Chapter 3 Waves in an Elastic Whole Space 8/28/2022

Equation of Motion of a Solid

Hopefully, many of the topics in this chapter are review. However, I find it useful to discuss some of the key characteristics of elastic continuous media. These concepts are critical for understanding both seismic waves in the Earth and also the response of engineered structures (e.g. buildings). I will assume that you already know what stress and strain are and I will begin with the equation of motion. I will use Einstein's summation convention that any repeated index signifies summation over three spatial coordinates.

In the first two chapters we considered dynamics problems in which time was the only dependent variable. However, in a continuum, the motion is a function of both time and space. Consider an infinitesimally small cube of elastic solid shown in Figure 3.1. Although this cube is surrounded by a continuous solid, we can ask about the net forces on the cube.

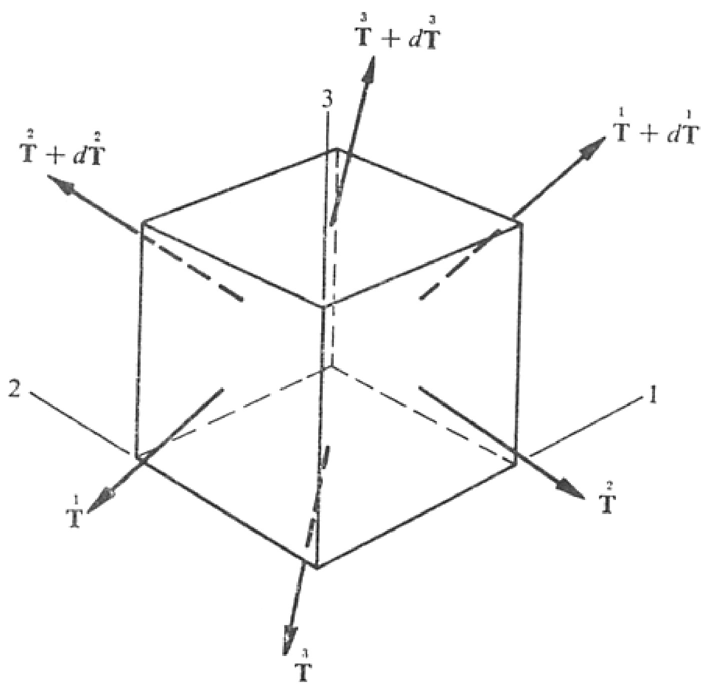


Figure 3.1. Distribution of tractions on the faces of an infinitesimal cube of matter. $\overset{i}{\mathbf{T}}$ is the vector traction (force per unit area) on the i^{th} face of the cube.

We inquire about the net force $\mathbf{F}$ on the cube. We begin by noting that the traction vector on the i^{th} face is given by

$$\overset{i}{\mathbf{T}} = \sigma_{ij}\mathbf{n}_j \tag{3.1}$$

where σ_{ij} is stress in cartesian corrdinates and $\mathbf{n}_j$ is unit normal vector to the j^{th} face of the cube. For example,

$$\overset{1}{\mathbf{T}} = \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3 \tag{3.2}$$

We begin by assuming that there is no net torque on the cube, otherwise it would start to spin. This condition is satisfied if and only if the stress tensor is symmetric; that is

$$\sigma_{ij} = \sigma_{ji} \tag{3.3}$$

We next employ Newton's 2nd law to derive the rectilinear acceleration of the mass,

$$\mathbf{F} = \dot{\mathbf{P}} \approx m\ddot{\mathbf{u}} \tag{3.4}$$

The cube is assumed to have a density of ρ and dimensions of dx_1, dx_2, dx_3. The $\approx$ becomes a true $=$ if we take $\mathbf{u}$ to be the position of the center of mass of our infinitesimal cube. The i^{th} component of net force on the cube is

$$F_i = d\overset{1}{T_i}\,dx_2dx_3 + d\overset{2}{T_i}\,dx_1dx_3 + d\overset{3}{T_i}\,dx_1dx_2 \tag{3.5}$$

Recognizing that

$$d\overset{j}{T}_i = \frac{\partial\sigma_{ij}}{\partial x_j}dx_j \quad \text{(no summation)} \tag{3.6}$$

we can rewrite Newton's law (3.4) for the i^{th} component of net force and acceleration as

$$\frac{\partial\sigma_{ij}}{\partial x_j}dx_1dx_2dx_3 = \rho\ddot{u}_i dx_1dx_2dx_3 \text{ (summation on } j) \tag{3.7}$$

or using the notation where $,i$ signifies differentiation with respect to i^{th} coordinate, this be written

$$\sigma_{ij,j} = \rho\ddot{u}_i \tag{3.8}$$

We can obtain a slightly more general expression by allowing there to be some external "body" force ***f*** that is acting on the cube (e.g. gravity) and we then obtain

$$\sigma_{ij,j} + f_i = \rho\ddot{u}_i \tag{3.9}$$

Equation (3.9) is the basic equation of motion of a solid continuum. Although we derived it from Newton's law, it is fundamentally different in that it contains a spatial derivative of forces as well as the time derivative of linear momentum. As we will see, this fundamentally changes the nature of the forces in the problem. In particular, it says that acceleration at a point is not related to stress at that point (force per unit area), but to the spatial derivative of stress. We can generalize (3.9) by noting that it can be written as

$$\underset{\rightarrow}{\nabla}\bullet\sigma + \mathbf{f} = \rho\ddot{\mathbf{u}} \tag{3.10}$$

Where $\nabla\bullet$ is the divergence operator (operating on the stress tensor). This operation is a 3-vector whose components are the divergence of the three columns of the stress tensor.

