the $\mathbb{Z}_\ell$ graded Floer homology is given by

$$\langle \partial_0 a, b \rangle = n^0_{ab}, \tag{45}$$

for a and b in $M_c(Y, s)$ with $\mu(a) - \mu(b) = 1 \bmod \ell$. Here the coefficient n^0_{ab} is obtained by counting with the orientation only the flow lines that belong to the component $\hat{\mathcal{M}}(a_{min}, b_{min})$. We use the notation $HF_*(Y, s, \mathbb{Z}_\ell)$ for the resulting $\mathbb{Z}_\ell$ graded Floer homology.

We still have to explain how to handle the bad case $b_1(Y) > 0$ and $c_1(L) = 0$. First of all notice that, in the good case $c_1(L) \neq 0$, we can refine the boundary operator (45) defined above as follows [7]

$$\langle \partial_k a, b \rangle = \sum_{k \geq 0} n^k_{ab} t^k, \tag{46}$$

for a and b in $M_c(Y, s)$ with $\mu(a) - \mu(b) = 1 \bmod \ell$. Here we have $n^k_{ab} = \#\hat{\mathcal{M}}(a_{min}, b_k)$ and the resulting version of the Floer homology $HF_*(Y, s, \mathbb{Z}[[t]])$ has coefficients in the ring $\mathbb{Z}[[t]]$. It can be thought of as a Novikov complex. We have the relation $\partial_0 = \partial_k|_{t=0}$. This construction can be extended to the bad case as follows.

In the case $b_1(Y) > 0$ and $c_1(L) = 0$, we can get rid of the reducibles by choosing a fixed non-trivial cohomology class for the perturbation ρ. This implies that the Chern–Simons–Dirac functional is $\mathbb{R}$-valued on a $\mathbb{Z}$-covering $\mathcal{B}_\rho$ of $\mathcal{B}$ obtained when considering the action of gauge transformations in

$$\mathcal{G}_\rho = \{\lambda | \langle [\lambda] \cup [\rho], [Y] \rangle = 0\}.$$

Given two elements a and b in $M_c(Y, s)$ with $c_1(L) = 0$, we do not have an analogue of the boundary operator ∂_0, since the flow lines in the various components $\hat{\mathcal{M}}(a_k, b_{k'})$ have the same energy, but the union $\cup_{k,k'} \hat{\mathcal{M}}(a_k, b_{k'})$ is non-compact. However, it is still possible to define the boundary operators ∂_k by fixing one representative a_0 and defining as before

$$\langle \partial_k a, b \rangle = \sum_{k \geq 0} n^k_{ab} t^k,$$

with $n^k_{ab} = \#\hat{\mathcal{M}}(a_0, b_k)$. This gives a Novikov-type Floer homology for the case $b_1(Y) > 0$ and $c_1(L) = 0$,

$$HF_*(Y, s, \mathbb{Z}[[t]]).$$

We can state the analogue of Taubes' result [35] on the Casson invariant and instanton Floer homology [20].

Proposition 7.12 *The invariant* $\chi(Y,s)$ *computes the Euler characteristic of the* $\mathbb{Z}_\ell$*-graded Seiberg–Witten Floer homology,*

$$\chi(Y,s) = \chi(HF_*(Y,s,\mathbb{Z}_\ell))$$

if $b_1(Y) > 0$ *and*

$$\chi_{g,\nu}(Y,s) = \chi(HF_*(Y,s,g,\nu))$$

when $b_1(Y) = 0$.

Proof: We have to check that the counting of orientations used to define the invariant $\chi(Y,s)$ agrees with the grading of the Floer complex.

We know that, under a generic choice of metric and perturbation, we have $Ker(T_{A,\psi}) = 0$ at all solutions (A,ψ) of (25) and (26). Suppose given a path $\gamma(t)$, $t \in [0,1]$ in a slice of the gauge action that locally describes $\mathcal{B}_L$, connecting two critical points a and b in $\tilde{M}_c(Y,s)$. Consider the set $\Omega = \gamma \times \mathbb{C}$. Since $Ker(T_{(A_a,\psi_a)}) = Ker(T_{(A_b,\psi_b)}) = 0$ the spectral flow of

$$T_t = T_{(A(t),\psi(t))}$$

along γ can be thought of (see [2]) as the algebraic sum of the intersections in Ω of the set

$$\mathcal{S} = \cup_{t\in\gamma} Spec(T_t)$$

with the line $\{(t,0) \mid t \in \gamma\}$. This counts the points where the discrete spectrum of the operator T crosses zero, with the appropriate sign. Up to perturbations we can make these crossings transverse.

We can express the same procedure in a different, yet equivalent, way. Consider T_t as a section over $\mathcal{B}_L$ of the bundle of index zero Fredholm operators, Fr_0. There is a first Stiefel-Whitney class w_1 in $H^1(Fr_0;\mathbb{Z}_2)$ that classifies the determinant line bundle of Fr_0 (see [35]).

The submanifold of codimension one in Fr_0 that represents the class w_1 is given by Fredholm operators of nonempty kernel, $Fr_0^1 \subset Fr_0$. This submanifold can be thought of as the zero set of a generic section of the determinant line bundle.

Given a path $\gamma : I \to \mathcal{B}_L$ joining two points a and b, the image of γ composed with a generic section σ of Fr_0 will meet Fr_0^1 transversely.

We call this intersection number $\delta_\sigma(a,b)$. This counts the points in

$$Det(\sigma \circ \gamma)^{-1}(0)$$

with the orientation. If we take the section $\sigma \circ \gamma$ to be T_t what we get is exactly the spectral flow between the critical points a and b.

Since, in the case $b_1(Y) > 0$, we are identifying generators up to the $\mathbb{Z}_\ell$ periodicity, this number is defined modulo ℓ, but the mod 2 spectral flow does not depend on the choice of the path. Now we want to show that this same intersection number measures the change of orientation between the corresponding two points a and b of $M_c(Y,s)$.

The orientation of the tangent space at a critical point is given by a trivialisation of $Det(T_{(A,\psi)})$ or equivalently of $Det(T_{(A,\psi)})$ at that point.

Up to a perturbation, we can assume that $Coker(\tilde{T}_t)$ is trivial along the path γ. Therefore we can think of $Det(T_t)$ as $\Lambda^{top}Ker(T_t)$, which specifies the orientation of the space $Ker(T_t)$. Thus we obtain that the change of orientation between two critical points is measured by the mod 2 spectral flow.

This means that the sign attached to the point in the grading of the Floer complex is the same as the sign that comes from the orientation of $M_c(Y,s)$.
QED

7.6 The cobordism argument

The Seiberg–Witten Floer homology is independent of the choice of the metric and perturbation in the case of manifolds with $b_1(Y) > 0$.

Theorem 7.13 *Suppose we have $b_1(Y) > 0$, $c_1(L) \neq 0$, and the perturbations are cohomologically trivial. Let (g_0,ρ_0) and (g_1,ρ_1) be two choices of the metric and perturbation, and (g_t,ρ_t) a generic path connecting them. The cobordism used in proving the metric independence of the invariant $\chi(Y,s)$ induces a morphism between the Floer complexes*

$$I_* : FC_*(Y,s,g_0,\rho_0) \to FC_*(Y,s,g_1,\rho_1).$$

The morphism I_* induces an isomorphism in homology.

Proof: Consider the complexes $FC_*(Y,s,g_0,\rho_0)$ and $FC_*(Y,s,g_1,\rho_1)$ that correspond to the choices (g_0,ρ_0) and (g_1,ρ_1) of metric and perturbation. The grading is determined by fixing arbitrarily the grading of one particular solution. Thus, in order to compare the Floer groups

$$FH_*(Y,s,g_0,\rho_0) \quad \text{and} \quad FH_*(Y,s,g_1,\rho_1)$$

we need a way to choose the grading consistently.

Suppose given two points a_0 and a_1 of $\mathcal{B}_L$ that lie in $\tilde{M}_c(Y,s,g_0,\rho_0)$ and in $\tilde{M}_c(Y,s,g_1,\rho_1)$ respectively. If the grading of $FC_*(Y,s,g_0,\rho_0)$ is determined by the choice $\mu(a_0) = 0$ we set the grading of $FC_*(Y,s,g_1,\rho_1)$ according to the rule

$$\mu(a_1) = SF(\Theta_{(A(t),\psi(t))}),$$

the spectral flow along a path in $\mathcal{B}_L$ from a_0 to a_1. This determines an absolute grading of the generators of $FC_*(Y,s,g_1,\rho_1)$ that is consistent with the relative grading, by the additivity of the spectral flow.

Once the choice of the grading is determined, we can construct a chain homomorphism as follows. Suppose given a path (g_t, ρ_t) between (g_0, ρ_0) and (g_1, ρ_1). Consider the manifold $Y \times \mathbb{R}$ endowed with the metric $g_t + dt^2$ extended to the product metric outside the interval $[0, 1]$. Consider on this manifold the perturbed gradient flow equations(that is the perturbed Seiberg–Witten equations in four dimensions). The techniques developed in the analysis of the moduli spaces of flow lines can be adapted with minor modifications to prove the following.

Proposition 7.14 *Let $\mathcal{M}(a, a')$ be the moduli space of solutions of the perturbed flow equations on the manifold $Y \times \mathbb{R}$ with metric $g_t + dt^2$ and perturbation in the class $\rho_t + \mathcal{P}$, with asymptotic values a in $\tilde{M}_c(Y, s, g_0, \rho_0)$ and a' in $\tilde{M}_c(Y, s, g_1, \rho_1)$. For a generic choice of the perturbation, the space $\mathcal{M}(a, a')$ is a smooth manifold of dimension $\mu(a) - \mu(a')$ which is cut out transversely by the equations. Moreover, when $\mu(a) = \mu(a')$ the manifold $\mathcal{M}(a, a')$ is compact (hence a finite set of points), and in the case with $\mu(a) - \mu(a') = 1$, $\mathcal{M}(a, a')$ has a compactification with boundary strata*

$$\begin{aligned} \partial\mathcal{M}(a, a') &= \bigcup_b \hat{\mathcal{M}}(a, b) \times \mathcal{M}(b, a') \cup \\ & \bigcup_{b'} -\mathcal{M}(a, b') \times \hat{\mathcal{M}}(b', a'), \end{aligned}$$

where $\mu(a) - \mu(b) = 1$, $b \in \tilde{M}_c(Y, s, g_0, \rho_0)$, and $\mu(b') - \mu(a') = 1$, $b' \in \tilde{M}_c(Y, s, g_1, \rho_1)$.

Thus, we can define a degree zero morphism

$$I_q : FC_q(Y, s, g_0, \rho_0) \to FC_q(Y, s, g_1, \rho_1)$$

by setting

$$< a', Ia >= N_{aa'},$$

where $N_{aa'}$ is the number of points counted with the orientation in the moduli space $\mathcal{M}(a, a')$.

Lemma 7.15 *The map I is a chain homomorphism, that is, we have*

$$\partial_q I_q - I_{q-1}\partial_q = 0.$$

Proof: This follows directly from proposition 7.14, since the expression

$$n_{ab}N_{ba'} - N_{ab'}n_{b'a'}$$

is the sum of points with orientation in the boundary $\partial\mathcal{M}(a, a')$.
QED

We can define an analogous chain homomorphism

$$J_q : FC_q(Y, s, g_1, \rho_1) \to FC_q(Y, s, g_0, \rho_0),$$

by choosing a path $(\tilde{g}_t, \tilde{\rho}_t)$, for $t \in [0,1]$, such that $(\tilde{g}_0, \tilde{\rho}_0) = (g_1, \rho_1)$ and $(\tilde{g}_1, \tilde{\rho}_1) = (g_0, \rho_0)$, and counting solutions modulo gauge of the flow equations on $Y \times \mathbb{R}$ with the metric $\tilde{g}_t + dt^2$ and a perturbation chosen in the space $\tilde{\rho}_t + \mathcal{P}$.

Suppose given a metric γ_1 on $Y \times \mathbb{R}$ that is of the form

$$\gamma_1 = \begin{cases} g_0 + dt^2 & t < -2 \\ g_{t+2} + dt^2 & t \in [-2,-1] \\ g_1 + dt^2 & t \in [-1,1] \\ \tilde{g}_{2-t} + dt^2 & t \in [1,2] \\ g_0 + dt^2 & t > 2. \end{cases}$$

Consider the manifold $Y \times \mathbb{R}$ with a family of metrics γ_σ, with $\sigma \in [0,1]$ that connects $\gamma_0 = g_0 + dt^2$ to γ_1, such that for all σ the metric γ_σ is the product metric $g_0 + dt^2$ outside a fixed large interval $[-T, T]$.

Let $\mathcal{M}^P(a,b)$ be the parametrised moduli space of $(A(t), \psi(t), \sigma)$, solutions modulo gauge of the perturbed gradient flow equations with respect to the metric γ_σ.

Again, the techniques developed in the analysis of the moduli spaces of flow lines can be adapted to prove the following result.

Proposition 7.16 *For generic choice of the perturbation, the space $\mathcal{M}^P(a,b)$ is a smooth manifold of dimension $\mu(a) - \mu(b) + 1$ that is cut out transversely by the equations. Moreover, when $\mu(b) = \mu(a) + 1$, the manifold $\mathcal{M}^P(a,b)$ is compact, hence a finite set of points. When $\mu(b) = \mu(a)$, the manifold $\mathcal{M}^P(a,b)$ has a compactification with boundary strata of the form*

$$\begin{aligned} \partial \mathcal{M}^P(a,b) &= \bigcup_c \hat{\mathcal{M}}(a,c) \times \mathcal{M}^P(c,b) \cup \\ & \bigcup_d -\mathcal{M}^P(a,d) \times \hat{\mathcal{M}}(d,b) \cup \\ & \bigcup_{a'} \mathcal{M}(a,a') \times \mathcal{M}(a',b), \end{aligned}$$

if $a \neq b$, where $\mu(a) - \mu(c) = 1$, $\mu(d) - \mu(b) = 1$ and $\mu(a) = \mu(a')$. For $a = b$ the boundary is given by

$$\begin{aligned} \partial \mathcal{M}^P(a,a) &= \bigcup_c \hat{\mathcal{M}}(a,c) \times \mathcal{M}^P(c,a) \\ & \cup \bigcup_d -\mathcal{M}^P(a,d) \times \hat{\mathcal{M}}(d,a) \\ & \cup -\{a\} \cup \bigcup_{a'} \mathcal{M}(a,a') \times \mathcal{M}(a',a). \end{aligned}$$

Thus, we have the following result.

Proposition 7.17 *There is a chain homotopy*

$$H_q : FC_q(Y, s, g_0, \rho_0) \to FC_{q+1}(Y, s, g_1, \rho_1).$$

The relation

$$id_q - J_q I_q = \partial_{q+1} H_q + H_{q-1} \partial_q$$

is satisfied, where id_q is the identity map on $FC_q(Y, s, g_0, \rho_0)$.

Proof: In fact, we have the identities

$$< b, (id_q - J_q I_q) a >= \delta_{ab} - N_{aa'} M_{a'b},$$

where $N_{aa'}$ and $M_{a'b}$ are the number of points counted with orientation in the moduli spaces $\mathcal{M}(a, a')$ and $\mathcal{M}(a', b)$ on $Y \times \mathbb{R}$ with metrics $g_t + dt^2$ and $\tilde{g}_t + dt^2$ respectively, and

$$< b, (\partial_{q+1} H_q + H_{q-1} \partial_q) a >= n_{ac} N^P_{cb} - N^P_{ad} n_{db}.$$

Here N^P_{ad} and N^P_{cb} are the number of points counted with the orientation in $\mathcal{M}^P(a, d)$ and $\mathcal{M}^P(c, b)$ respectively. The difference of the two expressions vanishes, since the number

$$n_{ac} N^P_{cb} - N^P_{ad} n_{db} + N_{aa'} M_{a'b} - \delta_{ab}$$

counts the sum with orientation of the points in the boundary $\partial \mathcal{M}^P(a, b)$.
QED

This proves the invariance of the Floer groups as stated in theorem 7.13.
QED

Now consider the case of a rational homology sphere. As in the case of the invariant $\chi(Y, s)$, we can expect a problem with the metric dependence generated by the condition $Ker(\partial^g_\nu) = 0$ in the case with $b_1(Y) = 0$.

There is a way to avoid the metric dependence problem, and construct a version of Seiberg–Witten Floer homology that is well defined and metric independent on all three-manifolds. This can be done via a $U(1)$-equivariant construction that takes into account all the generators [21]. The equivariant Seiberg–Witten Floer homology will be briefly introduced in the next section. It is isomorphic to the Floer homology defined in this section when the manifold has $b_1(Y) > 0$, and it is related to $FH_*(Y, s, g, \nu)$ by an exact sequence otherwise. The exact sequence contains in principle all the information about the wall crossing of $FH_*(Y, s, g, \nu)$. In fact, for instance the wall crossing of theorem 6.11 for the invariant $\chi_{g,\nu}(Y)$ of a homology sphere can be derived directly from these exact sequence [21].

7.7 Equivariant Floer homology and wall crossing

We introduce here a formulation of the Seiberg–Witten Floer homology which can be given for any closed three manifold Y and is independent of the choice of the metric and perturbation. In order to deal with the presence of reducible solutions of (21) and (22), the quotient is taken with respect to the action of a smaller gauge group.

Consider the subgroup $\mathcal{G}_b$ of based gauge transformations, that is the elements $\lambda \in \mathcal{G}_L$ with the condition that they act as the identity on the fibre over a fixed base point $b \in Y$, $\lambda(b) = 1$.

The subgroup $\mathcal{G}_b$ acts freely on the space $\mathcal{A}$. In fact, if (A,ψ) is fixed by some λ, then we have $\psi \equiv 0$ and λ a constant gauge transformation; but the base point condition implies that there are no non-trivial constants in $\mathcal{G}_b$.

Thus the quotient $\mathcal{B}_b = \mathcal{A}/\mathcal{G}_b$ is a manifold. We consider the *framed moduli space* $M_c^b(Y,s) \subset \mathcal{B}_b$ of solutions of the perturbed critical point equations (25) and (26) modulo based gauge transformations.

The framed moduli space has a residual $U(1)$ action, since the based gauge group is obtained as a quotient $\mathcal{G}_L/U(1)$, and the moduli space $\tilde{M}_c(Y,s)$ is the quotient $\tilde{M}_c(Y,s) = M_c^b(Y,s)/U(1)$. The orbits of the $U(1)$ action on $M_c^b(Y,s)$ are copies of S^1 that correspond to all the irreducible critical points in $\tilde{M}_c(Y,s)$. If $b_1(Y) > 0$ these are all the elements in $M_c^b(Y,s)$, which is therefore just $\tilde{M}_c(Y,s) \times U(1)$. If $b_1(Y) = 0$ there is also a fixed point in $M_c^b(Y,s)$ that corresponds to the reducible θ in $\tilde{M}_c(Y,s)$.

The compactness result applies to this case as well and shows that $M_c^b(Y,s)$ is compact if $b_1(Y) = 0$, or is a $\mathbb{Z}$-covering of a compact space in case $b_1(Y) > 0$. However, we also have to verify that the orbits and the fixed point are isolated in $\mathcal{B}_b$.

Proposition 7.18 *If $b_1(Y) = 0$, under a generic choice of the metric and of the perturbation (g,ν), the moduli space $M_c^b(Y,s)$ is a disjoint union of finitely many circles O_a and one point θ fixed by the $U(1)$ action. If $b_1(Y) > 0$, under a generic choice of metric and perturbation, $M_c^b(Y,s)$ consists of a $\mathbb{Z}$-covering of finitely many circles.*

Proof: In the case with $b_1(Y) > 0$ we can choose the perturbation in such a way that there are no reducibles in $\tilde{M}_c(Y,s)$. Thus, the framed moduli space $M_c^b(Y,s)$ is just the $\mathbb{Z}$-covering of a finite disjoint union of circles with a free $U(1)$ action. If $b_1(Y) = 0$, the virtual tangent space of $M_c^b(Y,s)$ at a reducible point is $H^1(Y,\mathbb{R}) \oplus Ker(\partial_A)$, that is, just $Ker(\partial_A)$. For a generic metric on Y the kernel of the Dirac operator twisted with a fixed flat connection is trivial, as we have already discussed. Assume that the metric is chosed away from the codimension one walls in the space of metrics for which the above condition fails. Then we can look at the local Kuranishi model of the moduli space M_c^b to show that no other non-trivial solutions can accumulate near the fixed point. This is the same argument we have seen before.
QED

Proposition 7.19 *Under a generic choice of the metric and perturbation, the perturbed functional (34) on $\mathcal{B}_b$ has the Morse-Bott property. Namely, the Hessian is non-degenerate in the directions orthogonal to the orbits.*

Proof: Consider the case $b_1(Y) = 0$. The other case follows similarly. The condition that the trivial solution is isolated implies that it is also non-degenerate. In fact the same argument proves that the Hessian $H_{(\nu,0)}$ is non-degenerate whenever the choice of (g,ν) satisfies $Ker(\partial_\nu^g) = 0$. Moreover, for a generic choice of

the perturbation, we obtain $Ker(H_{(A,\psi)}) = 0$ in the direction orthogonal to the $U(1)$ orbit at all the irreducible solutions (A, ψ).
QED

Thus, we can construct an equivariant Floer homology of Y following the model of equivariant cohomology of a finite dimensional manifold using a Morse-Bott function [4]. A similar construction was obtained in the case of instanton homology [3].

First we consider gradient flow lines in $\mathcal{B}_b$. The important point is that at least one of the asymptotic values of the flow lines is always an irreducible point, and this implies that there are generically no reducible flow lines [21]. Thus, the moduli spaces of flow lines all have a free $U(1)$-action. This basically reduces the analysis of the spaces of flow lines to the non-equivariant case, and Froyshov's type of perturbation is sufficient to give the expected result.

Proposition 7.20 *Let O_a and O_b be two critical orbits of the functional (34), one of them possibly equal to the fixed point θ in case $b_1(Y) = 0$. Under a generic choice of a perturbation of the gradient flow equations, in the class $\mathcal{P}$ of Froyshov's perturbations, the space $\mathcal{M}(O_a, O_b)$ of trajectories with asymptotic values*

$$\lim_{t\to-\infty}(A(t), \psi(t)) \in O_a \quad \lim_{t\to+\infty}(A(t), \psi(t)) \in O_b$$

is a smooth manifold of dimension

$$\dim \mathcal{M}(O_a, O_b) = \mu(a) - \mu(b) + 2 - \dim G_a,$$

which is cut out transversely. Here $\mu(a) - \mu(b)$ is the relative index of the corresponding critical points in $\tilde{M}_c(Y, s)$ and G_a is the stabiliser of the orbit O_a.

We have endpoint maps

$$e_a^+ : \mathcal{M}(O_a, O_b) \to O_a$$

and

$$e_b^- : \mathcal{M}(O_a, O_b) \to O_b$$

that are fibrations with boundaries. In fact, there is a natural compactification of the space $\hat{\mathcal{M}}(O_a, O_b) = \mathcal{M}(O_a, O_b)/\mathbb{R}$ with boundary strata

$$\hat{\mathcal{M}}(O_a, O_{c_1}) \times_{O_{c_1}} \hat{\mathcal{M}}(O_{c_1}, O_{c_2}) \times_{O_{c_2}} \cdots \times_{O_{c_k}} \hat{\mathcal{M}}(O_{c_k}, O_b),$$

and the endpoint maps restrict well to the boundary.

Thus we can define the complex that computes the equivariant Floer homology.

Definition 7.21 *Fix arbitrarily the Morse index of one of the critical orbits. Consider the complex*

$$FC_{q,U(1)}(Y) = \bigoplus_{\mu(a)=i, i+j=q} \Lambda_{j,U(1)}(O_a),$$

where $(\Lambda_{*,U(1)}(O_a), \partial_{U(1)})$ *is the equivariant complex (6) (actually the dual complex with currents instead of forms). The differential is defined as*

$$D_{a,b}\eta = \begin{cases} \partial_{U(1)}\eta & O_a = O_b \\ (-1)^{r(\eta)}(e_b^-)_*(e_a^+)^*\eta & \mu(a) > \mu(b). \end{cases}$$

Here $r(\eta)$ *is the maximal degree of* η *as a de Rham form in* $\Lambda_{*,U(1)}(O_a)$. *The equivariant Floer homology is*

$$FH_{*,U(1)}(Y) = H_*(FC_{*,U(1)}(Y), D).$$

Again it is necessary to prove that the boundary operator satisfies

$$D^2_{a,c}\eta = \sum_b D_{a,b}D_{b,c}\eta \equiv 0.$$

This is a consequence of Stokes' theorem for a fibration with boundary,

$$\pi_*(d\eta) = d(\pi_*\eta) + (-1)^l(\pi_\partial)_*\eta,$$

where π is the projection, π_∂ is the restriction to the boundary, and l depends on the fibre dimension. This, applied to the endpoint fibrations of $\mathcal{M}(O_a, O_c)$, gives that $D^2_{a,c} \equiv 0$.

We have the following theorem [21].

Theorem 7.22 *The equivariant Floer homology is metric independent. Moreover in the case of a manifold with* $b_1(Y) > 0$ *it is isomorphic to the ordinary Floer homology, whereas in the case of an integral homology sphere there is an exact sequence*

$$\cdots \to H_{k,U(1)}(\omega) \to FH_{k,U(1)}(Y) \xrightarrow{i} FH_k(Y) \to \cdots \tag{47}$$

where ω *is the unique reducible solution and this has equivariant homology*

$$H_{*,U(1)}(\omega) = H_{*,U(1)}(BU(1)) = \mathbb{R}[u], \tag{48}$$

with u *a generator of degree two.*

Sketch of the Proof: The proof of metric independence is a delicate argument involving gluing theorems. The main idea is to mimic the proof of theorem 7.13. The technical difficulties arise when stating the analogue of proposition 7.14. In fact, consider the case where O_a is the reducible point θ_0 for the metric and perturbation (g_0, ν_0) and $O_{a'}$ is the reducible point θ_1 for the metric and perturbation (g_1, ν_1). If we have a spectral flow $SF(\partial^{g_t}_{\nu_t}) = -2$, then the virtual dimension of the moduli space $\mathcal{M}(O_a, O_{a'})$ is negative, but the space is not generically empty, since there is one reducible solution of the four-dimensional Seiberg–Witten equations with asymptotic values θ_0 and θ_1. If this singular boundary component actually arises in the gluing theorem, the proof of 7.13 would not go through to this case. A more careful analysis [21] shows that we still have the inclusions

$$\begin{aligned} \partial\mathcal{M}(O_a,\theta_1) \subset \quad & \hat{\mathcal{M}}(O_a,\theta_0) \times \mathcal{M}(\theta_0,\theta_1) \cup \\ & \bigcup_{b\neq\theta_0} \hat{\mathcal{M}}(O_a,O_b) \times \mathcal{M}(O_b,\theta_1) \cup \\ & \bigcup_{b'} -\mathcal{M}(O_a,O_{b'}) \times \hat{\mathcal{M}}(O_{b'},O_{a'}), \end{aligned}$$

and

$$\begin{aligned} \partial\mathcal{M}(\theta_0,O_{a'}) \subset \quad & \bigcup_{b'\neq\theta_1} -\mathcal{M}(\theta_0,O_{b'}) \times \hat{\mathcal{M}}(O_{b'},O_{a'}) \cup \\ & -\mathcal{M}(\theta_0,\theta_1) \times \hat{\mathcal{M}}(\theta_1,O_{a'}) \cup \\ & \bigcup_{b} \hat{\mathcal{M}}(\theta_0,O_b) \times \mathcal{M}(O_b,O_{a'}). \end{aligned}$$

However, when checking the reverse inclusion, one finds that there is a non-trivial obstruction that prevents from gluing the components

$$\hat{\mathcal{M}}(O_a,\theta_0) \times \mathcal{M}(\theta_0,\theta_1)$$

and

$$-\mathcal{M}(\theta_0,\theta_1) \times \hat{\mathcal{M}}(\theta_1,O_{a'})$$

to form solutions in $\mathcal{M}(O_a,\theta_1)$ and $\mathcal{M}(\theta_0,O_{a'})$ respectively. This means that the reverse inclusion only holds for the irreducible components.

The obstruction technique required to obtain this result is based on Taubes' obstruction bundle [36], [37]. The details of the argument can be found in [21].

QED

The reason why there is an isomorphism in the case with $b_1(Y) > 0$ is that we can perturb away all the reducible solutions, hence all the critical orbits in $M^b_c(Y,s)$ are copies of S^1 with a free $U(1)$-action. The equivariant complex associated to each orbit gives rise to the homology of the quotient, hence what is left is just the ordinary Floer homology of $\tilde{M}_c(Y,s)$. In the case of a homology sphere, using (48) the exact sequence can be rewritten in the form

$$0 \to FH_{2k+1,U(1)}(Y) \to FH_{2k+1}(Y) \to \mathbb{R}u^k \to$$

$$FH_{2k,U(1)}(Y) \to FH_{2k}(Y) \to 0.$$

This and the metric independence of the equivariant cohomology give a wall crossing formula [21] for the Euler characteristic $\chi(FH_*(Y))$ of a homology sphere Y, when the metric crosses a wall in such a way that the relative Morse index with respect to the reducible solution jumps by 2,

$$\chi(FH_*(Y, g_1)) = \chi(FH_*(Y, g_0)) + 1.$$

QED

Another approach to the construction of the equivariant Seiberg–Witten Floer homology is possible, modeled on singular homology [43]. The construction is very similar to the one derived in [21] and briefly described in this section. In this approach, the moduli spaces $\hat{\mathcal{M}}(O_a, O_b)$ define boundary maps that send singular chains from one critical orbit to another, instead of pulling back and pushing forward equivariant differential forms. The approach of [43] to proving the diffeomorphism invariance of this singular model is also different from [21] in as it introduces a perturbation of the Dirac equation that destroys the reducible component $\mathcal{M}(\theta_1, \theta_0)$, with negative spectral flow, instead of using Taubes' obstruction bundle technique. The authors then rederive theorem 7.22 by showing that the equivariant Floer homology we presented above is isomorphic to their singular model. The singular model has the advantage that it is defined with any coefficients, hence it can detect torsion information that is lost in the de Rham model.

7.8 Exact triangles

We have seen in the previous section the analogue, at the level of Floer homology, of the cobordism and wall crossing phenomenon we previously discussed for the Seiberg–Witten invariant. In this section we discuss the question of surgery formulae that generalises theorem 6.14 at the level of Floer homology. A similar question for instanton Floer homology was addressed in [6].

The reader should be warned that, at the time when this book is being written, not all the technical analysis necessary to establish the results about exact triangles in Seiberg–Witten Floer theory [7] has yet been worked out. Thus, the content of this section should be consider largely conjectural.

The setup is as follows. Consider a knot K in a homology sphere Y. If we perform a $+1$-surgery on K, we obtain another homology sphere Y_1. If we perform 0-surgery on K, we obtain a three-manifold Y_0 that has the homology of $S^1 \times S^2$. We choose metrics on Y and Y_1 as in theorem 6.14, with long cylinders $[-r, r] \times T^2$ and positive scalar curvature on a tubular neighbourhood of the knot.

As we discussed previously, we have a family s_k $k \in \mathbb{Z}$ of $Spin_c$ structures on Y_0. These satisfy the condition $\langle c_1(L_k), \sigma\rangle = 2k$, where σ is the generator of $H_2(Y_0, \mathbb{Z})$. Only finitely many of these s_k correspond to non-empty moduli spaces $M_c(Y_0, s_k)$. We have corresponding Floer groups $FH_*(Y_0, s_k, \mathbb{Z}_\ell)$ with $\ell = 2k$.

Remember also that, according to theorem 6.14, there exists a perturbation μ of the Seiberg–Witten equations on Y such that we obtain an identification of the moduli spaces

$$M_c(Y, \mu) \cong M_c(Y_1) \cup \bigcup_k M_c(Y_0, s_k). \tag{49}$$

Lemma 7.23 *Under the correspondence (49), the grading $\mu_Y(a) = \mu_Y(a) - \mu_Y(\theta)$ of elements in $M_c(Y)$ induces a choice of $\mathbb{Z}$-grading in the elements of the spaces $M_c(Y_0, s_k)$. This corresponds to a lifting of the $\mathbb{Z}_\ell$-grading of $FH_*(Y_0, s_k, \mathbb{Z}_\ell)$ to a $\mathbb{Z}$-grading. Moreover, the grading $\mu_Y(a) = \mu_Y(a) - \mu_Y(\theta)$ of elements in $M_c(Y)$ induces a grading on $M_c(Y_1)$. We denote by $FC_*(Y_1)$ the Floer complex of Y_1, with the grading of the elements $a \in M_c(Y_1)$ determined accordingly.*

This result is based on the analysis of the splitting of the spectral flow under the decomposition $Y = V \cup_{T^2} [-r, r] \times T^2 \cup_{T^2} \nu(K)$ as in the proof of theorem 6.14.

We denote the resulting $\mathbb{Z}$-graded lift of $FH_*(Y_0, s_k, \mathbb{Z}_{2k})$ as $FH_{(*)}(Y_0, s_k)$. Notice that these are not the same Floer groups as the previous $FH_*(Y_0, s_k)$: the latter sits in infinitely many degrees, whereas $FH_{(*)}(Y_0, s_k)$ has at most $\ell = 2k$ possibly non-trivial groups. The relation between

$$FH_{(*)}(Y_0, s_k) \quad \text{and} \quad FH_*(Y_0, s_k, \mathbb{Z}_{2k})$$

is expressed by

$$FH_{q(mod\ 2k)}(Y_0, s_k, \mathbb{Z}_{2k}) = \oplus_j FH_{(q+2jk)}(Y_0, s_k).$$

The analogue of the statement 6.14 at the level of Floer homology is the following.

Conjecture 7.24 *Let $FC_*(Y_1, g_1, \nu_1)$ and $FC_*(Y, g, \mu)$ be defined as above, with the grading as in lemma 7.23. Then there are chain maps*

$$I_* : FC_*(Y_1, g_1, \nu_1) \to FC_*(Y, g, \mu)$$

and

$$J_* : FC_*(Y, g, \mu) \to \oplus_k FC_{(*)}(Y_0, s_k),$$

such that the sequence

$$0 \to FC_*(Y_1, g_1, \nu_1) \stackrel{I_*}{\to} FC_*(Y, g, \mu) \stackrel{J_*}{\to} \bigoplus_k FC_{(*)}(Y_0, s_k) \to 0$$

induces a long exact sequence of the Floer homologies

$$\cdots \stackrel{\Delta_*}{\to} FH_*(Y_1, g_1, \nu_1) \stackrel{I_*}{\to} FH_*(Y, g, \mu) \stackrel{J_*}{\to}$$

$$\to \bigoplus_k FH_{(*)}(Y_0, s_k) \stackrel{\Delta_*}{\to} FH_{*-1}(Y_1, g_1, \nu_1) \to \cdots$$

The term $FH_{(*)}(Y_0, s_k)$ corresponding to s_0 with $c_1(L_0) = 0$ is the evaluation of $FH_{(*)}(Y, s_0, \mathbb{Z}[[t]])$ at $t = 0$.

