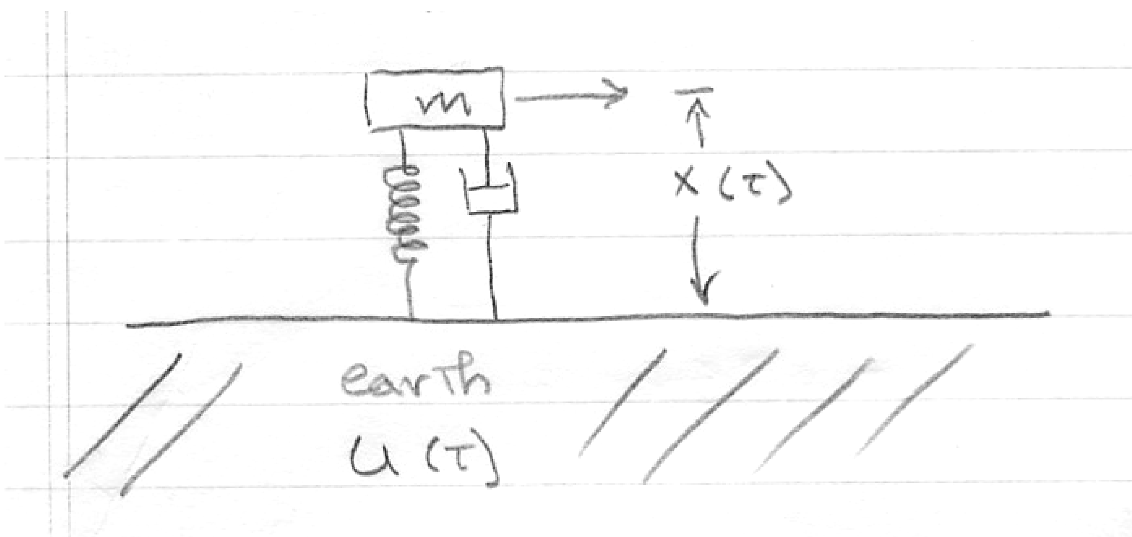
Chapter 1 08/28/2022

Single-Degree-of-Freedom Linear Oscillator (SDOF)

For many dynamic systems the relationship between restoring force and deflection is approximately linear for small deviations about some reference. If the system is complex (e.g., a building that requires numerous variables to describe its properties) it is possible to transform it (using the normal modes of the system) into a number of simple 1-dimensional linear oscillator problems (SDOF). The SDOF problem is also fundamental to understanding the principles of seismometers.

Consider the most fundamental of seismometers shown in Figure (1.1). In this case the ground moves with displacement $u(t)$; a mass m is supported by a spring of stiffness k; and there is a viscous damper that resists the relative velocity $\dot{x}$ of the mass with respect to the ground with force $-b\dot{x}$. $x(t)$ is sometimes measured directly using an optical transducer (e.g. a light beam deflected by a mirror on the mass).



the force on m is $-kx-b\dot{x}$ and the inertial force on m is $m(\ddot{x}+\ddot{u})$. The equation of motion of the system is then

$$m(\ddot{x}+\ddot{u})+kx+b\dot{x}=0, \tag{1.1}$$

which can be rewritten as

$$\ddot{x}+2\beta\dot{x}+\omega_0^2 x=-\ddot{u}\,, \tag{1.2}$$

where

$$\omega_0 = \sqrt{\frac{k}{m}} = \text{undamped natural frequency} \tag{1.3}$$

$$\beta \equiv \frac{b}{2m} = \text{ damping constant,} \tag{1.4}$$

which is related to the fraction of critical damping ς by

$$\beta = \omega_0 \varsigma. \tag{1.5}$$

Equation (1.2) is a 2[nd] order linear differential equation and its solution is widely known. In general the solution is broken into two parts. The homogeneous solution, which solves the equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{1.6}$$

Any solutions, $x_n(t)$, of the homogeneous equation (1.6) can be summed and they also solve the homogeneous equation since it is linear. The actual form of the solution to the homogeneous problem is determined by the initial conditions x_0 and $\dot{x}_0$. Solutions to the homogeneous equation can also be summed to solutions to the full inhomogeneous equation (1.2) and they will still solve the inhomogeneous equation. The homogeneous solutions typically represent the transient part of the response of the system. The homogeneous solution that matches the initial conditions is then added to the particular solution that solves equation (1.2). The particular solution often represents the steady-state part of problems.