Strain and Constitutive Laws

In order to actually solve elasticity problems, we must have some relationship between the deformation of the body and the internal stresses. If we consider our infinitesimal cube as shown in Figure 3.2, then we can describe the motion of the cube as a combination of a rigid body rotation and internal strain. We will keep track of the motions of our cube by characterizing the position $\mathbf{u}$ and the diagonal vector $\mathbf{R}$. We will call the diagonal of the unstrained element $\mathbf{R}$ and the diagonal of the element after straining $\mathbf{R}'$. We define the change in the diagonal element due as

$$\delta\mathbf{R} = \mathbf{R}' - \mathbf{R} \tag{3.11}$$

If the motion of the infinitesimal cube is small, then in component form

$$\delta R_i = u_{i,j} dx_j \tag{3.12}$$

which can be rewritten in the form of

$$\delta R_i = \omega_{ij} dx_j + \varepsilon_{ij} dx_j \tag{3.13}$$

where

$$\omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) \tag{3.14}$$

and

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{3.15}$$

ω_{ij} represents rigid body rotation and it is anti-symmetric. ε_{ij} is the infinitesimal strain tensor and it is symmetric.

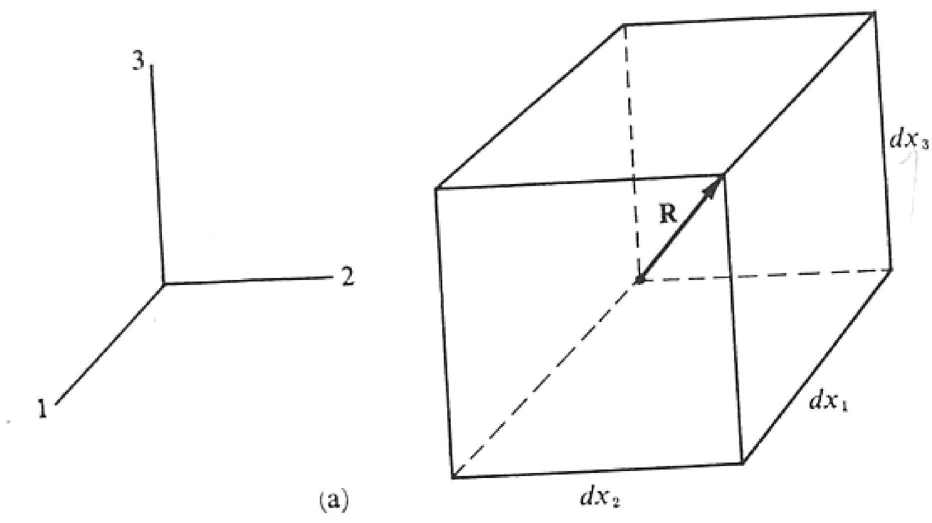


Figure 3.2. Deformation of an infinitesimal element.

The relationship between stress and strain is called the constitutive relation. For small strains, most materials exhibit a linear relationship between stress and strain that can be generally written as

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{3.16}$$

where there are 81 elastic coefficients C_{ijkl}. However, due the symmetry of the stress and strain tensor, and due to the requirement for a unique strain energy, there are at most 21 independent elastic coefficients. If the material is isotropic (no intrinsic directionality to the properties), then there are only 2 independent elastic coefficients. Table 3.1 provides a handy conversion between several different elastic coefficients for an isotropic solid. For our discussion we will use the 1st and 2nd Lame constants λ and μ. In this case (3.16) simplifies to

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \tag{3.17}$$

where

$$\delta_{ij} = \begin{Bmatrix} 0 & i \neq j \\ 1 & i = j \end{Bmatrix} \equiv \text{Kronecker delta} \tag{3.18}$$

	LAMÉ'S MODULUS λ	SHEAR MODULUS μ	YOUNG'S MODULUS η	POISSON'S RATIO ν	BULK MODULUS κ
λ, μ			$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{3\lambda+2\mu}{3}$
λ, η		irrational		irrational	irrational
λ, ν		$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$		$\frac{\lambda(1+\nu)}{3\nu}$
λ, κ		$\frac{3(\kappa-\lambda)}{2}$	$\frac{9\kappa(\kappa-\lambda)}{3\kappa-\lambda}$	$\frac{\lambda}{3\kappa-\lambda}$	
μ, η	$\frac{(2\mu-\eta)\mu}{\eta-3\mu}$			$\frac{\eta-2\mu}{2\mu}$	$\frac{\mu\eta}{3(3\mu-\eta)}$
μ, ν	$\frac{2\mu\nu}{1-2\nu}$		$2\mu(1+\nu)$		$\frac{2\mu(1+\nu)}{3(1-2\nu)}$
μ, κ	$\frac{3\kappa-2\mu}{3}$		$\frac{9\kappa\mu}{3\kappa+\mu}$	$\frac{3\kappa-2\mu}{2(3\kappa+\mu)}$	
η, ν	$\frac{\nu\eta}{(1+\nu)(1-2\nu)}$	$\frac{\eta}{2(1+\nu)}$			$\frac{\eta}{3(1-2\nu)}$
η, κ	$\frac{3\kappa(3\kappa-\eta)}{9\kappa-\eta}$	$\frac{3\eta\kappa}{9\kappa-\eta}$		$\frac{3\kappa-\eta}{6\kappa}$	
ν, κ	$\frac{3\kappa\nu}{1+\nu}$	$\frac{3\kappa(1-2\nu)}{2(1+\nu)}$	$3\kappa(1-2\nu)$		

Table 3.1. Relationship between elastic constants for an isotropic elastic medium

Navier's Equation

We are now in a position to write the equation of motion entirely in terms of displacement of the medium. Combining equations (3.9), (3.15), and (3.17), we obtain

$$\rho \ddot{u}_i = f_i + \mu u_{i,jj} + (\lambda + \mu) u_{j,ji} \tag{3.19}$$

This is Navier's equation and it is such an important equation that it is worth writing it out to see the terms more explicitly.

$$\rho \frac{\partial^2 u_i}{\partial t^2} = f_i + \sum_{j=1}^{3} \left[\mu \frac{\partial^2 u_i}{\partial x_j^2} + (\lambda + \mu) \frac{\partial^2 u_j}{\partial x_j \partial x_i} \right] \tag{3.20}$$

In Navier's equation 2nd derivatives of displacements with respect to time are linearly related to 2nd derivatives of displacement with respect to space. Everything that happens in an isotropic linearly-elastic solid is a solution to this equation.

