Lower bounds for the number of small convex k-holes☆

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Abstract

Let S be a set of n points in the plane in general position, that is, no three points of S are on a line. We consider an Erdős-type question on the least number $h_k(n)$ of convex k-holes in S, and give improved lower bounds on $h_k(n)$, for $3 \leq k \leq 5$. Specifically, we show that $h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7}$, $h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - o(n)$, and $h_5(n) \geq \frac{3n}{4} - o(n)$. We further settle several questions on sets of 12 points posed by Dehnhardt in 1987.

Keywords:
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Introduction

Let S be a set of n points in the plane in general position, that is, no three points of S lie on a common (straight) line. A *k-hole* of S is a simple polygon, P, spanned by k points from S, such that no other point of S is contained in the interior of P. A classical existence question raised by Erdős [10] is: "What is the smallest integer $h(k)$ such that any set of $h(k)$ points in the plane contains at least one convex k-hole?". Esther Klein observed that every set of 5 points contains a convex 4-hole, and Harborth [14] showed that every set of 10 points determines a convex 5-hole. Both bounds are tight w.r.t. the cardinality of S. Only in 2007 and 2008 Nicolás [16] and independently Gerken [13] proved that every sufficiently large point set contains a convex 6-hole. On the other hand, Horton [15] showed that there exist arbitrarily large sets which do not contain any convex 7-hole; see [1] for a brief survey.

A generalization of Erdős' question is: What is the least number $h_k(n)$ of convex k-holes determined by any set of n points in the plane? In this paper we concentrate on this question for $3 \leq k \leq 5$, that is, the number of empty triangles (3-holes), convex 4-holes, and convex 5-holes. We denote by $h_k(S)$ the number of convex k-holes determined by S, and by $h_k(n) = \min_{|S|=n} h_k(S)$ the number of convex k-holes any set of n points in general position must have. Throughout this paper let $\operatorname{ld} x = \frac{\log x}{\log 2}$ be the binary logarithm (logarithmus dualis). Furthermore, we denote with $\operatorname{CH}(S)$ the convex hull of S and with $\partial \operatorname{CH}(S)$ the boundary of $\operatorname{CH}(S)$.

We start in Section 1 by providing improved bounds on the number of convex 5-holes. In particular, we increase the previously best known bound $h_5(n) \geq \frac{n}{2} - O(1)$ by Valtr [18] to $h_5(n) \geq \frac{3n}{4} - n^{0.87447} + 1.875$. In Section 2 we combine these results with a technique recently introduced by García [11, 12], and improve the previously best bounds on the number of empty triangles and convex 4-holes, $h_3(n) \geq n^2 - \frac{37n}{8} + \frac{23}{8}$ and $h_4(n) \geq \frac{n^2}{2} - \frac{11n}{4} - \frac{9}{4}$ (both in [12]), to $h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7}$ and $h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - 1.2641\, n^{0.926} + \frac{199}{24}$, respectively. In Section 3 we use these results to answer several questions on sets of 12 points posed by Dehnhardt [8] in 1987.

1. Convex 5-holes

The currently best upper bound on the number of convex 5-holes, $h_5(n) \leq 1.0207n^2 + o(n^2)$, is by Bárány and Valtr [7], and it is widely conjectured that

$h_5(n)$ grows quadratically. Still, to this date not even a super-linear lower bound is known.

As early as in 1987 Dehnhardt presented a lower bound of $h_5(n) \geq 3\lfloor \frac{n}{12} \rfloor$ in his thesis [8]. Unfortunately, this result, published in German only, remained unknown to the scientific community until recently. Thus, the best known lower bound was $h_5(n) \geq \lfloor \frac{n}{10} \rfloor$, published by Bárány and Füredi [5] in 1987, later (in 2001) refined to $h_5(n) \geq \lfloor \frac{n-4}{6} \rfloor$ by Bárány and Károlyi [6]. Both bounds are derived from the result of Harborth [14]. In the presentation of [11] the lower bound was improved to $h_5(n) \geq \frac{2}{9}n - \frac{25}{9}$. A slightly better bound $h_5(n) \geq 3\lfloor \frac{n-4}{8} \rfloor$ was presented in [3], which was then sharpened to $h_5(n) \geq \lceil \frac{3}{7}(n-11) \rceil$ in [4]. The latest and so far best bound of $h_5(n) \geq \frac{n}{2} - O(1)$ is due to Valtr [18]. In this section we further improve this bound to $h_5(n) \geq \frac{3}{4}n - o(n)$.