As we pointed out, there are several subtle technical points in establishing this result. For instance, the analogue of the gluing theorem of 6.14 for the moduli spaces of flow lines requires a careful analysis of the four-dimensional Seiberg–Witten equations on manifolds of the form $(V \cup_{T^2} T^2 \times [0, \infty)) \times \mathbb{R}$. The analysis is complicated by the presence of two directions of non-compactness. Other complications arise, for instance the choice of a suitable perturbation for the flow lines equations that satisfies the properties of the class $\mathcal{P}$ and behaves well under surgery. The definition of the chain maps and the proof of exactness are also very subtle, since in the context of Seiberg-Witten theory we do not have the filtration induced by the Yang–Mills functional on the complex computing instanton homology [6]. It is clear that the decomposition (49) of the moduli spaces defines maps

$$0 \to FC_q(Y_1, g_1, \nu_1) \stackrel{i_q}{\to} FC_q(Y, g, \mu) \stackrel{\pi_q}{\to} \bigoplus_k FC_{(q)}(Y_0, s_k) \to 0.$$

These form an exact sequence at the level of groups, but they are not chain homomorphism. On the other hands there four manifolds W_1 and W_0 that give cobordisms with $\partial W_1 = Y_1 \cup Y$ and $\partial W_0 = Y \cup Y_0$. Upon suitably counting the contribution of different $Spin_c$ structures, it is possible to define chain maps

$$FC_*(Y_1, g_1, \nu_1) \stackrel{W_{1*}}{\to} FC_*(Y, g, \mu)$$

and

$$FC_*(Y, g, \mu) \stackrel{W_{0*}}{\to} \bigoplus_k FC_{(*)}(Y_0, s_k).$$

These count elements in the zero-dimensional components of the moduli spaces of solutions of the Seiberg–Witten equations on W_1 and W_0, with asymptotic values that define elements in $M_c(Y, \mu)$, $M_c(Y_1)$, and $M_c(Y_0, s_k)$, respectively. These maps, however, do not necessarily form an exact sequence of chain complexes. Thus, an interesting part of the argument is to show the interplay between these maps.

7.9 The relation to instanton Floer homology

The other natural question to ask is whether there is a relation between the Seiberg–Witten and instanton Floer homology that generalises the relations between the invariant $\chi(Y, s)$ and the classical Casson invariant and Milnor torsion.

We should recall that the instanton Floer homology was originally defined only for integral homology spheres [13], but was later extended to the case of circle bundles over a Riemann surface [38]. Since the Seiberg–Witten Floer homology is defined for all three-manifolds, one expects to find a relation in all cases where the instanton Floer homology is defined.

At the time when this book is being written, there is still no clear picture of this relation. Kronheimer and Mrowka have outlined a program [16] that relates exact triangles, the results on contact structures and Seiberg–Witten invariants, and the study of the relation between the two Floer homologies in the case of manifolds Y_0 of the homology type of $S^1 \times S^2$, in order to prove property P for knots. In this case, their program indicates that, if the Seiberg–Witten Floer homology is non-vanishing, then the instanton Floer homology is also non-vanishing, that is, the Seiberg–Witten Floer homology captures the information contained in a subcomplex of the instanton Floer homology.

In another direction, a relation can be seen in an indirect way when analysing the case where Y is the mapping cylinder on a Riemann surface. The Seiberg–Witten Floer homology in this case [9] is expected to be isomorphic to the quantum cohomology of a symmetric product $s^r(\Sigma)$ of the Riemann surface, where the power r is related to the choice of the $Spin_c$-structure on Y. In the part III of this book the reader can find a brief summary of the definition and a few properties of quantum cohomology. An explicit computation of the quantum cohomology of $s^r(\Sigma)$ has been worked out in [5].

Previous work of Dostoglou and Salamon [12] has described the instanton Floer homology of Y in terms of the quantum cohomology of the moduli space of flat connections on Σ. It is also known that the moduli space of flat connections on Σ and the symmetric products $s^r(\Sigma)$ are related by a "cobordism" construction [39]: this construction in fact provided the model for the expected relation between the Seiberg–Witten and Donaldson invariants of four-manifolds that will be discussed in the last part of this book.

This seems to lead to an indirect way of comparing the Seiberg–Witten and instanton Floer homologies of Y, in this particular case.

In the case of homology spheres Y, one can try to identify a metric dependent correction term, using the exact sequence and maps that connect $HF_*(Y)$ to the equivariant $HF_{*,U(1)}(Y)$. However, at the moment there is no precise understanding of the relation between this and the instanton Floer homology.

7.10 Summary

We can summarise the properties of the Seiberg–Witten invariants of three-manifolds in the following table.

3-manifold	$\chi(Y, s)$	$HF_*(Y, s)$ and $HF_{*,U(1)}(Y, s)$
$b_1(Y) > 1$ $c_1(L) \neq 0$	indep. of (g, ρ)	indep. of (g, ρ), $[*\rho] = 0$ $HF_*(Y, s) = HF_{*,U(1)}(Y, s)$ $\mathbb{Z}_\ell$-graded or $\mathbb{Z}$-graded
$b_1(Y) = 1$ $c_1(L) \neq 0$	wall crossing at $[*\rho] = \pi c_1(L)$	indep. of (g, ρ), $[*\rho] = 0$, $HF_*(Y, s) = HF_{*,U(1)}(Y, s)$ $\mathbb{Z}_\ell$-graded or $\mathbb{Z}$-graded
$b_1(Y) > 0$ $c_1(L) = 0$	indep. of (g, ρ) fixed $[*\rho] \neq 0$	(?) Novikov type $HF_*(Y, s, \mathbb{Z}[[t]])$
$b_1(Y) = 0$	wall crossing at $Ker(\partial^g_\nu) \neq 0$, $\rho = *d\nu$;	$HF_*(Y, s)$ dep. on (g, ν) $HF_{*,U(1)}(Y, s)$ indep. of (g, ν) exact sequence relates $HF_*(Y, s)$ and $HF_{*,U(1)}(Y, s)$

7.11 Exercises

Advice to the reader: some of the exercises in this section should be looked at after reading part III of the book on the topological and geometric applications of Seiberg–Witten theory. They have been reported here, since they refer closely to the three-dimensional theory.

- Denote by $\mathcal{P}(X)$ the space of paths on a manifold X. Let $\mathcal{A}_Y$ be the configuration space of $U(1)$-connections and spinors on the three-manifold Y, and $\mathcal{G}_Y = Maps(Y, U(1))$ What does the discussion about gauge actions on the spaces of flow lines say about the difference between the spaces $\mathcal{P}(\mathcal{A}_Y)/\mathcal{P}(\mathcal{G}_Y)$ and $\mathcal{P}(\mathcal{A}_Y/\mathcal{G}_Y)$?

- Complete the proof of point (3) of the Thom conjecture (see the following chapters): consider solutions (A_R, ψ_R) on the cylinder $Y \times [-R, R]$; show that the change in the functional

  $$C(A_R(R), \psi_R(R)) - C(A_R(-R), \psi_R(-R))$$

  is negative and uniformly bounded, independently of R. For the latter property consider gauge transformations such that $A_R - \lambda_R^{-1} d\lambda_R$ and the first derivatives are uniformly bounded. Show that the functional C changes monotonically along the cylinder, and deduce that for all N there is a solution on $Y \times [0, 1]$ for which the change of C is bounded by $1/N$. Complete the argument by showing that there is a translation invariant solution.

- How does the "stretching the neck" argument for the connected sum theorem work? Consider solutions on cylinders $S^3 \times [-R, R]$.

- Let Y_1 be the Poincaré homology sphere. This is obtained from S^3 by $+1$ surgery on the trefoil knot K, a genus 1 knot. It is also a quotient of S^3 by a free action of a finite group of isometries, so it carries a metric of positive scalar curvature. Thus, it has trivial Seiberg–Witten Floer homology. By the exact triangle, the Seiberg–Witten Floer homology for Y_0, obtained by zero surgery on K, should also be vanishing. However, Gabai's foliation is smooth for all genus one knots with fewer than 11 crossings [15], hence, by the results of Kronheimer and Mrowka about taut foliations, contact structures, and Seiberg–Witten invariants [16], the Seiberg–Witten Floer homology of Y_0 should be non-vanishing (see the following chapters). Explain how this example fits into the picture of conjecture 7.24.

References

[1] S. Akbulut, J.D. McCarthy, *Casson's invariant for oriented homology 3-spheres. An exposition*, Princeton University Press, 1990.

[2] M.F. Atiyah, V.K. Patodi, I.M. Singer, *Spectral asymmetry and Riemannian geometry*, I,II,III. Math. Proc. Cambridge Phil. Soc. 77 (1975) 43-69. 78 (1975) 405-432. 79 (1976) 71-99.

[3] D.M. Austin, P.J. Braam, *Equivariant Floer theory and gluing Donaldson polynomials*, Topology 35, N.1 (1996) 167-200.

[4] D.M. Austin, P.J. Braam, *Morse-Bott theory and equivariant cohomology*, in The Floer Memorial Volume, Progress in Mathematics, Birkhäuser 1995, 123-183.

[5] A. Bertram, M. Thaddeus, *On the quantum cohomology of a symmetric product of an algebraic curve*, preprint math.AG/9803026.

[6] P.J. Braam, S.K. Donaldson, *Floer's work on instanton homology, knots and surgery*, Floer Memorial Volume, Progress in Mathematics, Vol. 133. Birkhäuser 1995.

[7] A.L. Carey, M. Marcolli, B.L. Wang, *Exact triangles in Seiberg–Witten Floer theory*, preprint.

[8] A.L. Carey, B.L. Wang, *Seiberg–Witten-Floer homology and Gluing formulae*, preprint.

[9] A.L. Carey, B.L. Wang, *Seiberg–Witten Floer theory and holomorphic curves*, preprint.

[10] W. Chen, *Casson's invariant and Seiberg–Witten gauge theory*, Turkish J. Math. 21 (1997), no. 1, 61–81.

[11] W. Chen, *Dehn surgery formula for Seiberg–Witten invariants of homology 3-spheres*, preprint, dg-ga/9703009.

[12] S. Dostoglou, D. Salamon, *Self-dual instantons and holomorphic curves*, Ann. of Math. (2) 139 (1994) N.3, 581-640.

[13] A. Floer, *An Instanton-Invariant for 3-Manifolds*, Commun. Math. Phys. 118 (1988), 215-240.

[14] K.A. Froyshov, *The Seiberg–Witten equations and four-manifolds with boundary*, Math. Res. Lett. 3 (1996), no. 3, 373–390.

[15] D. Gabai, *Foliations and genera of links*, Topology 23 (1984) N.4 381-394.

[16] P.B. Kronheimer, *Embedded surfaces and gauge theory in three and four dimensions*, preprint.

[17] Ch. Lescop, *Global surgery formula for the Casson-Walker invariant*, Princeton 1996.

[18] Y. Lim, *The equivalence of Seiberg–Witten and Casson invariants for homology 3-spheres*, preprint.

[19] Y. Lim, *Seiberg–Witten invariants for 3-manifolds in the case* $b_1 = 0$ *or* 1, preprint.

[20] M. Marcolli, *Seiberg–Witten-Floer homology and Heegaard splittings*, Internat. J. Math. 7 (1996), no. 5, 671–696.

[21] M. Marcolli, B.L. Wang, *Equivariant Seiberg–Witten Floer homology*, preprint.

[22] G. Meng, C.H. Taubes, *SW=Milnor torsion*, Math. Res. Lett. 3 (1996) N.5, 661-674.

[23] J.W. Morgan, T. Mrowka, D. Ruberman, *The* L^2*-moduli space and a vanishing theorem for Donaldson polynomial invariants*, International Press, 1994.

[24] J.W. Morgan, Z. Szabó, C.H. Taubes, *A product formula for the Seiberg–Witten invariants and the generalized Thom conjecture*, J. Differential Geom. 44 (1996), no. 4, 706–788.

[25] T.S. Mrowka, P. Ozsváth, B. Yu, *Seiberg–Witten monopoles on Seifert fibered spaces.* Comm. Anal. Geom. 5 (1997), no. 4, 685–791.

[26] W. Neumann, F. Raymond, *Seifert manifolds, plumbing,* μ*-invariant, and orientation reversing maps*, in *Algebraic and Geometric Topology*, Lecture Notes in Math. 664, Springer 1978, 162-194.

[27] W. Neumann, J. Wahl, *Casson invariant of links of singularities*, Comment. Math. Helvetici 65 (1990) 58-78.

[28] L. I. Nicolaescu, *Lattice points, Dedekind-Rademacher sums and a conjecture of Kronheimer and Mrowka*, preprint, math.DG/9801030.

[29] L. I. Nicolaescu, *Eta invariants of Dirac operators on Circle bundles over Riemann surfaces and virtual dimension of finite energy Seiberg–Witten moduli spaces*, math.DG/9805046.

[30] Ch. Okonek, A. Teleman, *3-dimensional Seiberg–Witten invariants and non-Kählerian geometry*, preprint.

[31] V.V. Prasolov, A.B. Sossinsky, *Knots, Links, Braids, and 3-manifolds*, AMS 1997.

[32] M. Schwarz, *Morse Homology*, Birkhäuser 1993.

[33] P. Scott, The geometries of 3-manifolds, *Bull. Lond. Math. Soc.* **15** (1983) 401-487.

[34] L. Simon, *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Ann. of Math. 118 (1983) 525-571.

[35] C.H. Taubes, *Casson's Invariant and Gauge Theory*, J. Diff. Geom. 31 (1990) 547-599.

[36] C.H. Taubes, *The stable topology of self-dual moduli spaces*, J.Diff.Geom. 29 (1989) 163-230.

[37] C.H. Taubes, *Self-dual connections on 4-manifolds with indefinite intersection matrix*, J.Diff.Geom. 19 (1984) 517-560.

[38] C.H. Taubes, unpublished manuscript.

[39] M. Thaddeus, *Stable pairs, linear systems and the Verlinde formula*, Invent. Math. 117 (1994) N.2, 317-353.

[40] V.G. Turaev, *Reidemeister torsion in knot theory*, Uspekhi Mat. Nauk, 41 (1986) N.1 (247) 97-147.

[41] K. Walker, *An extension of Casson's invariant* Princeton Univ. Press, 1992.

[42] B.L. Wang, *Seiberg–Witten Floer theory for homology 3-spheres*, preprint dg-ga/9602003.

[43] G. Wang, R. Ye, *Equivariant and Bott-type Seiberg–Witten Floer homology: Part I* and *Part II*, preprint.

[44] R.G. Wang, *On Seiberg–Witten Floer invariants and the generalized Thom problem*, preprint.

Part III
Topology and Geometry

Now, here in this fort of brahman there is a small lotus, a dwelling-place, and within it, a small space. In that space there is something –and that's what you should try to discover, that's what you should seek to perceive.

Chāndogya Upaniṣad, 4.8.1

8 Computing Seiberg–Witten invariants

We provide here some concrete examples in which it is possible to compute the Seiberg–Witten invariants for some classes of four–manifolds.

8.1 Connected Sum theorem

As in the case of Donaldson invariants, it is possible to prove that Seiberg–Witten invariants vanish on connected sums. Thus the invariants provide a test of "irreducibility" of the manifold.

Theorem 8.1 *If a manifold X splits as a connected sum $X = X_1 \# X_2$, where both pieces are manifolds with boundary with $b_2^+(X_i) \geq 1$, then the Seiberg–Witten invariants vanish for all choices of the $Spin_c$-structure.*

Sketch of the Proof: Following Donaldson [4], consider metrics that have a neck near the S^3 where the connected sum is performed. If the metrics shrink the radius of the neck to zero, then a family of solutions (A, ψ) of the Seiberg–Witten equations for a given $Spin_c$ structure $s \in \mathcal{S}(X)$ converges smoothly on compact sets, away from the neck, to a pair of solutions (A_1, ψ_1) and (A_2, ψ_2) on X_1 and X_2, respectively. It is possible to show [4] that the index of the linearisation of the equations splits as

$$Ind(T_{A,\psi}) = Ind(T_{A_1,\psi_1}) + Ind(T_{A_2,\psi_2}) + 1,$$

where the last summand is due to the presence of a $U(1)$ action (gluing parameter). If the linearisation of the equations at the solution (A, ψ) has index zero, then one of the two linearisations on X_1 and X_2 has negative index, which implies that there is no irreducible solution.

8.2 The blowup formula

Another useful tool in computing Seiberg–Witten invariants of four-manifolds is the blowup formula [14].

Theorem 8.2 *Let X be a compact oriented smooth 4-manifold with $b_2^+ > 1$. If a $Spin_c$ structure $s \in \mathcal{S}$ satisfies*

$$\dim M_s(X) = r(r+1),$$

for a non-negative integer r, then we have

$$N_s(X) = N_{\tilde{s}}(X \# \overline{\mathbb{C}P^2}),$$

where $\tilde{s}$ and s are related by

$$c_1(\tilde{L}) = c_1(L) \pm (2r+1)E,$$

and E is the class of the exceptional divisor.

Sketch of the Proof: Following [14], we begin by observing that $\overline{\mathbb{C}P^2}$ has a metric of positive scalar curvature, hence it only admits reducible solutions to the Seiberg–Witten equations. Thus, one can try to form solutions on the blowup $X\# \overline{\mathbb{C}P^2}$ by gluing solutions on X with reducible solutions on $\overline{\mathbb{C}P^2}$. If $\dim M_s(X) = 0$ it is possible to show [14] that in fact we obtain

$$M_{\tilde{s}}(X\# \overline{\mathbb{C}P^2}) = (M^b_s(X) \times M^b(\overline{\mathbb{C}P^2}))/U(1),$$

where $c_1(\tilde{L}) = c_1(L) \pm E$.

In the case where $\dim M_s(X) > 0$, the situation is complicated by the fact that not all the approximate solutions of the form

$$(A, \psi)\#(A_0, 0),$$

with (A, ψ) a solutions on X and $(A_0, 0)$ a solution on $\overline{\mathbb{C}P^2}$, give rise to a solution on the blowup. In other words, we are in the presence of an obstruction. Taubes' obstruction bundle technique can be employed to show that there is a canonical bundle over the set of approximate solutions, with a section σ, such that

$$\sigma^{-1}(0) = M_{\tilde{s}}(X\# \overline{\mathbb{C}P^2}).$$

This means that the section σ precisely measures the obstruction to gluing approximate solutions to actual solutions.

It is shown in [14] that in this case the obstruction bundle is the restriction to the set of approximate solutions of the $\frac{r(r+1)}{2}$-th power of the line bundle $\mathcal{L}$ over the configuration space $\hat{\mathcal{B}} = \hat{\mathcal{A}}/\mathcal{G}$.

Thus, we obtain

$$\langle c_1(\mathcal{L})^{\dim M_{\tilde{s}}(X\# \overline{\mathbb{C}P^2})}, [M_{\tilde{s}}(X\# \overline{\mathbb{C}P^2})]\rangle =$$

$$\langle c_1(\mathcal{L})^{(\dim M_s(X) - r(r+1))/2}, [\sigma^{-1}(0)]\rangle = \langle c_1(\mathcal{L})^{\frac{\dim M_s(X)}{2}}, [M_s(X)]\rangle.$$

QED

8.3 Kähler Surfaces

Kähler surfaces provide a class of manifolds with non-trivial Seiberg–Witten invariants. Computations of Seiberg–Witten invariants in the Kähler case were initially carried out in [54]. A detailed exposition of the results for Kähler manifolds is now available in [37]. See also [2] and [17], where the argument is further developed with algebro–geometric techniques.

The first fact to mention is that a metric which is Kähler is non–generic, hence the computations that we have seen in the generic case do not hold here; in particular, the dimension of the moduli space can be larger than the virtual dimension prescribed by the index theorem 4.8, see proposition 8.5.

We describe here the computation of the invariants under the assumption that the $Spin_c$ structure is chosen in such a way that $c_1(L)^2 = 2\chi + 3\sigma$, i.e. that the virtual dimension is zero. However, we shall see that the moduli space of solutions of the unperturbed equations is not just a finite set of points.

Basic to the analysis of the Seiberg–Witten equations on a Kähler manifold is the following fact. As we recalled in our preliminary discussion of complex manifolds, there is a canonical $Spin_c$-structure s_0 on a Kähler manifold X, namely the spinor bundle S given by

$$S = \Lambda^{(0,*)}(X).$$

The Clifford multiplication on S is defined by

$$(\alpha^{(0,1)} + \alpha^{(1,0)}) \bullet \beta = \sqrt{2}(\alpha^{(0,1)} \wedge \beta - \alpha^{(1,0)} \rfloor \beta,$$

with $\alpha^{(1,0)} \rfloor \beta = \sum_k {\alpha^{(1,0)}}^k \beta_k$, where the covariant index k of the 1-form β is lowered by the metric tensor.

In this case the spinor bundle $S = \Lambda^{(0,*)}(X)$ splits as

$$S^+ = \Lambda^{(0,0)} \oplus \Lambda^{(0,2)} = 1 \oplus K^{-1},$$

$$S^- = \Lambda^{(0,1)}.$$

Note that $L = det\ S^+ = K^{-1}$. Any other $Spin_c$ structure is obtained by tensoring with a line bundle H, so that

$$S^+ = (\Lambda^{(0,0)} \oplus \Lambda^{(0,2)}) \otimes H$$

and

$$S^- = \Lambda^{(0,1)} \otimes H.$$

This $Spin_c$-structure has determinant $L = K^{-1} \otimes H^2$.

As we observed previously, this fact holds true for almost–complex structures as well: it is useful in the case of symplectic manifolds discussed in the next section.

In particular, the self dual part of the curvature decomposes as $F_A^+ = if\omega + \eta - \bar{\eta}$, with f a real valued function and $\eta \in \Lambda^{(0,2)}$. Thus, we can rewrite the equations according to this splitting:

$$\begin{aligned} \bar{\partial}_A \alpha - i\bar{\partial}_A^* \bar{\beta} &= 0, \\ {F^{(1,1)}}^+ &= i\tfrac{\omega}{2}(\mid \alpha \mid^2 - \mid \beta \mid^2), \\ F^{(0,2)} &= -i\bar{\alpha}\bar{\beta}, \end{aligned} \tag{50}$$

where $\psi = (\alpha, -i\bar{\beta})$, with $\alpha \in \Gamma(X, H)$ and $\beta \in \Gamma(X, \bar{H} \otimes K)$. We used the fact that, according to remark 2.14, ${F^{(0,2)}}^+ = F^{(0,2)}$. The Dirac operator has the form $D\psi = \sqrt{2}(\bar{\partial}\alpha - i\bar{\partial}^* \bar{\beta})$; full details can be found in [2] or [37].

The Seiberg–Witten functional (11) can also be rewritten as

$$S(A,\alpha,\beta) = \int_X (\mid F_A^+ \mid^2 + < \nabla_A \bar{\alpha}, \nabla_A \alpha > + < \nabla_A \bar{\beta}, \nabla_A \beta > \\ + \tfrac{1}{8}(\mid \alpha \mid^2 + \mid \beta \mid^2)^2 + \tfrac{\kappa}{4}(\mid \alpha \mid^2 + \mid \beta \mid^2)) dv. \tag{51}$$

Note that the functional (51) is invariant under the transformation given by

$$A \mapsto A, \quad \alpha \mapsto \alpha, \quad \beta \mapsto -\beta;$$

hence this transformation has to map solutions into other solutions. However, this change of variables into the first and third equations of (50) gives

$$F^{(0,2)} = 0, \quad \bar{\alpha}\bar{\beta} = 0. \tag{52}$$

Since $(\alpha, -i\bar{\beta})$ is in the kernel of the Dirac operator, it has some regularity properties (analytic continuation): in particular if α or β vanishes on an open set, then it vanishes identically. Hence half of the condition above reads $\alpha \equiv 0$ or $\beta \equiv 0$.

An equivalent way to obtain (52) is by applying $\bar{\partial}_A$ to the first equation of (50). Using the second and third equation in (50) we can write

$$\begin{aligned} \bar{\partial}_A^2 \alpha - i\bar{\partial}_A \bar{\partial}_A^* \bar{\beta} = 0 \\ = F^{(0,2)}\alpha - i\bar{\partial}_A \bar{\partial}_A^* \bar{\beta} \\ = -i|\alpha|^2 \bar{\beta} - i\bar{\partial}_A \bar{\partial}_A^* \bar{\beta}, \end{aligned}$$

hence the condition

$$\int_X \mid \alpha \mid^2 \mid \beta \mid^2 dv + \int_X \mid \bar{\partial}_A \bar{\beta} \mid^2 dv = 0, \tag{53}$$

where we use $*\beta = \beta$. This implies the condition (52). This point of view will be useful to compare the result of Kähler manifolds with the one for symplectic manifolds.

Lemma 8.3 *In the Seiberg–Witten equations (50), we have $\alpha \equiv 0$ or $\beta \equiv 0$, according to whether*

$$0 \leq \int_X \omega \wedge c_1(L)$$

or

$$0 \geq \int_X \omega \wedge c_1(L).$$

Proof: Because of the decomposition of remark 2.14, ω is orthogonal to the $(0,2)$ and to the $(2,0)$ components of F_A. Moreover, $c_1(L) = \frac{1}{4\pi} F_A$. Hence

$$\int_X \omega \wedge c_1(L) = -\frac{1}{8\pi} \int_X \omega \wedge \omega(| \alpha |^2 - | \beta |^2).$$

QED

Lemma 8.4 *The condition $F^{(0,2)} = 0$ implies that the connection A induces a holomorphic structure on H.*

Proof: According to remark 2.14, in holomorphic local coordinates on X we have a basis of self dual forms given by the Kähler form

$$\omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2),$$

and the forms

$$dz_1 \wedge dz_2, \quad d\bar{z}_1 \wedge d\bar{z}_2.$$

Hence $F^{(2,0)^+} = F^{(2,0)}$ and $F^{(0,2)^+} = F^{(0,2)}$. Thus $\bar{\partial}_A$ gives H a holomorphic structure.
QED

Suppose that $\beta \equiv 0$. Because of lemma 8.4, the equation (50) becomes

$$\bar{\partial}_A \alpha = 0,$$

i.e. α is a holomorphic section of H.

The original computation [54] of the moduli space of solutions from these data is based on adapting to the infinite dimensional quotient $\hat{\mathcal{A}}/\mathcal{G}$ some techniques of symplectic geometry. First there is an argument which shows that we need to consider only irreducible solutions $\psi \neq 0$, see e.g. [17]. Then we can proceed as follows.

A symplectic structure is defined on the space $\hat{\mathcal{A}}$ by:

$$\Omega(v_A, w_A) = \int_X \omega \wedge v_A \wedge w_A$$

and

$$\Omega(v_\alpha, w_\alpha) = \int_X \omega \wedge \omega(\bar{v}_\alpha w_\alpha - \bar{w}_\alpha v_\alpha).$$

Here v_A and w_A are 1–forms in the tangent space to the connection A, and v_α and w_α are in the tangent space to the section α.

The gauge group $\mathcal{G}$ acts symplectically on $\hat{\mathcal{A}}$. The moduli space M is the quotient of the fibre over zero of the moment map by this action,

$$M = \mu^{-1}(0)/\mathcal{G}.$$

The standard arguments connecting symplectic and geometric invariant theory quotients yield the following.

Proposition 8.5 *The moduli space of solutions of the Seiberg–Witten equations on a Kähler manifold is given by the set of possible choices of a holomorphic structure on $K^{1/2} \otimes L$ and the corresponding projectivisation of the space of holomorphic sections,*

$$\mathbf{P}H^0(X, K^{1/2} \otimes L).$$

As a consequence one gets a result on the Seiberg–Witten invariants.

Theorem 8.6 *If X is Kähler and $K \cdot \omega > 0$, where K is the class of the canonical line bundle, which determines the canonical $Spin_c$-structure s_0. Then the corresponding Seiberg–Witten invariant is*

$$N_{s_0} = \pm 1.$$

Sketch of the Proof: The first part of the statement can be proved by showing that the relevant projective space consists uniquely of the class of the constant section on a trivial line bundle. It is also necessary to show that this single point is a "smooth point", i.e. that the moduli space is cut out transversely. This is proved [37] by showing that the linearisation

$$Ds : \begin{array}{c} \Lambda^1(X; i\mathbb{R}) \\ \oplus \\ \Lambda^0(X;\mathbb{C}) \\ \oplus \\ \Lambda^{(0,2)}(X;\mathbb{C}) \end{array} \longrightarrow \begin{array}{c} \Lambda^0(X; i\mathbb{R}) \\ \oplus \\ \Lambda^{2+}(X;\mathbb{C}) \\ \oplus \\ \Lambda^{(0,1)}(X;\mathbb{C}) \end{array}$$

is surjective. This is obtained by considering this as a short bigraded complex and showing that the induced map

$$\begin{array}{c} H^1(X; i\mathbb{R}) \\ \oplus \\ H^0(X;\mathbb{C}) \\ \oplus \\ H^{(0,2)}(X;\mathbb{C}) \end{array} \longrightarrow \begin{array}{c} H^0(X; i\mathbb{R}) \\ \oplus \\ H^{2+}(X;\mathbb{C}) \\ \oplus \\ H^{(0,1)}(X;\mathbb{C}) \end{array}$$

is an isomorphism.

Notice how the theorem 8.6 only gives us a way of computing the invariant for the canonical class K. We shall discuss later how the Seiberg–Witten invariant can be defined for other classes for which the moduli space is of virtual dimension zero but of actual positive dimension, using the localised Euler class [2]. We shall discuss the localised (or regularised) Euler class at length in the chapter on the quantum field theory formalism.

8.4 Symplectic Manifolds

The same technique used above in attacking the problem of Kähler manifolds can be partially extended to symplectic manifolds. In fact, Taubes proved an analogue [46] of theorem 8.6. The result is in some sense surprising. In fact, the argument we described in the previous case of Kähler manifolds, seems to break down because of the presence of terms coming from the Nijenhuis tensor, for instance, as we are going to discuss in a moment, the relation (53) will be replace by the weaker (54) below. An excellent reference for Seiberg–Witten theory on symplectic manifolds is the survey paper of D. Kotschick [24].

On a symplectic manifold, a canonical class is defined as in the case of Kähler manifolds. The class is independent of the choice of an almost complex structure, since the family of compatible almost complex structures is a contractible set.

Theorem 8.7 *If X has a symplectic structure ω compatible with the orientation, and it has $b_2^+ > 1$, then the Seiberg–Witten invariant corresponding to the canonical $Spin_c$–structure s_0 is*

$$N_{s_0}(X) = \pm 1.$$

Moreover, if $N_s(X) \neq 0$, then necessarily $0 \leq c_1(L)[\omega] \leq K\omega$ is satisfied.

The proof of this theorem appeared in [46]. It has subsequently been simplified [48]. A brief sketch of the argument follows.

Sketch of the Proof: Consider first the case of the canonical $Spin_c$ structure s_0 with $L = K^{-1} = \Lambda^{0,2}$.

The main idea is to show that the Seiberg–Witten invariant can be computed by means of a one parameter family of perturbed equations.

These are of the form

$$D_A \psi = 0$$

and

$$F_A^+ = F_{A_0}^+ + \frac{1}{4} < e_i e_j \psi, \psi > e^i \wedge e^j - \frac{ir}{4}\omega.$$

One particular solution (A_0, u_0), which corresponds to the value $r = 0$ of the parameter, is constructed by taking A_0 to be a connection on K^{-1} and the section u_0 to be a covariantly constant norm 1 section of the trivial summand in $S^+ = 1 \oplus K^{-1}$. In the condition $\nabla_{A_0} u_0 = 0$ we are considering the induced covariant derivative $\tilde{\nabla}_{A_0}$ on $S^+ = 1 \oplus K^{-1}$, obtained from the exterior derivative d, and we set $\nabla_{A_0} = \frac{1}{2}(1 + i\omega)\tilde{\nabla}_{A_0}$.

In the perturbed equation above the section ψ is written as

$$\psi = r^{1/2}(\alpha u_0 + \beta),$$

with α a function and β a section of K^{-1}.

By successively projecting the second equation on the trivial summand and on K^{-1} a uniform bound (independent of r) is obtained on the expression

$$\int_X (\frac{4}{r} \mid F_a^+ \mid^2 + \mid \nabla_a \alpha \mid^2 - \frac{r}{8}(1- \mid \alpha \mid^2)^2) dv,$$

where $F_a = \frac{1}{2}(F_A - F_{A_0})$, and $\nabla_a = d + ia$ acting on complex valued functions. Thus, when considering a sequence of solutions (A_m, ψ_m) corresponding to a sequence of parameters $r_m \to \infty$, the above says that $\mid \alpha_m \mid \to 1$ and $\mid \nabla_a \alpha_m \mid \to 0$. Finally it is shown that the β_m vanish identically for large enough m; and consequently α_m is covariantly constant of norm 1.

The upshot is that (A_m, ψ_m), for large enough m, is gauge equivalent to (A_0, u_0); and therefore there is just one point in the moduli space, and the Seiberg–Witten invariant is ± 1.

In order to get the second part of the statement, it is useful to reformulate the previous argument as follows. There exists a solution (A_0, u_0) such that we have the Dirac operator of the form $D = \partial_a + \bar{\partial}_a^*$, with a as above, satisfying $F_a = \frac{1}{2}(F_A - F_{A_0})$. If one wants to derive an analogue of the expression (53), by the same procedure used in the Kähler case, there is an extra term which depends precisely on the non-integrability of J,

$$\int_X \mid \alpha \mid^2 \mid \beta \mid^2 dv + \int_X \mid \bar{\partial}_A \bar{\beta} \mid^2 dv = \frac{1}{4} \int_X < N_J(\partial_a \alpha), \beta > dv. \qquad (54)$$

For the equations perturbed with a large parameter r, this leads to an estimate

$$\int_X |d_a \alpha|^2 + \frac{1}{4}|\alpha|^2|\beta|^2 + \frac{1}{4}(|\alpha|^2 - r)^2 + \frac{r}{4}(|\alpha|^2 - r) = \int_X 2\langle \beta, N_J(\partial_a \alpha)\rangle$$

and subsequently to

$$-2\pi r c_1(L)[\omega] + \int_X (\frac{1}{2}|d_a \alpha|^2 + \frac{1}{4}|\alpha|^2|\beta|^2 + \frac{1}{4}(|\alpha|^2 - r)^2 + \frac{1}{4}|\beta|^2) \leq C \int_X |\beta|^2. \quad (55)$$

Thus, if $c_1(L) \cdot \omega < 0$ or $c_1(L) \cdot \omega > c_1(K) \cdot \omega$, one gets $N_s \equiv 0$, for s with $\det S^+ = L$. Moreover, only if $c_1(L) \cdot \omega \leq c_1(K) \cdot \omega$ one can have $N_s \neq 0$. If $c_1(L) \cdot \omega = 0$ all terms vanish: L must be trivial with a the trivial connection, $\beta \equiv 0$ and $|\alpha|^2 = r$. There is only one such pair (a, α) up to gauge. One can then show that the unique solution is a regular point in the moduli space as in the case of Kähler manifolds.
QED

As a consequence of Taubes' result that symplectic manifolds admit some non–trivial Seiberg–Witten invariants, it is natural to ask whether a converse might hold as well, i.e. whether any four-manifold with non-trivial invariants might be symplectic. This problem has been considered in [26], and solved negatively. There are 4-manifolds with non–trivial Seiberg–Witten invariants but without symplectic structure.