We can also write Navier's equation in vector form as

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \underset{\to}{\nabla} (\underset{\to}{\nabla} \bullet \mathbf{u}) + \mathbf{f} = \rho \ddot{\mathbf{u}} \tag{3.21}$$

Where Laplacian operator $\nabla^2 \mathbf{u} \equiv \underset{\to}{\nabla} \bullet (\underset{\to}{\nabla} \mathbf{u})$ is the divergence (a 3-vector) of the gradient of the displacement vector (a 3-tensor). The term $\underset{\to}{\nabla} \bullet \mathbf{u}$ is seen to be the *dilatation*, or the net volume change of our infinitesimal element. This vector form of the equation has the advantage that we can rewrite it in any type of coordinate frame for which we know the Laplacian operator, the gradient operator, and the acceleration vector. In particular, we can write these operators for

Cartesian coordinates

$$\ddot{\mathbf{u}} = \ddot{u}_i \mathbf{e}_i \tag{3.22}$$

$$\underset{\to}{\nabla} \cdot \mathbf{u} = u_{i,i} \tag{3.23}$$

$$\underset{\to}{\nabla} = \mathbf{e}_i \frac{\partial}{\partial x_i} \tag{3.24}$$

$$\nabla^2 \mathbf{u} = \frac{\partial^2 u_i}{\partial x_j \partial x_j} \mathbf{e}_i \quad \text{(note the double sum on } i \text{ and } j\text{)} \tag{3.25}$$

$$\underset{\to}{\nabla} \otimes \mathbf{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) \mathbf{e}_1 + \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) \mathbf{e}_2 + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \mathbf{e}_3 \tag{3.26}$$

Cylindrical coordinates

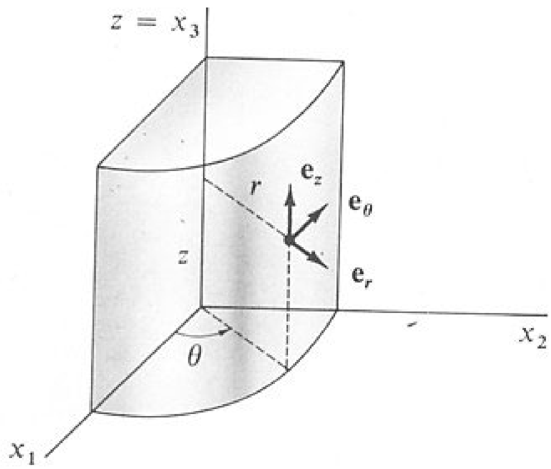
$$\ddot{\mathbf{u}} = \ddot{u}_r \mathbf{e}_r + \ddot{u}_\theta \mathbf{e}_\theta + \ddot{u}_z \mathbf{e}_\mathbf{z} \tag{3.27}$$

$$\underset{\to}{\nabla} \cdot \mathbf{u} = \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \tag{3.28}$$

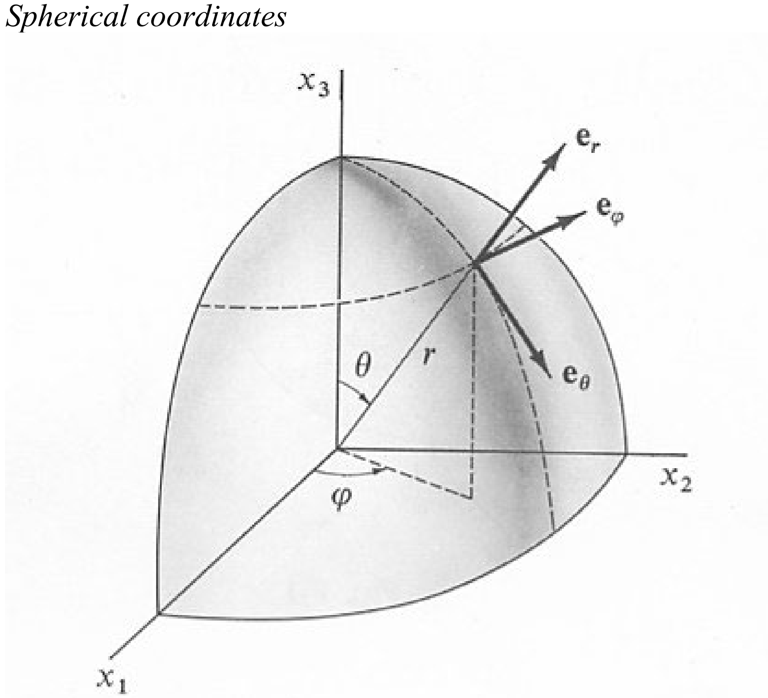
$$\underset{\to}{\nabla} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \tag{3.29}$$