It is particularly useful to represent the ground motion *u(t)* as a Fourier series, or

$$u(t) = \sum_{n=1}^{\infty} \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right] = \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \theta_n) \tag{1.7}$$

or

$$\ddot{u}(t) = -\omega_n{}^2 \sum_{n=1}^{\infty} \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right] = -\omega_n{}^2 \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \theta_n) \tag{1.8}$$

This representation is possible for functions that are periodic with a repeat time of T. In this case

$$\omega_n = \frac{2n\pi}{T} \tag{1.9}$$

Since our equation is linear, we can write the solution *x(t)* as the sum of solutions to individual harmonic problems, or

$$x(t) = \sum_{n=1}^{\infty} C_n x_n(t), \tag{1.10}$$

where x_n solves the equation

$$\ddot{x}_n + 2\beta\dot{x}_n + \omega_0^2 x_n = \omega_n^2 \cos\left(\omega_n t - \theta_n\right). \tag{1.11}$$

Since the cosine function is truly periodic and has no beginning or end, there are no initial conditions or transient solutions to deal with. That is, the solution consists of just the particular solution. As it turns out, when a linear system is harmonically forced at one frequency, then the resulting motions (except for transients) are also harmonic at that frequency. Therefore, let us guess that the solution of (1.11) is

$$x_n\left(t\right) = D_n \cos\left(\omega_n t - \delta_n - \theta_n\right) \tag{1.12}$$

substituting (1.12) into(1.11), we find that

$$D_n\left(\omega_0^2 - \omega_n{}^2\right)\cos\left(\omega_n t - \delta_n\right) - 2D_n\beta\omega_n \sin\left(\omega_n t - \delta_n\right) = \omega_n{}^2 \cos\left(\omega_n t\right). \tag{1.13}$$

We then utilize the following trig identities

$$\cos\left(\omega_n t - \delta_n\right) = \cos\left(\omega_n t\right)\cos\delta_n + \sin\left(\omega_n t\right)\sin\delta_n \tag{1.14}$$

$$\sin\left(\omega_n t - \delta_n\right) = \sin\left(\omega_n t\right)\cos\delta_n - \cos\left(\omega_n t\right)\sin\delta_n \tag{1.15}$$

substituting (1.14) and (1.15) into (1.13) we find that

$$\begin{aligned}&\left\{\omega_n{}^2 - D_n\left[\left(\omega_0^2 - \omega_n{}^2\right)\cos\delta_n + 2\omega_n\beta\sin\delta_n\right]\right\}\cos\left(\omega_n t\right)\\ &-D_n\left[\left(\omega_0^2 - \omega_n{}^2\right)\sin\delta_n - 2\omega_n\beta\cos\delta_n\right]\sin\left(\omega_n t\right) = 0\end{aligned} \tag{1.16}$$

Now since $\sin\left(n\omega t\right)$ and $\cos\left(n\omega t\right)$ are linearly independent functions, each term in equation (1.16) must be linearly independent and thus setting the second term to zero

$$\left(\omega_0^2 - \omega_n{}^2\right)\sin\delta_n - 2\omega_n\beta\cos\delta_n = 0 \tag{1.17}$$

or

$$\tan\delta_n = \frac{2\omega_n\beta}{\omega_0^2 - \omega_n{}^2} \tag{1.18}$$

Now setting the first term in(1.16) to 0 gives

$$D_n = \frac{\omega_n{}^2}{\left(\omega_0^2 - \omega_n{}^2\right)\cos\delta_n + 2\omega_n\beta\sin\delta_n} \tag{1.19}$$

If we make the following clever observation that equation (1.18) can be rewritten as the following two equations

$$\sin\delta_n = \frac{2\omega_n\beta}{\sqrt{\left(\omega_0^2 - \omega_n{}^2\right)^2 + 4\omega_n{}^2\beta^2}} \tag{1.20}$$

$$\cos\delta_n = \frac{\omega_0^2 - \omega_n{}^2}{\sqrt{\left(\omega_0^2 - \omega_n{}^2\right)^2 + 4\omega_n{}^2\beta^2}} \tag{1.21}$$

then we can substitute (1.20) and (1.21) into (1.19) to obtain

$$D_n = \frac{\omega_n{}^2}{\sqrt{\left(\omega_0^2 - \omega_n{}^2\right)^2 + 4\omega_n{}^2\beta^2}} \tag{1.22}$$

thus the steady-state solution of equation (1.11) is

$$x_n(t) = \frac{\omega_n{}^2}{\sqrt{\left(\omega_0^2 - \omega_n{}^2\right)^2 + 4\omega_n{}^2\beta^2}} \cos\left(\omega_n t - \delta_n - \theta_n\right) \tag{1.23}$$

where,

$$\delta_n = \tan^{-1}\left(\frac{2\omega_n\beta}{\omega_0^2 - \omega_n{}^2}\right) \tag{1.24}$$

Notice that

$$x_n(t) \approx -\cos\left(\omega_n t\right) = -U(t) \quad \text{when} \quad \omega_n \gg \omega_0 \tag{1.25}$$

and that

$$x_n(t) \approx -\frac{\omega_n^2}{\omega_0^2}\cos\left(\omega_n t\right) = -\frac{m}{k}\ddot{U}(t) \quad \text{when} \quad \omega_n \ll \omega_0 \tag{1.26}$$

That is the motion of the mass with respect to the case is proportional to ground displacement at high frequencies and it is proportional to acceleration at low frequencies (compared to the natural frequency).

Figure 1.2 shows the amplitude and phase of $x_n(t\text{-}\delta)$ as a function of frequency for an SDOF.

Notice that when the damping is $1/\sqrt{2}$, then there is the maximum response without having a peak in the response curve. Most manufacturers of seismometers attempt to achieve this level of damping. Figure 1.2 can be thought of as an amplification factor as a function of frequency for ground acceleration. That is the size of the record x(t) is the ground acceleration times the amplification factor. Notice that the instrument response is proportional to ground acceleration at low frequencies.