The strategy used to construct such examples is to prove two lemmata:

Lemma 8.8 *If X is symplectic and it decomposes as a smooth connected sum $X = X_1 \# X_2$, then one of the summands, say X_1, has negative definite intersection form (because of the connected sum theorem) and fundamental group $\pi_1(X_1)$ which does not admit any non-trivial finite quotient.*

Lemma 8.9 *If X_1 has some non-trivial Seiberg–Witten invariant and X_2 has $b_1(X_2) = b_2^+(X_2) = 0$, then the connected sum $X = X_1 \# X_2$ also has some non-trivial Seiberg–Witten invariant.*

The first lemma depends on the connected sum theorem 8.1, and on the behaviour of b_2^+ on finite covers; the second depends on an explicit construction of a $Spin_c$ structure on X, given the structure on X_1 with non-trivial invariants and a unique $Spin_c$ structure on X_2. A "stretching the neck" argument is required to glue solutions corresponding to the two separate structures, when performing the connected sum, [53]. lemma 8.8 is further refined in [25], where it is also proven that, under the same hypotheses, X_1 is an integral homology sphere. Manifolds with non-trivial invariants and with no symplectic structures are therefore given by the following class of examples.

Example 8.10 *Let X_1 be a symplectic manifold with $b_2^+(X_1) > 1$. By theorem 8.7, X_1 has a non-trivial invariant. Let X_2 be a manifold with $b_1(X_2) = b_2^+(X_2) = 0$, and with fundamental group that admits non-trivial finite quotients. Then $X = X_1 \# X_2$ has a non-trivial invariant, but it does not admit a symplectic structure.*

In [25] another interesting problem is considered. After the results of [46], [47], it was conjectured that the following decomposition result might hold:

Conjecture 8.11 *Every smooth, compact, oriented 4–manifold without boundary is a connected sum of symplectic manifolds, with either the symplectic or the opposite orientation, and of manifolds with definite intersection form.*

In [25] it is proven that this conjecture is false. The construction is similar to the arguments used in the two lemmata 8.8 and 8.9. The conjecture was then reformulated for simply connected manifolds.

Conjecture 8.12 *Every smooth, compact, oriented, simply connected four–manifold without boundary is a connected sum of symplectic manifolds, with both orientations allowed.*

Notice how this conjecture for simply connected manifolds resembles an older conjecture that proposed complex surfaces as the basic building blocks of differentiable four-manifolds, namely the conjecture that every smooth compact oriented simply connected 4-manifold without boundary would be obtained as a connected sum of complex surfaces. This older conjecture was disproved by Gompf and Mrowka [21] and the symplectic analogue 8.12 was disproved recently by Szabó [44], using Seiberg–Witten invariants.

This class of manifolds that violate conjecture 8.12 were reinterpreted in the framework of a more general construction by Fintushel and Stern [12]. The procedure is very interesting, and it also produces a new large class of exotic $K3$ surfaces. We summarise it here briefly. Suppose given a knot K in S^3 and let V_K be the knot complement. Start with a given simply connected four-manifold X. The construction produces a four-manifold X_K homotopy equivalent to X, obtained by removing a neighbourhood of a torus in X and replacing it with $V_K \times S^1$.

The Seiberg–Witten invariants of X are assembled in a Laurent series

$$N(X) = n_0 + \sum_{j=1}^{k} n_j (t_j + (-1)^{\frac{\chi(X)+\sigma(X)}{4}} t_j^{-1}),$$

in the variables t_j, with $j = 1, \dots, k$. Here $n_0 = N_{s_0}(X)$, with $c_1(L_0) = 0$, and $\{s_1, \dots, s_k\}$ are the remaining $Spin_c$-structures that give rise to basic classes. The variables t_j correspond to $t_j = \exp(c_1(L_j))$, and the coefficients n_j are the Seiberg–Witten invariants $n_j = N_{s_j}(X)$. The motivation for this definition is the symmetry $N_{\bar{s}}(X) = (-1)^{\frac{\chi(X)+\sigma(X)}{4}} N_s(X)$, where $\bar{s}$ is the conjugate $Spin_c$-structure.

With this notation the following result is proven.

Proposition 8.13 *Let T be the class in $H^2(X, \mathbb{Z})$ Poincaré dual to the torus in X on which surgery is performed to produce the manifold X_K. Under the assumption that the torus has two vanishing cycles (see [12]), the Seiberg–Witten invariants of the two manifolds X and X_K are related by*

$$N(X_K) = N(X)\Delta_K(t),$$

where $\Delta_K(t)$ is the Alexander polynomial of the knot, and the variable t represents $t = \exp(2T)$.

A combination of this result with Taubes' results on the Seiberg–Witten invariants of symplectic manifolds leads to the conclusion [12] of the following corollary.

Corollary 8.14 *If X is symplectic, the torus is symplectically embedded, and the Alexander polynomial of the knot K is monic, then X_K is also symplectic. However, if the Alexander polynomial of the knot K is not monic, then X_K does not admit a symplectic structure.*

The result of this corollary is used in [12] as a way to provide examples of simply connected irreducible four-manifolds that do not admit a symplectic structure.

8.5 Pseudo–holomorphic curves

In the world of symplectic manifolds, there are invariants, known as Gromov invariants [36], [43], defined in any dimension in terms of the *pseudo-holomorphic curves.* A pseudo-holomorphic curve is a 2-submanifold embedded in a $2n$-dimensional symplectic manifold X via a map which is holomorphic with respect to a Riemann surface structure on the domain and an almost-complex structure J on X tamed by the symplectic form.

Recent results by Taubes [48], [49], [50] have uncovered a deep relation between these and the Seiberg–Witten invariants in the case of four-dimensional symplectic manifolds.

We shall not attempt here to present any of the details of Taubes' work, since the technical difficulties involved are well beyond the purpose of this book. We recommend the interested reader to approach this topic by first reading the detailed research announcement [48] and the survey paper [24], and then the substantial part of the work [49], [50]. What we do in this section is just a brief exposition of the result, and some digressions on the theme of pseudo–holomorphic curves.

In our context, we consider the case of a 4-dimensional compact connected symplectic manifold without boundary. Given a homology class $A \in H_2(X; \mathbb{Z})$, let $\mathcal{H}(A)$ be the space of J-holomorphic curves that realise the class A.

Lemma 8.15 *For a generic choice of a tamed almost-complex structure J, the space $\mathcal{H}(A)$ is a smooth even dimensional manifold of dimension*

$$d_{\mathcal{H}} = < -c_1(K) \cup PD(A) + PD(A)^2, [X] >,$$

where K is the canonical line bundle defined by the symplectic structure and $PD(A)$ is the Poincaré dual of the class A.

Notice that the canonical class $c_1(K)$ (often simply denoted K) is well defined independently of the choice of the almost complex structure J, since the set $\mathcal{J}$ of ω-tamed almost complex structures is contractible.

This result is proven with the same technique we have used in computing the dimension of the Seiberg–Witten moduli space: in fact the equation describing the J-holomorphic condition linearises to a Fredholm operator.

When $d_{\mathcal{H}} > 0$, given a set Ω of $\frac{d_{\mathcal{H}}}{2}$ distinct points in X, we take $\mathcal{H}_\Omega$ to be the subspace of $\mathcal{H}(A)$ whose points are the curves that contain all of the points in Ω.

Theorem 8.16 *For a generic choice of J and of the points in X, $\mathcal{H}_\Omega$ is a compact zero–dimensional manifold endowed with a canonical orientation.*

Some details can be found in the first chapter of [36]. This allows us to define invariants as follows.

Definition 8.17 *We define the Gromov invariant to be a map*

$$\Phi : H_2(X; \mathbb{Z}) \to \mathbb{Z}$$

which is the sum with orientation of the points in $\mathcal{H}_\Omega(A)$, if $d_{\mathcal{H}} > 0$; the sum of points of $\mathcal{H}(A)$ with orientation, if $d_{\mathcal{H}} = 0$; and zero by definition when $d_{\mathcal{H}} < 0$.

The value of Φ is independent of the choice of a generic set of points and of the quasi-complex structure J. The result announced in Taubes' paper [48] is the following.

Theorem 8.18 *Given a $Spin_c$ structure s on a compact symplectic 4-manifold X, with $L = \det(S^+)$, we have*

$$N_s \equiv \Phi(PD(c_1(L))).$$

The strategy of the proof is to extend to other $Spin_c$-structures the asymptotic technique introduced in [45] to compute the Seiberg–Witten invariant of a symplectic manifold with respect to the canonical $Spin_c$-structure.

In this process, the asymptotic method shows how to associate a pseudo–holomorphic curve to a solution of the Seiberg–Witten equations, in the limit when the perturbation parameter satisfies $r \to \infty$. A different argument, which is briefly sketched in [48], provides a converse construction of a solution to the equations, given a J-holomorphic curve. A combined use of these constructions would complete the proof of the theorem.

A more precise way of stating the theorem is given in [49].

Theorem 8.19 *Let (X, ω) be a compact symplectic four-manifold with $b_2^+ > 1$. If the Seiberg–Witten invariant is non-zero $N_s \neq 0$ for a certain $Spin_c$-structure, then there exists a J-holomorphic curve C (which need not be connected and can have multiplicities $C = \cup_i m_i C_i$) such that*

(i) the class $[C] \in H_2(X; \mathbb{Z})$ is the Poincaré dual of the class of the line bundle L,

$$[C] = PD(c_1(L));$$

(ii) the intersection numbers satisfy

$$K \cdot C_i \leq C_i \cdot C_i,$$

where K is the class of the canonical line bundle;

(iii) the multiplicities m_i satisfy $m_i = 1$ unless the component C_i is a torus of self intersection zero.

This version of the theorem leads to a clear geometric interpretation of the result. It is known in fact that in the case of Kähler surfaces there are two equivalent ways of thinking of holomorphic curves: either as parametrised curves from a Riemann surface that satisfy the Cauchy-Riemann equations or as the zero set of a holomorphic section of a holomorphic line bundle. The first method extends to symplectic geometry via the notion of J-holomorphic curves, while there is no available notion of "holomorphic bundle" and "holomorphic section". A way of thinking of Taubes' result (as I first learnt from lectures of S. Bauer and D. Salamon) is precisely the construction of a similar notion in the symplectic category. In fact one can think of the equivalent of holomorphic line bundles as being those line bundles that correspond to Seiberg–Witten basic classes and the theorem says how to use the zero sets of sections that satisfy the Seiberg–Witten equations in order to obtain a J-holomorphic curve.

The steps that lead from solutions of the Seiberg–Witten equations to a pseudo-holomorphic curve are summarised in [24]. The aim is to show that, if X is a closed symplectic four-manifold with $b_2^+(X) > 1$, and if $s \in \mathcal{S}$ is a $Spin_c$-structure with $L \neq 0$ satisfying $N_s(X) \neq 0$, there exists a pseudo-holomorphic curve with C satisfying $[C] = PD(c_1(L))$.

Considering the estimate (55) in the case of a non-trivial line bundle L, that is when $c_1(L)[\omega] > 0$, we obtain a uniform bound for $\int_X (|\alpha|^2 - r^2)$, which implies that $\frac{\alpha}{\sqrt{r}} \to 0$ almost everywhere (not everywhere, since, L being a non-trivial bundle, sections must have a non-empty vanishing set). The bound on $\int_X (r - C)|\beta|^2$ still gives $\beta \to 0$, and by applying the Weitzenböck formula and the Dirac equation we also obtain the estimate $|\bar{\partial}_a \alpha| \to 0$, that is, the section tends to become holomorphic.

The most delicate part of the argument is then centred around pointwise estimates that give the convergence, in the sense of currents, of the zero set $\alpha^{-1}(0)$ to a pseudo-holomorphic curve C, when the parameter r is pushed to infinity.

Several applications of Taubes' result are summarised in [24].

8.6 Beyond the symplectic world

Many interesting questions arise in connection with Taubes' result. One of them is how much the results can be extended beyond the realm of symplectic manifolds. Taubes' pointed out that much of the analysis that goes into the proof of theorem 8.19 carries over to manifolds that are endowed with a non-trivial closed self-dual two form ω which is not necessarily non-degenerate. On a four manifold, such a two form will be degenerate along embedded circles. In this case, current work of Taubes' indicates that one can expect solutions of the Seiberg–Witten equations to produce pseudo-holomorphic curves away from the degenerate locus, with some regularity conditions on how they approach the circles where ω is degenerate.

Therefore, via this correspondence with Seiberg–Witten invariants, it may be possible to think of extensions of invariants defined by pseudo-holomorphic curves, to manifolds that are not symplectic.

We digress briefly, in order to describe one of these symplectic invariants defined via pseudo-holomorphic curves: Quantum Cohomology.

The quantum cup product is a deformed product in the cohomology ring of a symplectic manifold. Geometrically it can be thought of as a coarser notion of intersection of homology classes realised by embedded submanifolds. Instead of counting, with the orientation, the number of intersection points of two cycles in generic position, the counting is made over the J-holomorphic curves that touch the given cycles in generic points. Note that, in order to define this product, several technical hypotheses are introduced, which are explained in [36]. In our case, since we deal with the 4-dimensional case only, the situation is simpler.

More precisely, the quantum cohomology ring of X is

$$QH^*(X) = H^*(X) \otimes \mathbb{Z}[q, q^{-1}],$$

where q is a formal variable of degree $2N$, with N the minimal Chern number of X (see [36]). A class $a \in QH^k(X)$ splits as $a = \sum_i a_i q^i$, with $a_i \in H^{k-2iN}(X)$. There is a non-degenerate pairing

$$QH^*(X) \otimes QH^*(X) \to \mathbb{Z},$$

$$< a, b >= \sum_{2(i+j)N=k+l-2n} a_i b_j,$$

with $k = deg(a)$, $l = deg(b)$ in $QH^*(X)$, and $n = dim(X)$. The quantum cup product

$$QH^*(X) \otimes QH^*(X) \xrightarrow{\bullet} QH^*(X)$$

is defined by specifying the values of $< a \bullet b, c >$:

$$< a \bullet b, c >:= \sum_{i,j,k} \sum_{A} \Phi_A(\alpha_i, \beta_j, \gamma_k),$$

with $a = \sum a_i q^i$, $b = \sum b_j q^j$, $c = \sum c_k q^k$, $\alpha_i = PD(a_i)$, $\beta_j = PD(b_j)$, $\gamma_k = PD(c_k)$. A is a homology class realised by a J-holomorphic curve of genus zero, with $c_1(A) + (i + j + k)N = 0$, $c_1(A)$ being the evaluation over A of the Chern class of the restriction of the tangent bundle to A (the latter condition is imposed for dimensional reasons). The coefficient $\Phi_A(\alpha_i, \beta_j, \gamma_k)$ is a more general version of the Gromov invariants introduced in definition 8.17. Here, instead of imposing $d_{\mathcal{H}}/2$ points in $\mathcal{H}_\Omega$, we want a positive dimensional manifold. Hence $\Omega = \{p_1, \ldots, p_r\}$, $r < d_{\mathcal{H}}/2$. Now we would like to map this manifold to X^r, so that it gives rise to a pseudo-cycle (see [36]). Thus, we consider an evaluation map

$$\epsilon_r : \mathcal{H}_\Omega \to X^r$$

whose image is the space

$$\{(u(p_1), \ldots, u(p_r)) \mid u : \Sigma \to X, [u(\Sigma)] = A\},$$

where u is a J-holomorphic parametrisation of the curve. It is clear, however, that we need to use parametrised curves. Hence the space $\mathcal{H}_\Omega$ has to be intended rather as the moduli space of parametrised curves that touch the points in Ω, modulo reparametrisations, i.e. modulo automorphisms of the Riemann surface Σ.

The topology of this moduli space is more complicated than the previous case: in fact there are non-trivial problems related to the compactification. The theory works sufficiently well in the case of genus zero curves (see [36], chapter 5), but more serious technical complications arise when considering curves of genus $g > 0$. Once a pseudo-cycle of a certain dimension (which is computed in [36], chapter 7) is defined in X^r, the Gromov invariant is obtained by intersecting this cycle in X^r (with the usual intersection product) with a number of homology classes of X, so to reach the complementary dimension.

Although the definition of this invariant is different from the one used in [48], one could ask if there is any gauge theoretic description within the context of Seiberg–Witten theory. Such a formulation, together with the extension of Taubes' results to 4-manifolds with degenerate symplectic forms, may provide a formulation of quantum cohomology in the non-symplectic world. Moreover, one of the highly non-trivial results in quantum cohomology is the fact that the quantum cup product is associative. One can speculate on the existence of a different proof based on the gauge theoretic counterpart.

8.7 Algebraic Surfaces

In a different direction, Seiberg–Witten theory has been applied to obtain information on the diffeomorphism type of algebraic surfaces. Many results were previously obtained via Donaldson theory [16]. We should recall here some notions about algebraic surfaces: we follow strictly the overview given in the first chapter of [16].

A complex surface is *minimal* if it does not contain any holomorphic curve $C \cong \mathbb{C}P^1$ with self intersection $C \cdot C = -1$, that is, by the Castelnuovo criterion, if it cannot be blown down to another smooth complex surface. In this section we discuss some diffeomorphism properties of algebraic surfaces: we shall deal with minimal surfaces only.

Two surfaces X_1 and X_2 are *deformation equivalent* if there exists a family (not necessarily smooth) $\mathcal{X}$ of complex surfaces over the disk D, with $\pi : \mathcal{X} \to D$, such that there are two points p_1 and p_2 in D with $\pi^{-1}(p_1) = X_1$ and $\pi^{-1}(p_2) = X_2$.

Let K_X be the canonical line bundle as defined previously. Two deformation equivalent surfaces X_1 and X_2 are diffeomorphic through an orientation preserving diffeomorphism $f : X_1 \to X_2$ that satisfies $f^* K_{X_2} = K_{X_1}$ (we are using the same notation for the canonical line bundle and its Chern class).

The *plurigenera* of a complex surface X are the dimensions of the spaces of holomorphic sections of powers of the canonical line bundle, namely

$$P_n(X) = \dim_{\mathbb{C}} H^0(X, K_X^{\otimes n}).$$

It can be shown that the ratio $\frac{1}{n^2} P_n(X)$ is bounded as $n \to \infty$. This gives rise to the following definition of the *Kodaira dimension* $\kappa(X)$:

$$\kappa(X) = \begin{cases} -\infty & \text{if } P_n(X) = 0 \text{ for all } n \geq 1 \\ 0 & \text{if } \sup_n P_n(X) \text{ is bounded but some } P_n(X) \neq 0 \\ 1 & \text{if } \sup_n P_n(X) = \infty \text{ but } \sup_n \frac{1}{n} P_n(X) \text{ is bounded} \\ 2 & \text{if } \sup_n \frac{1}{n} P_n(X) = \infty \end{cases}$$

Geometrically, if $\kappa(X) \neq -\infty$, the value of $\kappa(X)$ is the dimension of the image of X in some $\mathbb{C}P^N$ under the rational map defined by the linear system $K_X^{\otimes n}$.

The following is a result in the theory of algebraic surfaces.

Proposition 8.20 *If X_1 and X_2 are deformation equivalent, then they have the same Kodaira dimension and the same plurigenera,*

$$\kappa(X_1) = \kappa(X_2) \quad \textit{and} \quad P_n(X_1) = P_n(X_2) \quad \textit{for all } n \geq 1.$$

For an overview of the Enriques–Kodaira classification of algebraic surfaces, we address the reader to the first chapter of [16]. We should just recall here that the "classification" is really a classification only for Kodaira dimension $\kappa(X) \leq 1$. In the case of *surfaces of general type* with $\kappa(X) = 2$ no classification of deformation types exists.

A natural question about algebraic surfaces is how much discrepancy there is between deformation equivalence and diffeomorphism. The Van de Ven conjectures were formulated within the context of this general question:

Conjecture 8.21 *Two diffeomorphic algebraic surfaces X_1 and X_2 have the same Kodaira dimension and the same plurigenera,*

$$\kappa(X_1) = \kappa(X_2) \quad \textit{and} \quad P_n(X_1) = P_n(X_2) \quad \textit{for all } n \geq 1.$$

Using Donaldson theory, the following results were obtained [16]

Theorem 8.22 *(1) There is a finite–to–one discrepancy between diffeomorphism and deformation equivalence.*

(2) If X_1 and X_2 are two diffeomorphic algebraic surfaces, which do not form a pair of a rational surface and a surface of general type, then they have the same Kodaira dimension, $\kappa(X_1) = \kappa(X_2)$.

The first computation of Seiberg–Witten invariants for algebraic surfaces confirms the fact that, even at the level of diffeomorphism type nothing more can be said for surfaces of general type. In fact, the following lemma shows that for such surfaces Seiberg–Witten theory only recovers the information given by the canonical class K.

Theorem 8.23 *If X is a minimal surface of general type, the Seiberg–Witten invariant satisfies $N_s = \pm 1$ if $s = s_0$ and $N_s \equiv 0$ otherwise.*

However, in the case of surfaces of smaller Kodaira dimension one can obtain the complete differentiable classification. The first step we are going to present is the resolution of the exceptional case in (2) of 8.22.

Theorem 8.24 *A surface X_1 diffeomorphic to a rational surface X_2 is also rational.*

This result was also obtained by means of Donaldson theory in [18] and [42]. It was independently obtained by means of Seiberg–Witten theory by [2], [17], and [41]. We describe briefly the strategy of the proof given in [41].

<u>Sketch of the Proof:</u> The first step consists of proving that a rational surface admits a Hitchin metric g_0, namely a Kähler metric with positive total scalar curvature. Then, assuming that X_1 has $\kappa(X_1)$ positive, and considering the case when X_1 can be non-minimal, it is possible to find a metric g on X_1 determined by a choice of an ample divisor on the *minimal model* (see [16]), and a $Spin_c$ structure s determined by the canonical class and by the exceptional divisors of the blowups, such that $N_s \neq 0$. The Seiberg–Witten moduli space M_s consists of a single smooth point. The final step is to show that if there is an orientation preserving diffeomorphism $f : X_1 \to X_2$, then the Hitchin metric $f^*(g_0)$ and the metric g are in the same chamber: this leads to a contradiction since the curvature constraint makes the Seiberg–Witten invariant for $f^*(g_0)$ vanish.

lemma 8.24 completes the proof of the Van de Ven conjecture for what concerns the Kodaira dimension. Seiberg–Witten gauge theory has proved to be useful in establishing the analogous result about plurigenera [2], [17].

The Van de Ven conjecture is proven in [2] as a consequence of the following stronger result.

Theorem 8.25 *The class K_X (or K_{min} of a minimal model of X in case X is non-minimal of positive Kodaira dimension) is determined by the diffeomorphism type of X.*

Sketch of the Proof: The first step is to rephrase the invariance of K_X, for X with $\kappa(X) \geq 0$, in terms of the existence of a set $\mathcal{K}$ of classes $K_i \in H^2(X, \mathbb{Z})$ of type $(1,1)$, satisfying $2g(H) - 2 \geq H^2 + |K_i \cdot H|$ for any ample divisor H, and such that $K_i \equiv w_2(X)$ mod 2. The set $\mathcal{K}$ is also required to satisfy $K_X \in \mathcal{K}$ and, under blowups $\sigma : \tilde{X} \to X$, $\sigma_*(\mathcal{K}(\tilde{X})) = \mathcal{K}(X)$.

Then Seiberg–Witten theory shows that the set of Seiberg–Witten basic classes on X gives the desired $\mathcal{K}$, and that such a set is non-empty. We have seen in proposition 8.5 that the moduli space of solutions of the Seiberg–Witten equations for the non-generic Kähler metrics can be positive dimensional even though the virtual dimension is zero. Thus, we need a different way to define the Seiberg–Witten invariant $N_s(X)$ in this case, in order to identify the set of basic classes. To this purpose [2] develops the technique of the localised Euler class, that we shall discuss at length later on. This provides a way of extending Fulton's intersection theory to the infinite dimensional Fredholm context and generates a homology class that sits in the degree prescribed by the virtual dimension and deals naturally with the excess intersection problem. We shall return to this later and clarify its relation with the partition function of the quantum field theoretic formalism. It is enough to mention here that the Seiberg–Witten invariant is then defined via the zero-dimensional moduli space defined by the localised Euler class, and this identifies the required set of basic classes.

9 Topology of embedded surfaces

An interesting question in low dimensional topology is to find sharp estimates for the genus of embedded surfaces. One can observe, for instance, that the failure of the Whitney lemma in dimension four is one of the reasons that make topology so different than in higher dimensions [15], [23]. This failure is measured precisely by the non-zero minimal genus of embedded surfaces representing a certain cohomology class. Gauge theory provided a useful tool in studying the topology of embedded surfaces [27], [30]. More recently, Seiberg–Witten theory has helped in determining adjunction inequalities that provide such lower bounds. As we are going to see, the Thom conjecture provides a sharp lower bound for the genus in the symplectic case. Bounds provided by Seiberg–Witten theory are far less sharp in non-symplectic cases, as discussed at length in [29], but interesting estimates can still be obtained via Seiberg–Witten theory as in [28].

The analogous question can be formulated in three-manifolds. The minimal genus of surfaces in a given homology class is measured by the Thurston norm. Several contributions [1] [29] [33] have linked Seiberg–Witten equations on three-manifolds with the Thurston norm. We shall recall briefly some of the results later in this section.

For a general overview of the subject, we recommend the beautiful survey article [29].

9.1 The Thom Conjecture

Various statements go under the name of Thom conjecture. In the more general form the statement can be phrased as follows.

Conjecture 9.1 *Let (X, ω) be a four-dimensional compact symplectic manifold. Let C be a symplectic 2-submanifold (namely an embedded submanifold such that $\omega|_C$ is an area form). If Σ is an embedded 2-submanifold such that*

$$[\Sigma] = [C] \in H_2(X; \mathbb{Z}),$$

then the genera are related by

$$g(\Sigma) \geq g(C).$$

Several weaker statements have been proved [27], [30], [40]. All of these require the extra assumption that the symplectic submanifold C has non-negative self intersection, $C \cdot C \geq 0$.

One of the first applications of the Seiberg–Witten gauge theory was a proof [31] of the Thom conjecture for $\mathbb{C}P^2$, see theorem 9.2 below. A more general version of this result was presented more recently by Morgan, Szabó, and Taubes [40]. Other results on the Thom conjecture have been obtained by R. Wang [52] using a version of Seiberg–Witten Floer homology. See also the related references [13], [14], [28], [29]. We reproduce here the proof that is given in [31].

Theorem 9.2 *An oriented two–manifold Σ that is embedded in $\mathbb{C}P^2$ and represents the same homology class as an algebraic curve of degree $d > 3$ has genus g such that*

$$g \geq \frac{(d-1)(d-2)}{2}.$$

<u>Sketch of the Proof:</u> The proof is obtained in several steps.

(1) We start by introducing the notion of good metrics.

Definition 9.3 *a metric on a 4-manifold X is "good" with respect to a certain choice of the line bundle L if the moduli space of solutions of the Seiberg–Witten equations is smooth. lemma 3.4 implies that this happens if $c_1(L)$ is not in $H^{2-}(X; \mathbb{R})$.*

We shall consider the manifold

$$X = \mathbb{C}P^2 \# n\, \overline{\mathbb{C}P^2},$$

the blowup (in the language of algebraic geometry) of the projective space at n points. Since $H^{2+}(X;\mathbb{R})$ is one-dimensional, there is, up to scalar, a unique self-dual harmonic form ω_g, which depends on the metric. We identify the good metrics on X in terms of a condition on the product of the first Chern class of the line bundle with ω_g, namely

$$\int_X c_1(L) \cup [\omega_g] \neq 0.$$

(2) If the product $\int_X c_1(L) \cup [\omega_g]$ is negative, the Seiberg–Witten moduli space is non empty. This is shown by first proving that the Seiberg–Witten invariant (mod 2) changes parity if $\int_X c_1(L) \cup [\omega_g]$ changes sign; and then proving that there is a particular choice of the metric g, the Hitchin metric, such that $\int_X c_1(L) \cup [\omega_g] > 0$ and the moduli space is empty.

(3) Suppose a four–manifold X splits along a three-manifold Y, so that the metric is a product on a neighbourhood $[-\epsilon, \epsilon] \times Y$. Consider the metric g_R given by inserting a flat cylinder $[-R, R] \times Y$. If the moduli space $M(g_R)$ is non-empty for all large R, then there exists a solution of the Seiberg–Witten equations which is "translation invariant in a temporal gauge" on the manifold $\mathbb{R} \times Y$. This means that the dt component of the connection A vanishes (see Definition 6.1).

(4) If there is a solution on $\mathbb{R} \times Y$ that is translation invariant in a temporal gauge and $Y = S^1 \times \Sigma$, where Σ is a surface of constant scalar curvature and genus $g \geq 1$, then there is an estimate

$$\mid \int_\Sigma c_1(L) \mid < 2g - 2.$$

(5) Let H be the generator of $H^2(\mathbb{C}P^2; \mathbb{Z})$; by assumption, we have an embedded surface Σ of genus g that determines the homology class dual to dH. Consider the blowup of $\mathbb{C}P^2$ at d^2 points that avoid Σ, and consider the embedding

$$\Sigma \hookrightarrow \mathbb{C}P^2 \# d^2\, \overline{\mathbb{C}P^2}.$$

The second cohomology of $X = \mathbb{C}P^2 \# d^2\, \overline{\mathbb{C}P^2}$ has generators H and E_i, $i = 1, \ldots, d^2$, with intersection form $Q_X = (1, d^2)$. Take $\tilde{\Sigma} = \Sigma \# d^2 S^2$, where $S^2 \subset \overline{\mathbb{C}P^2}$ is dual to $-E_i$. Thus the homology class $[\tilde{\Sigma}]$ is dual to $dH - E$, $E = \sum_i E_i$. Take a tubular neighbourhood T of $\tilde{\Sigma}$ and a metric g_0 on X such that $Y = \partial T = \tilde{\Sigma} \times S^1$ with a product metric and constant scalar curvature $-2\pi(4g-4)$ on $\tilde{\Sigma}$ (assume $\tilde{\Sigma}$ has unit area and use the Gauss–Bonnet theorem). Let L be the canonical line bundle K. This has Chern class $c_1(K) = 3H - E$. Insert a cylinder $[-R, R] \times Y$. Then

$$\int_X c_1(K) \cup [\omega_{g_R}] = [\tilde{\Sigma}][\omega_{g_R}] - (d-3) \int_X H \cup [\omega_{g_R}].$$

By normalising $1 = \int_X H \cup [\omega_{g_R}]$, and showing that $[\tilde{\Sigma}][\omega_{g_R}] \to 0$ as $R \to \infty$, the result of the theorem follows from the estimate of step (4).

Now we see in more details the various steps of the proof. Part of it is left as a series of exercises at the end of this section.

Step (1): The intersection form Q_X on the manifold $X = \mathbb{C}P^2 \# n\, \overline{\mathbb{C}P^2}$ has signature $(1, n)$; thus it determines a cone C in $H^2(X; \mathbb{R})$ where the form is positive.

If H is the generator of the cohomology ring of $\mathbb{C}P^2$, then $H \in C$. Call C^+ the nappe of the cone that contains H.

The manifold X has $b_2^+ = 1$; hence for a chosen metric there exists a unique harmonic self-dual form ω_g such that the corresponding cohomology class $[\omega_g] \in C^+$.

So we have that the metric g is good in the sense of definition 9.3 iff

$$\int_X c_1(L) \wedge \omega_g \neq 0.$$

Step (2): $\int_X c_1(L) \wedge \omega_g = 0$ detects the presence of some singular point in the moduli space. By perturbing the equation with a small $\eta \in \Lambda^{2+}$, we can assume that on a given path of metrics $\{g_t \mid t \in [0, 1]\}$ the expression

$$f(t) = \int_X c_1(L) \wedge \omega_{g_t} + 2\pi \int_X \eta \wedge \omega_{g_t}$$

changes sign transversely at $t = 0$.

Hence the parametrised moduli spaces M_{g_t} look like a family of arcs; we want to prove that at a singular point an odd number of arcs meet. Thus the invariant computed mod 2 changes parity.

This requires the analysis of a local model of the moduli space around a singular point. A model of Donaldson's can be adapted to this case [7].

Step (3): This is the part of the proof where the gauge theoretic techniques have a prominent role. Since the argument involves the dimensional reduction of Seiberg–Witten theory to three dimensions, we postpone the proof of this step until after the section that deals with three–manifold applications of the theory.

Step (4): Assuming Σ to be of unit area and with constant scalar curvature, the Gauss–Bonnet theorem implies that the scalar curvature $\kappa = -4\pi(2g - 2)$. From the estimate on the spinor ψ given in lemma 3.10, we have

$$|\psi|^2 \leq 4\pi(2g - 2).$$

But, from the equation (9), since

$$|< e_i e_j \psi, \psi >|^2 = 2 |\psi|^4,$$

we get an estimate $|F_A^+| \leq \sqrt{2}\pi(2g - 2)$.

Since the solution is translation invariant in a temporal gauge, F_A is the pullback on $\mathbb{R} \times Y$ of a form on Y. Hence $F_A \wedge F_A = 0$, and this means that $|F_A^+| = |F_A^-|$, since $F_A \wedge F_A = (|F_A^+|^2 - |F_A^-|^2) dv$.

Thus the resulting estimate on F_A is

$$|F_A| \leq 2\pi(2g-2). \tag{56}$$

Therefore

$$\left| \frac{1}{2\pi} \int_\Sigma F_A \right| = | c_1(L^2)[\Sigma] | \leq 2g-2.$$

9.2 Contact structures

Notice that the stretching argument used in the proof of the Thom conjecture [31] proves that if $c_1(L)$ is a basic class on a four-manifold X (i.e. the invariant $N_s(X) \neq 0$ for all metrics), then its restriction to an embedded three-manifold corresponds to a non-trivial moduli space $M_c(s|_Y) \neq \emptyset$. This condition is weaker than having non-vanishing three-dimensional Seiberg–Witten invariant: it can be regarded as a condition on the presence of generators of the Floer homology, as opposed to the stronger non-vanishing of the Euler characteristic of the Floer complex. Classes $c_1(L) \in H^2(Y, \mathbb{Z})$ that correspond to $Spin_c$-structures $s \in \mathcal{S}(Y)$ satisfying the condition $M_c(s) \neq \emptyset$ for a generic metric are called [29] *monopole classes*, as opposed to the more restrictive condition of *basic classes* that satisfy $\chi(HF_*^{SW}(Y,s)) \neq 0$.