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{3.30}$$

$$\underset{\rightarrow}{\nabla} \otimes \mathbf{u} = \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \mathbf{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \mathbf{e}_\theta + \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r u_\theta \right) - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right] \mathbf{e}_z \qquad (3.31)$$



Spherical coordinates



$$\ddot{\mathbf{u}} = \ddot{u}_r \mathbf{e}_r + \ddot{u}_\theta \mathbf{e}_\theta + \ddot{u}_\varphi \mathbf{e}_\varphi \qquad (3.32)$$

$$\underset{\rightarrow}{\nabla} \cdot \mathbf{u} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 u_r \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(u_\theta \sin\theta \right) + \frac{1}{r \sin\theta} \frac{\partial u_\varphi}{\partial \varphi} \qquad (3.33)$$

$$\underset{\rightarrow}{\nabla} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{1}{r \sin\theta} \frac{\partial}{\partial z} \qquad (3.34)$$

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \qquad (3.35)$$

$$\underset{\to}{\nabla} \otimes \mathbf{u} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(u_\varphi \sin\theta\right) - \frac{\partial u_\theta}{\partial\varphi}\right]\mathbf{e}_r + \frac{1}{r\sin\theta}\left[\frac{\partial u_r}{\partial\varphi} - \sin\theta\frac{\partial}{\partial r}\left(r u_\varphi\right)\right]\mathbf{e}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r u_\theta\right) - \frac{\partial u_r}{\partial\theta}\right]\mathbf{e}_\varphi \tag{3.36}$$

There are infinitely many solutions to Navier's equation and the solution to any individual problem is the one that has the correct initial conditions and boundary conditions for any particular problem. In general, it is not possible for humans to analytically solve 3.18 for all classes of three-dimensional solutions to (3.20). However, there are a number of analytic solutions to (3.20) if the problem is assumed to be uniform in one direction (two-dimensional). This is ultimately due to the fact that division is defined for two dimensional vectors (the same as division by complex numbers) but it cannot be defined for higher dimension vectors. Therefore, there are analytic (well mostly analytic) solutions to problems in which the elastic media is described by a stack of horizontal plane layer, but entirely numerical procedures (finite-element or finite-difference) must be used to solve problems in which the structure is truly three dimensional. The techniques for solving general layer problems often rely on expressing the displacement vector field as the sum of potentials (Helmholtz decomposition). That is, we can decompose the displacement as

$$\mathbf{u} = \underset{\to}{\nabla}\phi + \underset{\to}{\nabla} \otimes \underset{\to}{\psi} \tag{3.37}$$

where φ and $\underset{\to}{\psi}$ are scalar and vector functions of time and space. If we make this change of variables, then Navier's equation separates into several wave equations as follows.

$$\nabla^2\phi = \frac{1}{\alpha^2}\ddot{\phi} \tag{3.38}$$

$$\nabla^2\psi_i = \frac{1}{\beta^2}\ddot{\psi}_i \tag{3.39}$$

Of course the boundary conditions must also be transformed into potential form. These potential forms can be used in any coordinate system as long as you know how to compute the Laplacian, the gradient and the curl.

It is beyond the scope of this class to demonstrate general solution techniques for Navier's equation (see Achenbach for a nice treatment), but we can demonstrate several simple solutions which have attributes similar to those of solutions encountered in the real world. **Since Navier's equation is linear, any solution that is added to any other solution is also a solution.** Therefore, we can often build the appropriate solution by adding together known simple solutions in such a way that they produce the desired stresses or displacements on the boundary of a domain; that is they match boundary conditions. When a domain contains layers, the solutions apply inside the individual layer and they are constructed to produce continuous displacement at the boundaries and balanced tractions on the boundaries.

Plane P-waves

Suppose that we consider a motion defined by

$$u_1\left(x_1, x_2, x_3, t\right) = f\left(t - \frac{x_1}{\alpha}\right) \tag{3.40}$$

and

$$u_2 = u_3 = 0 \tag{3.41}$$

then it is a simple matter of substituting (3.40) and (3.41) into (3.20) to show that this is a valid solution for any single-variable function f, that is twice differentiable, and provided that

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{3.42}$$

We could have alternatively chosen the potentials,

$$\phi = -\alpha \int f\left(\xi\right) d\xi; \qquad \xi \equiv t - \frac{x_1}{c} \tag{3.43}$$

$$\underset{\rightarrow}{\psi} = \mathbf{0} \tag{3.44}$$

It is a trivial matter to show that its gradient is the displacement field given by (3.40) and (3.41), and that it satisfies the wave equations (3.38) and (3.39).

This is the equation of a planar P-wave traveling at velocity α in the positive x_1 direction. Since the material is isotropic, this direction is arbitrary and it could just as well be traveling in the negative x_1 direction. Note that the shape of the waveform is unchanged as it propagates through the medium. This property is called **nondispersive** and it contrasts with some other solutions that we will explore later where the wave velocity depends on the frequency of the oscillation.

Since the equation is linear, we could write a more general solution that has different P-waves traveling in both positive and negative directions as

$$u_1 = f\left(t - \frac{x_1}{\alpha}\right) + g\left(t + \frac{x_1}{\alpha}\right) \tag{3.45}$$

where *g* is some other twice differentiable function. **P-waves** are also called **longitudinal waves** since their particle motion is in the same direction as the wave propagates. They are also called **compressional waves,** although they have both compressional and shear stresses as shown by the computing the strain and stress tensor for (3.40) as follows.

$$
\begin{aligned}
\varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} \\
&= -\frac{f'\left(t-\frac{x_1}{\alpha}\right)}{\alpha} = -\frac{\dot{f}\left(t-\frac{x_1}{\alpha}\right)}{\alpha} \\
&= -\frac{\dot{u}\left(t-\frac{x_1}{\alpha}\right)}{\alpha}
\end{aligned}
\tag{3.46}
$$

and all other strain components are zero. Don't be confused by the f' , it simply means differentiation with respect to the argument, $\left(t-\frac{x_1}{\alpha}\right)$. We see that the **strain in this wave is proportional to the particle velocity divided by the wave speed**. This will be a recurring theme for other solutions of Navier's equation.