The amplification can be determined as a function of the amplitude of the ground displacement by simply multiplying the response by ω^2. In this case the instrument response looks like Figure 1.3.

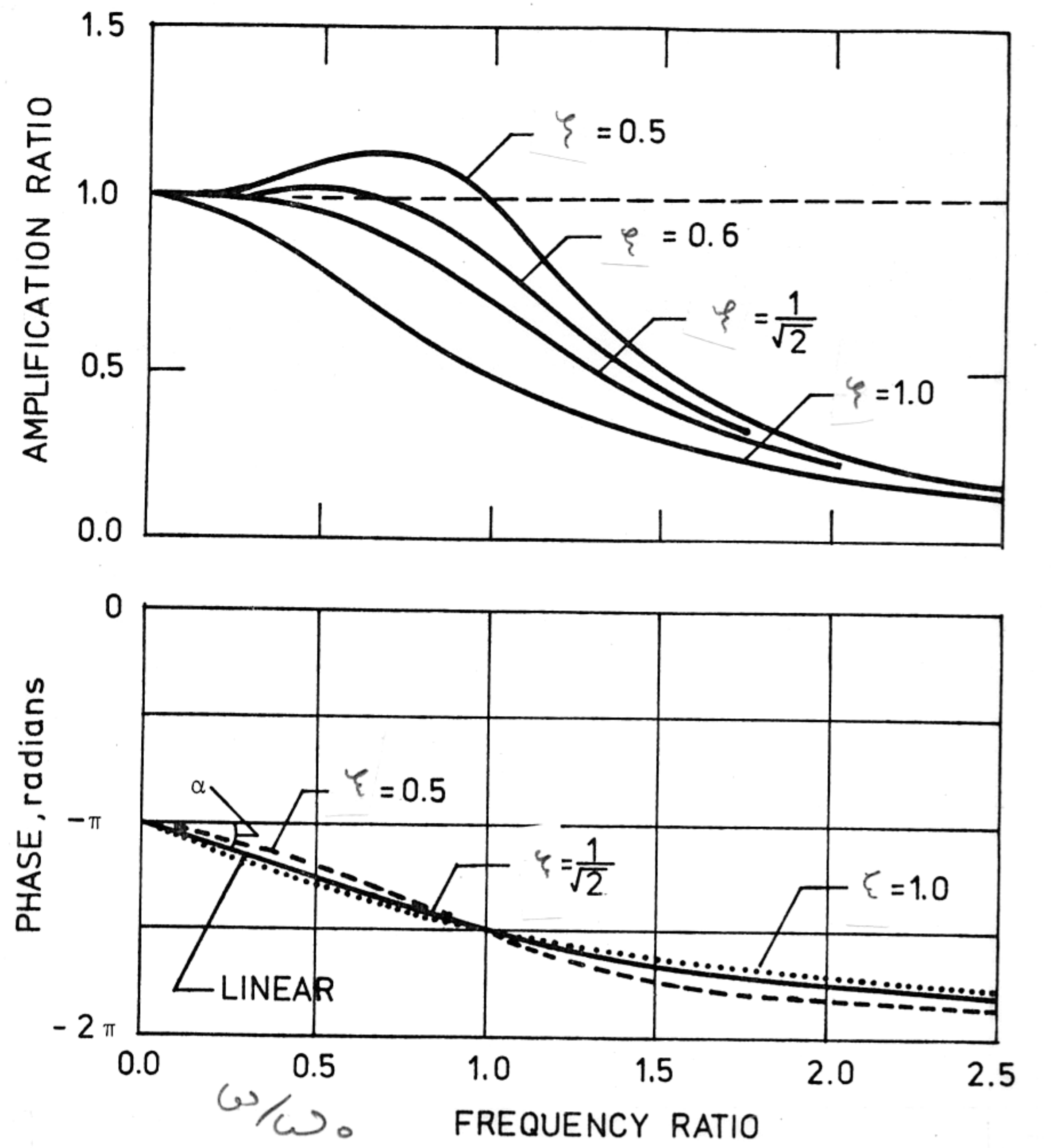


Figure 1.2 Amplification ratio is $X/\ddot{U}$

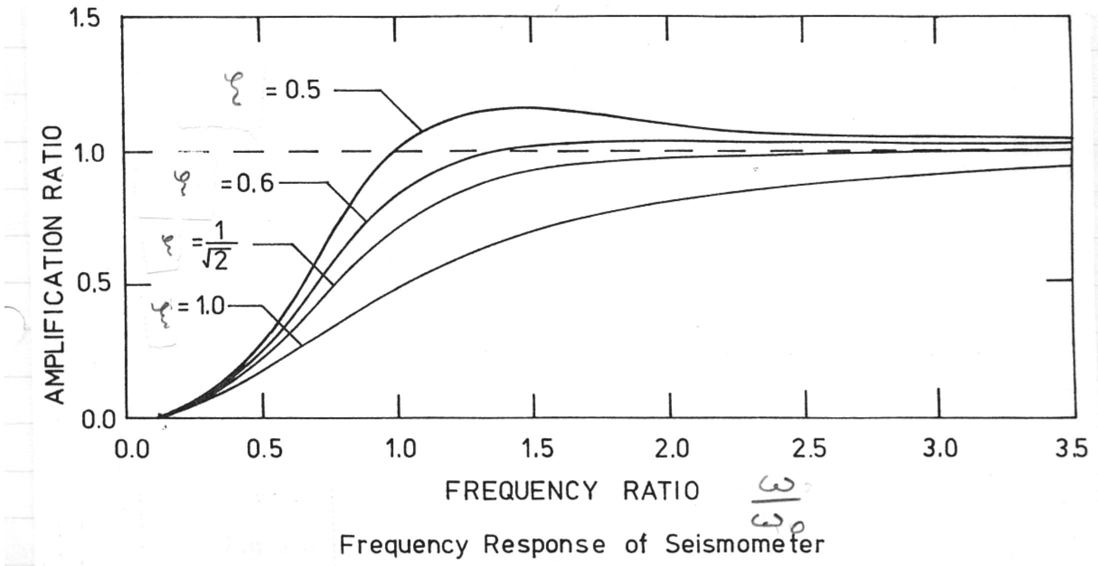


Figure 1.3 Amplification ratio is X/U

We can also find the frequency at which the SDOF has its maximum amplitude response to a forced vibration by finding the minimum of the response as follows.

$$\left.\frac{dx_n}{d\omega_n}\right|_{\omega_n=\omega_R} = 0 \tag{1.27}$$

where ω_r is the frequency of maximum resonance. Performing this differentiation on equation (1.23) gives

$$\omega_R = \sqrt{\omega_0^2 - 2\beta^2} = \omega_0\sqrt{1-2\zeta^2} \tag{1.28}$$

To add to the things to remember, there is yet another way to describe the system damping called the "quality factor" Q, which is defined as

$$Q \equiv \frac{\omega_R}{2\beta} = \frac{\sqrt{1-2\varsigma^2}}{2\varsigma} \tag{1.29}$$

For lightly damped systems, it can be shown that

$$Q \approx 2\pi\left(\frac{\text{Total Energy in One Cycle}}{\text{Energy Loss During One Cycle}}\right) \approx \frac{1}{2\varsigma} \tag{1.30}$$

Q for can also be approximated for lightly damped systems from the resonance curve of the system.

$$Q \approx \frac{\omega_0}{\Delta\omega} \tag{1.31}$$

where $\Delta\omega$ is the frequency interval between the points at which the amplitude of x_n is $\frac{1}{\sqrt{2}}x(\omega_R)$, its maximum.

Impulse Responses