We start by fine-tuning the proof from [4], showing $h_5(n) \geq \lceil \frac{3}{7}(n-11) \rceil$, by utilizing the results $h_5(10) = 1$ [14], $h_5(11) = 2$ [8], and $h_5(12) \geq 3$ [8]. Although this does not lead to an improved lower bound of $h_5(n)$ for large n, it provides better lower bounds for small values of n, $17 \leq n \leq 56$; see Table 1.

n	10	11	12	13	14	15	16	17	18
$h_5(n)$	1	2	3	3..4	3..6	3..9	≥ 3	≥ 4	≥ 5
n	19..23	24	25	26..30	31	32	33..37	38	39
$h_5(n)$	≥ 6	≥ 7	≥ 8	≥ 9	≥ 10	≥ 11	≥ 12	≥ 13	≥ 14
n	40..44	45	46	47..50	51	52	53	54..56	57
$h_5(n)$	≥ 15	≥ 16	≥ 17	≥ 18	≥ 19	≥ 19	≥ 20	≥ 21	≥ 22

Table 1: The updated bounds on $h_5(n)$ for small values of n.

Lemma 1. *Every set S of n points in the plane in general position with $n = 7 \cdot m + 9 + t$ (for any natural number $m \geq 0$ and $t \in \{1, 2, 3\}$) contains at least $h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$ convex 5-holes.*

Proof. Because of $h_5(10) = 1$, $h_5(11) = 2$, and $h_5(12) \geq 3$ this is true for $m = 0$. Obviously $h_5(n) \geq h_5(n-1)$. Hence, $h_5(n) \geq 3$ for any $n \geq 12$.

If there exists a point $p \in (\partial \operatorname{CH}(S) \cap S)$ that is a point of a convex 5-hole, then $h_5(S) \geq 1 + h_5(S \backslash \{p\}) \geq 1 + h_5(n-1)$. In this case, the lemma is true by induction, as for $t = 1$ and $m > 0$, $h_5(n-1) = h_5(7 \cdot$

$m+9) \geq h_5(7 \cdot (m-1)+9+3)$. (The case $t \in \{2,3\}$ follows trivially, as $h_5(n-1) = h_5(7 \cdot m + 9 + (t-1))$ and $(t-1) \in \{1,2\}$.)

Otherwise, no point $p \in (\partial \operatorname{CH}(S) \cap S)$ is a point of a convex 5-hole. For $m > 0$ choose one such point p (e.g. the bottom-most one) and successively partition $S \backslash \{p\}$ (in clockwise order around p) into the following (disjoint) subsets: S_0 containing the first 7 points; S_0' containing the next 4 points; $(m-1)$ pairs of subsets S_i, S_i': S_i containing 3 points and S_i' containing 4 points $(1 \leq i \leq (m-1))$; and the subset S_{rem} containing the remaining $(t+4)$ points. See Figure 1 for a sketch.

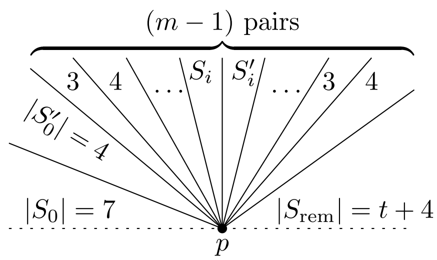

Figure 1: Partition of $S \backslash \{p\}$ clockwise around an extreme point p: starting with the pair S_0, S_0'; continuing with $(m-1)$ pairs of sets S_i, S_i', for $1 \leq i \leq (m-1)$, with $|S_i| = 3$ and $|S_i'| = 4$; and ending with the remainder set S_{rem}.