We can substitute (3.46) into (3.17) to obtain the stress, which gives

$$
\begin{aligned}
\sigma_{11} &= \lambda\left(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}\right)+2\mu\varepsilon_{11} \\
&= \left(\lambda+2\mu\right)\varepsilon_{11}
\end{aligned}
\tag{3.47}
$$

and

$$\sigma_{22} = \sigma_{33} = \lambda\varepsilon_{11} \tag{3.48}$$

$$\sigma_{12} = \sigma_{23} = \sigma_{13} = 0 \tag{3.49}$$

Substituting (3.42) and(3.46) into (3.47) and (3.48) we find that

$$\sigma_{11} = -\rho\alpha\dot{u} \tag{3.50}$$

and

$$\sigma_{22} = \sigma_{33} = \frac{\lambda}{\lambda+2\mu}\sigma_{11} \tag{3.51}$$

Equation (3.50) tells us that the **stress in this wave is related to the particle velocity times the product of the density and the wave speed.** The ratio of the stress to the particle velocity $\sigma/\dot{u} = \rho\alpha$ is called the **mechanical impedance**; it measures the stress that is needed to make a particular ground motion. In our particular example,

$$\text{mecahnical impedance} = \frac{\sigma_{11}}{\dot{u}_1} = \rho\alpha = \sqrt{\rho\left(\lambda+2\mu\right)} \tag{3.52}$$

Notice that although there are no explicit shear stresses in this coordinate frame (which is the principal coordinate frame for this problem), there are shear stresses in other coordinate frames. The maximum shear stress is in the frame rotated 45 degrees from the principal frame and in this frame the maximum shear stress is

$$\sigma_{1'2'} = \frac{1}{2}\left(\sigma_{11}-\sigma_{22}\right) = \frac{2\mu}{\lambda+2\mu}\sigma_{11} \tag{3.53}$$

Therefore there are shear stresses associated with these P-waves.

We can also calculate the **power** $P(x_1,t)$ associated with this wave as the energy flux in the x_1 direction. This energy flux is the rate of work per unit area done by the traction vector on a plane perpendicular to the velocity of propagation. This rate of work (power P) per infinitesimal unit area dS is the stress times the particle velocity, or

$$\frac{P(x_1,t)}{dS} = -\sigma_{11}\dot{u}_1 = \rho\alpha\dot{u}_1^2 \tag{3.54}$$

The energy per unit volume $E(x_1,t)$ associated with the wave is just the energy flux divided by the wave velocity, or

$$\frac{E(x,t)}{dV} = \rho\dot{u}^2 \tag{3.55}$$

As is the case for all linear dynamic systems, this energy is evenly divided between kinetic energy and potential (strain) energy if averaged throughout the system.

Finally we can inquire about the maximum accelerations that can occur in an elastic continuum. We can differentiate equation (3.50) to obtain

$$\ddot{u}_1(x_1,t) = -\frac{\dot{\sigma}_{11}\left(t-\frac{x_1}{\alpha}\right)}{\rho\alpha} \tag{3.56}$$

That is the acceleration of a point scales like the time derivative of the compressive stress. If a finite compressive stress were suddenly applied to a surface then it would generate a P-wave whose acceleration would be described by a Dirac-delta function, which has infinite acceleration. That is, if

$$\sigma_{11} = \sigma_0 H\left(t-\frac{x_1}{\alpha}\right) \tag{3.57}$$

where *H(t)* is a Heaviside step function, then

$$\ddot{u}_i = \frac{\sigma_0}{\rho\alpha}\delta\left(t-\frac{x_1}{\alpha}\right) \tag{3.58}$$

Notice that the acceleration is infinite, whereas the stress is finite.

Plane Shear Waves

Another important solution to Navier's equation can be expressed as

$$u_2 = f\left(t-\frac{x_1}{\beta}\right) \tag{3.59}$$

$$u_1 = u_3 = 0 \tag{3.60}$$

It is again a simple matter to substitute (3.59) and (3.60) into Navier's equation (3.20) to find that this is a solution so long as

$$\beta = \sqrt{\frac{\mu}{\rho}} \tag{3.61}$$

As before, we could have used the displacement potentials

$$\phi = 0 \tag{3.62}$$

$$\psi_1 = \psi_2 = 0 \tag{3.63}$$

$$\psi_3 = \beta f\left(t - \frac{x_1}{\beta}\right) \tag{3.64}$$

where the curl of $\underset{\rightarrow}{\psi}$ is the displacement and (3.64) solves the scalar wave equation (3.39).

This is the description of a planar shear wave (S-wave) traveling in the positive x_1 direction with velocity β. The particle motion is in the x_2 direction and it is parallel to the wave front and perpendicular to the direction of motion. As was the case with P-waves, *f(t)* is any function with a finite 2nd derivative. Like the planar P-wave, planar S-waves are also nondispersive.