We just saw how we can derive the response to an arbitrary periodic function by decomposing the function into a sum of sinusoids and cosines. However, there are other alternative decompositions that can be quite useful. For example we can approximate any function by a series of stair steps up and down in time. We define a Heaviside step function as

$$H(t) \equiv \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \tag{1.32}$$

Now suppose that we have the problem of a SDOF subjected to a step function in acceleration with the condition that the mass starts at rest. We can write the equation of motion as

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = H(t) \tag{1.33}$$

with the initial conditions that

$$x(t=0) = \dot{x}(t=0) = 0 \tag{1.34}$$

Unlike the problem of the harmonically driven oscillator, for which the solution was entirely the particular solution, the complementary solution to the homogeneous equation (the transient solution) is very important. The general solution can be written

$$x(t) = x_C(t) + x_P(t) \tag{1.35}$$

If $\beta < \omega_0$ (underdamping), then the complementary solution can be written as

$$x_C(t) = e^{-\beta t}\left(A\cos\omega_1 t + B\sin\omega_1 t\right) \tag{1.36}$$

where

$$\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2} \tag{1.37}$$

The particular solution, which is the steady-state solution, can be found by inspection.

$$x_P(t) = \frac{H(t)}{\omega_0^2} \tag{1.38}$$

Substituting (1.38) and (1.36) into (1.35), and applying boundary conditions (1.34) leads to

$$x(t) = \frac{1}{\omega_0^2}\left(1 - e^{-\beta t}\cos\omega_1 t - \frac{\beta e^{-\beta t}}{\omega_1}\sin\omega_1 t\right)H(t) \tag{1.39}$$

which is shown in figure 1.4.