The union $S_0 \cup S_0' \cup \{p\}$ (of disjoint subsets) has cardinality 12 and thus contains at least 3 convex 5-holes [8]. The same is true for each union $S_{i-1}' \cup S_i \cup S_i' \cup \{p\}$ $(1 \leq i \leq (m-1))$. Finally, the union $S_{m-1}' \cup S_{\text{rem}} \cup \{p\}$ has cardinality $(9+t)$ and therefore contains at least t convex 5-holes [8, 14]. Note that we count every convex 5-hole at most once, as the considered unions which are sets of 10, 11, and 12 points, respectively, overlap in 4 points plus p, and p is not a point of a convex 5-hole. In total this gives at least $3 + (m-1) \cdot 3 + t = 3 \cdot \frac{n-9-t}{7} + t = \frac{3n-27+4t}{7}$ convex 5-holes. □

We state a special case of the preceding lemma for later use in the proof of Theorem 3.

Corollary 2. *Every set S of 17 points in the plane in general position contains at least 4 convex 5-holes, i.e., $h_5(17) \geq 4$.*

Table 1 shows the bounds on $h_5(n)$ obtained by Lemma 1, for some small values of n. By Harborth [14] we have $h_5(10) = 1$, and Dehnhardt [8] shows $h_5(11) = 2$ and $h_5(12) \geq 3$. The bound for $n = 51$ and for $57 \leq n < 62250$

(of which only $n = 57$ is shown in the table) are due to $h_5(n) \geq \lceil \frac{n}{2} \rceil - 7$ from Valtr [18]. The bound for $n \geq 62250$ is due to Theorem 3. The bounds $h_5(12) \leq 3$, $h_5(13) \leq 4$, $h_5(14) \leq 6$, and $h_5(15) \leq 9$ are from [4, 19].

In the following theorem we present an improved lower bound on $h_5(n)$ for larger n.

Theorem 3. *Every set S of $n \geq 12$ points in the plane in general position contains at least $\frac{3n}{4} - n^{\mathrm{ld}\,\frac{11}{6}} + \frac{15}{8} = \frac{3n}{4} - o(n)$ convex 5-holes, i.e., $h_5(n) \geq \frac{3n}{4} - o(n)$.*

Proof. For $12 \leq n < 17$ we count three convex 5-holes for S. For $17 \leq n < 24$ we can count four convex 5-holes for S by Corollary 2.

If $n \geq 24$ consider an (almost) halving line ℓ of S which splits S into S_L ($|S_L| = \lceil \frac{n}{2} \rceil$) and S_R ($|S_R| = \lfloor \frac{n}{2} \rfloor$) and does not contain any point of S. See Figure 2.

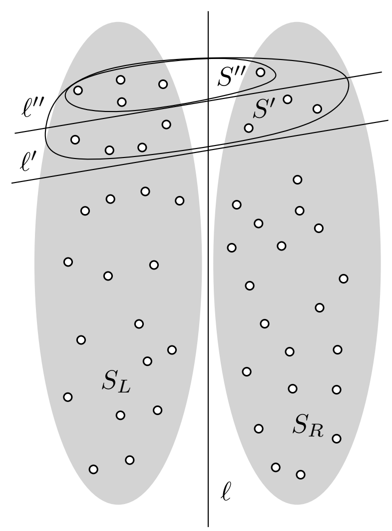

Figure 2: A point set S split by a halving line ℓ into two point sets $S_L, S_R \subset S$. The line ℓ' cuts off a set $S' \subseteq S$, consisting of 8 points of S_L and 4 points of S_R. The line ℓ'' is parallel to ℓ' and halves $S_L \cap S'$.

Furthermore, consider a line ℓ' that intersects ℓ and cuts off a set $S' \subseteq S$, consisting of eight points from S_L and four points from S_R. That this is in fact possible is folklore, see e.g. Exercise 4.5 (b) in [9]. Let a line ℓ'' be parallel to ℓ' and split $S' \cap S_L$ into two groups of four points, and let $S'' \subset S'$

be the set which is cut off by ℓ''. Note that neither ℓ' nor ℓ'' contain any points of S.

As $|S'| = 12$ we have that S' contains at least three convex 5-holes. We distinguish two cases.

Case 1: S' contains at least three convex 5-holes which are not intersected by ℓ. Then each of these 5-holes contains only points from S_L and thus at least one point above ℓ''. We count the three convex 5-holes for the set S_L and continue on $S \setminus S''$.

Case 2: S' contains at most two convex 5-holes which are not intersected by ℓ. Then at least one convex 5-hole in S' is intersected by ℓ. We count one convex 5-hole for the halving line ℓ and continue on $S \setminus S'$.