This observation [29] is the starting point for understanding the work of Kronheimer and Mrowka on Seiberg–Witten equations and contact structures [32]. We do not recall here the notion of contact structure and the many aspects of contact geometry on three-manifolds. We refer the reader to [9], [10], [11], [20], [51]. In [32] the following result is proven.

Theorem 9.4 *Let Y be a compact oriented three-manifold with an oriented contact structure ξ, let X be a four-manifold such that $\partial X = Y$ with compatible orientation. The contact structure ξ determines a $Spin_c$ structure on a collar neighbourhood of Y. Moreover, the manifold X can be completed with a symplectic cone $Y \times [0, \infty)$, so that the symplectic form ω and the induced $Spin_c$ structure s_ω on the cone are compatible with ξ. On this completed manifold $\tilde{X}$ there is a compact smooth Seiberg–Witten moduli space that is cut out transversely by the equations. Moreover, if X has a symplectic form compatible with the contact structure ξ on the boundary, the corresponding invariant on $\tilde{X}$ satisfies $N_{s_\omega,\xi}(\tilde{X}) = \pm 1$.*

A brief summary of the argument can be given as follows. First of all, on $\tilde{X}$ it is convenient to choose perturbed Seiberg–Witten equations, where the perturbation is given by the canonical solution on the symplectic cone with unit length spinor (extended arbitrarily on the non-symplectic part X) and a self-dual 2-form η that decays exponentially along the cone,

$$D_A\psi = 0,$$

$$F_A^+ - \sigma(\psi,\psi) = F_{A_0}^+ - \sigma(\psi_0,\psi_0) + i\eta,$$

where (A_0,ψ_0) is the canonical solution on the symplectic cone, as discussed previously in the review of Seiberg–Witten theory on symplectic manifolds.

The configuration space on $\tilde{X}$ is defined as the set of pairs (A,ψ) that are asymptotic in the L^2_k norm to the canonical pair (A_0,ψ_0) along the conical end. The gauge group of L^2_{k+1} gauge transformations that decay to the identity acts on this configuration space, and the quotient contains the moduli space of solutions of the perturbed equations. Since this is a problem on a non-compact manifold, the Fredholm property of the linearisation is not a consequence of ellipticity: a delicate analysis shows how the fact that the canonical section ψ_0 is nowhere vanishing on the cone plays an essential role in proving the desired Fredholm property.

This shows that the moduli space is finite dimensional and, under a generic choice of the perturbation, cut out transversely by the equations. The proof of compactness is more subtle and technically demanding than in the case of a compact manifold. The key step is to prove a uniform exponential decay to the asymptotic value (A_0,ψ_0) along the conical end. The analysis is similar to the one used in [49].

A consequence of theorem 9.4 is the following.

Proposition 9.5 *Suppose given a compact oriented three-manifold Y with a contact structure ξ. If there is a symplectic four-manifold X with boundary, such that Y is one of the boundary components with the boundary orientation compatible with ξ, then there is a non-trivial element in the Seiberg–Witten Floer homology, that is, $HF_*^{SW}(Y) \neq 0$.*

Sketch of the Proof: A stretching argument like the one used in the proof of the Thom conjecture provides the existence of at least one generator $a = (A_a,\psi_a)$ of the Floer complex, i.e. $M_c(Y,s_\xi) \neq \emptyset$. We need to show that in fact we also have a cycle which is not a boundary. In order to show that we have a cycle, consider the manifold $Y \times \mathbb{R}$, with a cylindrical metric on $Y \times (-\infty,-1]$ and the conical metric and compatible symplectic form on $Y \times [1,\infty)$, patched smoothly on $Y \times [-1,1]$. On this manifold consider the configuration space of elements (A,ψ) that are asymptotic to the canonical solution (A_0,ψ_0) on the conical end and to a solution a of the three-dimensional Seiberg–Witten equations on the cylindrical end. To simplify the argument, if the class of s_ξ is trivial, then it is better to assume here that the elements $a = (A_a,\psi_a)$ are non-degenerate, $\psi_a \neq 0$: under this assumption, gauge transformations that are asymptotic to the identity act freely on the configuration space and we can form the quotient and define the moduli space $M(\xi, a)$ of solutions of the perturbed Seiberg–Witten equations as in [32]. Combining [3] with [32], we can show that $M(\xi, a)$ is a smooth manifold cut out transversely by the equations. The zero-dimensional component $M^0(\xi, a)$ is compact by the argument of [32], hence we can take the algebraic sum $n(\xi, a) = \# M^0(\xi, a)$. Now we can form the linear combination $C_\xi = \sum_{a \in M_c(Y, s_\xi)} n(\xi, a) a$. This is the candidate for the Floer cycle. To see that it is killed by the Floer boundary, it is enough to prove that we have a gluing map

$$\# : \cup_{a \in M_c(Y, s_\xi), \mu(a) - \mu(b) = 1} M^0(\xi, a) \times \hat{\mathcal{M}}(a, b) \to M^1(\xi, b),$$

where $M^1(\xi, b)$ is the 1-dimensional component of $M(\xi, b)$, and $\hat{\mathcal{M}}(a, b)$ is the moduli space of unparametrised flow lines connecting the critical points a and b. We shall not present the proof of this gluing theorem here. However, the technique employed is very similar to the one used in the construction of the Floer homology and can be obtained by combining [3] and [32].

We still need to prove that the class $[C_\xi]$ in $HF_*^{SW}(Y, s_\xi)$ is non-trivial, i.e. that C_ξ is not a boundary. One way to attack this problem is to show that there is a non-trivial pairing of $[C_\xi]$ with some other class under the natural pairing of Floer homology and cohomology. For this we need the suitable gluing theorem that reconstructs Seiberg–Witten invariants of four-manifolds that split along a three-manifold by pairing relative invariants in the Floer homology of the three-manifold. Formulations of such gluing theorems can be found in [3], [38], In our case, consider the manifold $X \cup Y \times [0, \infty)$ with a cylindrical metric on the end $Y \times [0, \infty)$ and define the moduli space $M(X, a)$ with asymptotic condition $a = (A_a, \psi_a)$ in $M_c(Y, s_\xi)$, as in [3]. The argument has to be modified since the manifold X can have other boundary components that can be completed with a cylindrical end, so that the Fredholm analysis carries over from [32]. We are not going to enter into the details here. We want to obtain the invariant $N_{s_\omega}(\tilde{X})$ of theorem 9.4 from a pairing of relative invariants, that is, we want a description of the moduli space $M(\tilde{X}, s_\omega)$ of [32] as a fibred product of the $M(X, a)$ and of the $M(\xi, a)$ described above. We shall not attempt here to prove such a decomposition. It is enough to point out that this will lead to a proof of the non-triviality of $[C_\xi]$.

QED

In [29] some results of Gabai, Eliashberg, and Thurston are presented, which give conditions under which the three-manifold Y satisfies the hypothesis of 9.4 and 9.5. The key ingredient is the fact that, if the three-manifold Y is not $S^1 \times S^2$ and it admits a smooth taut foliation by oriented leaves, then the foliation can be deformed by a small deformation to a contact structure, and the four-manifold $[-1, 1] \times Y$ admits a symplectic form compatible with the contact structures $\xi_\pm$ on the oriented boundary. The results of [32] and 9.5 then apply to this case.

Notice, however, that not all contact structures arise as a small deformation of a taut foliation, as follows from results of Lisca and Matić, [34] and [35]. It should be mentioned, although we shall not discuss any of this in detail, that the connection between Seiberg–Witten invariants and contact structures on three-manifolds has very interesting geometric consequences. Contact structures are distinguished as *tight* and *overtwisted*, see [9]. A classification is available for overtwisted contact structures which is based on their homotopy type, whereas not much is known about the tight case. The results of [34] and [35], based on Seiberg–Witten theory, provide a construction of a class of homology 3-spheres that have an arbitrarily large number of homotopic non-isomorphic tight contact structures.

9.3 Three-manifolds: Thurston norm

Since Seiberg–Witten theory helped in finding estimates on the minimal genus of embedded surfaces in four-manifold, it is natural to attempt similar constructions that estimate the minimal genus in three-manifolds, i.e. the Thurston norm.

Let us recall briefly how the Thurston norm is defined. Again, we follow closely [29], where the reader can find a much more detailed exposition of these results.

Definition 9.6 *Suppose given a fixed homology class*

$$\sigma \in H_2(Y, \mathbb{Z}).$$

The Thurston norm is the minimum of the quantity

$$2g(\Sigma) - 2$$

(or of the sum of such quantities over connected components of genus at least one, if Σ is not connected) over all embedded surfaces Σ in the three-manifold Y of positive genus, realising the class σ.

The dual Thurston norm, defined on classes α in $H^2(Y, \mathbb{R})$, is the supremum over all connected oriented embedded surfaces Σ of the (possibly infinite) quantity

$$\frac{\langle \alpha, [\Sigma] \rangle}{2g(\Sigma) - 2}.$$

The first result that used non-vanishing of the Seiberg–Witten invariant to provide an estimate on the Thurston norm is [1]. By techniques similar to the proof of the Thom conjecture, it is shown that the following holds.

Proposition 9.7 *If $c_1(L)$ is a basic class on Y, i.e. $\chi(HF_*^{SW}(Y, L)) \neq 0$, then the Thurston norm of a class σ is bounded below by*

$$\|\sigma\| \geq \langle c_1(L), \sigma \rangle.$$

This in particular implies that the dual Thurston norm of a basic class is finite.

A simple combination of the this result with the Weitzenböck formula provides a bound on the dual Thurston norm in terms of the scalar curvature as in [33].

The result can be improved, by replacing the strong condition of basic classes with the weaker condition of monopole classes. In fact, as explained in [29], results of Gabai and Thurston give a characterisation of the Thurston norm of a homology class σ as the maximum of the quantities $\langle e(\mathcal{F}), \sigma \rangle$, where $\mathcal{F}$ runs over smooth taut foliations of Y. One needs the hypothesis that $H_2(Y, \mathbb{Z})$ is not generated by tori to ensure the existence of such foliations. This result can be combined with theorem 9.4 and proposition 9.5 about contact structures and Seiberg–Witten invariants, and with the observation made at the end of the previous section about taut foliations. These results imply the following refinement of Auckly's result derived in [29].

Proposition 9.8 *If $c_1(L)$ is a monopole class on Y, i.e. $M_c(Y, L) \neq \emptyset$, then the Thurston norm of a class σ is bounded below by*

$$\|\sigma\| \geq \langle c_1(L), \sigma \rangle.$$

10 Further applications

There are several other directions in which Seiberg–Witten gauge theory has been exploited to provide topological and geometric results. For reasons of space we are unable to present all the interesting contributions. The reader is referred to the bibliographical section at the end of this volume for a list of the literature available at the time when these notes were being collected. We can spend a few words pointing out some particular aspects. The choice reflects the taste of the author.

Among the topological results obtained via Seiberg–Witten theory we should mention work of Morgan and Szabó [39] on the complexity of cobordisms. Gauge theory had already been used in relation to the failure of the h-cobordism theorem in dimension four, in a paper by Donaldson [6]. Morgan and Szabó introduce a notion of complexity that measures the amount by which the h-cobordism theorem fails. Using Seiberg–Witten theory, they are then able to construct a family of examples with arbitrarily large complexity.

In a different direction, various authors have re-obtained results of Donaldson theory via the new Seiberg–Witten invariants. Among these we should mention that Donaldson's theorem on smooth 4-manifolds with definite intersection form [5] has been rederived with Seiberg–Witten techniques in [22], and in [19] in the more general context of 4-manifolds with boundary a disjoint union of rational homology 3-spheres. In this case the Seiberg–Witten moduli spaces have asymptotic values in the Seiberg–Witten Floer homology.

In the field of differential geometry it is worth mentioning some results on Einstein metrics. A uniqueness theorem for such metrics on compact quotients of irreducible 4-dimensional symmetric spaces of non-compact type was proven by LeBrun using Seiberg–Witten theory. Several more recent results were obtained by similar techniques: see the references listed in the bibliographical section at the end of the volume.

10.1 Exercises

- Fill in the details of the proof of step (1) above.
- Check that the argument given in [7] can be adapted to the proof of step (2).
- To complete step (5): check carefully the proof that $[\tilde{\Sigma}][\omega_{g_R}] \to 0$ as $R \to \infty$, as given in [31].
- Complete the proof of point (3) of the Thom conjecture: following [31], consider solutions (A_R, ψ_R) on the cylinder $Y \times [-R, R]$; show that the change in the functional $C(A_R(R), \psi_R(R)) - C(A_R(-R), \psi_R(-R))$ is negative and uniformly bounded, independently of R. For the latter property consider gauge transformations such that $A_R - \lambda_R^{-1} d\lambda_R$ and the first derivatives are uniformly bounded. Show that the functional C changes monotonically along the cylinder, and deduce that for all N there is a solution on $Y \times [0, 1]$ for which the change of C is bounded by $1/N$. Complete the argument by showing that there is a translation invariant solution.

References

[1] D. Auckly, *The Thurston norm and three-dimensional Seiberg–Witten theory*, Osaka J. Math. 33 (1996), no. 3, 737–750.

[2] R. Brussee, *Some C^∞–properties of Kähler surfaces*, New York J. Math. 2 (1996), 103–146.

[3] A.L. Carey, B.L. Wang, *Seiberg–Witten-Floer homology and gluing formulae*, preprint.

[4] S. Donaldson, *The Seiberg–Witten equations and 4-manifold topology*, Bull. AMS, Vol.33 N.1 (1996) 45-70.

[5] S.K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Diff. Geom. 18 (1983) N.2, 279-315.

[6] S.K. Donaldson, *Irrationality and the h-cobordism conjecture*, J. Diff. Geom. 26 (1987) N.1 141-168.

[7] S. Donaldson, *Connections, cohomology, and the intersection forms of four–manifolds*, J. Diff. Geom. 24 (1986) 275–341.

[8] S. Donaldson, P. Kronheimer, *The Geometry of Four-Manifolds*, Oxford 1990.

[9] Y. Eliashberg, *Contact* 3*-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier (Grenoble) 42 (1992), no. 1-2, 165–192.

[10] Y. Eliashberg, W.P. Thurston, *Contact structures and foliations on* 3*-manifolds*, Turkish J. Math. 20 (1996), no. 1, 19–35.

[11] Y. Eliashberg, W.P. Thurston, *Confoliations*. University Lecture Series, 13. AMS 1998.

[12] R. Fintushel, R.J. Stern, *Knots, links, and four-manifolds*, preprint, dg-ga/9612014.

[13] R. Fintushel, R.J. Stern, *Surfaces in* 4*-manifolds*, Math. Res. Lett. 4 (1997) N.6, 907-914.

[14] R. Fintushel, R.J. Stern, *Immersed spheres in* 4*-manifolds and the immersed Thom conjecture*, Turkish J. Math. 19 (1995), no. 2, 145–157.

[15] M.H. Freedman, F. Quinn, *Topology of 4-manifolds*, Princeton 1990.

[16] R. Friedman, J.W. Morgan, *Smooth four-manifolds and complex surfaces*, Springer 1994.

[17] R. Friedman, J.W. Morgan, *Algebraic Surfaces and Seiberg–Witten Invariants*, J. Alg. Geom. 6 (1997) N.3 445-479.

[18] R. Friedman, Zh. Qin, *On complex surfaces diffeomorphic to rational surfaces*, Invent. Math. 120 (1995), no. 1, 81–117.

[19] K.A. Froyshov, *The Seiberg–Witten equations and four-manifolds with boundary*, Math. Res. Lett. 3 (1996), no. 3, 373–390.

[20] D. Gabai, *Foliations and the topology of* 3*-manifolds*, J. Differential Geom. 18 (1983), no. 3, 445–503. *Foliations and the topology of* 3*-manifolds, II*, J. Differential Geom. 26 (1987), no. 3, 461–478. *Foliations and the topology of* 3*-manifolds, III*, J. Differential Geom. 26 (1987), no. 3, 479–536.

[21] R. Gompf, T.S. Mrowka, *Irreducible 4-manifolds need not be complex*, Ann. Math. 138 (1993) 61-111.

[22] M. Katz, *A proof via the Seiberg–Witten moduli space of Donaldson's theorem on smooth* 4*-manifolds with definite intersection forms.* R.C.P. 25, Vol. 47 (Strasbourg, 1993–1995), 269–274, Pripubl. Inst. Rech. Math. Av., 1995/24.

[23] R.C. Kirby, *The topology of 4-manifolds*, Springer LNM 1374, 1989.

[24] D. Kotschick, *The Seiberg–Witten invariants of symplectic four-manifolds (after C. H. Taubes)*, Seminaire Bourbaki, Vol. 1995/96. Asterisque No. 241 (1997), Exp. No. 812, 4, 195–220.

[25] D. Kotschick, *On Irreducible Four–Manifolds*, preprint.

[26] D. Kotschick, J.W. Morgan, C.H. Taubes, *Four manifolds without symplectic structures but with non-trivial Seiberg–Witten invariants*, Math. Research Letters 2 (1995) 119-124.

[27] P. Kronheimer, *The genus-minimizing property of algebraic curves*, Bull. Amer. Math. Soc. 29, N.1 (1993) 63-69.

[28] P.B. Kronheimer, *Minimal genus in* $S^1 \times M^3$, preprint.

[29] P.B. Kronheimer, *Embedded surfaces and gauge theory in three and four dimensions*, preprint.

[30] P. Kronheimer, T. Mrowka, *Gauge theory for embedded surfaces. I*, Topology 32, N. 4 (1993) 773-826. *Gauge theory for embedded surfaces. II*, Topology 34, N.1 (1995) 37-97.

[31] P.B. Kronheimer, T.S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Research Letters 1 (1994) 797-808.

[32] P.B. Kronheimer, T.S. Mrowka, *Monopoles and contact structures*, Invent. Math. 130 (1997), no. 2, 209–255.

[33] P.B. Kronheimer, T.S. Mrowka, *Scalar curvature and the Thurston norm.* Math. Res. Lett. 4 (1997), no. 6, 931–937.

[34] P. Lisca, G. Matić, *Tight contact structures and Seiberg–Witten invariants*, Invent. Math. 129 (1997), no. 3, 509–525.

[35] P. Lisca, G. Matić, *On homotopic, non-isomorphic tight contact structures on* 3*-manifolds*, Turkish J. Math. 20 (1996), no. 1, 37–45.

[36] D. McDuff, D. Salamon, *J-holomorphic curves and quantum cohomology*, University Lecture Series 6, AMS 1994.

[37] J.W. Morgan, *The Seiberg–Witten equations and applications to the topology of smooth four-manifolds*, Princeton University Press, 1996.

[38] J.W. Morgan, T.S. Mrowka, Z. Szabó, *Product formulas along* T^3 *for Seiberg–Witten invariants*, Math. Res. Lett. 4 (1997) N.6 915-929.

[39] J.W. Morgan, Z. Szabó, *On h-cobordisms and Seiberg-Witten invariants*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), 117–124, First Int. Press Lect. Ser., I, Internat. Press 1998 .

[40] J.W. Morgan, Z. Szabó, C.H. Taubes, *A product formula for the Seiberg–Witten invariants and the generalized Thom conjecture*, J. Differential Geom. 44 (1996), no. 4, 706–788.

[41] Ch. Okonek, A. Teleman, *Seiberg–Witten invariants and rationality of complex surfaces*, Math. Z. 225 (1997), no. 1, 139–149.

[42] V. Ya. Pidstrigatch *Some gluing formulas for spin polynomials and a proof of the Van de Ven conjecture* (Russian) Izv. Ross. Akad. Nauk. Ser. Mat. 58 (1994) N.6 105-122 (transl. in Russ. Acad. Sci. Izv. Mat. (1995) N.3, 529-543).

[43] Y. Ruan, G. Tian, *A mathematical theory of quantum cohomology*, J. Differential Geom. 42 (1995), no. 2, 259–367.

[44] Z. Szabó, *On irreducible simply-connected* 4*-manifolds* Turkish J. Math. 21 (1997), no. 1, 41–46.

[45] C.H. Taubes, *Casson's Invariant and Gauge Theory*, J. Diff. Geom. 31 (1990) 547-599.

[46] C.H. Taubes, *The Seiberg–Witten invariants and symplectic forms*, Math. Research Letters, 1 (1994) 809-822.

[47] C.H. Taubes, *More constraints on symplectic manifolds from Seiberg–Witten invariants*, Math. Research Letters, 2 (1995), 9-13.

[48] C.H. Taubes, *The Seiberg–Witten invariants and The Gromov invariants*, Math. Research Letters 2 (1995) 221-238; and Geometry and physics (Aarhus, 1995), 591–601, Lecture Notes in Pure and Appl. Math., 184, Dekker, New York, 1997.

[49] C.H. Taubes, $SW \Rightarrow Gr$. *From the Seiberg–Witten equations to pseudo–holomorphic curves*, J. Amer. Math. Soc. 9, N.3 (1996) 845-918.

[50] C.H. Taubes, *Counting pseudo-holomorphic submanifolds in dimension* 4, J. Differential Geom. 44 (1996), no. 4, 818–893.

[51] C.B. Thomas, Y. Eliashberg, E. Giroux, 3-*dimensional contact geometry*, Contact and symplectic geometry (Cambridge, 1994), 48–65, Publ. Newton Inst., 8, Cambridge 1996.

[52] R.G. Wang, *On Seiberg–Witten Floer invariants and the generalized Thom problem*, preprint.

[53] S. Wang, *A Vanishing theorem for Seiberg–Witten Invariants*, Math Res. Lett. 2 (1995), no. 3, 305–310.

[54] E. Witten, *Monopoles and Four–Manifolds*, Math. Research Letters 1 (1994) 769-796.

Part IV

Seiberg–Witten and Physics

Tell me, Kāpya —–do you know the string on which this world and the next, as well as all beings, are strung together? 'That, my lord, I do not know'.

Bṛhadāraṇyaka Upaniṣad, 3.7.1

11 Mathai-Quillen formalism and Euler numbers

In this chapter we discuss a certain unified approach to different problems arising in Gauge Theory. The approach we present is well known in Theoretical Physics where most of the gauge theoretic problems originated and where they can be formulated in terms of the non-rigorous "functional integration". To a certain extent the problem of functional integration can be overcome by translating the Physics formulation into the more appealing mathematical language of some homology classes of Banach manifolds. The vocabulary that would allow a satisfactory restatement of the Physics results in topological terms is far from being fully developed. We recall briefly some of the known results related to the construction of the Mathai–Quillen form and to infinite dimensional generalisations and we discuss the connection between the regularised Euler numbers of certain Banach bundles and Floer homology.

11.1 The finite dimensional case

We begin by recalling a well known construction, due to V. Mathai and D. Quillen [28], of a de Rham representative of the Thom class of a vector bundle E over a manifold X of dimension $n = 2m$. Having a de Rham representative provides a way of computing Euler numbers as integrals over X. We should think of this operation as our finite dimensional model for the physicist's functional integration.

The object that arises from this construction has three different descriptions: the form (a de Rham representative of an equivariant cohomology class), the (co)-homology class (the Euler class), and the zero set of a generic section of the bundle E.

The advantage of the homological definition over the explicit description in terms of de Rham representative is that it allows easier generalisations to the case when the zero set of the generic section is not a smooth manifold. Under suitable hypotheses, intersection theoretic methods can be used to describe the singular zero set.

On the other hand it is clear that the representative contains more information than the equivalence class, hence it is advisable to seek for generalisations that carry "the same amount" of information. We shall discuss this point later.

11.1.1 The Mathai-Quillen form

Suppose given a real oriented vector bundle E of rank $2m$ over a compact oriented manifold X of dimension $n = 2m$. In this case the Euler number of the bundle is given as

$$\chi(E) = \int_X e(E).$$

The differential form $e(E)$ defines the Euler class,

$$\epsilon(E) = [e(E)] \in H^{2m}(X; \mathbb{R}).$$

This class is the pullback via the zero section of the Thom class $\tau(E)$ that lives in the compactly supported cohomology of the total space of E.

The pullback via any other section would define the same class in cohomology, since any two sections of the vector bundle E are homotopic. Therefore the same Euler number can be also defined by means of the expression

$$\int_X e_s(E), \tag{57}$$

where the form $e_s(E)$ is the pullback via the section s of a form representing the Thom class.

If s has isolated zeroes, we expect (57) to reproduce the Hopf theorem, namely the fact that the Euler number is the algebraic sum of zeroes of a generic section.

It is known from the Chern–Weil theory of characteristic classes that a representative of the Euler class can be given in terms of the Pfaffian of a curvature form on E as

$$e(E) = \frac{1}{(2\pi)^m} Pf(\Omega),$$

where Ω is the curvature 2-form of a connection compatible with an inner product. For a given antisymmetric matrix A, the Pfaffian $Pf(A)$ satisfies $Pf(A)^2 = \det(A)$. The Pfaffian of an antisymmetric matrix is given explicitly as

$$\frac{1}{m!}(\frac{1}{2}w^i A_{ij} w^j)^m = Pf(A) w^1 \wedge \cdots \wedge w^n,$$

where $n = 2m$. Here the product on the left hand side is taken in the exterior algebra generated by the w^i. Notice that $Pf(A) = Pf({}^tTAT)$ for any $T \in SO(2m)$.

It is convenient to express the Pfaffian in terms of the so called Berezin or fermionic integral. This is a functional on a $\mathbb{Z}_2$-graded algebra (superalgebra). In the case where the superalgebra is the exterior algebra $\Lambda[w]$ in the generators w^i, the Berezin integral is defined as

$$\alpha \mapsto \int \mathcal{D}w\ \alpha,$$

where $\int \mathcal{D}w\ \alpha$ is the coefficient of $w^1 \wedge \cdots \wedge w^n$ in the expression of α. In terms of the Berezin integral we can write the Pfaffian of an antisymmetric matrix as

$$Pf(A) = \int \mathcal{D}w \exp(\frac{1}{2}w^i A_{ij} w^j). \tag{58}$$

Thus the Euler class is represented by a form

$$e(E) = \frac{1}{(2\pi)^m} \int \mathcal{D}w \exp(\frac{w_i \Omega^{ij} w_j}{2}). \tag{59}$$

It is clear that, in order to have a representative of the Thom class, we should construct a closed n-form on E that is rapidly decreasing along the fibre direction ([28] pg. 98-99), such that integration along the fibres (which is an isomorphism of degree $-n$) gives 1 in cohomology, and moreover such that, when pulled back along the zero section, reproduces the form (59). Mathai and Quillen gave an explicit form of a representative that satisfies all the expected properties, in terms of $G = SO(n)$ equivariant differential forms on $V = \mathbf{R}^n$. The construction is obtained using the de Rham model (6) of equivariant cohomology. Given a vector bundle E on X, we can write a form in $\Lambda^*(E)$ as a form in $\Lambda^*(P \times_G V)$, and this corresponds to the equivariant complex $\Lambda^*_G(V)$ under the Weil homomorphism. Thus we can describe the Mathai-Quillen form as an object in $\Lambda^*_G(V)$, by choosing generators Ω_{ij} of degree two and the degree one elements $d\xi^j + \theta_{jk}\xi^k$, where the θ_{jk} are the degree one generators.

Consider the extension of the Berezin integral to a map

$$\Lambda[J] \otimes \Lambda[w] \to \Lambda[J],$$

with the variables J to be $J^j = d\xi^j + \theta_{jk}\xi^k$. In this way we can make sense of the expression

$$\int \mathcal{D}w \ \exp(1/2({}^t wAw) + {}^t Jw).$$

This defines a form $\Phi(E)$ as

$$\Phi(E) = \frac{e^{-\xi^2/2}}{(2\pi)^m} \int \mathcal{D}w \exp((w_i \Omega^{ij} w_j)/2 + i\nabla\xi^i w_i). \tag{60}$$

When pulled back to X via the zero section (i.e. with $\xi \equiv 0$), the expression (60) reproduces the representative (59) of the Euler class. On the other hand, integration along the fibres of the component of $\Phi(E)$ which is a $2m$-form in the fibre direction gives 1. The fact that $\Phi(E)$ is closed is not immediate and is proven in [28]. Thus (60) is a representative of the Thom class, $\tau(E) = [\Phi(E)]$.

The form (60) can be pulled back to X via any other section s, and still gives a representative of the Euler class, which is of the form

$$e(E) = \frac{e^{-s^2/2}}{(2\pi)^m} \int \mathcal{D}w \exp((w_i F^{ij} w_j)/2 + i\nabla s^i w_i). \tag{61}$$

Remark 11.1 *The representative of the Euler class (61) has Gaussian decay centred at the zeroes of the section. If we introduce a homotopy parameter t and consider the asymptotic behaviour for $t \to \infty$, the form is centred in a small* *neighbourhood of the zeroes of s, and with the stationary phase approximation this reproduces the Hopf theorem (since we are assuming that the rank of E equals the dimension of X the set of zeroes will be generically a discrete set of points).*

When the rank $2m$ of the bundle E is smaller than the dimension n of the manifold X, the above no longer gives an Euler number. However, it is possible to evaluate against the fundamental class of X the Euler class $e(E)$ cupped with cohomology classes that live in $H^{n-2m}(X)$. The numbers obtained in this way are the intersection numbers of X associated with the bundle E.

Remark 11.2 *In the case $2m < n$ the Euler form can still be thought of as concentrated near the zeroes of a section s (as the parameter $t \to \infty$). Now the zero set of a transverse section will be a manifold of dimension $n - 2m$.*

11.1.2 Intersection theoretic approach

If we restrict ourselves to consider classes instead of representatives, we can give a generalisation of the Euler class in the case when the section is not generic and therefore the zero set is not a smooth manifold of dimension $n - 2m$, but rather some singular space. This is possible under more restrictive hypotheses which make the algebro–geometric machinery of intersection theory available.

In a compact smooth manifold X we can define an Euler class in homology, by taking the cap product of the Euler class $\epsilon_s(E) = s^*\tau(E)$ with the fundamental class of X, i.e. by Poincaré duality. The Thom class $\tau(E)$ can be thought of as a class $\tau(E) \in H^n(E, E_0)$, where E_0 is E with the zero section removed.

Again we see from this approach [30] that the Euler class (as an element in homology) is localised on the zeroes of the section s. In fact we can think of s as a map

$$s : (X, X - \{s = 0\}) \to (E, E_0),$$

and the pullback $s^*\tau(E)$ as living in $H^*(X, X - \{s = 0\})$. The evaluation against X defines a map

$$H^*(X, X - \{s = 0\}) \to H_*(\{s = 0\}).$$

Thus we get that $\epsilon_{*,s}(E) = \epsilon_s(E) \cap [X]$ lives in $H_*(\{s = 0\})$. When the section s_g is generic, i.e. it is transverse to the zero-section (and therefore the zero set of the section s_g is a smooth embedded submanifold), the homological Euler class is the fundamental class of the manifold $Z_g = \{s_g = 0\}$ in $H^{n-2m}(Z_g)$. In the more general case the zero set Z might be singular. Under the assumption that the bundle has a complex structure and the section is holomorphic, Z will be an analytic set. It is still possible in this case to give a notion of homological Euler class by means of Segre classes [15].

We get an expression ([15], [30]) of the form

$$\epsilon_{*,s}(E) = [c^*(E \mid_Z - TX \mid_Z) \cap c_*(Z)]_{n-2m}. \qquad (62)$$

Here the homology class $c_*(Z)$ is Fulton's class, defined for the case of a singular scheme in terms of the Segre classes [15]. It has components in degrees from zero up to the dimension of Z. In the case of a smooth submanifold Fulton's class would correspond just to the expression $c^*(TZ_g) \cap [Z_g]$.

Remark 11.3 *We can think of (62) as a good generalisation of*

$$\epsilon_{*,s_g}(E) = \epsilon_{s_g}(E) \cap [X] = [Z_g].$$

In fact when we consider the section s_g transverse to the zero section we can rewrite (62) as

$$[\frac{c^*(E)}{c^*(TX)} \cap c_*(Z_g)]_{n-2m} = [\frac{c^*(E)}{c^*(N_{Z_g}X)} \cap [Z_g]]_{n-2m},$$

where $N_{Z_g}X$ is the normal bundle of Z_g in X. By the transversality assumption $N_{Z_g}X \simeq E$, therefore in the above we are left just with $[Z_g]$ in dimension $n-2m$.

In general (62) always represents the fundamental class $[Z_g]$ even if computed by means of a section s which is not transverse or such that Z_s is not a smooth manifold.

When s is not transverse to the zero section but, nevertheless, the zero scheme Z is a non-singular embedded submanifold, we can write (62) as

$$[\frac{c^*(E)}{c^*(N_Z X)} \cap [Z]]_{dim(Z_g)} = [c^*(E) \cap ([Z] - c_1(N_X Z) + \cdots)]_{dim(Z_g)},$$

where the contribution in degree $dim(Z_g)$ is exactly $[Z_g]$, whereas the highest degree term would give $[Z]$.

11.2 The infinite dimensional case

The purpose of this section is to discuss extensions of the Mathai-Quillen formalism to the case of an infinite dimensional bundle $\mathcal{E}$ over an infinite dimensional manifold $\mathcal{X}$. This is the situation that arises in the framework of Topological Quantum Field Theory, where the topological Lagrangian is written as the functional integral of an effective action which is, formally, the Mathai-Quillen form of an infinite dimensional vector bundle.

A very clear description of the quantum field theoretic point of view can be found in [7] and [8]. Our purpose is to rephrase as much as possible the results of these references in a more rigorous mathematical language.

Starting with Witten's work [35] and the Atiyah-Jeffrey description [4], it became clear that the partition function of certain $N = 1$ supersymmetric gauge theories can be described as a formal (functional integral) expression

$$\int_{\mathcal{X}} e_s(\mathcal{E}),$$

where s is a section of an infinite dimensional bundle $\mathcal{E}$ over an infinite dimensional manifold $\mathcal{X}$, and $e_s(\mathcal{E})$ is the formal analogue of the Mathai-Quillen representative of the Euler class.

Under suitable hypotheses on the bundle and the section, it is possible to make sense of the above expression in a rigorous way in terms of the Euler number of a finite rank vector bundle over the zero set of the section s. This was pointed out in [7], [8], where this procedure of reduction to a finite dimensional subbundle gives rise to an object that is referred to as the "regularised Euler class". However in these references the "proof" relies on functional integral techniques. We describe in the following how the argument can be made mathematically precise. Our description will also point out the essential difference between a (co)-homological formulation and a formulation at the level of representatives.