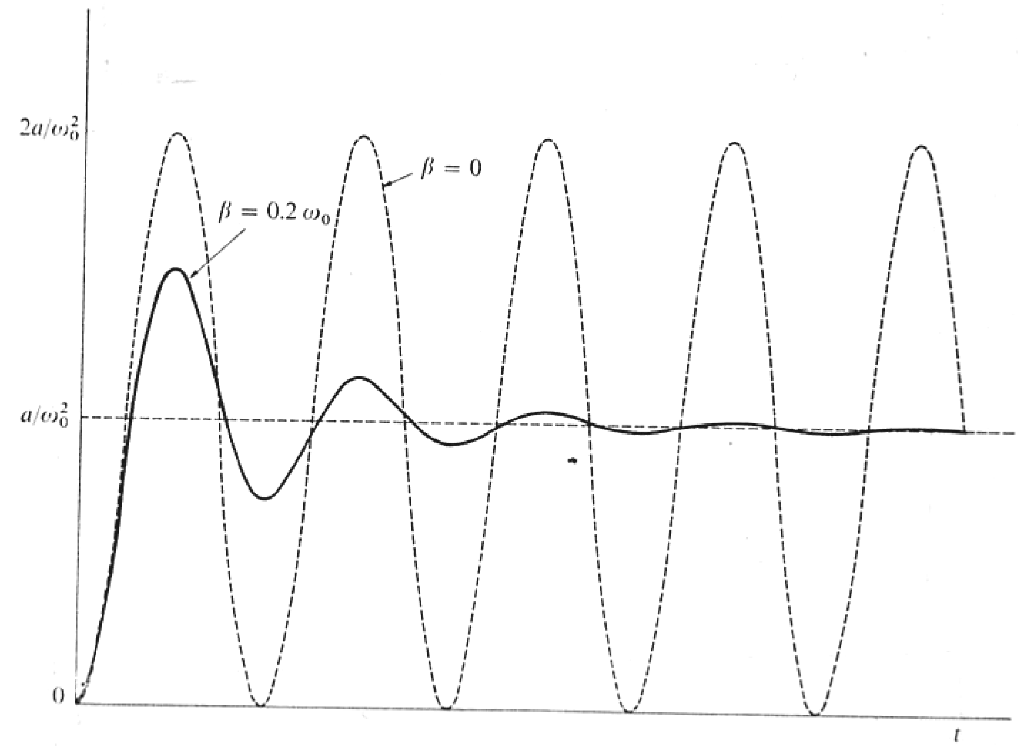


Figure 1.4 shows the *x(t)* for 20% damping and no damping.

We can take the time derivative of the solution for a step in acceleration, equation (1.39), to derive the solution for an impulse in time (delta function, $\delta(t)$).

$$G(t) = H(t)\frac{1}{\omega_1}e^{-\beta t}\sin\omega_1 t \tag{1.40}$$

where *G(t)* is now called a Green's function for this system. The Green's function for 20% damping is shown in Figure 1.5. Although we solved this Green's function problem as a forced vibration problem, we would have gotten the same answer if we had solved a free vibration problem (the homogeneous problem) but with initial conditions of zero displacement and a velocity of unity. Since the integral of the delta function acceleration is a step in velocity, this problem could have been solved in this alternative fashion. In this case it is easy to see that equation (1.40) consists entirely of the transient response.

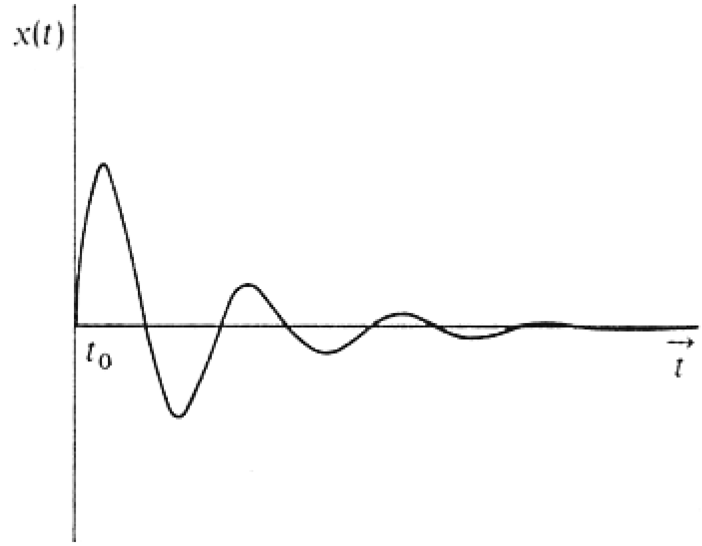


Figure 1.5. The response of an SDOF to an impulse of acceleration.

Convolution

With our Green's function in hand, we can find the response to any ground motion by use of the convolution operator, which is defined as follows.

$$x(t) = \ddot{u}(t) * G(t) \equiv \int_{-\infty}^{\infty} \ddot{u}(\tau) G(t-\tau)\, d\tau \qquad (1.41)$$

There are several ways to view the convolution operator. Taken literally, it is the integral of the product of two functions as they slide past each other. This can be shown graphically in a Figure 1.6, which is taken from Bracewell.

Another way to view convolutions is that the Green's function is added to itself many times, but shifted in time τ and with an amplitude that is given by *x(τ)*. This is the same as saying that the source consists of a continuous sequence of delta functions. Each one excites the response given by (1.40). Then all of these delta functions are added together.