Note that in both cases we cut off at least four points from S_L, but at most four points from S_R. Thus, we can repeat this process until we have processed all $\lceil \frac{n}{2} \rceil$ points of S_L (except for a possible remainder of less than 8 points). Let c_L be the number of convex 5-holes counted for ℓ when processing S_L. Hence, Case 2 appeared c_L times, and Case 1 appeared at least $\lfloor \frac{1}{4} \cdot \left(\lceil \frac{n}{2} \rceil - 8c_L \right) \rfloor - 1$ times (the correction term of -1 takes care of the possible remainder). Therefore, the number of convex 5-holes we counted in S_L (i.e., not intersecting ℓ) is $h_5(S_L) \geq 3 \left(\lfloor \frac{1}{4} \left(\lceil \frac{n}{2} \rceil - 8c_L \right) \rfloor - 1 \right)$.

Repeating the same procedure for S_R (exchanging the roles of S_L and S_R), we obtain $h_5(S_R) \geq 3 \left(\lfloor \frac{1}{4} \left(\lfloor \frac{n}{2} \rfloor - 8c_R \right) \rfloor - 1 \right)$, where c_R is the number of convex 5-holes which we counted for ℓ when processing S_R. Note that any convex 5-hole intersected by ℓ, which we counted while processing S_L, might have occurred again when processing S_R. Thus, the total number c of convex 5-holes intersected by ℓ is at least $\max\{c_L, c_R\} \geq \frac{c_L + c_R}{2}$. As $h_5(S) = h_5(S_L) + h_5(S_R) + c$, we obtain

$$h_5(S) \geq 3 \left(\left\lfloor \frac{1}{4} \left(\left\lceil \frac{n}{2} \right\rceil - 8c_L \right) \right\rfloor - 1 \right) + 3 \left(\left\lfloor \frac{1}{4} \left(\left\lfloor \frac{n}{2} \right\rfloor - 8c_R \right) \right\rfloor - 1 \right) + \frac{c_L + c_R}{2} .$$

Considering that

$$\left\lfloor \frac{\lceil \frac{n}{2} \rceil}{4} \right\rfloor + \left\lfloor \frac{\lfloor \frac{n}{2} \rfloor}{4} \right\rfloor = \begin{cases} 2 \cdot \left\lfloor \frac{\frac{n}{2}}{4} \right\rfloor \geq \frac{n}{4} - \frac{6}{4} & \dots \text{ if } n \text{ is even} \\ \left\lfloor \frac{\frac{n+1}{2}}{4} \right\rfloor + \left\lfloor \frac{\frac{n-1}{2}}{4} \right\rfloor \geq \frac{n}{4} - \frac{6}{4} & \dots \text{ if } n \text{ is odd} \end{cases}$$

careful transformation gives

$$h_5(S) \geq \frac{3n}{4} - 11 \cdot \frac{c_L + c_R}{2} - \frac{21}{2} \tag{1}$$

as a first lower bound for the number of convex 5-holes in S.

Using $h_5(S) = c + h_5(S_L) + h_5(S_R)$, and the fact that the (almost) halving line ℓ splits S such that $|S_L| = \lceil \frac{n}{2} \rceil$ and $|S_R| = \lfloor \frac{n}{2} \rfloor$, we get $h_5(S) \geq \frac{c_L + c_R}{2} + h_5(\lceil \frac{n}{2} \rceil) + h_5(\lfloor \frac{n}{2} \rfloor) \geq \frac{c_L+c_R}{2} + h_5(\lceil \frac{n-1}{2} \rceil) + h_5(\lceil \frac{n-1}{2} \rceil)$, and hence, a second lower bound for $h_5(S)$:

$$h_5(S) \geq \frac{c_L + c_R}{2} + 2 \cdot h_5\left(\left\lceil \frac{n-1}{2} \right\rceil\right) . \tag{2}$$

Combining this with the bound (1), we obtain

$$\begin{aligned} h_5(S) \geq \max \Bigg\{ &\left(\frac{3n}{4} - 11 \cdot \frac{c_L + c_R}{2} - \frac{21}{2}\right), \\ &\left(\frac{c_L + c_R}{2} + 2 \cdot h_5\left(\left\lceil \frac{n-1}{2} \right\rceil\right)\right)\Bigg\} . \end{aligned} \tag{3}$$