Remark 11.4 *In the finite dimensional case the Euler number does not depend on the choice of the section s. In the infinite dimensional setup, when $\chi_s(\mathcal{E})$ can be defined (in a sense that will be made precise in the following), this number depends on the choice of the section s. Actually it will depend essentially on the index of the Fredholm operator Ds that linearises the section.*

11.2.1 The localised homological Euler class

The construction we illustrate in this section is due to R. Brussee [9]; a similar construction was also introduced by V. Pidstrigatch and A. Tyurin, [30], [31]. In a different context, the localised Euler class has been introduced by J. Li and G. Tian [26].

In order to make sense of a homological Euler class for the infinite dimensional case we need to assume certain hypotheses on the bundle and the section.

Definition 11.5 *A Fredholm bundle $(\mathcal{E}, \mathcal{X}, s)$ is a Banach bundle*

$$\mathcal{E} \stackrel{\pi}{\to} \mathcal{X}$$

over a Banach manifold $\mathcal{X}$, endowed with a section s which satisfies the conditions:
(i) the linearisation Ds is a Fredholm operator of index d,
(ii) the determinant line bundle $\det(Ds)$ is trivialised over $Z_s = s^{-1}(0)$.

The linearisation Ds is defined over the zero set Z_s as

$$Ds : T\mathcal{X} \,|_{Z_s} \stackrel{Ts}{\to} s^*T\mathcal{E} \,|_{Z_s} \cong s_0^*T\mathcal{E} \,|_{Z_s} \to \mathcal{E} \,|_{Z_s}, \tag{63}$$

where s_0 is the zero-section, via the splitting

$$0 \to T\mathcal{X} \,|_{Z_s} \stackrel{Ts_0}{\to} s_0^*T\mathcal{E} \,|_{Z_s} \cong s^*T\mathcal{E} \,|_{Z_s} \to \mathcal{E} \,|_{Z_s} \to 0$$

of the exact sequence

$$0 \to \mathcal{E} \to s^*T\mathcal{E} \xrightarrow{\pi} T\mathcal{X} \to 0.$$

Notice that if the bundle $\mathcal{E}$ has a connection the splitting extends off the zero set Z_s.

The result according to Brussee [9] is the following.

Proposition 11.6 *A Fredholm bundle $(\mathcal{E}, \mathcal{X}, s)$ with $Ind(Ds) = d$ has a well defined localised (or regularised) Euler class*

$$\epsilon_{*,s}(\mathcal{E}) \in \check{H}_d(Z_s; \mathbb{Z}).$$

If we want to set the construction in such a way as to deal with situations in which the zero set is not a compact set, then the appropriate homology is an inverse limit of the Čech homology groups over a cofinal family of compact sets,

$$\check{H}^{cl}_*(X; \mathbb{Z}) = \lim_K \check{H}_*(U_K, U_K - K; \mathbb{Z}),$$

and we obtain

$$\epsilon_{*,s}(\mathcal{E}) \in \check{H}^{cl}_d(Z_s; \mathbb{Z}).$$

Here the sets K are compact and the sets U_K are open neighbourhoods that embed in some Euclidean space, with $K \subset U_K$. The technical details of the construction are worked out in [9]. The class in this case is defined through a limiting process over compact sets, with a compatibility condition for $K \subset K'$ of the form

$$\epsilon_{*,K',s}(\mathcal{E}) \mid_{Z_s - K'} = \epsilon_{*,K,s}(\mathcal{E}).$$

The situation is technically less difficult if we restrict our attention only to cases in which Z_s is compact. That will be the case here, since we are mainly concerned with examples taken from Seiberg–Witten gauge theory, where the necessary compactness result holds.

The class $\epsilon_{*,K,s}(\mathcal{E})$ should be thought of as the correct mathematical formulation of the regularised Euler class of [7], [8], defined in terms of a finite rank subbundle of $\mathcal{E}$ over a finite dimensional submanifold of $\mathcal{X}$.

The essential idea of the construction consists of taking a finite rank subbundle E of $\mathcal{E}$ such that on the quotient bundle $\tilde{\mathcal{E}}$ the induced section $\tilde{s}$ has surjective linearisation $D\tilde{s}$. This works over Z_s by taking E spanned by sections that generate $Coker(Ds)$. Upon choosing a sufficiently small neighbourhood U of K in $\mathcal{X}$, $D\tilde{s}$ is also surjective on $Z_{\tilde{s}}$. Thus by the implicit function theorem in Banach spaces $Z_{\tilde{s}}$ is a finite dimensional smooth manifold which is cut out transversely. The dimension of $Z_{\tilde{s}}$ is $N + d$, where N is the rank of E and $d = Ind(Ds)$. Over the zero set $Z_{\tilde{s}}$ there is an induced section $\hat{s}$ of E and the localised Euler class is defined as

$$\epsilon_{*,K,s}(\mathcal{E}) = \epsilon^*_{\hat{s}}(E \mid_{Z_{\tilde{s}}}) \cap [Z_{\tilde{s}}],$$

where $\epsilon^*_{\tilde{s}}(E\mid_{Z_{\tilde{s}}})$ is the usual cohomological Euler class of the finite rank bundle E over the finite dimensional manifold $Z_{\tilde{s}}$.

There is a compatibility condition which ensures that the resulting class is independent of all choices, namely if E' is another finite rank subbundle of $\mathcal{E}$ with the same properties and with rank $N' > N$, then the localised Euler class satisfies

$$\epsilon'_{*,K,s}(\mathcal{E}) = \epsilon^*_{\tilde{s}'}(E'\mid_{Z_{\tilde{s}'}}) \cap [Z_{\tilde{s}'}] =$$
$$= \epsilon^*_{\tilde{s}}(E\mid_{Z_{\tilde{s}}}) \cap \epsilon^*_{\tilde{s}}(E'/E\mid_{Z_{\tilde{s}}}) \cap [Z_{\tilde{s}'}] = \epsilon^*_{\tilde{s}}(E\mid_{Z_{\tilde{s}}}) \cap [Z_{\tilde{s}}] = \epsilon_{*,K,s}(\mathcal{E}).$$

Remark 11.7 *If the linearisation Ds is surjective then the localised Euler class gives, as expected, the fundamental class of the zero set Z_s.*

In the finite dimensional case we have seen that, with some additional structure, it is possible to extend the construction to the case of a singular zero set with intersection theoretic techniques. This extension also carries over to the infinite dimensional case as proved in [9], [30], and [31]. In this case the intersection theoretic formula which corresponds to (62) is

$$\epsilon_{*,s}(\mathcal{E}) = [c^*(Ind(Ds))^{-1} \cap c_*(Z_s)]_{2d}, \tag{64}$$

for the case of a holomorphic bundle $\mathcal{E}$ and a holomorphic section s with the linearisation Ds of complex index d.

11.2.2 Equivariant homology and the Atiyah-Jeffrey formalism

Unfortunately the precise mathematical formulation works at the level of classes, whereas at the level of representatives one can only proceed formally as in the very enlightening introduction to the Mathai–Quillen formalism in Quantum Field Theory, given in [4].

Some manipulations of the expression (61) that are fully explained in [4] lead to another expression for a de Rham representative of the regularised Euler class of a bundle E. This expression can be translated into a formal infinite dimensional integration that describes a "de Rham representative" of the regularised Euler class of a Fredholm bundle $\mathcal{E}$. An accurate analysis of the conditions under which the formal expression can be given a more precise mathematical sense is beyond the scope of this book. It is the author's belief, however, that a construction of a regularised Euler class in the sense of Brussee that exists directly at the level of representatives may be sought within the context of the theory of characteristic currents of Harvey and Lawson [17].

Proposition 11.8 *A representative of the Euler class of a (finite dimensional) bundle E is computed in terms of the Mathai–Quillen form as the integral over the principal bundle P of the form*

$$\begin{array}{c} 2^{-d}\pi^{-d-m}\int \exp(-\mid s\mid^2 + \frac{{}^t w\rho(q)w}{4} + i^t dsw \\ -i < d\nu, h > +i(q,Rh) + < dy, \gamma(f) >) \mathcal{D}f\mathcal{D}w\mathcal{D}q\mathcal{D}h. \end{array} \tag{65}$$

In the above expression $d = dim(G)$; q, h, and f are Lie algebra variables that arise in the use of a Fourier transform and in expressing the invariant volume in terms of the Killing form; the y's are coordinates on P; and ν is a canonical 1–form with values in $\mathcal{L}(G)$, defined by the action of G on P. The operators γ and R arise in the definition of ν as

$$\nu_\xi(v) =< \gamma(\xi), v >,$$

where v is a tangent vector, and $\xi \in \mathcal{L}(G)$; $\gamma(\xi)$ is the vector field given by the infinitesimal action of ξ (like in definition 2.12). The operator R is defined as $R = \gamma^*\gamma$, i.e. $< \gamma(\xi), \gamma(\zeta) >= (R\xi, \zeta)$, where the inner product $(,)$ is the one given by the Killing form.

A very nice description of the representative (65) in terms of equivariant homology can be found in [5]. In the case of a free G-action on a manifold M, the form (65) is exactly the current that represents the fundamental class of the quotient M/G in the dual equivariant de Rham complex of M.

The argument given here carries over, at least formally, to some infinite dimensional cases, [4], [28], and it provides therefore the right mathematical setup in which the topological Lagrangian introduced by Witten for Donaldson theory [35] lives. The analogous construction works for Seiberg–Witten theory, as shown in [10] or [25]. We are going to discuss this in detail in the following.

11.3 Euler numbers in Seiberg–Witten theory

The Fredholm map Ds given by the linearisation (63) is the operator that specifies the deformation complex of the gauge theory. Typically the Banach manifold $\mathcal{X}$ is some configuration space $\mathcal{A}/\mathcal{G}$ of connections (or connections and sections) modulo gauge transformations. It is often better to think of framed configuration spaces in order to avoid singularities that otherwise occur at the points where the action of the gauge group is not free. Alternatively, one can restrict the action only to the irreducible elements in $\mathcal{A}$. The section of the Banach bundle is given by the differential equations of the gauge theory and the operator Ds corresponds to the linearisation of the equations, with some gauge fixing condition. The zero set Z_s is the moduli space $\mathcal{M}$ of solutions of the equations modulo gauge.

The deformation complex can be thought of as the short complex C^* given by

$$0 \to T\mathcal{X} \stackrel{Ds}{\to} \mathcal{E} \to 0,$$

where $H^0(C^*) = Ker(Ds)$ and $H^1(C^*) = Coker(Ds)$. However it is often the case that C^* can be written as the assembled complex of a longer deformation complex involving the infinitesimal action of the gauge group $\mathcal{G}$.

Consider the example of Seiberg–Witten gauge theory on four-manifolds. Let X be a compact oriented four-manifold without boundary, with $b_2^+(X) > 1$. The Banach manifold $\hat{\mathcal{A}}$ is the manifold of pairs (A, ψ) with a non-trivial spinor section $\psi \in \Gamma(X, W^+)$. Upon topologising the space $\hat{\mathcal{A}}$ and the group of gauge transformations $\mathcal{G}$ with the appropriate Sobolev norms, the quotient $\mathcal{X} = \mathcal{A}/\mathcal{G}$ can be made into a Banach manifold. The fact that we only consider non-trivial sections ensures that the action of $\mathcal{G}$ is free, hence no singularities occur in $\mathcal{X}$. The bundle $\mathcal{E}$ over $\mathcal{X}$ has fibre $\Lambda^{2+}(X, i\mathbb{R}) \oplus \Lambda^0(X, i\mathbb{R}) \oplus \Gamma(X, W^-)$. The section s of $\mathcal{E}$ is given by the Seiberg–Witten equations with a gauge fixing condition

$$s(A, \psi) = (F_A^+ - \sigma(\psi, \psi), d^*(A - A_0), D_A\psi), \tag{66}$$

with $\sigma(\psi, \psi) = < e_i e_j \psi, \psi > e^i \wedge e^j$.

The linearisation Ds at a point (A, ψ) in Z_s is the Fredholm operator

$$Ds : \Lambda^1(X, i\mathbb{R}) \oplus \Gamma(X, W^+) \to \Lambda^{2+}(X, i\mathbb{R}) \oplus \Lambda^0(X, i\mathbb{R}) \oplus \Gamma(X, W^-)$$

given by

$$Ds \mid_{(A,\psi)} (\alpha, \phi) = \begin{cases} d^+\alpha - \frac{1}{2} Im(< e_i e_j \psi, \phi >) e^i \wedge e^j \\ d^*\alpha - i < \psi, \phi > \\ D_A \phi + i\alpha \cdot \psi. \end{cases}$$

This can be rewritten in terms of the long deformation complex

$$0 \to \Lambda^0 \xrightarrow{G} \Lambda^1 \oplus \Gamma(X, W^+) \xrightarrow{T} \Lambda^{2+} \oplus \Gamma(X, W^-) \to 0,$$

where the operator G is the linearisation of the action of the gauge group $G(f) = (-idf, if\psi)$ and T is the linearisation of the section s regarded as a map from $\mathcal{A}$ to $\mathcal{E}$,

$$T \mid_{(A,\psi)} (\alpha, \phi) = (d^+\alpha - \frac{1}{2} Im(< e_i e_j \psi, \phi >) e^i \wedge e^j, D_A\phi + i\alpha \cdot \psi).$$

Thus we obtain $Ds = T + G^*$.

In the case of the long deformation complex we have $H^0(C^*) = 0$ (the infinitesimal action of the gauge group in an injective operator whenever ψ is not identically zero). We also have $H^1(C^*) = Ker(T)/Im(G) \cong Ker(Ds)$ and $H^2(C^*) = Coker(T) \cong Coker(Ds)$. Thus we get the exact same information. In particular notice that the $H^2(C^*)$ is an obstruction to Z_s being a smooth and cut out transversely. Under the assumption that the four-manifold has $b_2^+(X) > 1$ it is possible to perturb the section by adding a self dual 2-form and make $H^2(C^*)$ trivial, as we discussed previously. The zero set Z_s is the Seiberg–Witten moduli space, which in this case is a compact smooth manifold of dimension given by the index of the operator Ds,

$$Ind(Ds) = \frac{c_1(L)^2 - (2\chi + 3\sigma)}{4}.$$

Another example is the dimensional reduction of the Seiberg–Witten theory on three-manifolds. Let Y be a closed oriented three-manifold. Let $\tilde{W}$ be the spinor bundle associated to a $Spin_c$-structure. If Y has $b_1(Y) > 1$, then we can consider the infinite dimensional manifold $\mathcal{A}$ of $U(1)$ connections and non-trivial spinors, and the quotient $\mathcal{X}$ by the action of the gauge group. The bundle $\mathcal{E}$ has fibre $\Lambda^1(Y, i\mathbb{R}) \oplus \Lambda^0(Y, i\mathbb{R}) \oplus \Gamma(Y, \tilde{W})$ and the section s is given by

$$s(A, \psi) = (*F_A - \tau(\psi, \psi), d^*(A - A_0), \partial_A \psi), \tag{67}$$

where $\tau(\psi, \psi)$ is the 1-form given in local coordinates as $< e_i \psi, \psi > e^i$ and ∂_A is the Dirac operator on $\tilde{W}$ twisted with the connection A.

The linearisation Ds determines the short deformation complex

$$0 \to \Lambda^0(Y) \oplus \Lambda^1(Y) \oplus \Gamma(Y, \tilde{W}) \stackrel{Ds}{\to} \Lambda^0(Y) \oplus \Lambda^1(Y) \oplus \Gamma(Y, \tilde{W}) \to 0,$$

with

$$Ds \mid_{(A,\psi)} (f, \alpha, \phi) = \begin{cases} *d\alpha + 2Im(\tau(\psi, \phi)) - df \\ -\partial_A \phi - i\alpha\psi + if\psi \\ d^*\alpha - if < \psi, \phi > . \end{cases}$$

The space $\Lambda^0(Y) \oplus \Lambda^1(Y) \oplus \Gamma(Y, \tilde{W})$ is the tangent space $T\mathcal{X}$ at the point (A, ψ) of Z_s. Since we are assuming $b_1(Y) > 1$, we can guarantee that under a suitable perturbation of the section by a 1-form the zero set will not contain points with trivial ψ.

The more general case can be worked out in the equivariant setup [27], by considering $\mathcal{X} = \mathcal{A}/\mathcal{G}_b$ where $\mathcal{G}_b$ is the group of based gauge transformations, i.e. those maps that act as the identity on a preferred fibre of $\tilde{W}$. In this case the framed configuration space $\mathcal{X}$ is a manifold, even though the action of the full gauge group is not free.

11.3.1 Atiyah-Jeffrey description

Some of the references available on the Mathai-Quillen formalism in Seiberg–Witten gauge theory are [25], [10]. A different construction that uses the BRST model of equivariant cohomology (see [19], [12], [13]) can be found for instance in [16].

Consider the case of perturbed Seiberg–Witten equations on four-manifolds. The section $s(A, \psi)$ given in (66) satisfies

$$\mid s(A, \psi) \mid^2 = S(A, \psi),$$

where $S(A, \psi)$ is the Seiberg–Witten functional defined in (11).

We can identify the various terms of the Euler class, as given in proposition 11.8 following the case of Donaldson theory analysed in [4]: this has been done in [25], [10].

Note that, according to (17), we have variables $dy = (\alpha, \phi)$, with $\alpha \in \Lambda^1(X)$ and $\phi \in \Gamma(X, W^+)$, that represent a basis of forms on $\hat{\mathcal{A}}$; and the variable f which counts the "gauge directions": $f \in \Lambda^0(X)$ satisfies $e^{if} \in \mathcal{G}$. Similarly q and h are in the Lie algebra, i.e. in $\Lambda^0(X)$. $w = (\beta, \chi) \in \Lambda^{2+}(X) \oplus \Gamma(X, W^-)$ is the variable along the fibre.

The term ds, therefore, is just the linearisation Ds of the Seiberg–Witten equations given in lemma 4.2:

$$ds_{(A,\psi)}(\alpha, \phi) = (D_A\phi + i\alpha \cdot \psi, d^+\alpha - \frac{1}{2}Im(< e_i e_j \psi, \phi >)e^i \wedge e^j).$$

Moreover, it is clear that the operator γ is the map G of the complex (17) that describes the infinitesimal action of the gauge group. Hence

$$\gamma(f) = G(f) = (-idf, if\psi) \in \Lambda^1(X) \oplus \Gamma(X, W^+)$$

and

$$\gamma^*(\alpha, \phi) = -d^*\alpha + \frac{1}{2}Im(< e_i e_j \psi, \phi >).$$

Thus the operator R is given by

$$R = d^*d + \mid \psi \mid^2 .$$

Thus all the terms in (65) can be computed:

$$\frac{1}{4}{}^t w\rho(q)w = \frac{-i}{4}q \mid \chi \mid^2 dv,$$

$$i^t ds\ w = i\beta \wedge *d^+\alpha - \frac{i}{2}(\beta, Im(< \psi, \phi >)e^i \wedge e^j)dv + < D_A\phi + i\alpha\psi, \chi > dv,$$

$$-i < d\nu, h >= -ih \mid \beta \mid^2,$$

$$i(q, Rh) = iq(d^*dh + \mid \psi \mid^2 h),$$

$$< dy, \gamma(f) >= (\gamma^* dy, f) = df \wedge *d\alpha + \frac{f}{2}Im(< e_i e_j \psi, \phi >)dv.$$

The only expression that requires some more comments is the third above, where $d\gamma^*(dy_1, dy_2)$ is computed [25] using

$$d\gamma^*(dy_1, dy_2) = dy_1(\gamma^*(dy_2)) - dy_2(\gamma^*(dy_1)) - \gamma^*([dy_1, dy_2])$$

and constant vector fields dy_1 and dy_2, whose Lie bracket vanishes identically.

Thus we have constructed a "topological Lagrangian"; we want to recover the Seiberg–Witten invariants as correlation functions.

11.3.2 Seiberg–Witten Invariants Revisited

There is an essentially unique possible definition of the Seiberg–Witten invariants when the moduli space M is zero-dimensional, which we introduced in definition 4.11. As already mentioned, there is a natural choice of how to extend the definition of the invariants for a higher dimensional moduli space. In fact we have seen that the choice of the line bundle of lemma 4.12 is preferable for homotopy-theoretic reasons. Nevertheless, there are other constructions that can be considered and that bear a certain naturality within the context of Quantum Field Theory. We describe one in this section and another, related to the concept of regularised Euler characteristic, in the following.

What follows here is partially just a formal argument, as the computations have to be carried out in an infinite dimensional setup. In the previous paragraph we have shown that the Mathai–Quillen form of the Euler class of the bundle E on $\hat{\mathcal{A}}/\mathcal{G}$ is

$$e = 2^{-d}\pi^{-m-d} \int \exp(- \mid s \mid^2 + (\frac{-i}{4} q \mid \chi \mid^2 - \frac{i}{2}(\beta, Im < \psi, \phi > e^i \wedge e^j)$$

$$+ < D_A\phi + i\alpha\psi, \chi > -ih \mid \beta \mid^2 + iq(\Delta h + \mid \psi \mid^2 h) + \frac{f}{2} Im < e_i e_j \psi, \phi >)dv$$

$$+ i\beta \wedge *d^+\alpha + df \wedge *d\alpha)\mathcal{D}f\mathcal{D}h\mathcal{D}q\mathcal{D}\beta\mathcal{D}\chi.$$

When the moduli space is zero dimensional the Seiberg–Witten invariant is obtained as the Euler number of the bundle, [4], i.e. by integrating the above class over the total space P (here $\mathcal{D}\alpha\mathcal{D}\phi$ is just a formal measure, or a "functional integration" in the language of physics, since the form is being integrated over an infinite dimensional manifold).

The Euler class above formally has "codimension" equal to the dimension of the moduli space, since that is $-Ind(C^*)$ where C^* is the chain complex obtained by linearising the Seiberg–Witten equations [4].

Hence, when the moduli space is of positive dimension, in order to obtain numerical invariants, we need to cup the class above with other cohomology classes of $\hat{\mathcal{A}}/\mathcal{G}$.

The following is a possible construction, mimicking Donaldson's construction of the polynomial invariants. The idea is to choose a bundle over $\hat{\mathcal{A}}/\mathcal{G} \times X$, integrate a characteristic class of this bundle against a homology class of X, and then, restricting to $M \times X \hookrightarrow \hat{\mathcal{A}}/\mathcal{G} \times X$, get a map

$$\omega : H_k(X) \to H^{i-k}(\hat{\mathcal{A}}/\mathcal{G}),$$

where $i-k$ is the dimension of M, and i is the degree of the characteristic class.

There is a $U(1)$-bundle $\hat{\mathcal{A}}/\mathcal{G}_0 \to \hat{\mathcal{A}}/\mathcal{G}$, where $\mathcal{G}_0$ is the gauge group of base point preserving maps. In fact $\mathcal{G}/\mathcal{G}_0 = U(1)$. The restriction of this bundle to $M \hookrightarrow \hat{\mathcal{A}}/\mathcal{G}$ is the $U(1)$ bundle used in definition 4.13.

We can form the bundle $\mathcal{Q} = \hat{\mathcal{A}} \times_{\mathcal{G}_0} L^2$ over $\hat{\mathcal{A}}/\mathcal{G} \times X$ and consider its first Chern class $c_1(\mathcal{Q})$.

Then take the pullback via the inclusion of the moduli space $M \hookrightarrow \hat{\mathcal{A}}/\mathcal{G}$; this gives $i^*\mathcal{Q} \to M \times X$.

The construction above gives a map

$$\omega : H_k(X;\mathbb{Z}) \to H^{2-k}(M;\mathbb{Z}).$$

In fact, take a class $\alpha \in H_k(X;\mathbb{Z})$, and its Poincaré dual $a = PD(\alpha)$; take the first Chern class

$$c_1(\mathcal{Q}) \in H^2(\hat{\mathcal{A}}/\mathcal{G} \times X;\mathbb{Z})$$

and decompose it according to the Künneth formula

$$H^2(\hat{\mathcal{A}}/\mathcal{G} \times X;\mathbb{Z}) = \oplus_j H^j(X;\mathbb{Z}) \otimes H^{2-j}(\hat{\mathcal{A}}/\mathcal{G};\mathbb{Z}),$$

as

$$c_1(\mathcal{Q}) = \oplus_j c_1(Q)^j_{2-j}.$$

Evaluate against X to get a class

$$\int_X c_1(\mathcal{Q})^k_{2-k} \wedge a \in H^{2-k}(\hat{\mathcal{A}}/\mathcal{G};\mathbb{Z}).$$

The pullback via i^* defines a class in $H^2(M;\mathbb{Z})$:

$$\omega(\alpha) \equiv i^* \int_X c_1(\mathcal{Q}) \wedge a.$$

This defines maps

$$q_d : H_{k_1}(X) \times \cdots \times H_{k_r}(X) \to \mathbb{Z}$$

with $\sum_{j=1}^r (2 - k_j) = d$

$$q_d(\alpha_1, \cdots, \alpha_r) = \int_M \omega(\alpha_1) \wedge \cdots \wedge \omega(\alpha_r),$$

where d is the dimension of the moduli space M. These play the role, in our construction, of Donaldson's polynomial invariants.

Thus, according to the quantum field theoretic formalism, we obtain the invariants by evaluating over $\hat{\mathcal{A}}/\mathcal{G}$ the Euler class cupped with classes $\omega(\alpha)$:

$$N \equiv \int_{\hat{\mathcal{A}}/\mathcal{G}} s^*(e) \wedge \omega(\alpha_1) \wedge \cdots \wedge \omega(\alpha_r),$$

with $d = dim(M)$.

This should be considered as the definition that corresponds to the description of the invariants as expectation values of the operators obtained by the formalism of Quantum Field Theory, [35].

The operators constructed in [25], or [10] following [35], are:

$$W_{k,0} = \frac{f^k}{k!},$$
$$W_{k,1} = \alpha W_{k-1,0},$$
$$W_{k,2} = FW_{k-1,0} - \frac{1}{2}\alpha \wedge \alpha W_{k-2,0},$$
$$W_{k,3} = F \wedge \alpha W_{k-2,0} - \frac{1}{3!}\alpha \wedge \alpha \wedge \alpha W_{k-3,0},$$
$$W_{k,4} = \frac{1}{2}F \wedge FW_{k-2,0} - \frac{1}{2}F \wedge \alpha \wedge \alpha W_{k-3,0} - \frac{1}{4!}\alpha \wedge \alpha \wedge \alpha \wedge \alpha W_{k-4,0}.$$

The choice of different k should correspond [35] to different choices of characteristic classes to pair with the homology classes of X in the construction of the invariants. In particular we should obtain a relation between some of these operators and the polynomial invariants constructed above, adapting to the present case the argument given for Donaldson theory in [4].

The Chern class $c_1(\mathcal{Q})$ should be interpreted as a curvature on the infinite dimensional bundle $\mathcal{Q}$ in such a way that the components $c_1(\mathcal{Q})_i^{2-i} \in \Lambda^i(X) \otimes \Lambda^{2-i}(\hat{\mathcal{A}}/\mathcal{G})$ should be written in terms of the operators W as

$$\int_{\hat{\mathcal{A}}/\mathcal{G}} s^*(e) \prod_{i=1}^{r} \int_{\alpha_i} W_i = \int_M \int_{\alpha_1} c_1(\mathcal{Q})_{i_1}^{2-i_1} \wedge \cdots \wedge \int_{\alpha_r} c_1(\mathcal{Q})_{i_r}^{2-i_r},$$

integrated over submanifolds α_i of X of the proper dimension.

It is not clear that the polynomial invariants defined in this section are non-trivial for positive dimensional moduli spaces. In fact it is conjectured that for simple type manifolds (see the section on Seiberg–Witten and Donaldson theory) the only non-trivial invariants are associated to zero-dimensional moduli spaces.

11.3.3 Remarks

There are many other cases of interest in which the Mathai-Quillen formalism and the mathematical formulation of regularised Euler classes can be applied. Most of them present technical problems due to the non-compactness of the moduli space. The case of Donaldson theory was considered with similar techniques in [4], [7], [8], [30], [31], [35]. The case of $SU(2)$ gauge theory on three-manifolds was considered in [7], [8]. Another interesting case is the moduli space of J-holomorphic curves on a symplectic manifold. A construction via the regularised Euler class has been recently introduced by Tian [26]. Interesting insight on the use of analogous techniques for moduli spaces of curves can be found in [1], [20], [36].

11.4 $N = 2$ symmetry and the Euler characteristic

It was already noticed in [7] and [8] that, in the formulation of gauge theories via the localised Euler class, the examples on four-manifolds and the dimensional reductions on three-manifolds present different behaviour. This phenomenon is described in [7] and [8] as essentially related to the index of the operator Ds being zero or not. The index is, in fact, the virtual dimension of the moduli space or, in other words, the homological degree in which the regularised Euler class sits.

To have positive index means that the Euler class has no component in degree zero. From the viewpoint of Physics the degree zero term of the regularised Euler class is the partition function of the field theory. Therefore in this case they say [7], [8] that in the presence of zero-modes (the non-trivial kernel of Ds) the partition function vanishes identically.

In gauge theory this problem is overcome by capping the regularised Euler class with other classes until the resulting class sits in degree zero. This method is used to produce numerical invariants when the moduli space is of positive dimension, as in the case of Donaldson polynomials. In Seiberg–Witten theory there is a canonical choice of a class which is used to lower the homological degree of the Euler class, namely the Chern class of the line bundle

$$Z_s^b \overset{U(1)}{\to} Z_s,$$

where Z_s^b is the zero set of the section s defined on the framed configuration space $\mathcal{X} = \mathcal{A}/\mathcal{G}_b$, with $\mathcal{G}_b$ the group of based gauge transformations that act as the identity on a preferred fibre of $\tilde{W}$. It is a canonical choice, since Z_s is a model of the classifying space of the group $\mathcal{G}$ and it is therefore homotopy equivalent to $\mathbb{C}P^\infty \times K(H^1(X, \mathbb{Z}), 1)$. The circle bundle $Z_s^b \overset{U(1)}{\to} Z_s$ is a principal $U(1)$-bundle with first Chern class given by the generator of the homology of the $\mathbb{C}P^\infty$ factor, as we discussed in the first part of these notes.

When the index of Ds is zero the Euler class lives in degree zero (or degree zero and higher in the singular case) and it computes the regularised Euler number of the bundle $\mathcal{E}$. In this case, even when the section is not transverse to the zero section and the set Z_s can be a positive dimensional manifold, the partition function does not vanish and we can still get a non-trivial invariant.

This simple observation leads to what the physicists call the $N = 2$ supersymmetric formulation of regularised Euler numbers. We state some mathematical results and then we try to explain the meaning of the physical interpretation.

Proposition 11.9 *Suppose given a Fredholm bundle $(\mathcal{E}, \mathcal{X}, s)$ such that the operator Ds is surjective. Under this assumption s induces a section s_T of the tangent bundle $T\mathcal{X}|_{Z_s}$. Thus we can associate to the bundle $(\mathcal{E}, \mathcal{X}, s)$ a regularised Euler characteristic, which is in fact the Euler characteristic of Z_s and which we can regard as a regularised Euler characteristic of the infinite dimensional manifold $\mathcal{X}$.*

Proof: Since we assume that $Coker(Ds) = 0$, the zero set Z_s is a smooth manifold which is cut out transversely and $Ker(Ds)$ is the tangent bundle of Z_s. Thus we have a splitting

$$0 \to Ker(Ds)|_{Z_s} \to T\mathcal{X}|_{Z_s} \stackrel{Ds}{\to} \mathcal{E}|_{Z_s} \to 0.$$

The restriction to $Ker(Ds) = TZ_s$ of the induced section s_T of $T\mathcal{X}|_{Z_s}$ gives us the Euler characteristic of Z_s, computed as the pairing of the cohomological Euler class of the bundle TZ_s with Z_s.
QED

Thus we can define a regularised Euler characteristic of the Fredholm bundle $\mathcal{E}$ with a transverse section s as the topological Euler characteristic of the zero-set Z_s, which can be obtained in terms of regularised Euler classes,

$$\chi_s(\mathcal{E}) = \chi^{topol}(Z_s) = \epsilon_{*,s_T}(T\mathcal{X}|_{Z_s}) = \epsilon^*_{s_T}(Ker(Ds)) \cap [Z_s].$$

The hypothesis that Ds is surjective is essential in 11.9 in order to be able to lift the section s to a section s_T. Morevor, if Ds has a non-trivial cokernel, in general $Ker(Ds)$ would not be a bundle, hence a different formulation is needed.

We can expect the definition of the regularised Euler characteristic to generalise to the singular case. The construction in that case would require an accurate analysis of degeneracy loci. A particular case has been considered in [36].

One can seek an intersection theoretic formulation for the Euler characteristic as well. In fact, in the finite dimensional case we have

$$\chi(Z_g) = [c^*(TX|_Z - E|_Z) \cap [c^*(E|_Z - TX|_Z) \cap c_*(Z)]_{\dim Z_g}]_0,$$

where Z_g is the zero set of a generic (transverse) section, whereas we would have

$$\chi(Z) = [c^*(TX|_Z - E|_Z)c^*(E|_Z - TX|_Z) \cap c_*(Z)]_0 = [c_*(Z)]_0$$

for $Z = Z_s$ the smooth zero set of a non-transverse section.

Notice that, under the hypothesis of 11.9, the construction provides a different gauge theoretical invariant associated to a positive dimensional moduli space, namely its Euler characteristic. In a sense this is more canonical than capping with cohomology classes, since it does not depend on the choice of the classes.

We digress briefly to explain the relation of all this to supersymmetry. The $N = 1$ supersymmetry is already intrinsic in the construction of the Mathai-Quillen form [28], in as it enters the definition of the Berezin integral. This relation has been exploited in Witten's interpretation of Donaldson's theory as a Topological Quantum Field Theory [36] and in the subsequent work of Atiyah and Jeffrey [4]. Now consider the expression

$$\chi(Z_s) = \epsilon^*_{s_T}(Ker(Ds)) \cap [Z_s]$$

that we obtained as the regularised Euler characteristic, under the strong assumption that Ds is surjective. In this case we can rewrite the fundamental class $[Z_s]$ as the regularised Euler class of $\mathcal{E}$,

$$\chi(Z_s) = \epsilon^*_{s_T}(Ker(Ds)) \cap [\epsilon^*_{\tilde{s}}(E|_{Z_{\tilde{s}}}) \cap [Z_{\tilde{s}}]].$$

The $N = 2$ supersymmetry enters the picture when we want to interpret the extra cohomological factor $\epsilon^*_{s_T}(Ker(Ds))$ that appears in this formula as another fermionic integral. This is related to the Riemannian structure of the moduli space Z_s [6] and to the presence of a compatible complex structure. An exposition of this in the case of Donaldson moduli spaces can be found for instance in [18].

After the brief digression into Physics, we would like to come back to our favourite gauge theoretic examples. Consider again Seiberg–Witten theory on three-manifolds. The section $s(A, \psi) = (*F_A - \tau(\psi, \psi), d^*(A - A_0), \partial_A \psi)$ can be thought of as being a section of the tangent bundle of $\mathcal{X}$.

Since the index of the linearisation is zero, the regularised Euler class sits in degree zero and already computes the Euler characteristic of the moduli space. Thus we have the following result.