Notice that the S-wave is slower than the P-wave and that the ration of the velocities is

$$\frac{\alpha}{\beta} = \sqrt{\frac{\lambda + 2\mu}{\mu}} \tag{3.65}$$

This can be expressed in terms of Poisson's ratio ν by using Table 3.1. In this case,

$$\frac{\alpha}{\beta} = \sqrt{\frac{2 - 2\nu}{1 - 2\nu}} \tag{3.66}$$

So the ratio of P- to S-wave velocities depends only on Poisson's ratio. For many solids, $\lambda \approx \mu$, or $\nu \approx 1/4$, in which case we call the solid Poissonian and $\alpha/\beta \approx \sqrt{3} = 1.717$. The typical P- and S-wave speeds in the Earth's crust are 4 km/s and 6.5 km/s, respectively. A handy trick is estimated the distance between an earthquake and a seismic station by the following simple formula

$$\Delta \approx \left(t_S - t_P\right) \cdot 7\left(\mathrm{km}/\mathrm{s}\right) \tag{3.67}$$

There are important cases where the P-wave speed is much higher than the S-wave speed. In particular, the types of water saturated muds found in coastal areas can have P-wave speeds that are more than 10 times the S-wave speed. In this case Poisson's ratio approaches its upper limit of $\frac{1}{2}$.

We can also compute strain, stress, and energy flux for the S-wave wave as we did for the planar P-wave. In this case,

$$\varepsilon_{12} = -\frac{1}{2}\frac{\dot{u}_2}{\beta} \tag{3.68}$$

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0 \tag{3.69}$$

$$\sigma_{12} = \rho\beta\dot{u}_2 \tag{3.70}$$

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{13} = \sigma_{23} = 0 \tag{3.71}$$

$$\frac{P(x_1,t)}{dS} = -\sigma_{12}\dot{u}_2 = \rho\beta\dot{u}_2^2 \tag{3.72}$$

Diagrams of the motion of Planar P- and S-waves are shown in Figure 3.3.

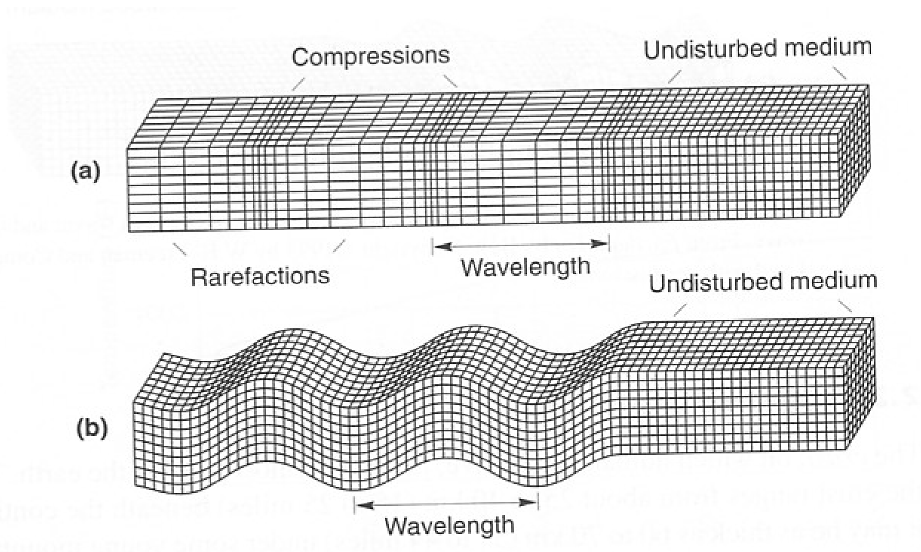


Figure 3.3. a) longitudinal P-wave, b) Transverse S-wave

Harmonic Plane Waves

While planar P- and S-waves can be expressed for any function of the variable, $\left(t - \frac{x}{c}\right)$, where c is the wave velocity, it is instructive to investigate the solution if the function is harmonic, a sinusoid or cosine. That is, there are many instances in which the superposition of harmonic solutions can be used to construct solutions to more general problems. To demonstrate, let's consider the planar S-wave in the previous section, but we will assume that our function is a cosine. That is,

$$\begin{aligned} u_2 &= \cos\left[\omega\left(t - \frac{x_1}{\beta}\right)\right] \\ &= \cos\left(kx_1 - \omega t\right) \end{aligned} \tag{3.73}$$

where k is spatial wavenumber given by

$$k = \frac{\omega}{\beta} = \frac{2\pi}{\Lambda} \tag{3.74}$$

and Λ is the wavelength. We can now consider what happens when two harmonic plane waves of identical strength and frequency, but traveling in opposite directions are added together. We can use standard trigonometric identities to easily show that.

$$
\begin{aligned}
u_2 &= \cos(kx_1 - \omega t) + \cos(kx_1 + \omega t) \\
&= 2\cos(kx_1)\cos(\omega t)
\end{aligned}
\tag{3.75}
$$

Equation (3.75) is therefore a **standing wave** with the same frequency and wavenumber as the two traveling waves. Since Navier's equation is linear, and since the waves traveling in each direction are solutions, then their sum (the standing wave) is also a solution of Navier's equation. Obviously, standing wave solutions are natural when identical waves are traveling in opposite directions. This is a common occurrence when harmonic waves are reflected off of an interface. It also happens in our spherical Earth when waves that travel around the Earth in opposite directions meet. In this case the interference makes the free oscillations of the Earth.

In a similar fashion, it is possible to add two harmonic standing waves together to produce a single harmonic traveling wave. Again we can use standard trig identities to show that

$$
\begin{aligned}
u_2 &= \cos(kx_1)\cos(\omega t) + \sin(kx_1)\sin(\omega t) \\
&= \cos(kx_1 - \omega t)
\end{aligned}
\tag{3.76}
$$

We have shown that we can represent **any** harmonic plane wave as **either** the sum of traveling waves **or** the sum of standing waves. Obviously it works for P-waves too, since we use the same trig identities. As it turns out, this duality of representations is far more general and can be applied to a variety of more complex problems. These two solutions are sometimes referred to as **characteristic** solutions and **mode** solutions. Figure 3.4 shows a schematic of how sinusoids traveling in opposite directions sum to make a standing wave.