Note that the first term in inequality (3) is strictly decreasing in $\frac{c_L+c_R}{2}$, while the second term is strictly increasing in $\frac{c_L+c_R}{2}$. Thus, the minimum of the lower bound in (3) is reached if both bounds are equal.

$$\begin{aligned} \frac{3n}{4} - 11 \cdot \frac{c_L + c_R}{2} - \frac{21}{2} &= \frac{c_L + c_R}{2} + 2 \cdot h_5\left(\left\lceil \frac{n-1}{2} \right\rceil\right) \\ \frac{3n}{4} - \frac{21}{2} - 2 \cdot h_5\left(\left\lceil \frac{n-1}{2} \right\rceil\right) &= 12 \cdot \frac{c_L + c_R}{2} \\ \frac{c_L + c_R}{2} &= \frac{n}{16} - \frac{7}{8} - \frac{1}{6} \cdot h_5\left(\left\lceil \frac{n-1}{2} \right\rceil\right) \end{aligned}$$

Plugging this result for $\frac{c_L+c_R}{2}$ into the lower bound (2) for $h_5(S)$, we obtain a lower bound for $h_5(S)$ for any S with n points. Therefore, this also leads to a lower bound for $h_5(n)$.

$$\begin{aligned} h_5(n) &\geq \frac{n}{16} - \frac{7}{8} - \frac{1}{6} \cdot h_5\left(\left\lceil \frac{n-1}{2} \right\rceil\right) + 2 \cdot h_5\left(\left\lceil \frac{n-1}{2} \right\rceil\right) \\ &= \frac{n}{16} - \frac{7}{8} + \frac{11}{6} \cdot h_5\left(\left\lceil \frac{n-1}{2} \right\rceil\right) . \end{aligned} \tag{4}$$

We show by induction that this recursion resolves to $h_5(n) \geq \frac{3n}{4} - n^{\mathrm{ld}\,\frac{11}{6}} + \frac{15}{8}$, for $n \geq 12$. For the base case of the presented counting approach we know that $h_5(12), \ldots, h_5(16) \geq 3$ and $h_5(17), \ldots, h_5(23) \geq 4$ (see first paragraph of this proof). As $\frac{3n}{4} - n^{\mathrm{ld}\,\frac{11}{6}} + \frac{15}{8}$ is monotonically increasing for $12 \leq n \leq 23$, it is sufficient to check the induction base for $n = 16$ and $n = 23$: $h_5(16) \geq 3 \geq 2.578 \geq \frac{3\cdot 16}{4} - 16^{\mathrm{ld}\,\frac{11}{6}} + \frac{15}{8}$ and $h_5(23) \geq 4 \geq 3.609 \geq \frac{3\cdot 23}{4} - 23^{\mathrm{ld}\,\frac{11}{6}} + \frac{15}{8}$.

For $n \geq 24$, the induction step, we insert the claim into the recursive formula:

$$\begin{aligned} h_5(n) &\geq \frac{n}{16} - \frac{7}{8} + \frac{11}{6} \cdot h_5\left(\left\lceil \frac{n-1}{2} \right\rceil\right) \\ &\geq \frac{n}{16} - \frac{7}{8} + \frac{11}{6} \cdot \left(\frac{3\frac{n-1}{2}}{4} - \left(\frac{n-1}{2}\right)^{\mathrm{ld}\,\frac{11}{6}} + \frac{15}{8} \right) \\ &= \frac{3n}{4} + \frac{15}{8} - \frac{11}{6} \cdot \frac{1}{2^{\mathrm{ld}\,\frac{11}{6}}} \cdot (n-1)^{\mathrm{ld}\,\frac{11}{6}} \geq \frac{3n}{4} - n^{\mathrm{ld}\,\frac{11}{6}} + \frac{15}{8} \ . \end{aligned}$$

The last inequality is true because $(n-1)^{\mathrm{ld}\,\frac{11}{6}} < n^{\mathrm{ld}\,\frac{11}{6}}$. This proves the claim and the theorem as we have:

$$h_5(n) \geq \frac{3n}{4} - n^{0.87447} + 1.875 = \frac{3n}{4} - o(n) \ . \tag{5}$$