Proposition 11.10 *Assume that $b_1(Y) > 0$. In this case the section s can be perturbed in such a way that Z_s contains no reducible point and the linearisation Ds is surjective. The Seiberg–Witten invariant in three dimensions then represents the regularised Euler characteristic of the Fredholm bundle $(\mathcal{E}, \mathcal{X}, s)$ with $\mathcal{X} = \hat{\mathcal{A}}/\mathcal{G}$ and fibre $\Lambda^1(Y, i\mathbb{R}) \oplus \Lambda^0(Y, i\mathbb{R}) \oplus \Gamma(Y, \tilde{W})$.*

11.4.1 Three dimensional Atiyah-Jeffrey formalism

Using the complex (23) and the Mathai–Quillen formalism, it is possible to construct a topological Lagrangian for the three-dimensional theory as well. This is dealt with in [10]. The upshot is that the invariant can be recovered as a partition function of the QFT. This has been explained in the language of regularised Euler classes by saying that the properties that $Ind(Ds) = 0$ and that s is transverse give rise to a regularised Euler class that lives in dimension zero and is just the sum of the oriented points in the (compact) moduli space. This is the analogue of the gauge theoretic description of the Casson invariant [32].

Again we face the problem of having a mathematical formulation which is rigorous but which only makes sense at the level of classes, whereas the language spoken in the Physics literature tends to represent the partition function as the formal integral of a de Rham representative of the regularised Euler class. The result would then be rephrased by saying that the integral is localised at the points of Z_s.

The formal expression of the de Rham representative, as we have already seen in the four dimensional case, is given following the Atiyah-Jeffrey formulation. The case of three dimensional gauge theory has been analysed in [10].

Proposition 11.11 *The expression of the formal de Rham representative of the regularised Euler number associated to the three-dimensional Seiberg–Witten gauge theory is of the form (65) with s the section (67), ds the corresponding linearisation $Ds_{(A,\psi)}(f, \alpha, \phi)$, and*

$$\gamma(f) = G(f) = (-idf, if\psi) \in \Lambda^1(Y) \oplus \Gamma(Y, \tilde{W}),$$
$$\gamma^*(\alpha, \phi) = G^*_{(A,\psi)}(\alpha, \phi),$$
$$R(f) = d^*d + \mid \psi \mid^2,$$
$$\frac{1}{4} w^t \rho(q) w = \frac{-i}{4} q \mid \chi \mid^2 \, dv,$$
$$-i < d\nu, h >= -ih \mid \beta \mid^2,$$
$$i(q, Rh) = iq(d^*dh + \mid \psi \mid^2 \, h),$$
$$< dy, \gamma(f) >= (\gamma^* dy, f) = i\alpha \wedge *df + < \phi, if\psi > dv.$$

The formal functional integral (65) computes in this case the regularised Euler characteristic of the Fredholm bundle $\mathcal{E}$ with the section $s(A, \psi)$ given in (67), that is, the Seiberg–Witten invariant of three-manifolds. The mathematician might think that the use of expression (65) to define the invariant rather than the sum of gauge inequivalent solutions of equations (21) and (22) is a useless complication, especially given that it is expressed in terms of non-rigorous functional integration. However, it must be understood that the quantum field theoretical formulation provides the physicist with appropriate tools and formal rules designed to the purpose of computing these functional integrals. Often it is precisely the fact that one can rephrase the computation as an integration over all the "unconstrained fields", rather than just the sum over the fields "constrained" by the differential equations, that makes it computable. It is the author's belief that a further study of the properties of the regularised Euler classes of Fredholm bundles will help to make some of these tools available to mathematicians as well.

11.5 Quantum Field Theory and Floer homology

The three dimensional invariant and the Floer homology were first introduced within the quantum field theoretic formalism [10]. It is well known from Atiyah's formulation of quantum field theory [2],[3] that we can think of a quantum field theory as a functor which associates to a closed three manifold a vector space and to a four-manifold with boundary an element in the vector space attached to the boundary. A pairing of the vector space with its dual corresponds to gluing two four-manifolds along their boundaries. The numbers that results via this pairing are invariants of the differentiable structure of a closed 4-manifold. In our case the Floer homology is the vector space associated to a three-manifold. However, our case does not entirely fit into Atiyah's definition. In fact we have seen that there is a subtle problem of metric dependence in the Seiberg–Witten Floer homology: a phenomenon that did not appear in Donaldson theory.

Instanton homology also didn't quite fit into Atiyah's formulation, but for a different reason. In that case the main difficulty was to extend the definition of the Floer homology from the case of homology spheres to all closed three-manifolds. To a large extent this problem was overcome following two different strategies: the Fukaya-Floer [14] homology or the equivariant Floer homology [5]. The latter can be used also in Seiberg–Witten theory to deal with a similar problem; and the equivariant formulation turns out to be effective also in dealing with the metric dependence problem. A Fukaya-Floer complex for Seiberg–Witten theory has been considered in [11] in connection to the formulation of relative invariants.

The problem of the metric dependence also has a physical formulation.

Definition 11.12 *A quantum field theory is determined by a manifold X and a Fredholm bundle $(\mathcal{E}, s)$ defined by means of geometric data on the manifold X (metrics, connections and sections of some vector bundles, etc.). The "expectation values" of a quantum field theory are the Euler numbers obtained by capping the regularised Euler class $\epsilon_s(\mathcal{E})$ with cohomology classes of total degree $Ind(Ds)$, which also encode some geometric data of the manifold X. A quantum field theory is called "topological" if the expectation values are independent of the metric on X.*

We distinguish two kinds of topological quantum field theories (see the overview [23], [24]).

Definition 11.13 *A topological quantum field theory is said to be of Schwarz type if it satisfies the condition that the variation of the section s of the Fredholm bundle $\mathcal{E}$ with respect to a one parameter family of metrics on X is zero and that the cohomology classes that are capped with the regularised Euler class $\epsilon_s(\mathcal{E})$ are also chosen in a way that is independent of the metric on X.*

A typical example of topological quantum field theory of Schwarz type is Chern-Simons theory.

Definition 11.14 *A topological quantum field theory is said to be of Witten type if it satisfies the condition that the cohomology classes capped with $\epsilon_s(\mathcal{E})$ are independent of the metric on X and the variation of the Mathai–Quillen form (65) with respect to a one parameter family of metrics on X is an exact form.*

An example of topological quantum field theory of Witten type is the twisted Yang-Mills theory that reproduces Donaldson polynomials as expectation values [35]. Seiberg–Witten gauge theory also fits into this second type. In the three-dimensional case when $b_1(Y) > 0$ the invariant is metric independent even if the section (67) and the functional (32) depend on the metric. This result can be rephrased exactly in terms of the property that defines topological quantum field theories of Witten type. This corresponds, in our more general picture of Floer-type homologies, to the independence with respect to perturbations that preserve the index of the linearisation Ds and the compactness of Z_s. On a homology sphere, however, there is a problem of metric dependence and this means that Seiberg–Witten theory is not a topological quantum field theory. Nevertheless a topological theory can be obtained by factoring the Floer homology through the equivariant complex via the map (47) and its dual map.

11.5.1 Relative Seiberg–Witten invariants

If we try to fit the construction of Seiberg–Witten-Floer homology into the framework of axiomatic topological quantum field theory [3], we need relative invariants. In fact we want to associate to a four-manifold with boundary an element that lives in the vector space (the Floer homology) of the boundary Y. The construction of relative Seiberg–Witten invariants has not yet appeared in full details. A construction based on a Fukaya-Floer complex has been considered in [11], where the corresponding gluing theorems are stated. Relative Seiberg–Witten invariants are also discussed in [22], [21], and [29].

Suppose that X is a four-manifold with boundary Y, endowed with a metric with a cylindrical end $[0,\infty)\times Y$ and a $Spin_c$-structure that coincides along the cylinder with the pullback of a $Spin_c$-structure on Y. Solutions of the Seiberg–Witten equations on a cylinder with the temporal gauge condition have an asymptotic value that is a critical point of the functional (32). Moreover, if the critical point is non-degenerate, the decay is exponential at a rate determined by the first non-trivial eigenvalue of the Hessian. This implies that for each fixed critical point a it is possible to define a moduli space $\mathcal{M}(X,a)$ of solutions (A,ψ) of the Seiberg–Witten equations on X that are in a temporal gauge on the cylinder $[0,\infty)\times Y$, with the asymptotic condition

$$\lim_{t\to\infty}(A(t),\psi(t)) = a$$

on $[0,\infty)\times Y$.

There is a compactification of $\mathcal{M}(X,a)$ with strata of the form

$$\cup_{\mu(b)-\mu(a)=1}\mathcal{M}(X,b)\times\hat{\mathcal{M}}(b,a). \tag{68}$$

Thus, if the moduli space $\mathcal{M}(X,a)$ is zero-dimensional, one can define the relative invariant $N(X,a)$ as the sum of points in $\mathcal{M}(X,a)$ with the orientation. As in the case of a compact manifold, the invariant depends on the choice of the $Spin_c$-structure.

The compactification (68) ensures that the expression

$$N(X,Y) = \sum_{a \in M_c} N(X,a)a,$$

which is an element in $FC_*(Y)$, is in fact a cycle. Thus $N(X,Y)$ defines an element in the Floer homology $HF_*(Y)$.

11.6 Exercises

- This and the following are not really thought of as exercises. They are just meant to address some questions that arise from the QFT formalism introduced above. The first problem is to describe more precisely the relation between the infinite dimensional bundle used to compute the invariants in this context, and the choice made by [33] (definition 4.13). The two following problems deal with supersymmetry.

- The formalism above that produces the topological Lagrangian can be rewritten in terms of superalgebras, [35]. Try to follow the argument given in [25], [10].

- We already know that the Seiberg–Witten functional provides other critical points that are non–minimising. Therefore, they do not correspond to solutions of the Seiberg–Witten equations, but rather of the second order variational problem (14), (15). Thus, we would like to modify the functional (11) in such a way that it still encodes all the information concerning the solutions of the Seiberg–Witten equations, but also in such a way to get rid of all non–minimising critical points. In Physics this kind of problem is taken care of in the supersymmetric formulation of gauge theories. How does this relate to the results of this section?

References

[1] P.S. Aspinwall, D.R. Morrison, *Topological field theory and rational curves*, Comm. Math. Phys. 151 (1993) 2, 245-262.

[2] M.F. Atiyah, *New invariants of three and four dimensional manifolds*, Proc. Symp. Pure Math. 48 (1988) 285-299.

[3] M.F. Atiyah, *Topological quantum field theory*, Inst. Hautes Études Sci. Publ. Math. No. 68 (1988) 175–186.

[4] M.F. Atiyah, L. Jeffrey, *Topological Lagrangians and Cohomology*, J. Geom. Phys. 7 (1990), 119-136.

[5] D.M. Austin, P.J. Braam, *Equivariant homology*, Math. Proc. Camb. Phil. Soc. 118 (1995) 125-139.

[6] O. Babelon, C.M. Viallet, *The Riemannian geometry of the configuration space of gauge theories*, Comm. Math. Phys. 81 (1981) 515-525.

[7] M. Blau, *The Mathai-Quillen formalism and topological field theory*, J. Geom. Phys. 11 (1993) 95-127.

[8] M. Blau, G. Thompson, *N=2 Topological Gauge Theory, the Euler Characteristic of Moduli Spaces, and the Casson Invariant*, Commun. Math. Phys. 152 (1993) 41-71.

[9] R. Brussee, *Some C^∞–properties of Kähler surfaces*, New York J. Math. 2 (1996), 103–146.

[10] A.L. Carey, J. McCarthy, B.L. Wang, R.B. Zhang, *Seiberg–Witten monopoles in three dimensions*, Lett. Math. Phys. 39 (1997), no. 3, 213–228.

[11] A.L. Carey, B.L. Wang, *Seiberg–Witten-Floer homology and gluing formulae*, preprint.

[12] S. Cordes, G. Moore, S. Ramgoolam, *Lectures on 2D Yang-Mills theory, equivariant cohomology, and topological field theories*, Nuclear Phys. B Proc. Suppl. 41 (1995), 184-244.

[13] S. Cordes, G. Moore, S. Ramgoolam, *Large N 2D Yang-Mills theory and topological string theory*, Comm. Math. Phys. 185 (1997), no. 3, 543–619.

[14] K. Fukaya, *Floer homology for oriented 3-manifolds*, in Aspects of low-dimensional manifolds, Adv. Stud. Pure Math. 20, Tokyo, 1992, 1-92.

[15] W. Fulton, *Intersection Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer 1984.

[16] R. Gianvittorio, I. Martin, A. Restuccia, *The gauge invariant Lagrangian for Seiberg–Witten topological monopoles*, preprint, hep-th 9504146.

[17] F.R. Harvey, H.B. Lawson, *A theory of characteristic currents associated with a singular connection*, Astérisque No. 213 (1993).

[18] J.A. Harvey, A. Strominger, *String theory and the Donaldson polynomial*, Commun. Math. Phys. 151 (1993) 221-232.

[19] J. Kalkman, *BRST model for equivariant cohomology and representatives for the equivariant Thom class*, Commun. Math. Phys. 153 (1993) 447-463.

[20] M. Kontsevich, *Enumeration of rational curves via torus action*, in "The Moduli Space of Curves", Progress in Mathematics, Birkhäuser, 1995, 335-368.

[21] P.B. Kronheimer, *Embedded surfaces and gauge theory in three and four dimensions*, preprint.

[22] P.B. Kronheimer, T.S. Mrowka, *Monopoles and contact structures*, Invent. Math. 130 (1997), no. 2, 209–255.

[23] J.M.F. Labastida, *Quantum field theory: a progress report*, Differential geometry and its applications (Santiago de Compostela, 1995), 101–124, An. Fms. Monogr., 3, CIEMAT, Madrid.

[24] J.M.F. Labastida, C. Lozano *Lectures on topological quantum field theory*, Trends in theoretical physics (La Plata, 1997), 54–93, AIP Conf. Proc., 419, Amer. Inst. Phys. 1998

[25] J.M.F. Labastida, M. Mariño, *A topological Lagrangian for monopoles on four manifolds*, Phys. Lett. B 351, N.1-3 (1995) 146-152.

[26] J. Li, G. Tian, *Virtual moduli cycles and Gromov–Witten invariants of general symplectic manifolds*, Topics in symplectic 4-manifolds (Irvine, CA, 1996) 47-83, First Internat. Press Lect. Ser., Internat. Press, 1998.

[27] M. Marcolli, B.L. Wang, *Equivariant Seiberg–Witten Floer homology*, preprint.

[28] V. Mathai, D. Quillen, *Superconnections, Thom classes and equivariant differential forms*, Topology 25 (1986), 85-110.

[29] J.W. Morgan, T.S. Mrowka, Z. Szabó, *Product formulas along T^3 for Seiberg-Witten invariants*, Math. Res. Lett. 4 (1997), no. 6, 915–929.

[30] V.Ya. Pidstrigatch, *Deformation of Instanton Surfaces*, Math. USSR Izv. 38 (1992) 313-331.

[31] V.Ya. Pidstrigatch, A.N. Tyurin, *Invariants of the smooth structure of an algebraic surface arising form the Dirac operator*, Russian Acad. Sci. Izv. Math. 40 (1993) 267-351.

[32] C.H. Taubes, *Casson's Invariant and Gauge Theory*, J. Diff. Geom. 31 (1990) 547-599.

[33] C.H. Taubes, *The Seiberg–Witten invariants and The Gromov invariants*, Math. Research Letters 2 (1995) 221-238.

[34] E. Witten, *Supersymmetry and Morse theory*, J. Diff. Geom. 17 (1982) 661-692.

[35] E. Witten, *Topological Quantum Field Theory*, Commun. Math. Phys. 117 (1988) 353-386.

[36] E. Witten, *The N matrix model and gauged WZW models*, Nuclear Phys. B 371, N.1-2 (1992) 191-245.

12 Seiberg–Witten and Donaldson theory

In this chapter we would like to outline briefly the expected relation between Seiberg–Witten and Donaldson theory, and the different approach that physicists and mathematicians follow in intertwining the two theories.

In a fundamental paper [16], Kronheimer and Mrowka gave a description of a relation that constrains the values of the Donaldson invariants for a manifold of *Donaldson finite type*. The finite type assumption is a technical hypothesis on the behaviour of the polynomial invariants, which is satisfied by all presently known examples of simply connected 4-manifolds with $b_2^+ > 1$.

Let us recall that the polynomial invariants are defined in terms of homology classes $\alpha \in H_2(X;\mathbb{Z})$ and the moduli space $\mathcal{M}_k^{asd}$ of anti-self-dual $SU(2)$-connections on a bundle E with instanton number $k = c_2(E)$. The moduli space $\mathcal{M}_k^{asd}$ is of dimension $8k - 3(b_2^+ + 1)$. As we already discussed in relation with the Mathai-Quillen formalism, one can think of the space $X \times \mathcal{M}_k^{asd}$ inside the infinite dimensional space $X \times \hat{\mathcal{A}}/\mathcal{B}$, where the second factor is the space of irreducible $SU(2)$-connections modulo gauge [17], [18]. One can form a bundle $\mathcal{E}$ over this base space and integrate the second Chern class of this bundle against a homology class of X. This gives a map

$$\mu : H_i(X) \to H^{4-i}(\hat{\mathcal{A}}/\mathcal{G}).$$

Thus, given $\alpha = (\alpha_1, \dots, \alpha_d)$ with $\alpha_i \in H_2(X;\mathbb{Z})$, one gets a polynomial invariant [17]

$$q_d(\alpha) = \langle \mu(\alpha_1) \wedge \cdots \wedge \mu(\alpha_d), [\mathcal{M}^{asd}] \rangle.$$

We denote the term $\mu(\alpha_1) \wedge \cdots \wedge \mu(\alpha_d)$ as $\mu(\alpha)$ for simplicity.

It has been shown [16] that the polynomial invariants satisfies the relation

$$q_{d-2}(\alpha) = \langle \mu(\alpha) \wedge \nu, \mathcal{M}^{asd} \rangle,$$

where $\alpha = (\alpha_1, \dots, \alpha_{d-2})$. Here ν is the class in $H_4(X;\mathbb{Z})$ obtained from the element $\nu \otimes 1$ in the Künneth decomposition of the second Chern class $c_2(\mathcal{E})$. A detailed discussion of Donaldson polynomial invariants can be found in chapter 9 of [18]. The Donaldson simple type condition can be formulated as follows.

Definition 12.1 *X is of* Donaldson simple type *if the polynomial invariant satisfies*

$$q_{d-4}(\alpha) = 4\langle \mu(\alpha) \wedge \nu^2, \mathcal{M}^{asd} \rangle,$$

where $\alpha = (\alpha_1, \dots, \alpha_{d-4})$.

The following result, due to Kronheimer and Mrowka [16], is proved for manifolds of Donaldson simple type.

Theorem 12.2 *Let X be a simply connected manifold of Donaldson simple type, with $b_2^+ > 1$ odd. Combine the Donaldson polynomial invariants q_d in the expression*

$$q = \sum_d \frac{q_d}{d!}.$$

Then this expression satisfies

$$q(\alpha) = \exp(\frac{Q(\alpha)}{2}) \sum_k a_k e^{x_k \cdot \alpha}.$$

Here Q is the intersection form of the four-manifold X and the classes

$$x_k \in H^2(X; \mathbb{Z})$$

are called the Kronheimer–Mrowka basic classes *and they are subject to the constraint that the mod 2 reduction of each x_k is the Stiefel-Whitney class $w_2(X)$, and the corresponding coefficients a_k are non-zero rational numbers.*

The conjecture formulated by Witten in [29] is the following.

Conjecture 12.3 *In the expression*

$$q = \exp(\frac{Q}{2}) \sum_k a_k e^{x_k}$$

the basic classes $x_k \in H^2(X; \mathbb{Z})$ are exactly the Seiberg–Witten basic classes, namely those that satisfy

$$x_k^2 = c_1(\sqrt{L}_k)^2 = \frac{2\chi + 3\sigma}{4},$$

and correspond to a Spin$_c$ structure s_k with non-trivial Seiberg–Witten invariant,

$$N_{s_k}(X) \neq 0.$$

Moreover, the corresponding coefficient a_k is exactly, up to a topological factor, the Seiberg–Witten invariant N_{s_k}, that is, we have

$$a_k = 2^{2+(7\chi(X)+11\sigma(X))} N_{s_k}(X).$$

The Physics underlying this conjecture is sketched in the next section: the reader can consult [1], [29], or [30]. More detailed references are [26], [27], see also the bibliographic section at the end of the book.

From the conjecture 12.3 it seems that the Seiberg–Witten invariants should contain more information than the Donaldson invariants. In fact all the Donaldson polynomials can be recovered from the knowledge of the Seiberg–Witten invariants associated to zero dimensional moduli spaces. Thus, in principle, the Seiberg–Witten invariants associated to positive dimensional moduli spaces might give more information.

A related conjecture, which we mentioned already, is the following.

Conjecture 12.4 *A simply connected manifold X is of Donaldson simple type if and only if it is of Seiberg–Witten simple type, namely if the only non-trivial Seiberg–Witten invariants correspond to a choice of $Spin_c$-structure such that* $\dim(M) = 0$.

12.1 The Physics way: S-duality

The physicists' approach to the equivalence of Seiberg–Witten and Donaldson theory is based on Witten's interpretation of Donaldson's theory as a twisted supersymmetric Quantum Field Theory [31] and on the concept of electromagnetic duality. We attempt here a very rough overview of some of these topics. From the mathematician's point of view this concept of "duality" is rather mysterious; however, we'll try to present the basic ideas, mainly based on [1], [30], and on the exposition [2]. We especially recommend the very nice introduction to S-duality given in [5].

12.1.1 Maxwell equations

The first appearance of electromagnetic duality is in Maxwell equations. It is well known that the Maxwell equations in vacuum can be written as

$$dF = 0 \quad d^*F = 0,$$

where $F = dA$ is an imaginary 2-form, the curvature of a $U(1)$ bundle with connection A. In Physics notation one would write

$$E_k = -iF_{k4}$$

for the electric field, and

$$B^k = -i\frac{1}{2}\epsilon^{kpq}F_{pq}$$

for the magnetic field. The symbol ϵ^{kpq} is ± 1 according to the sign of the permutation $\{k, p, q\}$ of $\{1, 2, 3\}$ and zero if any two indices are equal.

It is clear that there is a symmetry given by the Hodge $*$-operator

$$F \mapsto *F$$

that preserves the equations and interchanges electric and magnetic fields.

The Maxwell equations are no longer invariant under the $*$-operator if one considers the presence of electric charges and electromagnetic currents, unless one postulates the existence of isolated magnetic charges, namely magnetic monopoles.

Magnetic monopoles satisfy a quantisation condition which states that the magnetic and electric charges are related by

$$m = \frac{2\pi}{e}.$$

There is an elegant topological motivation for this quantisation condition which is beautifully explained by Raoul Bott in [3].

There is an analogue of electromagnetic duality for monopoles in non-abelian field theory, where again one can see that electric and magnetic charges live in dual lattices and the magnetic charge can be given a topological meaning.

The electric charge enters the Lagrangian as a coupling constant (as we are going to discuss in a moment). Thus, one can see how electromagnetic duality interchanges weak and strong coupling (a small with a large coupling constant). Interchanging a weak with a strong coupling means to exchange the range in which perturbative theory can be applied with one in which it cannot. This will be discussed in the following.

12.1.2 Modular forms

In the abelian context, that is, with structure group $U(1)$, we can write the Lagrangian density on a four-manifold X as

$$\mathcal{L} = \frac{1}{8\pi} \int_X \left(\frac{4\pi}{e^2} F \wedge *F + \frac{i\theta}{2\pi} F \wedge F\right).$$

The second part of the Lagrangian density is a topological term,

$$\frac{i\theta}{2\pi} c_1(L)^2,$$

where L is the chosen line bundle on which the Maxwell equations are considered. The angle θ is the $U(1)$-symmetry of the vacuum state.

Upon setting

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2},$$

one can rewrite $\mathcal{L}$ in terms of τ,

$$\mathcal{L} = \frac{1}{8\pi} \int_X (\bar{\tau}(F^+)^2 - \tau(F^-)^2) dv.$$

The partition function, formally written as an infinite dimensional integral

$$Z \sim \int e^{-\mathcal{L}} \mathcal{D}A,$$

is invariant under the transformation

$$\tau \mapsto \tau + 2.$$

Under the transformation

$$\tau \mapsto \tau + 1$$

we have

$$Z \mapsto Z \cdot e^{\pi i c_1(L)^2}.$$

In the case of a *Spin*-manifold $c_1(L)^2$ is an even integer, hence there is an invariance under $\tau \mapsto \tau + 1$. A more complicated computation with the formal rules of infinite dimensional integrals "shows" that there is also an invariance under

$$\tau \mapsto -\frac{1}{\tau}.$$

This can be viewed as a consequence of a Poisson summation formula applied formally to the infinite dimensional integrals, which leads to the result

$$Z(-\frac{1}{\tau}) = \tau^{\frac{1}{4}(\chi(X)-\sigma(X))}\bar{\tau}^{\frac{1}{4}(\chi(X)+\sigma(X))}Z(\tau).$$

This implies that $Z(\tau)$ behaves like a modular form under the action of $SL(2,\mathbb{Z})$. This fact is an appearance of the phenomenon known as Montonen-Olive duality. It is related to electromagnetic duality, since the transformation

$$\tau \mapsto -\frac{1}{\tau}$$

corresponds to

$$F \mapsto *F,$$

in the sense that all the expectation values are preserved under the combined action of the transformations together. Thus, the modularity can be thought of as a refined version of the Hodge duality which manifests itself at the quantum level.

We should remark, however, that the picture presented here is quite incomplete. In fact it ignores the essential role of supersymmetry.

In the case of non-abelian monopoles the analogous phenomenon happens if one considers the Lagrangian density

$$\mathcal{L} = \frac{1}{g^2}\int_X Tr(F \wedge *F) + \frac{i\theta}{8\pi^2}\int_X Tr(F \wedge F).$$

The presence of the coefficient $\frac{1}{g^2}$ depends on the fact that the Killing form on the compact Lie group G is only defined up to a scalar multiple which is usually set equal to one in the mathematical literature, while it appears in Physics as a coupling constant. The second term represents the second Chern class of the vector bundle E on X on which the connection and curvature $F = dA + A \wedge A$ are considered. The fact that this topological term appears explicitly in the Lagrangian is already an effect of the presence of $N = 1$ supersymmetry. In fact also other terms appear in the partition function that contain the "auxiliary fields" introduced by the supersymmetry. These are the analogue of the elements of the algebra $\Lambda[w]$ in our definition of the fermionic integral in relation to the Mathai-Quillen formalism. The fact that the vacuum state (that is, the minimum of the classical potential) has a $U(1)$-symmetry which explains the presence of the angle θ is also an effect of the presence of the "unbroken" supersymmetry.

Thus the partition function can be formally written as

$$\int e^{-\mathcal{L}} \mathcal{D}A = \sum_{r=c_2(E)} e^{ir\theta} \int e^{-\mathcal{L}_r} \mathcal{D}A,$$

where $\mathcal{L}_r = \frac{1}{g^2} \int_X Tr(F \wedge *F)$ on the fixed bundle E.

Again one can introduce the variable

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}.$$

The modularity in this context can be formulated in a different way, which leads to an interesting conjecture [28].

Conjecture 12.5 *consider the expression*

$$Z_G(\tau) = q^c \sum_{r=0}^{\infty} \chi_r q^r,$$

where $q = e^{2\pi i \tau}$ and χ_r is some suitable regularised Euler characteristic of the moduli space of G instantons on the four-manifold X with instanton number $r = c_2(E)$. There is an action of $SL(2, \mathbb{Z})$ and Z_G transforms like

$$Z_G(-\frac{1}{\tau}) \sim Z_{\tilde{G}}(\tau),$$

where $\tilde{G}$ is the Langlands dual of G.

We do not discuss this statement any further, but just mention that the Langlands dual interchanges the torus lattice with its dual. It is thus related to electromagnetic duality for non-abelian monopoles.

12.1.3 Weak and strong coupling

As we have seen in the example of Maxwell theory, the interchanging of electric and magnetic charges due to Hodge duality also interchanges weak and strong coupling in the action. When the coupling constant is small, one can formally compute the infinite dimensional integral by means of a stationary phase approximation [3], [30]. The model is the finite dimensional situation in which one has a function $F(x)$ with an isolated minimum at $x = 0$. We can write $F(x) = F(0) + \frac{1}{2}Q(x) + \cdots$ and for a small coupling constant we can approximate the integral

$$Z = \int e^{-\frac{1}{\lambda}F(x)} \frac{dx_1 \cdots dx_n}{(2\pi)^{n/2}} \sim Z_0 = \int e^{-\frac{1}{\lambda}(F(0)+\frac{1}{2}Q(x))} \frac{dx_1 \cdots dx_n}{(2\pi)^{n/2}}.$$

The latter can be computed exactly and it gives

$$Z_0 = e^{-\frac{1}{\lambda}F(0)} \frac{\lambda^{n/2}}{\det(Q)^{1/2}}.$$

The finite dimensional computation can be easily related to the computation of the Pfaffian that we presented in relation to the Mathai-Quillen formalism, with the only difference that the matrix Q is symmetric instead of antisymmetric. This explains why one gets $\det(Q)^{-1/2}$ instead of $\det(Q)^{1/2} = Pf(Q)$.

In order to generalise this argument to the infinite dimensional context, the problem is reformulated in terms of a functional F with non-degenerate minima. The approximation of the partition function in this case can be taken to be the well defined mathematical object $\det(Q)^{-1/2}$, where Q is a positive elliptic operator (the Hessian of the functional F at a minimum) and the determinant is the Ray-Singer determinant [24], [25]. If the coupling constant is large this approximation method no longer works and the partition function is in general no longer computable.

12.1.4 The u-plane

In the case of $N = 2$ supersymmetry, the auxiliary fields that are introduced can be described as two independent variables of the type of the $\Lambda[w]$ used in the definition of the fermionic integral, and a field ϕ which is a section of the adjoint bundle of E. The classical potential can be written as a function $V(\phi)$ and as mentioned before the supersymmetry imposes that in the vacuum state $V(\phi) = 0$. This allows for certain symmetries of the vacuum. This means that the vacuum state is not an isolated point but there is some parametrisation of a certain manifold of possible vacuum states. In our case the parameter that classifies inequivalent vacua is $Tr(\phi^2)$.

This is better said by introducing a variable $u = \langle Tr(\phi^2)\rangle$ which is the expectation value (with respect to the partition function Z) of $Tr(\phi^2)$. The expectation value of the field ϕ is proportional to a variable a, $\langle\phi\rangle \sim a$. In the classical limit, that is, when the coupling is weak, one has the relation $u \sim \frac{1}{2}a^2$. In the strong coupling range the relation is more complicated.

In terms of the parameter a one has the corresponding modulus

$$\tau(a) = \frac{\theta(a)}{2\pi} + \frac{4\pi i}{g^2(a)}.$$

The symmetry of the action under the transformation $\tau \mapsto \frac{-1}{\tau}$ can be formally described in terms of a Legendre transformation over a potential (called *prepotential* in the Physics literature) $\mathcal{F}$. In fact, a dual variable a_D is introduced by the relation

$$a_D = \frac{\partial \mathcal{F}(a)}{\partial a}, \tag{69}$$

and a dual field ϕ_D is defined by $\langle \phi_D \rangle \sim a_D$. Here "dual" is intended in analogy to coordinates and moments in classical mechanics that are related by a Legendre transform similar to (69). The transformation $\tau \mapsto \frac{-1}{\tau}$ exchanges the action Z with a dual action Z_D where the field ϕ is replaced with ϕ_D and a with a_D. This exchanges weak and strong coupling.

The reason why this can be still thought of as electromagnetic duality is that one thinks of the purely electric or purely magnetic charge as quantities $q_e = n_e a$ and $q_m = n_m a_D$, for a pair of integers (n_e, n_m). One can also consider states (which are called *dyons* in the literature) that have both electric and magnetic charge $q = n_e a + n_m a_D$. The group $SL(2;\mathbb{Z})$ acts by mixing the electric and the magnetic charge

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}.$$

If one wants to express the variables a and a_D as functions of the parameter that determines the vacuum state, $a(u)$ and $a_D(u)$, one gets two multivalued functions, defined for $u \in \mathbb{C}$ with branch cuts. In particular one can compute the monodromy at the branch points [2]. One point is certainly the one at infinity, where the weak coupling range is attained. In this case the prepotential takes the form $\mathcal{F}(a) \sim \frac{i}{2\pi} a^2 \ln \frac{a^2}{\Lambda^2}$ and as $u \mapsto e^{2\pi i} u$ one has $a \mapsto -a$ and $a_D \mapsto -a_D + 2a$. Thus the monodromy at $u = \infty$ is

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}.$$

An argument depending on the factorisation of the matrix M_∞ in $SL(2;\mathbb{Z})$ shows that there are other two branching points. Up to the choice of a normalising constant these can be taken to be $u = \pm 1$. As $u \to \pm 1$ the strong coupling range is attained. The corresponding monodromies [2] are

$$M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

and

$$M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}.$$

The physical interpretation of the eigenvalues of the monodromy matrices leads to interpreting these branch points as the vacua at which a magnetic monopole (when $u = 1$) or a $(1, -1)$-dyon (when $u = -1$) become massless.

12.1.5 Elliptic curves

Given the data obtained above by physical arguments, namely the punctured sphere $\mathbb{C}P^1 - \{\infty, 1, -1\}$ (or u-plane) and the prescribed monodromies at the punctures, it is possible to proceed with a rigorous construction. The monodromies obtained above span a subgroup $\Gamma(2)$ in $SL(2; \mathbb{Z})$. The u-plane is equivalent to the quotient of the upper half plane with respect to the group $\Gamma(2)$. This gives the moduli of the family of elliptic curves

$$y^2 = (x^2 - 1)(x - u)$$

that becomes singular at the points $u = \pm 1$.

The functions $a(u)$ and $a_D(u)$ can be interpreted within this geometric picture as the periods

$$a = \int_{\gamma_1} \lambda \qquad a_D = \int_{\gamma_2} \lambda \ ,$$

with

$$\lambda = \frac{\sqrt{2(x - u)}}{2\pi\sqrt{x^2 - 1}} dx.$$

The relation of all this with the Witten conjecture comes when one reads the weak coupling limit of Z as Donaldson theory (that is twisted $N = 2$ supersymmetric Yang-Mills theory) and the strong coupling limit of Z as the Seiberg–Witten theory. Then the idea that leads to the equivalence of the two theories is that the geometric data encoded in this family of elliptic curves should provide "the gluing instructions" of how to interpolate for all values of u knowing the asymptotic behaviour at the singular points. The relation obtained would then be in the form given by Kronheimer and Mrowka, as in theorem 12.2.