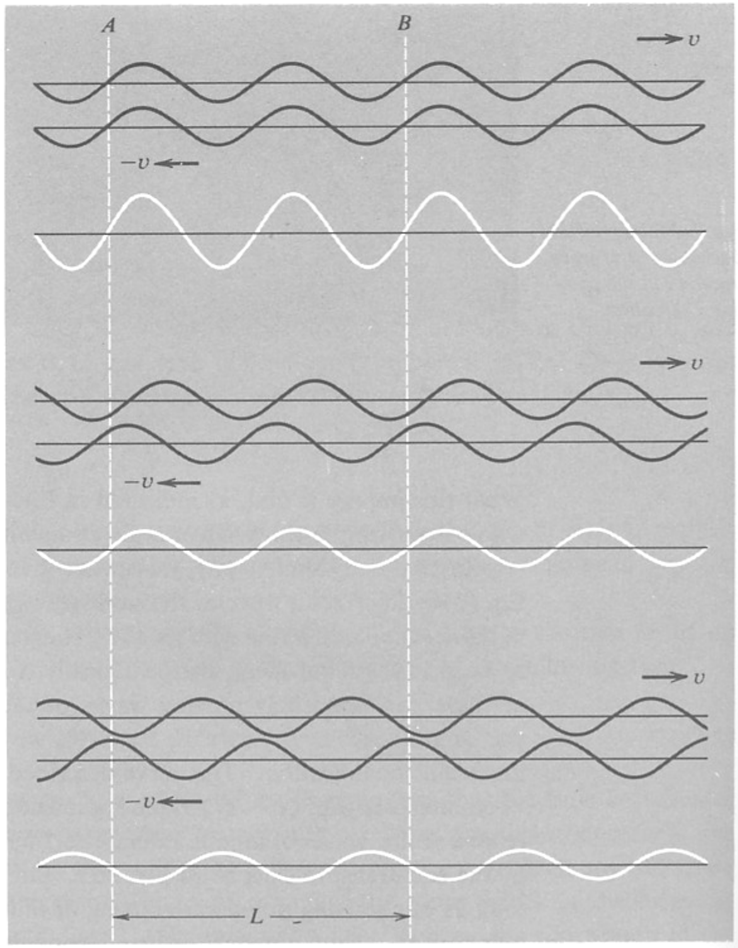


Fig. 7–3 Two exactly similar sinusoidal waves traveling in opposite directions and the resultant standing waves.

Figure 3.4. From "Vibration and Waves" by A. P. French. W.W. Norton and Co., 1971.

Spherical Waves

Many problems that we encounter concern the radiation of waves from a point in the medium. These waves spread spherically through the medium and their representation with Cartesian coordinates is awkward. In a homogeneous whole space it is usually most natural to solve these problems in spherical coordinates. However, if there are layers in the medium, then it usually is more convenient to solve these problems in cylindrical coordinates. General solutions for these problems are quite complex and beyond the scope of this class. However, we can consider the following potential in spherical coordinates. This potential has radial symmetry.

$$\varphi(r,t) = \frac{1}{r} f\left(t - \frac{r}{\alpha}\right) + \frac{1}{r} g\left(t + \frac{r}{\alpha}\right) \tag{3.77}$$

This solves the transformed form of Navier's equation given by (3.35) and (3.38). When the problem is radially symmetric, this can be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) = \frac{1}{\alpha^2}\ddot{\varphi} \tag{3.78}$$

The displacement that results from this is

$$
\begin{aligned}
u_r = \frac{\partial \varphi}{\partial r} &= \frac{-1}{r^2}\left[f\left(t-\frac{r}{\alpha}\right)+g\left(t+\frac{r}{\alpha}\right)\right]-\frac{1}{\alpha r}\left[f'\left(t-\frac{r}{\alpha}\right)-g'\left(t+\frac{r}{\alpha}\right)\right] \\
&= \frac{-1}{r^2}\left[f\left(t-\frac{r}{\alpha}\right)+g\left(t+\frac{r}{\alpha}\right)\right]-\frac{1}{\alpha r}\left[\dot{f}\left(t-\frac{r}{\alpha}\right)-\dot{g}\left(t+\frac{r}{\alpha}\right)\right]
\end{aligned}
\qquad (3.79)
$$

I have chosen a solution with waves that travels both radially outward (the *f* terms) and inwards (the *g* terms). Each of these has terms that decay with distance as both r^{-1} and r^{-2}; these are called far-field and near-field terms, respectively. They are both required to solve Navier's equation for this radial wave problem. Notice that the far-field term has a time dependence that looks like the time derivative of the near-field term. Also notice that the far-field term is scaled by the factor α^{-1}.

We can enquire about the energy in the spherically symmetric P-wave by integrating the power that is exerted on a shell of radius, r . Recall that power per unit area is given by equation (3.54), or

$$
P(t) = \left(4\pi r^2\right)\rho\alpha\dot{u}^2 \qquad (3.80)
$$

Inserting the far-field term from (3.79) into (3.80), we obtain

$$
P(t) = 4\pi\frac{\rho}{\alpha}\ddot{f}^2(t) \qquad (3.81)
$$

That is, the energy in the radiated far-field P-wave is the same as it passes through any spherical shell at any distance; the wave energy of far-field waves is conserved.