We can solve the integral(1.41) as a discrete problem in the following way. Define

$$x_i = x(i\Delta t); \quad i = 1, n \qquad (1.42)$$

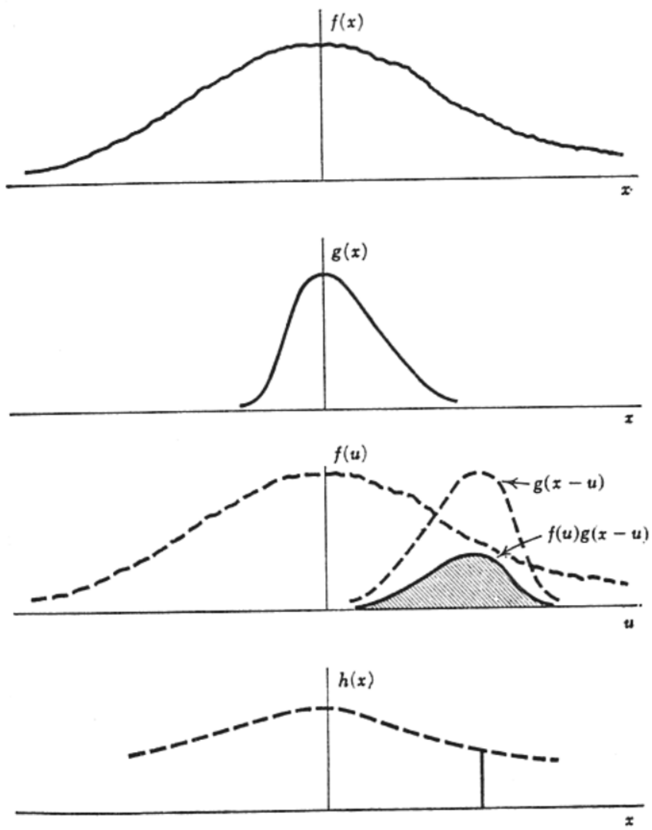
$$\ddot{u}_i = \ddot{u}(i\Delta t); \quad i = 1, n \qquad (1.43)$$

$$G_i = G(i\Delta t); \quad i = 1, n \qquad (1.44)$$

then

$$x_i = (\ddot{u} * G)_i = \Delta t \sum_{j=1}^{i} \ddot{u}_j G_{i-j} \qquad (1.45)$$

which is the serial product of $\left[u_1,...,u_n\right]$ and $\left[G_1,....,G_n\right]$. Bracewell discusses the details.



?. 3.1 *The convolution integral* $h(x) = f(x) * g(x)$ *represented by a shaded area.*

Figure 1.6 A graphical representation of the convolution of two functions (from Bracewell).

Another approach to convolution (and to solving linear differential equations) is to use Fourier transforms. In practice, Finite-Fourier transforms (FFT), which are actually Fourier series, are used in numerical calculations. We define the Fourier transform of *x(t)* as follows.

$$\tilde{x}(\omega) \equiv FT\left[x(t)\right] \equiv \int_{-\infty}^{\infty} x(t) e^{i\omega t} dt \tag{1.46}$$

with an inverse transform given by

$$x(t) = FT^{-1}\left[\tilde{x}(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{x}(\omega) e^{i\omega t} d\omega \tag{1.47}$$

An important aspect of Fourier transforms is that differentiation in the time domain is equivalent to multiplying by $i\omega$ in the frequency domain. This allows us to solve differential equations algebraically. That is,

$$\tilde{\dot{x}}(\omega) = i\omega\tilde{x}(\omega) \tag{1.48}$$

Another important property of Fourier transforms is the stretch rule.

$$FT\left[x(at)\right] = \frac{1}{|a|}\tilde{x}\left(\frac{\omega}{a}\right) \tag{1.49}$$

Now since the Fourier transform is itself a linear operator, we can take the Fourier transform of our entire differential equation (1.2).

$$-\omega^2\tilde{x}(\omega) + 2i\beta\omega\tilde{x}(\omega) + \omega_0^2\tilde{x}(\omega) = \tilde{\ddot{u}}(\omega) \tag{1.50}$$

or

$$\tilde{x}(\omega) = \frac{\tilde{\ddot{u}}(\omega)}{\omega_0^2 - \omega^2 + 2i\omega\beta} \tag{1.51}$$

or

$$x(t) = FT^{-1}\left[\frac{\tilde{\ddot{u}}(\omega)}{\omega_0^2 - \omega^2 + 2i\omega\beta}\right] \tag{1.52}$$

It just so happens that the Fourier transform of *G(t)* , which is given by equation (1.40) is

$$\tilde{G}(\omega) = FT\left[H(t)\frac{1}{\omega_1}e^{-\beta t}\sin\omega_1 t\right] = \left[\frac{1}{\omega_0^2 - \omega^2 + 2i\omega\beta}\right] \tag{1.53}$$

Therefore, we can now write

$$x(t) = \ddot{u}(t) * G(t) = FT^{-1}\left[\tilde{\ddot{u}}(\omega)\cdot\tilde{G}(\omega)\right] \tag{1.54}$$

That is, convolution in the time domain is equivalent to simple multiplication in the frequency domain.

In practice, the discrete values of *x(t)* are usually calculated by taking the inverse FFT of the product of the FFT's of *G(t)* and $\ddot{u}(t)$.