More recently, the conjecture 12.3 has been extended by Moore and Witten [21] to the case of manifolds with $b_2^+(X) = 1$. In this case a correction term to the relation 12.3 comes from integration over the u-plane.

12.2 The Mathematics way

A detailed strategy for the proof of the Witten conjecture 12.3, using a rather different approach, which does not involve functional integration, has been outlined by Pidstrigatch and Tyurin [23].

The main idea of the strategy is to relate Donaldson and Seiberg–Witten theory within a "mixed theory" of non-abelian monopoles, designed in such a way that the Donaldson and the Seiberg–Witten moduli spaces appear as singular submanifolds of the moduli space $\mathcal{M}$ of non-abelian monopoles. In this way the larger moduli space describes a cobordism between the links of the two types of moduli, thus defining a relation between the invariants, which turns out to be the one prescribed by the Witten conjecture.

As we pointed out in the brief remarks about the conjectural relation between Seiberg–Witten and instanton Floer homology, the program outlined by Pidstrigatch and Tyurin can be thought of as the four-dimensional analogue of Thaddeus' construction that links the moduli space of stable bundles over a Riemann surface Σ to the symmetric products $s^r(\Sigma)$. The latter, in fact, are moduli spaces of vortex equations on Σ and can be thought of as the two-dimensional reduction of Seiberg–Witten theory, whereas the moduli space of stable bundles over Σ, according to the results of Narasimhan and Seshadri [22], can be identified with the two dimensional reduction of Donaldson theory.

Although the idea is rather elegant and clear, the actual construction of the cobordism presents enormous analytical difficulties. These are being attacked and conquered, with a large display of technical skills, by Feehan and Leness in a long series of papers [7], [8], [9], [10], [11], [12], [13], [14].

In the following we summarise the various technical steps involved in the program of Pidstrigatch and Tyurin and of Feehan and Leness. The reader who is more seriously interested in the results and technical issues is advised to read the introductory paper [7], that we mostly follow here, and then the other references where the various technical problems are attacked separately.

Mostly the difficulty lies in the lower strata of the moduli space of non-abelian monopoles. The compactness argument fails in the non-abelian case. This is easily seen, in fact, the whole Donaldson moduli space $\mathcal{M}^{asd}$, which is itself non-compact, is recovered as a singular submanifold of the non-abelian monopole moduli space, corresponding to solutions with vanishing spinor.

There is an Uhlenbeck compactification $\bar{\mathcal{M}}$ of the moduli space of non-abelian monopoles, obtained by adding lower dimensional strata. Each of these strata may itself contain reducibles.

There are invariants associated to the non-abelian monopoles. These are obtained as in the Donaldson case, by integrating some cohomology classes over the fundamental class of $\mathcal{M}$, or equivalently by intersecting their homology representatives with $\mathcal{M}$. The problem arises when there is a non-trivial intersection with the links of the reducibles in the lower strata. In this case, an analogue of the Kotschick-Morgan conjecture is needed in order to compute the integrals of the cohomology classes over these links in the lower strata.

The Kotschick–Morgan conjecture was formulated for the case of Donaldson invariants of four-manifolds with $b_2^+(X) = 1$, where there is a chamber structure and a phenomenon of metric dependence [15]. The conjecture states that the wall crossing terms only depend on the homotopy type of the manifold X. The relevance, in our context, lies in the fact that the Kotschick-Morgan conjecture is in fact a problem of describing the links of reducibles in lower level strata in the Uhlenbeck compactification of $\mathcal{M}^{asd}$ and computing the integrals of the Donaldson cohomology classes over these links.

A substantial part of the work of Feehan and Leness goes into proving this analogue of the Kotschick–Morgan conjecture for the non-abelian monopoles. Most of the technical issues involved had not previously been worked out even in the 'simpler' context of the anti-self-dual moduli spaces.

12.2.1 Non–abelian Monopoles

Generalisations of the monopole equations to the case of a non-abelian structure group have been investigated by various authors. There are many more contributions in the Physics literature where several versions of non-abelian monopoles are considered. The reader should consult the bibliographical appendix for a partial list of references. The important remark is that the generalisation of the Seiberg–Witten equations to non-abelian groups is not unique. A nice introduction to the subject that compares various possible generalisations is [4]. Some of these extensions seem to have a natural interpretation [4] when restricted to the case of Kähler manifolds. In fact the usual Seiberg–Witten equations on a Kähler manifold take a particular form (which we discussed in Part III), which is a slightly modified version of equations known as the *vortex equations*. There are various known generalisations of the vortex equations in non-abelian context. These are analysed in [4]. The authors consider that, as the usual Seiberg–Witten equations can be though of as a Riemannian generalisation of the vortex equation, so in the non-abelian context the possible extensions of the vortex equation in the world of Kähler manifolds serve as a model for possible extensions of the Seiberg–Witten equations for more general Riemannian manifolds.

Here we want to give an idea of how one can construct these non-abelian monopole equations and why they contain information of both the Seiberg–Witten and the Donaldson theory. We follow [7].

We consider a compact oriented four-manifold X endowed with a $Spin_c$ structure $W^{\pm}$, as in Part I. Consider a Hermitian rank two vector bundle E on X, and a fixed connection A_0 on the determinant line bundle $Det(E)$. Given any connection A on E inducing the fixed determinant connection A_0, we can form the twisted Dirac operator

$$D_A : \Gamma(X, W^+ \otimes E) \to \Gamma(X, W^- \otimes E). \tag{70}$$

We can consider the traceless part $(F_A^+)_0$ of the self-dual component of the curvature $F_A \in \Lambda^2 \otimes u(E)$. Given a section $\psi \in \Gamma(X, W^+ \otimes E)$, we denote $\frac{1}{4} < e_i e_j \psi, \psi >_0 e^i \wedge e^j$ the component in $\Lambda^+(su(E))$. Under the map

$$\rho \otimes id_{su(E)} : \Lambda^+(su(E)) \to su(W^+) \otimes su(E)$$

this is mapped to an element, denoted by $(\psi \otimes \psi^*)_{00}$, which is the component in $su(W^+) \otimes su(E)$ of the Hermitian endomorphism $\psi \otimes \psi^*$ of $W^+ \otimes E$, as in [7].

Thus, we have the equations

$$D_A \psi = 0 \tag{71}$$

and

$$(F_A^+)_0 - \frac{1}{4} < e_i e_j \psi, \psi >_0 e^i \wedge e^j = 0. \tag{72}$$

On the configuration space $\mathcal{A}_E = \mathcal{C}_E \oplus \Gamma(X, W^+ \otimes E)$ we have an action of the gauge group $\mathcal{G}_E$ of unitary transformations of E with determinant one. The moduli space $\mathcal{M}$ lies in the quotient $\mathcal{B}_E$.

There is a deformation complex for the equations (71) and (72),

$$0 \to \Lambda^0(su(E)) \stackrel{G}{\to} \Lambda^1(su(E)) \oplus \Gamma(X, W^+ \otimes E) \stackrel{T}{\to}$$
$$\Lambda^{2+}(su(E)) \oplus \Gamma(X, W^- \otimes E) \to 0.$$

This computes the virtual dimension of the moduli space $\mathcal{M}$,

$$\begin{aligned} \dim \mathcal{M} = \quad & -\tfrac{3}{2}(c_1(E)^2 - 4c_2(E) + \chi(X) + \sigma(X)) \\ & +\tfrac{1}{2}((c_1(W^+) + c_1(E))^2 - \sigma(X)) - 1. \end{aligned}$$

The transversality result is one of the most delicate technical issues. In fact, it is necessary [8] to develop a suitable class of perturbations that make the linearisation surjective. This is achieved by constructing a sequence of holonomy perturbations, supported on the balls of a covering of X, with a universal energy bound, such that if the curvature gets concentrated over one of the balls, thus exceeding the energy bound, the corresponding perturbation vanishes. This ensures continuity over the Uhlenbeck compactification. The use of a unique continuation argument then shows vanishing of the cokernels of the perturbed linearisations at an irreducible solution (A, ψ).

The proof of orientability carries over following the case of Donaldson and of Seiberg–Witten theory.

The reducibles belong to two classes, denoted in [7] respectively as *zero section pairs* and *reducible pairs*.

The first case corresponds to solutions with trivial spinor. These are solutions of the anti-self-dual equations. The singular stratum gives a copy of the Donaldson moduli space $\mathcal{M}^{asd}$, of dimension

$$\dim \mathcal{M}^{asd} = -2(c_1(E)^2 - 4c_2(E)) - \frac{3}{2}(\chi(X) + \sigma(X)).$$

The other case corresponds to a splitting $E = L_1 \oplus L_2$ as a sum of two line bundles, with $L_2 = Det(E) \otimes L_1^*$. A solution

$$(A, \psi) = (A_1 \oplus (A_0 \otimes A_1^*), \psi_1)$$

has a $U(1)$-stabiliser and can be regarded as a solution of a perturbed version of the Seiberg–Witten equations on X with respect to the $Spin_c$ structure with spinor bundle $W^+ \otimes L_1$. In this case we obtain as singular strata the moduli spaces M_{s_1} of solutions of Seiberg–Witten equations for $Spin_c$ structures s_1 with determinant $\det(W^+) \otimes L_1$.

Upon passing to a blowup $\hat{X} = X \# \bar{\mathbb{C}P}^2$, it is possible to guarantee that the two types of singular strata do not intersect. Thus the irreducible part $\mathcal{M}^*$ gives a cobordism between the links of the two types of singular strata.

The space $\mathcal{M}^* \cup \mathcal{M}^{asd} \cup_{s_1} M_{s_1}$ is still non-compact. There are lower dimensional strata $\mathcal{M}_\ell$, for $\ell > 0$ an integer, in the Uhlenbeck compactification. These contain the classes of the Uhlenbeck limits (A, ψ, x) with (A, ψ) in

$$\mathcal{A}_{E_\ell} = \mathcal{C}_{E_\ell} \oplus \Gamma(X, W^+ \otimes E_\ell)$$

and $x \in Sym^\ell(X)$. The $U(2)$-bundle E_ℓ has $Det(E_\ell) = Det(E)$ and $c_2(E_\ell) = c_2(E) - \ell$.

The work of Feehan and Leness then proceeds by constructing the links of the strata $\mathcal{M}^{asd}$ and M_{s_1} in the top Uhlenbeck level and their intersection with the geometric representatives of the cohomology classes

$$\mu : H_*(X, \mathbb{Q}) \to H^{2-*}(\mathcal{B}_E, \mathbb{Q}).$$

If there were no reducibles in the lower strata, this would be enough to recover the relation given in 12.3. Unfortunately, the lower strata also contain reducibles, hence a detailed analysis of the intersection of the geometric representatives with the lower strata is necessary.

The authors consider tubular neighbourhood of the lower strata defined by the gluing maps. This requires a very careful and technically demanding analysis of the relevant gluing theorems [10], [11].

It should be mentioned that the approach of Feehan and Leness has led to a rigorous mathematical proof of some results derived from the physical theory of S-duality, besides the original goal of establishing the Witten conjecture. An example is the proof of a conjecture of Mariño, Moore, and Peradze [19], derived by Feehan, Kronheimer, Leness, and Mrowka [6]. In [19] the authors consider the Seiberg–Witten invariants of a smooth oriented 4-manifold X with $b_2^+ > 1$, assembled in the expression

$$SW_X^w(h) = \sum_{s \in \mathcal{S}(X)} (-1)^{\frac{1}{2}w^2 + c_1(L)w} N_s(X) e^{\langle c_1(L) \cup h, [X] \rangle}.$$

Here L is the determinant line bundle of the $Spin_c$-structure as we discussed in Part I, and h varies in $H^2(X, \mathbb{R})$ and w is any integral lift of the Stiefel–Whitney class $w_2(X)$. They introduce the notion of "superconformal simple type" to denote a class of compact oriented smooth 4-manifolds with $b_2^+ > 1$

and of Seiberg–Witten simple type, such that $SW_X^w(h)$ has a zero at $h = 0$ of order at least

$$c(X) - 3,$$

where

$$c(X) = -\frac{1}{4}(7\chi(X) + 11\sigma(X)).$$

In [20] it is then shown that all known 4-manifolds with $b_2^+ > 1$ are of superconformal simple type: in fact, it is shown that the superconformal simple type property is preserved under blowup, fibre sum along embedded tori, knot surgery, and generalised log transforms. Moreover, it is shown that compact complex surfaces with $b_2^+ > 1$ are of superconformal simple type. This leads to the following conjecture [20].

Conjecture 12.6 *All compact oriented smooth 4-manifolds with $b_2^+ > 1$ are of superconformal simple type.*

Using the analysis of $PU(2)$-monopoles of Feehan and Leness, the conjecture is reduced in [6] to the technical hypothesis that reducibles which appear in the lower levels of the Uhlenbeck compactification of the moduli space of $PU(2)$-monopoles do not contribute any non-trivial Seiberg–Witten invariants. This is possible under the assumption that the 4-manifold is of Seiberg–Witten simple type and is *abundant*, that is, the intersection form restricted to the orthogonal complement of the basic classes contains a hyperbolic sublattice. The latter is a condition which guarantees that the index of the twisted Dirac operator (70) is positive, $Ind(D_A) > 0$. This is a technical condition that is needed in order to apply the results of [9].

References

[1] M.F. Atiyah, *Duality and quantum field theory*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), 1–7, First Int. Press Lect. Ser., I, Internat. Press, 1998.

[2] A. Bilal, *Duality in N=2 SUSY SU(2) Yang-Mills Theory: A pedagogical introduction to the work of Seiberg and Witten*, preprint hep-th/9601007.

[3] R. Bott, *On some recent interactions between Mathematics and Physics*, Canad. Math. Bull. 28 N.2 (1985) 129-164.

[4] S. Bradlow, O. García-Prada, *Non-abelian monopoles and vortices*, Geometry and physics (Aarhus, 1995), 567–589, Lecture Notes in Pure and Appl. Math., 184, Dekker, New York, 1997.

[5] R. Dijkgraaf, *Fields, Strings, and Duality*, Symétries quantiques (Les Houches, 1995), 3–147, North-Holland, 1998.

[6] P.M.N. Feehan, P.B. Kronheimer, T.G. Leness, T.S. Mrowka, *PU(2) monopoles and a conjecture of Marino, Moore, and Peradze*, math.DG/9812125.

[7] P.M.N. Feehan, T.G. Leness, *PU(2) monopoles and relations between four-manifold invariants*, preprint, dg-ga/9709022.

[8] P.M.N. Feehan, T.G. Leness, *PU(2) Monopoles, I: Regularity, Uhlenbeck Compactness, and Transversality.* preprint, dg-ga/9710032.

[9] P.M.N. Feehan, T.G. Leness, *PU(2) monopoles, II: Highest-level singularities and relations between four-manifold invariants*, preprint, dg-ga/9712005.

[10] P.M.N. Feehan, T.G. Leness, *PU(2) monopoles, III: Existence of gluing and obstruction maps*, preprint.

[11] P.M.N. Feehan, T.G. Leness, *PU(2) monopoles, IV: Surjectivity of gluing maps*, preprint.

[12] P.M.N. Feehan, *Generic metrics, irreducible rank-one PU(2) monopoles, and transversality*, preprint, math.DG/9809001.

[13] P.M.N. Feehan, T.G. Leness, *Donaldson invariants and wall-crossing formulas. I: Continuity of gluing maps*, preprint, math.DG/9812060.

[14] P.M.N. Feehan, *A Kato-Yau inequality and decay estimate for harmonic spinors*, preprint, math.DG/9903021.

[15] D. Kotschick, J.W. Morgan, *$SO(3)$-invariants for four-manifolds with $b^+ = 1$, II*, J. Diff. Geom. 39 (1994) 433-456.

[16] P. Kronheimer, T. Mrowka, *Recurrence relations and asymptotics for four-manifold invariants*, Bull. Amer. Math. Soc. 30 (1994) 215-221 ; and *Embedded surfaces and the structure of Donaldson's polynomial invariants*, J. Diff. Geom. 41, No. 3 (1995) 573-734.

[17] S. Donaldson, *Polynomial invariants for smooth four-manifolds*, Topology 29 (1990) 257-341.

[18] S. Donaldson, P. Kronheimer, *The Geometry of Four-Manifolds*, Oxford 1990.

[19] M. Mariño, G. Moore, G. Peradze, *Superconformal invariance and the geography of four-manifolds*, preprint, hep-th/9812055.

[20] M. Mariño, G. Moore, G. Peradze, *Four-manifold geography and superconformal symmetry*, preprint math.DG/9812042.

[21] G. Moore, E. Witten, *Integration over the u-plane in Donaldson theory*, Adv. Theor. Math. Phys. 1 (1997) N.2, 298-387.

[22] M.S. Narasimhan, C.S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. Math. 82 (1965) 540-567.

[23] V.Ya. Pidstrigatch, A.N. Tyurin, *Localisation of the Donaldson's Invariants along Seiberg–Witten Classes*, preprint, dg-ga/9507004.

[24] D.B. Ray, I.M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Advances in Math. 7 (1971) 145-210.

[25] A.S. Schwarz, *The partition function of degenerate quadratic functional and Ray-Singer invariants*, Lett. Math. Phys. 2, N.3 (1977/78) 247–252.

[26] N. Seiberg, E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, Nucl. Phys. B431 (1994) 484-550.

[27] N. Seiberg, E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nuclear Phys. B 426, N.1 (1994) 19-52.

[28] C. Vafa, E. Witten, *A Strong Coupling Test of S-Duality*, Nucl. Phys. B431 (1994) 3-77.

[29] E. Witten, *Monopoles and Four–Manifolds*, Math. Research Letters 1 (1994) 769-796.

[30] E. Witten, lectures given at the Algebraic Geometry meeting in Santa Cruz, July 1995.

[31] E. Witten, *Topological Quantum Field Theory*, Commun. Math. Phys. 117 (1988) 353-386.

Part V
Appendix: a bibliographical guide

by Erion J. Clark[3] and Matilde Marcolli

We have decided to complete this volume with a separate Appendix that collects a reasonably updated bibliography on Seiberg–Witten theory. The bibliography is updated as of May 1998. In order to give this bibliography in a more user-friendly appearance, the items have been divided roughly into various denominations. *General Introduction* collects works that are meant to give a broad overview on Seiberg–Witten theory and are often of an expository nature. Then *Four-manifold topology*, and *Three-manifolds and Floer theory* collect references that cover the material discussed in part I and II of the book, including many more results that have not found space in the text. The references on *Non-abelian monopoles* cover the mathematical approach to the equivalence of Seiberg–Witten and Donaldson theory. The references on *Symplectic Geometry* and *Kähler surface; algebraic geometry*, cover the results discussed in the beginning of Part III, and much more. Most of the papers quoted directly use techniques of Seiberg–Witten theory, but we have also listed some references whose relevance to Seiberg–Witten theory is more indirect. The list of references on *Einstein metrics* will lead the reader through an interesting topic that has not been covered in the book. Moreover, there are lists of references that more properly belong to the Physics literature. We have decided that, in order to give as complete as possible a picture of the current status of the field, it is necessary to present both the results of physicists and mathematicians. Thus, we have collected the Physics references under the denominations *Quantum field theory*, *String theory and duality*, and *Integrable systems*, somewhat following the logical order of Part IV of the book. Often the same paper or book is quoted under different classifications, and clearly, for many of the references, the attribution to one or another denomination is largely arbitrary. Therefore, the reader should take such a subdivision only as a guideline. Another disclaimer: the bibliographical guide contains mostly papers that have appeared in print (as of May 1998) or that are easily available on electronic preprint archives. Some interesting contributions may not be listed here, simply because they have not yet been made available by the authors. We apologise anyway to all the authors whose work is not mentioned in this list. Despite the possible omissions, we

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hope that this bibliographical guide provides a useful tool and gives a fairly accurate picture of these four years of development of Seiberg–Witten theory from its first appearance in 1994, and of how fast the current research develops.

I: General Introduction

References

[1] S. Akbulut, *Lectures on Seiberg–Witten Invariants*, Turkish J. Math. 20, N.1 (1996) 95-118.

[2] M.F. Atiyah, *Geometry and physics: where are we going?*, Geometry and physics (Aarhus, 1995), 1–7, Lecture Notes in Pure and Appl. Math., 184, Dekker, New York, 1997.

[3] M. Audin, *Du nouveau en dimension* 4, Gaz. Math. No. 64 (1995), 43–56.

[4] A. Bilal, *Duality in N=2 SUSY SU(2) Yang-Mills Theory: A pedagogical introduction to the work of Seiberg and Witten*, preprint, hep-th/9601007.

[5] S. Donaldson, *The Seiberg–Witten equations and 4-manifold topology*, Bull. AMS, Vol.33 N.1 (1996) 45-70.

[6] R. Flume, L. O'Raifeartaigh, I. Sachs, *Brief Resume of Seiberg–Witten Theory*, preprint, hep-th/9611118.

[7] R. Fintushel, *New directions in* 4*-manifold theory*, Nonlinear partial differential equations in geometry and physics (Knoxville, TN, 1995), 1–28, Progr. Nonlinear Differential Equations Appl., 29, Birkhduser, Basel, 1997.

[8] Th. Friedrich, *Dirac-Operatoren in der Riemannschen Geometrie. Mit einem Ausblick auf die Seiberg–Witten-Theorie*, Advanced Lectures in Mathematics. Friedr. Vieweg and Sohn, Braunschweig, 1997.

[9] D. Kotschick, *Gauge theory is dead!—Long live gauge theory!*, Notices Amer. Math. Soc. 42 (1995), no. 3, 335–338.

[10] M. Kreck, *"Ist 4 denn noch normal?"*, Duration and change, 237–253, Springer, Berlin, 1994.

[11] P.B. Kronheimer, *Embedded surfaces and gauge theory in three and four dimensions*, preprint.

[12] W. Lerche, *Introduction to Seiberg–Witten Theory and its Stringy Origin*, Nucl. Phys. Proc. Suppl. 55B (1997) 83-117. Fortsch. Phys. 45 (1997) 293-340.

[13] M. Marcolli, *Notes on Seiberg–Witten gauge theory*, dg-ga/9509005.

[14] J.D. Moore, *Lectures on Seiberg–Witten invariants*, Lecture Notes in Mathematics, 1629. Springer-Verlag, Berlin, 1996

[15] J.W. Morgan, *The Seiberg–Witten equations and applications to the topology of smooth four-manifolds*, Princeton University Press, 1996.

[16] K. Paranjape, V. Pati, *Seiberg–Witten Invariants, an Expository Account*, preprint.

[17] D. Salamon, *Spin geometry and Seiberg–Witten invariants*, in preparation.

[18] Z. Szabó, *Notes on Seiberg–Witten invariants*, Geometry and physics (Aarhus, 1995), 59–70, Lecture Notes in Pure and Appl. Math., 184, Dekker, New York, 1997.

[19] E. Witten, *Monopoles and Four–Manifolds*, Math. Research Letters 1 (1994) 769-796.

[20] J.V. Yang, *Introduction to Seiberg–Witten's Invariants, Part I: Theory of Spinors*, preprint dg-ga/9508005.

II: Four-manifold topology

References

[1] S. Akbulut, R. Matveyev, *Exotic structures and adjunction inequality*, Turkish J. Math. 21 (1997), no. 1, 47–53.

[2] D. Auckly, *Homotopy $K3$ surfaces and gluing results in Seiberg–Witten theory*, Lecture Notes Series, 37. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[3] D. Bennequin, *Monopôles de Seiberg–Witten et conjecture de Thom (d'après Kronheimer, Mrowka et Witten)*, Séminaire Bourbaki, Vol. 1995/96. Astérisque No. 241 (1997), Exp. No. 807, 3, 59–96.

[4] J. Bryan, *Seiberg–Witten à la Furuta and genus bounds for classes with divisibility*, Turkish J. Math. 21 (1997), no. 1, 55–59.

[5] J. Bryan, *Seiberg–Witten Theory and $Z/2^p$ actions on spin 4-manifolds*, preprint, dg-ga/9704010.

[6] Y.S. Cho, *Seiberg–Witten invariants on non-symplectic 4-manifolds*, Osaka J. Math. 34 (1997), no. 1, 169–173.

[7] Y.S. Cho, *Seiberg–Witten invariants on connected sums of 4-manifolds*, Commun. Korean Math. Soc. 11 (1996), no. 2, 463–469.

[8] T. Draghici, *Seiberg–Witten invariants when reversing orientation*, Turkish J. Math. 21 (1997) 83-86.

[9] A. L. Edmonds, *Tori in certain aspherical four-manifolds*, Proc. Amer. Math. Soc. 126 (1998), no. 4, 1253–1255.

[10] R. Fintushel, R.J. Stern, *Knots, links, and four-manifolds*, preprint dg-ga/9612014.

[11] R. Fintushel, R.J. Stern, *Surfaces in* 4*-manifolds*, Math. Res. Lett. 4 (1997) N.6, 907-914.

[12] R. Fintushel, R.J. Stern, *Immersed spheres in* 4*-manifolds and the immersed Thom conjecture*, Turkish J. Math. 19 (1995), no. 2, 145–157.

[13] R. Fintushel, R.J. Stern, *Rational Blowdowns of Smooth 4-Manifolds*, preprint, alg-geom/9505018.

[14] R. Fintushel, R.J. Stern, *The blowup formula for Donaldson invariants*, Ann. of Math. (2) 143 (1996), no. 3, 529–546.

[15] K.A. Froyshov, *The Seiberg–Witten equations and four-manifolds with boundary*, Math. Res. Lett. 3 (1996), no. 3, 373–390.

[16] M. Katz, *A proof via the Seiberg–Witten moduli space of Donaldson's theorem on smooth* 4*-manifolds with definite intersection forms.* R.C.P. 25, Vol. 47 (Strasbourg, 1993–1995), 269–274, Pripubl. Inst. Rech. Math. Av., 1995/24.

[17] D. Kotschick, *On Irreducible Four–Manifolds*, preprint.

[18] D. Kotschick, J.W. Morgan, C.H. Taubes, *Four manifolds without symplectic structures but with non-trivial Seiberg–Witten invariants*, Math. Research Letters 2 (1995) 119-124.

[19] P.B. Kronheimer, *Minimal genus in* $S^1 \times M^3$, preprint.

[20] P.B. Kronheimer, *Embedded surfaces and gauge theory in three and four dimensions*, preprint.

[21] P.B. Kronheimer, T.S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Research Letters 1 (1994) 797-808.

[22] T.J. Li, A. Liu, *General wall crossing formula.* Math. Res. Lett. 2 (1995), no. 6, 797–810.

[23] M. Mariño, G. Moore, G. Peradze, *Superconformal invariance and the geography of four-manifolds*, preprint, hep-th/9812055.

[24] J.W. Morgan, T.S.Mrowka, Z. Szabó, *Product formulas along T^3 for Seiberg–Witten invariants*, Math. Res. Lett. 4 (1997) N.6 915-929.

[25] J.W. Morgan, Z. Szabó, *Homotopy $K3$ surfaces and mod 2 Seiberg–Witten invariants*, Math. Res. Lett. 4 (1997), no. 1, 17–21.

[26] J.W. Morgan, Z. Szabó, C.H. Taubes, *A product formula for the Seiberg–Witten invariants and the generalized Thom conjecture*, J. Differential Geom. 44 (1996), no. 4, 706–788.

[27] V. Muñoz, *Constraints for Seiberg–Witten basic classes of glued manifolds*, preprint dg-ga/9511012

[28] V. Muñoz, *Gluing formulae for Donaldson invariants for connected sums along surfaces*, preprint, dg-ga/9702002.

[29] L. I. Nicolaescu, *Finite energy Seiberg–Witten moduli spaces on 4-manifolds bounding Seifert fibrations*, preprint, dg-ga/9711006.

[30] Ch. Okonek, A. Teleman, *Seiberg–Witten invariants for manifolds with* $b_+ = 1$, preprint, alg-geom/9603025.

[31] Ch. Okonek, A. Teleman, *Seiberg–Witten invariants for manifolds with $b_+ = 1$ and the universal wall crossing formula*, Internat. J. Math. 7 (1996), no. 6, 811–832.

[32] V. Pati, *The Theorem of Kronheimer-Mrowka*, preprint, math.DG/9809045.

[33] D. Ruberman, *The minimal genus of an embedded surface of non-negative square in a rational surface*, Turkish J. Math. 20 (1996), no. 1, 129–133;

[34] Z. Szabó, *Exotic 4-manifolds with $b_2^+ = 1$*, Math. Res. Lett. 3 (1996), no. 6, 731–741.

[35] Z. Szabó, *On irreducible simply-connected 4-manifolds* Turkish J. Math. 21 (1997), no. 1, 41–46.

[36] H. D. Cao, J. Zhou, *Equivariant Cohomology and Wall Crossing Formulas in Seiberg–Witten Theory*, math.DG/9804134.

III: Three-manifolds and Floer Theory

References

[1] D. Auckly, *Surgery, Knots and the Seiberg–Witten Equations. Lectures for the 1995 TGRCIW*, preprint.

[2] D. Auckly, *The Thurston norm and three-dimensional Seiberg–Witten theory*, Osaka J. Math. 33 (1996), no. 3, 737–750.

[3] B. Broda, M. Bakalarska, *Topology of 3-dimensional Seiberg–Witten Theory*, hep-th/9802002.

[4] A.L. Carey, B.L. Wang, *Seiberg–Witten Floer theory and holomorphic curves*, preprint.

[5] A.L. Carey, B.L. Wang, *Seiberg–Witten-Floer homology and gluing formulae*, preprint.

[6] W. Chen, *Casson's invariant and Seiberg–Witten gauge theory*, Turkish J. Math. 21 (1997), no. 1, 61–81.

[7] W. Chen, *Dehn surgery formula for Seiberg–Witten invariants of homology 3-spheres*, preprint, dg-ga/9703009.

[8] K.A. Froyshov, *The Seiberg–Witten equations and four-manifolds with boundary*, Math. Res. Lett. 3 (1996), no. 3, 373–390.

[9] M. Hutchings, Y.J. Lee, *Circle-Valued Morse Theory and Reidemeister Torsion*, preprint, dg-ga/9706012.

[10] M. Hutchings, Y.J. Lee, *Circle-valued Morse theory, Reidemeister torsion, and Seiberg–Witten invariants of 3-manifolds*, preprint, dg-ga/9612004.

[11] P.B. Kronheimer, *Minimal genus in* $S^1 \times M^3$, preprint.

[12] P.B. Kronheimer, *Embedded surfaces and gauge theory in three and four dimensions*, preprint.

[13] P.B. Kronheimer, T.S. Mrowka, *Monopoles and contact structures*, Invent. Math. 130 (1997), no. 2, 209–255.

[14] P.B. Kronheimer, T.S. Mrowka, *Scalar curvature and the Thurston norm.* Math. Res. Lett. 4 (1997), no. 6, 931–937.

[15] Y.J. Lee, *Seiberg–Witten Equations on Asymptotically-Flat Three-Manifolds*, preprint, dg-ga/9706013.

[16] Y. Lim, *Seiberg–Witten invariants for three-manifolds and product formulae*, preprint.

[17] Y. Lim, *The equivalence of Seiberg–Witten and Casson invariants for homology 3-spheres*, preprint.

[18] Y. Lim, *Seiberg–Witten invariants for 3-manifolds in the case* $b_1 = 0$ *or* 1, preprint.

[19] P. Lisca, G. Matić, *Tight contact structures and Seiberg–Witten invariants*, Invent. Math. 129 (1997), no. 3, 509–525.

[20] P. Lisca, G. Matić, *On homotopic, non-isomorphic tight contact structures on* 3*-manifolds*, Turkish J. Math. 20 (1996), no. 1, 37–45.

[21] M. Marcolli, *Seiberg–Witten-Floer homology and Heegaard splittings*, Internat. J. Math. 7 (1996), no. 5, 671–696.

[22] M. Marcolli, B.L. Wang, *Equivariant Seiberg–Witten Floer homology*, preprint (a preliminary incomplete version available on dg-ga/9606003).

[23] G. Meng, C.H. Taubes, *$\underline{SW}$=Milnor torsion*, Math. Res. Lett. 3 (1996) N.5, 661-674.

[24] T.S. Mrowka, P. Ozsváth, B. Yu, *Seiberg–Witten monopoles on Seifert fibered spaces.* Comm. Anal. Geom. 5 (1997), no. 4, 685–791.

[25] L.I. Nicolaescu, *Adiabatic limits of Seiberg–Witten equations on Seifert manifolds*, preprint dg-ga/9601007.

[26] L. I. Nicolaescu, *Lattice points, Dedekind-Rademacher sums and a conjecture of Kronheimer and Mrowka*, preprint, math.DG/9801030.

[27] L. I. Nicolaescu, *Finite energy Seiberg–Witten moduli spaces on 4-manifolds bounding Seifert fibrations*, preprint, dg-ga/9711006.

[28] L. I. Nicolaescu, *Seiberg–Witten equations on tubes*, preprint, dg-ga/9701001.

[29] L. I. Nicolaescu, *Eta invariants of Dirac operators on circle bundles over Riemann surfaces and virtual dimension of finite energy Seiberg–Witten moduli spaces*, math.DG/9805046.

[30] B.L. Wang, *Seiberg–Witten Floer theory for homology 3-spheres*, preprint dg-ga/9602003.

[31] R.G. Wang, *On Seiberg–Witten Floer invariants and the generalized Thom problem*, preprint.

[32] G. Wang, R. Ye, *Bott-type and equivariant Seiberg–Witten Floer homology I*, preprint, dg-ga/0701010.

IV: Non-abelian monopoles

References

[1] D. Anselmi, P. Fré, *Gauged Hyperinstantons and Monopole Equations*, Phys. Lett. B 347 (1995) 247-254.

[2] Ph. C. Argyres, A. E. Faraggi, *The Vacuum Structure and Spectrum of N=2 Supersymmetric SU(N) Gauge Theory*, Phys. Rev. Lett. 74 (1995) 3931-3934.

[3] S. Bradlow, O. García-Prada, *Non-abelian monopoles and vortices*, Geometry and physics (Aarhus, 1995), 567–589, Lecture Notes in Pure and Appl. Math., 184, Dekker, New York, 1997.

[4] P.M.N. Feehan, P.B. Kronheimer, T.G. Leness, T.S. Mrowka, *PU(2) monopoles and a conjecture of Marino, Moore, and Peradze*, math.DG/9812125.

[5] P.M.N. Feehan, T.G. Leness, *PU(2) monopoles, II: Highest-level singularities and relations between four-manifold invariants*, preprint, dg-ga/9712005.

[6] P.M.N. Feehan, T.G. Leness, *PU(2) Monopoles, I: Regularity, Uhlenbeck Compactness, and Transversality.* preprint, dg-ga/9710032.

[7] P.M.N. Feehan, T.G. Leness, *PU(2) monopoles and relations between four-manifold invariants*, preprint, dg-ga/9709022.

[8] P.M.N. Feehan, T.G. Leness, *PU(2) monopoles, III: Existence of gluing and obstruction maps*, preprint.