Pressure Step in a Spherical Cavity

We can explore this difference between near-field and far-field terms by investigating the exact solution to the problem of a step change in pressure p_0 inside a spherical cavity of radius *a*. The derivation is somewhat lengthy and is given by Achenbach. The answer for a Poisson solid is

$$
u_r = p_0 H\left(\hat{t}\right)\frac{a^3}{4\mu r^2}\left\{1+\left[\left(\frac{r}{a}-\frac{1}{2}\right)\sqrt{2}\sin\omega_1\hat{t}-\cos\omega_1\hat{t}\right]e^{-b\hat{t}}\right\} \qquad (3.82)
$$

where

$$
\hat{t} \equiv t - \frac{r}{\alpha} \qquad (3.83)
$$

$$
\omega_1 = \frac{\alpha 2\sqrt{2}}{3a} \qquad (3.84)
$$

$$
b = \frac{2\alpha}{3a} \qquad (3.85)
$$

At the surface of the cavity the displacement is

$$
u_r\big|_{r=a} = p_0\frac{a}{4\mu}\left(1+e^{-bt}\frac{b}{\omega_1}\sin\omega_1 t - e^{-bt}\cos\omega_1 t\right) \qquad (3.86)
$$

This looks like the pressure rate convolved with the solution of damped harmonic oscillator problem subjected to a step in force (see equation 1.39). The period of the undamped oscillator is given by (1.37), which when combined with (3.84) and (3.85) gives

$$\omega_0^2 = \omega_1^2 + b^2 = 3b^2 \tag{3.87}$$

The fraction of critical damping of this system is given by (1.5) and is equal to

$$\zeta = \frac{b}{\omega_0} = \frac{1}{\sqrt{3}} = 0.58 \tag{3.88}$$

So the surface of the cavity is a 58% damped oscillator that settles about its new static equilibrium position. With each harmonic swing, it radiates wave energy to the far-field term, which at large r become.

$$u_r\big|_{r>>a} \approx p_0 \frac{a^2}{4\mu r} e^{-b\hat{t}} \sqrt{2} \sin \omega_1 \hat{t} \tag{3.89}$$

The damping of the oscillating cavity is sometimes referred to as **radiation damping** and since it is linear and depends on the velocity at the source, it is very analogous to viscous damping discussed in the SDOF problem of chapter 1. The concept of radiation damping can become useful when investigating the damping of an oscillating building that excites seismic waves as it oscillates.

Of course a spherical cavity has many other modes besides the radially symmetric mode just described. Each mode has its own natural frequency, mode shape, and radiation damping. The mode shapes are best described with spherical harmonics. Since the pressure problem is radially symmetric, we only need the fundamental mode solution that is given by (3.82).

Point Force

The displacement in the *i* direction from a point force in the *k* direction with time history *f(t)* was given by Love (The mathematical theory of elasticity, Dover Pubs., 1944) and is

$$u_i = \frac{1}{4\pi}\left\{ \frac{\partial^2}{\partial x_i \partial x_k} \frac{1}{r} \int_{r/\alpha}^{r/\beta} \tau f\left(t-\tau\right) d\tau + \frac{1}{2}\left(\frac{\partial r}{\partial x_i}\right)\left(\frac{\partial r}{\partial x_k}\right)\left[\frac{1}{r\alpha^2} f\left(t-\frac{r}{\alpha}\right) - \frac{1}{r\beta^2} f\left(t-\frac{r}{\beta}\right)\right] + \frac{\delta_{ik}}{r\beta^2} f\left(t-\frac{r}{\beta}\right)\right\} \tag{3.90}$$

where

$$r^2 = x_i x_i \tag{3.91}$$

This is an important building point in seismology, since it allows us to calculate the wave field that results from distributions of forces. Although this solution is relatively compact, it is written in terms of both Cartesian coordinates and radial distance. It is easier to write the full solution in spherical coordinates in which case all of the spatial derivatives turn into a relatively complex set of sines and cosines of the angular geometric parameters. This is called “radiation pattern” and an example will be given in Chapter 7 (Sources).

Anelastic Attenuation of a Traveling Wave

The solutions discussed above are for an elastic medium. However, it is useful to introduce the concept that their energy slowly decays as they travel due to some inelastic response of the medium. In addition, there are basic physical considerations that require that waves eventually attenuate. One convenient approach to this problem is to break a waveform into its harmonic constituent parts and to then introduce the following definition of Q which is entirely analogous to the one that we used in Chapter 1 for the SDOF problem. Recall that for a lightly damped oscillator (equation 1.30),

$$Q \approx -2\pi \frac{E}{\Delta E} \tag{3.92}$$

where E and ΔE are the total energy and energy lost per cycle. We can also define the logarithmic decrement of the amplitude lost per cycle as

$$\delta \equiv \ln\left(\frac{A_1}{A_2}\right) \tag{3.93}$$

since energy is proportional to the square of amplitude,

$$\ln A = \frac{1}{2} \ln E \tag{3.94}$$

from which it follows that

$$Q \approx \frac{\pi}{\delta} \tag{3.95}$$

We can now write the expression for the amplitude A of a harmonic wave as a function of distance traveled r as

$$A(r) = A_0 e^{-\left(\omega/2Qc\right)r} \tag{3.96}$$

where c is the velocity of the wave. Sometimes the attenuation is described by the parameter t^* which is defined to be

$$t^* = \frac{r}{cQ} = \frac{\text{travel time}}{\text{quality factor}} \tag{3.97}$$

Homework for Chapter 3

1. Show that (3.40) and (3.59) are solutions to Navier's equation.

2. Show that (3.77) is a solution to Navier's equation.

3. If a plane harmonic wave with a frequency of 1 Hz and a propagation velocity of 3 km/sec is ½ the amplitude after traveling 100 km through an attenuating medium, then what is the Q and t^*?