Properties of Convolution

Convolution is ubiquitous to linear vibrational problems. In this section I summarize some of the useful properties of the convolution of functions. In the following, *f, g,* and *h* are all arbitrary functions of the same real variable (usually time for our problems).

$$FT(f * g) = FT(f)\cdot FT(g) \tag{1.55}$$

$$f * g = g * f \tag{1.56}$$

$$(f * g) * h = f * (g * h) \tag{1.57}$$

$$f * (g + h) = f * g + f * h \tag{1.58}$$

$$\frac{\partial(f*g)}{\partial t} = \dot{f}*g = f*\dot{g} \tag{1.59}$$

$$f(t)*g(t-\tau) = f(t-\tau)*g(t) \tag{1.60}$$

$$f*\delta = f, \quad \text{where } \delta \text{ is a dirac-delta function} \tag{1.61}$$

$$f*H = \int f dt, \quad \text{where } H \text{ is a Heaviside step function} \tag{1.62}$$

Convolution is the operator that is used in linear filtering. For example a Rectangle function (sometimes called a boxcar) is defined as

$$\Pi(t) = \begin{cases} 0 & t < -\frac{1}{2} \\ 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & t > \frac{1}{2} \end{cases} \tag{1.63}$$

and its Fourier transform is

$$\tilde{\Pi}(\omega) = \frac{2\sin\frac{\omega}{2}}{\omega} \equiv \text{sinc}\left(\frac{\omega}{2}\right) \tag{1.64}$$

Convolution with this function is identical to taking the running mean of a function where the width of the running mean is unity in the time domain. It is also the same thing as multiplying by a sinc function in the frequency domain. A sinc function is shown in figure1.7. The sinc function has the property that its amplitude decays as ω^{-1} at high frequencies and it approaches a value of 1 at low frequencies. Thus filtering with a rectangle function (a running mean) causes the signal to decay as ω^{-1} at high frequencies compared to the unfiltered signal.

Figure 1.8 shows a sinc function plotted on a log-log scale. Log-log plots are useful for recognizing power-law relationships since if

$$y = x^{\alpha} \tag{1.65}$$

then

$$\log y = \alpha \log x \tag{1.66}$$

which is a linear relationship on a log-log plot. The slope of the relationship is the exponent of the power law. Notice the linear slope in Figure 1.8 that corresponds to a spectral amplitude decay of ω^{-1}. This decay rate is sometimes referred to as 6 dB per octave, since each octave refers to a factor of 2 in frequency and 6 dB refers to a factor of 2 in amplitude. A filter with an ω^{-2} spectral decay (e.g., sinc^2) has a spectral decay of 12 db per octave.

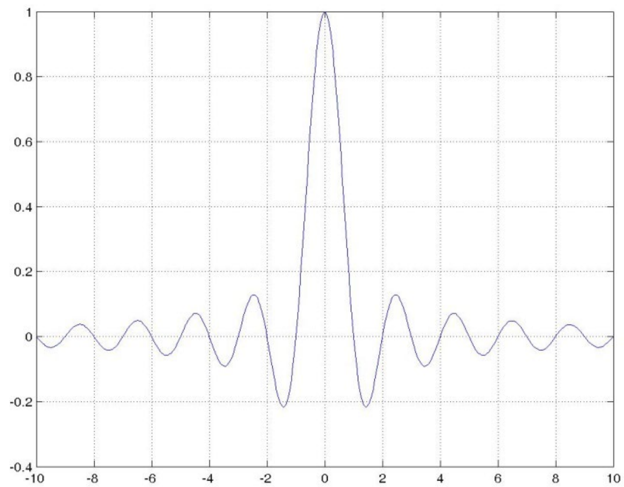


Fig. 1.7 plot of sinc(ω).

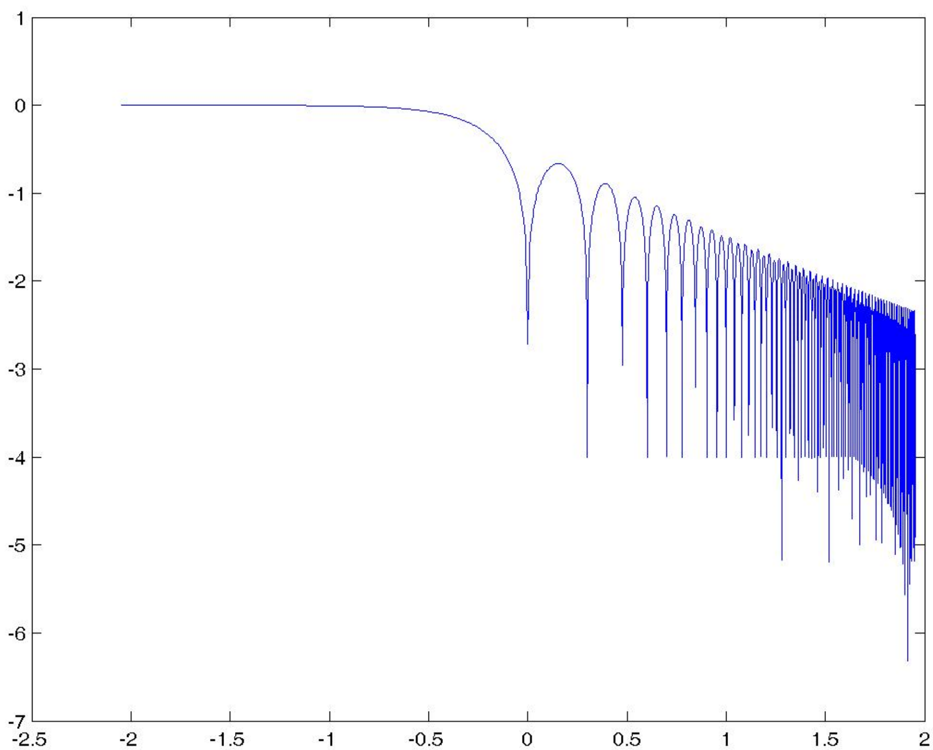


Fig. 3.1 log-log plot of |sinc(ω)|.