[9] P.M.N. Feehan, T.G. Leness, *PU(2) monopoles, IV: Surjectivity of gluing maps*, preprint.

[10] P.M.N. Feehan, *Generic metrics, irreducible rank-one PU(2) monopoles, and transversality*, preprint, math.DG/9809001.

[11] P.M.N. Feehan, T.G. Leness, *Donaldson invariants and wall-crossing formulas. I: Continuity of gluing maps*, preprint, math.DG/9812060.

[12] P.M.N. Feehan, *A Kato-Yau inequality and decay estimate for harmonic spinors*, preprint, math.DG/9903021.

[13] O. Garcia-Prada, *The Seiberg–Witten invariants and the vortex equations*, Differential geometry and its applications (Santiago de Compostela, 1995), 167–182; An. Fms. Monogr., 3.

[14] S. Hyun, J. Park, J.S. Park, *Topological QCD*, Nucl. Phys. B 453 N. 1-2 (1995) 199-224.

[15] J.M.F. Labastida, M. Mariño, *Non-abelian monopoles on four-manifolds*, Nuclear Phys. B 448, N. 1-2 (1995) 373-395.

[16] J.M.F. Labastida, M. Mariño, *Polynomial invariants for* SU(2) *monopoles*, Nuclear Phys. B 456, N.3 (1995) 633-668.

[17] M. Mariño, G. Moore, G. Peradze, *Superconformal invariance and the geography of four-manifolds*, preprint, hep-th/9812055.

[18] Ch. Okonek, A. Teleman, *Quaternionic Monopoles*, C. R. Acad. Sci. Paris Sir. I Math. 321 (1995), no. 5, 601–606, and Comm. Math. Phys. 180 (1996), no. 2, 363–388.

[19] Ch. Okonek, A. Teleman, *The coupled Seiberg–Witten equations, vortices, and moduli spaces of stable pairs*, Internat. J. Math. 6 (1995), no. 6, 893–910.

[20] V.Ya. Pidstrigatch, A.N. Tyurin, *Localisation of the Donaldson's Invariants along Seiberg–Witten Classes*, preprint, dg-ga/9507004.

[21] A. Teleman, *Moduli spaces of PU(2)-monopoles*, preprint, dg-ga/9702006.

[22] A. Teleman, *Non-abelian Seiberg–Witten theory and projectively stable pairs*, preprint, alg-geom/9609020.

[23] A. Teleman, *Non-abelian Seiberg–Witten theory and stable oriented pairs*, Internat. J. Math. 8 (1997), no. 4, 507–535.

[24] N. A. Tyurin, *A necessary condition and a sufficient condition for the deformation of a B-monopole to an instanton*, Izv. Ross. Akad. Nauk Ser. Mat. 60 (1996), no. 1, 211–224.

V: Symplectic Geometry

References

[1] A.L. Carey, B.L. Wang, *Seiberg–Witten Floer theory and holomorphic curves*, preprint.

[2] R.E. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. (2) 142 (1995), no. 3, 527–595.

[3] E.N. Ionel, T.H. Parker, *The Gromov Invariants of Ruan-Tian and Taubes*, preprint, alg-geom/9702008.

[4] D. Kotschick, *The Seiberg–Witten invariants of symplectic four-manifolds (after C. H. Taubes)*, Séminaire Bourbaki, Vol. 1995/96. Astérisque No. 241 (1997), Exp. No. 812, 4, 195–220.

[5] D. Kotschick, J.W. Morgan, C.H. Taubes, *Four-manifolds without symplectic structures but with non-trivial Seiberg–Witten invariants*, Math. Research Letters 2 (1995) 119-124.

[6] P.B. Kronheimer, *Some non-trivial families of symplectic structures*, preprint.

[7] F. Lalonde, D. McDuff, *The classification of ruled symplectic 4-manifolds*, Math. Res. Lett. 3 (1996), no. 6, 769–778.

[8] F. Lalonde, D. McDuff, *J-curves and the classification of rational and ruled symplectic 4-manifolds*, Contact and symplectic geometry (Cambridge, 1994), 3–42, Publ. Newton Inst., 8, Cambridge Univ. Press, Cambridge, 1996.

[9] D. McDuff, *Lectures on Gromov invariants for symplectic surfaces*, Preprint Nr.267 Institut für Mathematik, Universität Hannover, alg-geom/9507007.

[10] D. McDuff, D. Salamon, *A survey of symplectic 4-manifolds with $b^+ = 1$*, Turkish J. Math. 20 (1996), no. 1, 47–60.

[11] D. McDuff, D. Salamon, *Introduction to symplectic topology*, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995

[12] G. Mikhalkin, *J-holomorphic curves in almost complex surfaces do not always minimize the genus*, Proc. Amer. Math. Soc. 125 (1997), no. 6, 1831–1833.

[13] K. Mohnke, *On Seiberg–Witten equations on symplectic 4-manifolds*, Symplectic singularities and geometry of gauge fields (Warsaw, 1995), Banach Center Publ., 39, 89–104, Polish Acad. Sci., Warsaw, 1997.

[14] H. Ohta, K. Ono, *Symplectic 4-manifolds with $b_2^+ = 1$*, Geometry and physics (Aarhus, 1995), 237–244, Lecture Notes in Pure and Appl. Math., 184, Dekker, New York, 1997.

[15] H. Ohta, K. Ono, *Notes on symplectic 4-manifolds with $b_2^+ = 1$. II*, Internat. J. Math. 7 (1996), no. 6, 755–770.

[16] H. Ohta, K. Ono, *Notes on symplectic 4-manifolds with $b_2^+ = 1$*, Moduli of vector bundles (Sanda, 1994; Kyoto, 1994), 209–212, Lecture Notes in Pure and Appl. Math., 179, Dekker, New York, 1996.

[17] A. Stipsicz, *A note on the geography of symplectic manifolds*, Turkish J. Math. 20 (1996), no. 1, 135–139.

[18] C.H. Taubes, *The Seiberg–Witten invariants and symplectic forms*, Math. Research Letters, 1 (1994) 809-822.

[19] C.H. Taubes, *More constraints on symplectic manifolds from Seiberg–Witten invariants*, Math. Research Letters, 2 (1995), 9-13.

[20] C.H. Taubes, *The Seiberg–Witten invariants and The Gromov invariants*, Math. Research Letters 2 (1995) 221-238. and Geometry and physics (Aarhus, 1995), 591–601, Lecture Notes in Pure and Appl. Math., 184, Dekker, New York, 1997.

[21] C.H. Taubes, $SW \Rightarrow Gr$. *From the Seiberg–Witten equations to pseudo–holomorphic curves*, J. Amer. Math. Soc. 9, N.3 (1996) 845-918.

[22] C.H. Taubes, *Counting pseudo-holomorphic submanifolds in dimension* 4, J. Differential Geom. 44 (1996), no. 4, 818–893.

VI: Kähler surfaces; Algebraic Geometry

References

[1] D. Auckly, *Homotopy $K3$ surfaces and gluing results in Seiberg–Witten theory*, Lecture Notes Series, 37. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[2] S. Bauer, *Diffeomorphism types of elliptic surfaces with $p_g = 1$*, J. Reine Angew. Math. 451 (1994), 89–148.

[3] R. Brussee, *Some C^∞–properties of Kähler surfaces*, New York J. Math. 2 (1996), 103–146.

[4] J. Bryan, R. Wentworth, *The multi-monopole equations for Kähler surfaces*, Turkish J. Math. 20 (1996), no. 1, 119–128.

[5] R. Y. Donagi, *Seiberg–Witten integrable systems*, Algebraic geometry—Santa Cruz 1995, 3–43, Proc. Sympos. Pure Math., 62, Part 2, AMS 1997.

[6] N.D. Elkies, *A characterization of the $\mathbf{Z}^n$ lattice*, Math. Res. Lett. 2 (1995), no. 3, 321–326.

[7] R. Fintushel, R. Stern, *Rational Blowdowns of Smooth 4-Manifolds*, J. Differential Geom. 46 (1997), no. 2, 181–235.

[8] R. Fintushel, R. Stern, *The blowup formula for Donaldson invariants*, Ann. of Math. (2) 143 (1996), no. 3, 529–546.

[9] R. Friedman, *Donaldson and Seiberg–Witten invariants of algebraic surfaces*, Algebraic geometry—Santa Cruz 1995, 85–100, Proc. Sympos. Pure Math., 62, Part 1, AMS 1997.

[10] R. Friedman, J.W. Morgan, *Algebraic Surfaces and Seiberg–Witten Invariants*, J. Alg. Geom. 6 (1997) N.3 445-479.

[11] R. Friedman, J.W. Morgan, *Obstruction bundles, semiregularity, and Seiberg–Witten invariants*, preprint, alg-geom/9509007.

[12] T. Fuller, *Distinguishing embedded curves in rational complex surfaces*, Proc. Amer. Math. Soc. 126 (1998), no. 1, 305–310.

[13] O. García-Prada, *The Seiberg–Witten invariants and the vortex equations*, Differential geometry and its applications (Santiago de Compostela, 1995), 167–182. An. Fms. Monogr., 3.

[14] L. Göttsche, D. Zagier, *Jacobi forms and the structure of Donaldson invariants for 4-manifolds with* $b_+ = 1$, Selecta Math. (N.S.) 4 (1998), no. 1, 69–115.

[15] Y. Kametani, *The simple invariant and differentiable structures on the Horikawa surface*, Tôhoku Math. J. (2) 47 (1995), no. 4, 541–553.

[16] D. Kotschick, *Orientations and geometrisations of compact complex surfaces*, Bull. London Math. Soc. 29 (1997), no. 2, 145–149.

[17] N. C. Leung, *Uniformization of four manifolds*, preprint, dg-ga/9705001.

[18] N.C. Leung, *Seiberg–Witten Invariants and Uniformization*, Math. Ann. 306 (1996), no. 1, 31–46.

[19] T. J. Li, A. Liu, *Symplectic structure on ruled surfaces and a generalized adjunction formula*, Math. Res. Lett. 2 (1995), no. 4, 453–471.

[20] M. Lönne, *On the diffeomorphism groups of elliptic surfaces*, Preprint Nr.267 Institut fuer Mathematik, Universitaet Hannover, alg-geom/9507007.

[21] G. Mikhalkin, *Surfaces of small genus in connected sums of* $\mathbf{C}\mathrm{P}^2$ *and real algebraic curves with many nests in* $\mathbf{R}\mathrm{P}^2$, Real algebraic geometry and topology (East Lansing, MI, 1993), 73–82. Contemp. Math., 182, Amer. Math. Soc., Providence, RI, 1995.

[22] J.W. Morgan, Z. Szabó, *Homotopy* $K3$ *surfaces and mod* 2 *Seiberg–Witten invariants*, Math. Res. Lett. 4 (1997), no. 1, 17–21.

[23] J.W. Morgan, T.S. Mrowka, *The smooth classification of elliptic surfaces*, Geometry, topology, and physics, 246–292. Conf. Proc. Lecture Notes Geom. Topology, VI, Internat. Press, Cambridge, MA, 1995.

[24] Ch. Okonek, A. Teleman, *Recent Developments in Seiberg–Witten Theory and Complex Geometry*, preprint, alg-geom/9612015.

[25] Ch. Okonek, A. Teleman, *The coupled Seiberg–Witten equations, vortices, and moduli spaces of stable pairs*, Internat. J. Math. 6 (1995), no. 6, 893–910.

[26] Ch. Okonek, A. Teleman, *Les invariants de Seiberg–Witten et la conjecture de van de Ven*, C. R. Acad. Sci. Paris Sir. I Math. 321 (1995), no. 4, 457–461.

[27] Ch. Okonek, A. Teleman, *Seiberg–Witten invariants and rationality of complex surfaces*, Math. Z. 225 (1997), no. 1, 139–149.

[28] A. Stipsicz, Z. Szabó, *The smooth classification of elliptic surfaces with* $b^+ > 1$, Duke Math. J. 75 (1994), no. 1, 1–50.

VII: Einstein metrics

References

[1] O. Biquard, *Metriques d'Einstein a cusps et equations de Seiberg–Witten* (French) [Einstein metrics with cusps and Seiberg–Witten equations], J. Reine Math. 490 (1997).

[2] M. J. Gursky, C. LeBrun, *Yamabe Invariants and Spin^c Structures*, preprint, dg-ga/9708002.

[3] C. LeBrun, *Einstein metrics on complex surfaces*, Geometry and physics (Aarhus, 1995), Lecture Notes in Pure and Appl. Math., 184, 167–176.

[4] C. LeBrun, *On the scalar curvature of complex surfaces*, Geom. Funct. Anal. 5 (1995), no. 3, 619–628.

[5] C. LeBrun, *Einstein Metrics and Mostow Rigidity*, Math. Research Letters 2 (1995), 1-8.

[6] C. LeBrun, *Polarized 4–manifolds, extremal Kähler metrics, and Seiberg–Witten theory*, Math. Res. Lett. 2, N.5 (1995) 653-662.

[7] C. LeBrun, *Four-manifolds without Einstein metrics*, Math. Res. Lett. 3, N.2 (1996) 133-147.

[8] C. LeBrun, *Yamabe Constants and the Perturbed Seiberg–Witten Equations*, preprint dg-ga/9605009.

[9] C. LeBrun, *Weyl Curvature, Einstein Metrics, and Seiberg–Witten Theory*, math.DG/9803093

VIII: Quantum Field Theory

References

[1] R. Aldrovandi, J.G. Pereira, *An introduction to geometrical physics*, World Scientific Publishing Co., Inc., River Edge, NJ, 1995. ISBN: 981-02-2232-7.

[2] M. Atiyah, *Duality and quantum field theory*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), 1–7, First Int. Press Lect. Ser., I, Internat. Press, 1998.

[3] P. Bandyopadhyay, *Geometry, topology and quantization*, Mathematics and its Applications, 386. Kluwer Academic Publishers Group, Dordrecht, 1996, ISBN: 0-7923-4305-0.

[4] B. Broda, *Topological contents of 3D Seiberg–Witten theory*, talk at the NATO Advanced Research Workshop: "Recent Developments in Quantum Field Theory", June 14-20, 1997, Zakopane, Poland.

[5] B. Broda, M. Bakalarska, *Topology of 3-dimensional Seiberg–Witten theory*, presented at: "Gauge Theories of Gravitation", Jadwisin, Poland, 4-10 September 1997.

[6] R. Brooks, A. Lue, *The monopole equations in topological Yang-Mills*, J. Math. Phys. 37, N.3 (1996) 1100-1105.

[7] A.L. Carey, J. McCarthy, B.L. Wang, R.B. Zhang, *Seiberg–Witten monopoles in three dimensions*, Lett. Math. Phys. 39 (1997), no. 3, 213–228.

[8] E. D'Hoker, I. M. Krichever, D. H. Phong, *The renormalization group equation in $N = 2$ supersymmetric gauge theories*, Nuclear Phys. B 494 (1997), no. 1-2, 89–104.

[9] J. Eichhorn, T. Friedrich, *Seiberg–Witten theory. Symplectic singularities and geometry of gauge fields (Warsaw, 1995)*, 231–267, Banach Center Publ., 39, Polish Acad. Sci., Warsaw, 1997.

[10] R. Flume, M. Magro, L. O'Raifeartaigh, I. Sachs, O. Schnetz, *Uniqueness of the Seiberg–Witten Effective Lagrangian*, Nucl.Phys. B494 (1997) 331-345.

[11] R. Flume, L. O'Raifeartaigh, I. Sachs, *Brief Resume of Seiberg–Witten Theory*, preprint, hep-th/9611118.

[12] P.G.O. Freund, *Dirac monopoles and the Seiberg–Witten monopole equations*, J. Math. Phys. 36 (1995), no. 6, 2673–2674.

[13] Ch. Fronsdal, *A model for QCD in three dimensions*, Modern group theoretical methods in physics (Paris, 1995), 129–142, Math. Phys. Stud., 18, Kluwer Acad. Publ., Dordrecht, 1995.

[14] R. Gianvittorio, I. Martin, A. Restuccia, *The gauge invariant Lagrangian for Seiberg–Witten topological monopoles*, preprint, Lett. Math. Phys. 39 (1997), no. 1, 51–58.

[15] W. Greiner, J. Reinhardt, *Field quantization*. Translated from the German. With a foreword by D. A. Bromley. Springer-Verlag, Berlin, 1996 ISBN:3-540-59179-6.

[16] J. M. Guilarte, *Light monopoles, electric vortices and instantons*, Phys. Lett. B. 366 (1996), no. 1-4, 174–180.

[17] K. Ito, S.K. Yang, *Flat Coordinates, Topological Landau-Ginzburg Models and the Seiberg–Witten Period Integrals*, hep-th/9708017.

[18] S. V. Ketov, *Solitons, monopoles and duality: from sine-Gordon to Seiberg–Witten*, Fortsch.Phys. 45 (1997) 237-292.

[19] J.M.F. Labastida, *Quantum field theory: a progress report*, Differential geometry and its applications (Santiago de Compostela, 1995), 101–124, An. Fms. Monogr., 3, CIEMAT, Madrid.

[20] J.M.F. Labastida, C. Lozano *Lectures on topological quantum field theory*, Trends in theoretical physics (La Plata, 1997), 54–93, AIP Conf. Proc., 419, Amer. Inst. Phys. 1998.

[21] J.M.F. Labastida, M. Mariño, *A topological Lagrangian for monopoles on four manifolds*, Phys. Lett. B 351, N.1-3 (1995) 146-152.

[22] Y. Ohta, *Langevin approach for abelian topological gauge theory*, J. Math. Phys. 38 (1997), no.6, 2954–2962.

[23] K. Takasaki, T. Nakatsu, *Isomonodromic deformations and supersymmetric gauge theories* Internat. J. Modern Phys. A 11 (1996, no. 31, 5505–5518.

[24] E. Witten, *Some questions for constructive field theorists*, Constructive physics (Palaiseau, 1994), 1–5, Lecture Notes in Phys., 446, Springer, Berlin, 1995.

[25] S. Wu, *On the Mathai-Quillen formalism of topological sigma models*, J. Geom. Phys. 17 (1995), no. 4, 299–309.

[26] A. Yung, *Instanton-induced Effective Lagrangian in the Seiberg–Witten Model*, Nucl.Phys. B485 (1997) 38-62.

IX: String Theory and Duality

References

[1] M. Alishahiha, *Simple Derivation of the Picard-Fuchs equations for the Sei-berg–Witten Models*, preprint, hep-th/9703186.

[2] L. Alvarez-Gaumé, C. Kounnas, J. Distler, M. Mariño, *Softly broken $N = 2$ QCD*, Internat. J. Modern Phys. A 11 (1996), no. 26, 4745–4777.

[3] D. Anselmi, P. Fré, *Gauged Hyperinstantons and Monopole Equations*, Phys. Lett. B 347 (1995) 247-254.

[4] Ph. C. Argyres, A. E. Faraggi, *The Vacuum Structure and Spectrum of N=2 Supersymmetric SU(N) Gauge Theory*, Phys. Rev. Lett. 74 (1995) 3931-3934.

[5] O. Bergman, A. Fayyazuddin, *String Junctions and BPS States in Seiberg–Witten Theory*, hep-th/9802033.

[6] N. Berkovits, *Super-Maxwell actions with manifest duality*, Phys. Lett. B 398 (1997), no. 1-2, 79–82.

[7] A. Bilal, *Duality in N=2 SUSY SU(2) Yang-Mills Theory: A pedagogical introduction to the work of Seiberg and Witten*, preprint, hep-th/9601007.

[8] A. Brandhuber, J. Sonnenschein, S. Theisen, S. Yankielowicz, *M Theory And Seiberg–Witten Curves: Orthogonal and Symplectic Groups*, Nucl.Phys. B504 (1997) 175-188.

[9] A. Cappelli, P. Valtancoli, L. Vergnano, *Isomonodromic Properties of the Seiberg–Witten Solution*, preprint, hep-th/9710248.

[10] C. Csaki, W. Skiba, *Classification of the N=1 Seiberg–Witten Theories*, preprint, hep-th/9801173.

[11] G. Curio, *Topological partition function and string-string duality* Phys. Lett. B 366 (1996), no. 1-4, 131–133.

[12] E. D'Hoker, I. M. Krichever, D. H. Phong, *The effective prepotential of $N = 2$ supersymmetric $\mathrm{SU}(N_c)$ gauge theories*, Nuclear Phys. B 489 (1997), no. 1-2, 179–210.

[13] E. D'Hoker, D.H. Phong, *Strong Coupling Expansions of SU(N) Seiberg–Witten Theory*, Phys.Lett. B397 (1997) 94-103.

[14] E. D'Hoker, D.H. Phong, *Calogero-Moser Systems in SU(N) Seiberg–Witten Theory*, preprint, hep-th/9709053.

[15] M. Di Pierro, K. Konishi, *Mass, Confinement and CP Invariance in the Seiberg–Witten Model*, Phys.Lett. B388 (1996) 90-96.

[16] R. Dijkgraaf, *Lectures on four-manifolds and topological gauge theories. String theory, gauge theory and quantum gravity (Trieste, 1995)*, Nuclear Phys. B Proc. Suppl.45B,C (1996), 29–45.

[17] R. Dijkgraaf, *Fields, Strings, and Duality*, Symétries quantiques (Les Houches, 1995), 3–147, North-Holland, 1998.

[18] R. Donagi, A. Grassi, E. Witten, *A Non-Perturbative Superpotential With E_8 Symmetry*, Modern Phys. Lett. A 11 (1996), no. 27, 2199–2211.

[19] N. Dorey, V.V. Khoze, M.P. Mattis, *Multi-instanton check of the relation between the prepotential F and the modulus u in $N=2$ $SUSY$ Yang-Mills theory*, Phys. Lett. B 390 (1997), no. 1-4, 205–209.

[20] N. Dorey, V.V. Khoze, M.P. Mattis, *Multi-instanton calculus in $N=2$ supersymmetric gauge theory*, Phys. Rev. D (3) 54 (1996), no. 4, 2921–2943.

[21] N. Dorey, V.V. Khoze, M.P. Mattis, *Multi-instanton calculus in $N=2$ supersymmetric gauge theory. II. Coupling to matter*, Phys. Rev. D (3) 54 (1996), no. 12, 7832–7848.

[22] M.R. Douglas, S.H. Shenker, *Dynamics of* $\mathrm{SU}(N)$ *supersymmetric gauge theory*, Nuclear Phys. B 447 (1995), no. 2-3, 271–296.

[23] I. P. Ennes, S. G. Naculich, H. Rhedin, H. J. Schnitzer. *One Instanton Predictions of a Seiberg–Witten curve from M-theory: the Symmetric Representation of SU(N)*, hep-th/9804151.

[24] A. Fayyazuddin, M. Spalinski, *The Seiberg–Witten Differential From M-Theory*, Nucl.Phys. B508 (1997) 219-228.

[25] S. Ferrara, J.A. Harvey, A. Strominger, C. Vafa, *Second-quantized mirror symmetry*, Phys. Lett. B 361 (1995), no. 1-4, 59–65.

[26] F. Ferrari, A. Bilal, *The strong-coupling spectrum of the Seiberg–Witten theory*, Nuclear Phys. B 469 (1996), no. 3, 387–402.

[27] R. Flume, *Some remarks on N=2 extended supersymmetric Yang-Mills theories and Seiberg–Witten duality*, preprint, hep-th/9702192.

[28] P. Fré, *Lectures on special Kähler geometry and electric-magnetic duality rotations. String theory, gauge theory and quantum gravity (Trieste, 1995)*, Nuclear Phys. B Proc. Suppl. 45B,C (1996), 59–114.

[29] S.Gukov, *Seiberg–Witten Solution from Matrix Theory*, preprint, hep-th/9709138.

[30] T. Harano, M. Sato, *Multi-instanton calculus versus exact results in $N = 2$ supersymmetric QCD*, Nuclear Phys. B 484 (1997), no. 1-2, 167–195.

[31] S. Hyun, J. Park, J.S. Park, *N=2 Supersymmetric QCD and Four Manifolds. (I) the Donaldson and the Seiberg–Witten Invariants*, preprint hep-th/9508162.

[32] S. Hyun, J. Park, J.S. Park, *Topological QCD*, Nucl. Phys. B 453 N. 1-2 (1995) 199-224.

[33] J. M. Isidro, A. Mukherjee, J. P. Nunes, H. J. Schnitzer, *A Note on the Picard-Fuchs Equations for massive N=2 Seiberg–Witten Theories*, Nuclear Phys. B 502 (1997), no. 1-2, 363–382.

[34] S. Kachru, A. Klemm, W. Lerche, P. Mayr, C. Vafa, *Non-perturbative results on the point particle limit of $N = 2$ heterotic string compactifications*, Nuclear Phys. B 459 (1996), no. 3, 537–555.

[35] A. Klemm, W. Lerche, P. Mayr, C. Vafa, N. Warner, *Self-dual strings and $N = 2$ supersymmetric field theory*, Nuclear Phys. B 477 (1996), no. 3, 746–764.

[36] K. Konishi, *Confinement, Supersymmetry Breaking and Theta Parameter Dependence in the Seiberg–Witten Model*, Phys.Lett. B392 (1997) 101-105.

[37] K. Konishi, H. Terao, *CP, Charge Fractionalizations and Low Energy Effective Actions in the SU(2) Seiberg–Witten Theories with Quarks*, preprint, p-th/9707005.

[38] K. Landsteiner, E. Lopez, D. A. Lowe. *N=2 Supersymmetric Gauge Theories, Branes and Orientifolds*, Nucl.Phys. B507 (1997) 197-226.

[39] B.K. Lee, S. Nam, *Instanton-induced Effective Vertex in the Seiberg–Witten Theory with Matter*, preprint, hep-th/9703171.

[40] W. Lerche, *Introduction to Seiberg–Witten Theory and its Stringy Origin*, Nucl.Phys.Proc.Suppl. 55B (1997) 83-117. Fortsch.Phys. 45 (1997) 293-340.

[41] W. Lerche, P. Mayr, N.P. Warner, *Non-Critical Strings, Del Pezzo Singularities And Seiberg–Witten Curves*, Nucl.Phys. B499 (1997) 125-148.

[42] U. Lindstrom, M. Rocek, *A note on the Seiberg–Witten solution of N=2 Super Yang-Mills Theory*, Phys. Lett. B355 (1995) 492.

[43] A. Lossev, N. Nekrassov, S. Shatashvili, *Testing Seiberg–Witten Solution*, preprint, hep-th/9801061.

[44] G. Lopez Cardoso, D. Lust, T. Mohaupt, *Non-perturbative monodromies in $N = 2$ heterotic string vacua*, Nuclear Phys. B 455 (1995), no. 1-2, 131–164.

[45] Zh.Q. Ma, B.Y. Hou, *The modified Seiberg–Witten monopole equations and their exact solutions*, preprint, hep-th/9711141.

[46] A. Marshakov, A Mironov, A Morozov, *WDVV-like equations in $N = 2$ SUSY Yang-Mills theory.* Phys. Lett. B 389 (1996), no. 1, 43–52.

[47] A. Marshakov, A. Mironov, A. Morozov, *WDVV equations from algebra of forms*, Modern Phys. Lett. A 12 (1997), no. 11, 773–787.

[48] A. Marshakov, *Seiberg–Witten Theory, Integrable Systems and D-branes*, hep-th/9709001.

[49] J.A. Minahan, D. Nemeschansky *An $N = 2$ superconformal fixed point with E_6 global symmetry*, Nuclear Phys. B 482 (1996), no. 1-2, 142–152.

[50] G. Moore, E. Witten, *Integration over the u-plane in Donaldson theory*, Adv. Theor. Math. Phys. 1 (1997) N.2, 298-387.

[51] D.R. Morrison, C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. II*, Nuclear Phys. B 476 (1996), no. 3, 437–469.

[52] S. G. Naculich, H. Rhedin, H. J. Schnitzer. *One-Instanton Test of a Seiberg–Witten Curve from M-theory: the Antisymmetric Representation of SU(N)*, hep-th/98040105.

[53] W. Nahm, *On the Seiberg–Witten Approach to electric-magnetic Duality*, preprint, hep-th/9608121.

[54] S. Nergiz. C. Saçlioğlu, *Liouville vortex and ϕ^4 kink solutions of the Seiberg–Witten equations*, J. Math. Phys. 37 (1996), no. 8, 3753–3759.

[55] K. Olsen, *Dimensional reduction of dual topological theories*, Modern Phys. Lett. A 11 (1996), no. 22, 1777–1784.

[56] T. Ortín, *The genus-two heterotic string cosmological constant*, Nuclear Phys. B 387 (1992), no. 2, 280–314.

[57] J. Pawelczyk, *From Seiberg–Witten invariants to topological Green-Schwarz string*, hep-th/9707005.

[58] Sh. Ryang, *The Picard-Fuchs equations, monodromies and instantons in the $N = 2$ SUSY gauge theories*, Phys. Lett. B 365 (1996), no. 1-4, 113–118.

[59] N. Seiberg, E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, Nucl. Phys. B431 (1994) 484-550.

[60] N. Seiberg, E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nuclear Phys. B 426, N.1 (1994) 19-52.

[61] A. Sen, *F-theory and orientifolds*, Nuclear Phys. B 475 (1996), no. 3, 562–578.

[62] H. Suzuki, *Evaluation of Periods via Fibrations in Seiberg–Witten Theories and in Type-II String*, Mod.Phys.Lett. A12 (1997) 2847-2858.

[63] S. Terashima, S.K. Yang, *Exceptional Seiberg–Witten Geometry with Massive Fundamental Matters*, hep-th/9803014.

[64] C. Vafa, E. Witten, *A Strong Coupling Test of S-Duality*, Nucl. Phys. B431 (1994) 3-77.

X: Integrable Systems

References

[1] R. Y. Donagi, *Seiberg–Witten integrable systems*, Algebraic geometry—Santa Cruz 1995, 3–43, Proc. Sympos. Pure Math., 62, Part 2, AMS 1997.

[2] R. Donagi, E. Witten, *Supersymmetric Yang-Mills Systems And Integrable Systems*, Nuclear Phys. B 460, N.2 (1996) 299-334.

[3] A.Gorsky, I.M. Krichever, A. Marshakov, A. Mironov, A Morozov, *Integrability and Seiberg–Witten exact solution*, Phys. Lett. B 355 (1995), no. 3-4, 466–474.

[4] A. Gorsky, A. Marshakov, A. Mironov, A. Morozov, *$N = 2$ supersymmetric QCD and integrable spin chains: rational case $N_f < 2N_c$*, Phys. Lett. B 380 (1996), no. 1-2, 75–80.

[5] H. Itoyama, A. Morozov, *Integrability and Seiberg–Witten theory*, Frontiers in quantum field theory (Osaka, 1995), 301–324.

[6] H. Itoyama, A. Morozov, *Prepotential and the Seiberg–Witten Theory*, Nuclear Phys. B 491 (1997), no. 3, 529–573.

[7] H. Itoyama, A. Morozov, *Integrability and Seiberg–Witten Theory: Curves and Periods*, Nucl. Phys. B477 (1996) 855-877.

[8] A.Marshakov, *Seiberg–Witten Theory, Integrable Systems and D-branes*, hep-th/9709001.

[9] A. Marshakov, *Exact solutions to quantum field theories and integrable equations*, Modern Phys. Lett. A 11 (1996), no. 14, 1169–1183.

[10] E.J. Martinec, N.P. Warner, *Integrable systems and supersymmetric gauge theory*, Nuclear Phys. B 459 (1996), no. 1-2, 97–112.

[11] P. Sutcliffe, *Seiberg–Witten theory, monopole spectral curves and affine Toda solitons*, to appear in Phys. Lett. B.

XI: Various

References

[1] A. H. Bilge, T. Dereli, S.Kocak, *Seiberg–Witten Equations on* R^8, Turk.J.Math 21 (1997) 87-92.

[2] J. Fang *Solutions of the Seiberg–Witten equation on the flat Euclidean space* $\mathbf{R}^8$ (Chinese), Sichuan Daxue Xuebao 34 (1997), no. 6, 746–751.

[3] F.J. Flaherty, *Dirac operators for Finsler spaces*, Finsler geometry (Seattle, WA, 1995), 59–66. Contemp. Math., 196, Amer. Math. Soc., Providence, RI, 1996.

[4] T. Friedrich, *Dirac-Operatoren in der Riemannschen Geometrie, Mit einem Ausblick auf die Seiberg–Witten-Theorie*, Advanced Lectures in Mathematics. Friedr. Vieweg and Sohn, Braunschweig, 1997 ISBN: 3-528-06926-0.

[5] P. Gauduchon, *Hermitian connections and Dirac operators*, Boll. Un. Mat. Ital. B (7) 11 (1997), no. 2, suppl., 257–288.

[6] Y.Zh. Huang, J. Lepowsky, *Affine Lie algebras and vertex tensor categories*, preprint, q-alg/9706028.

[7] J. Jost, X. Peng, G. Wang, *Variational Aspects of The Seiberg–Witten Functional*, Calc. Var. Partial Differential Equations 4, N.3 (1996) 205-218.

[8] I. M. Krichever, D.H. Phong, *On the integrable geometry of soliton equations and $N = 2$ supersymmetric gauge theories.* J. Differential Geom. 45 (1997), no. 2, 349–389.

[9] T. Masuda, T. Sasaki, H. Suzuki, *Seiberg–Witten Theory of Rank Two Gauge Groups and Hypergeometric Series*, preprint, hep-th/9705166.

[10] A. Mironov, *WDVV Equations in Seiberg–Witten theory and associative algebras*, preprint, hep-th/9704205.

[11] M. Nagase, Spin^q *structures*, J. Math. Soc. Japan 47 (1995), no. 1, 93–119.

[12] T. Nitta, T. Taniguchi, *Quaternionic Seiberg–Witten equation*, Internat. J. Math. 7 (1996), no. 5, 697–703.

[13] Ch. Okonek, A. Teleman, *Quaternionic Monopoles*, C. R. Acad. Sci. Paris Sir. I Math. 321 (1995), no. 5, 601–606, and Comm. Math. Phys. 180 (1996), no. 2, 363–388.

[14] A. Pajitnov, *Simple homotopy type of Novikov Complex for closed 1-forms and Lefschetz ζ-functions of the gradient flow*, dg-ga/9706014.

[15] F. Quinn, *Problems in low-dimensional topology. Surgery and geometric topology (Sakado, 1996)*, Sci. Bull. Josai. Univ. 1997, Special issue no. 2, 97–104.

[16] C. Saçlioğlu, S. Nergiz, *Seiberg–Witten Monopole Equations And Riemann Surfaces*, Nucl.Phys. B503 (1997) 675-687.

[17] D. Salamon, *Removable singularities and a vanishing theorem for Seiberg–Witten invariants*, Turkish J. Math. 20 (1996), no. 1, 61–73.

[18] S. Wang, *A Vanishing theorem for Seiberg–Witten Invariants*, Math Res. Lett. 2 (1995), no. 3, 305–310.

Index