Deconvolution

A common problem in seismology is that of determining the ground motion *u(t)* that produced a particular seismogram *x(t)*. We can invert the convolution process. That is, if

$$\tilde{x}(\omega) = \tilde{\ddot{u}}(\omega)\tilde{G}(\omega) \tag{1.67}$$

then

$$\ddot{u}(t) = FT^{-1}\left[\frac{\tilde{x}(\omega)}{\tilde{G}(\omega)}\right] \tag{1.68}$$

This operation is typically performed using an FFT. However, as was the case with convolution, it is possible to do the calculation recursively in the time domain using serial division. That is the inverse of equation (1.45) is

$$\ddot{u}_i = \frac{1}{\Delta t G_0}\left(x_i - \sum_{j=0}^{i-1}\ddot{u}_j G_{i-j}\right) \tag{1.69}$$

In practice, it is often not feasible to perform an exact deconvolution since it often involves a division by zero or a very small number in the frequency domain. That is, Most deconvolutions are only valid over a limited frequency band.

One approach to deconvolution is pose it as a linear inverse problem. That is, if we are dealing with the discrete problem, then we can write $x(t) = \ddot{u}(t) * G(t)$ as the following matrix equation (this is a serial product).

$$\mathbf{G}\cdot\ddot{\mathbf{u}} = \mathbf{x} \tag{1.70}$$

Where

$$\mathbf{G} \equiv \begin{bmatrix} G_1 & G_n & G_{n-1} & G_{n-2} & . & G_2 \\ G_2 & G_1 & G_n & G_{n-1} & . & G_3 \\ G_3 & G_2 & G_1 & G_n & . & G_4 \\ . & . & G_2 & G_1 & . & . \\ G_{n-1} & G_{n-2} & . & . & . & G_n \\ G_n & G_{n-1} & G_{n-2} & . & G_2 & G_1 \end{bmatrix} \tag{1.71}$$

$$\mathbf{x} \equiv (x_1, x_2, x_3, ..., x_n) \tag{1.72}$$

$$\ddot{\mathbf{u}} \equiv (u_1, u_2, u_3, ..., u_n) \tag{1.73}$$

This means that we can if we know $\mathbf{G}$ and $\mathbf{x}$, then we can determine $\ddot{\mathbf{u}}$ as

$$\ddot{\mathbf{u}} = \mathbf{G}^{-1}\cdot\mathbf{x} \tag{1.74}$$

Where

$$\mathbf{G}^{-1}\cdot\mathbf{G} = \mathbf{I} \tag{1.75}$$

While this formulation is exactly equivalent to using an FFT and division in the frequency domain, there are stabilization techniques (e.g. singular value decomposition) in linear inverse problems that can help to reduce instabilities due to ill-conditioned decovolutions (i.e. division by small numbers in the frequency domain). Since the SDOF oscillator problem is linear in all aspects, it is not surprising that it is equivalent to a linear algebra problem, namely the inversion of the matrix **G** can be done by finding eigenvectors and eigenvalues.

Direct Solution

The solution techniques that we have shown explicitly, or implicitly, assume that we are dealing with periodic functions. That is, we are usually assuming that the record time series repeats itself indefinitely. This can be seen directly in the structure of the matrix **G** in (1.71). This can cause complications if there is a discontinuous jump from the end of a record and the beginning. That is, there is often a step change in the value at the beginning/end of the repeating function. Of course, this repeating jump is unphysical and is simply the result of truncating our analysis after some finite time. One simple way to deconvolve the record that avoids this problem is direct integration of equation(1.2).

$$\ddot{u} = -\ddot{x} - 2\beta\dot{x} - \omega_0^2 x \tag{1.76}$$

Integrating once, we obtain

$$\begin{aligned}\dot{u}(t) &= -\int_0^t \ddot{x}dt - 2\beta\int_o^t \dot{x}dt - \omega_0^2\int_0^t xdt \\ &= -\dot{x} - 2\beta x - \omega_0^2\int_0^t xdt + C\end{aligned} \tag{1.77}$$

Integrating a 2nd time, we obtain

$$u(t) = -x - 2\beta\int_0^t xdt - \omega_0^2\int_0^t\int_0^t xdt + Ct + D \tag{1.78}$$

Where C and D are constants of integration.

If $u(t=0) = \dot{u}(t=0) = x(t=0) = \dot{x}(t=0) = 0$, then $C = D = 0$.

Chapter 1. Homework

Problem 1.1 Derive equation (1.30).

Problem 1.2 Calculate and sketch the function given by $\Pi\left(t/T_1\right) * \Pi\left(t/T_2\right)$. What does the Fourier amplitude spectrum look like?

Problem 1.3 Derive the response of an SDOF to ground motion described by

$$U(t) = \begin{Bmatrix} 0, \; t < 0 \\ \dfrac{t}{T_1}, \; 0 < t < T_1 \\ 1, \; t > T_1 \end{Bmatrix}$$