□

2. Empty triangles and convex 4-holes

The currently best upper bounds on the number of empty triangles and convex 4-holes, $h_3(n) \leq 1.6196n^2 + o(n^2)$ and $h_4(n) \leq 1.9397n^2 + o(n^2)$, respectively, are by Bárány and Valtr [7]. Unlike for convex 5-holes, the lower bounds for empty triangles and convex 4-holes are also known to be quadratic. The previously best known lower bounds are $h_3(n) \geq n^2 - \frac{37n}{8} + \frac{23}{8}$ and $h_4(n) \geq \frac{n^2}{2} - \frac{11n}{4} - \frac{9}{4}$, shown by García [12]. Using the new lower bound on the number of convex 5-holes in Theorem 3 we improve the second terms of the lower bounds on the number of empty triangles and convex 4-holes.

For consistency with previous publications, we use the same definitions and notation as in [17, 11, 12]. We further will recall (and slightly adapt) some statements and proofs from [12] to keep our paper self-contained. Let S be a set of n points in the plane in general position. We will have to define a total order on the points of S, such that this order allows to define a line ℓ_q through every point $q \in S$, so that each point $r \in S$ is either in the closed halfplane "below" ℓ_q, i.e., $q \geq r$, or in the open halfplane "above" ℓ_q, i.e., $q < r$. In [12] the points of S are sorted in increasing order of the y-coordinate (with the additional restriction that no two points have equal y-coordinate). Of course any fixed direction gives a valid order for the points of S. Furthermore, a cyclic order around some point $p \in (\partial\,\mathrm{CH}\,(S) \cap S)$ is a valid order for the points of $S\setminus\{p\}$, as there exists a line ℓ through p, such that all points of $S\setminus\{p\}$ are in an open halfplane bounded by ℓ. This will be crucial for the proof of Lemma 6, where we will order the points of a set $S\setminus\{p\}$ around such a point p. Because of the general position assumption for S, no two points in $S\setminus\{p\}$ are equivalent in this order. Anyhow, for simplicity, and apart from the aforementioned exception for the proof of Lemma 6, we will use the order along the y-coordinate, as in [12].

Let P be a convex 5-hole spanned by points of S and let v be the *top vertex* of P, i.e., the vertex of P with highest order. We name an empty triangle *generated by* P if it is spanned by v and the two vertices of P that are not adjacent (on the boundary of P) to v. For an example of a triangle ($\triangle$) generated by a convex 5-hole $(p_i, p_j, p_L, p_k, p_R)$ see Figure 3 (left). Let $h_{3|5}(S)$ be the number of such triangles determined by S, and let $h_{3|5}(n) = \min_{|S|=n} h_{3|5}(S)$ be the least number of empty triangles generated by convex 5-holes that every set of n points spans. Likewise, we name a convex 4-hole *generated by* P if it is spanned by all vertices of P except for one of the two vertices of P that are adjacent (on the boundary of P) to v. Observe that each convex 5-hole generates two convex 4-holes by this definition. Let $h_{4|5}(S)$ be the number of such 4-holes determined by S, and let $h_{4|5}(n) = \min_{|S|=n} h_{4|5}(S)$ be the least number of convex 4-holes generated by convex 5-holes that every set of n points spans. Note that an empty triangle (or convex 4-holes) can be generated by more than one convex 5-hole, see Figure 3. In fact, the number of convex 5-holes generating the same empty triangle can be quadratic. For an example consider the convex set and let the empty triangle be spanned by the top-most vertex and two neighbored vertices on the convex hull, such that on both sides of the triangle are approximately $\frac{n}{2}$ vertices.

García [12] recently proved that $h_3(S) = n^2 - 5n + H + 4 + h_{3|5}(S) \geq n^2 - 5n + H + 4 + h_{3|5}(n)$ and $h_4(S) = \frac{n^2}{2} - \frac{7n}{2} + H + 3 + h_{4|5}(S) \geq \frac{n^2}{2} - \frac{7n}{2} + H + 3 + h_{4|5}(n)$, where H is the number of points of $(\partial\,\mathrm{CH}\,(S) \cap S)$. Consequently, this gives $h_3(n) \geq n^2 - 5n + 7 + h_{3|5}(n)$ and $h_4(n) \geq \frac{n^2}{2} - \frac{7n}{2} + 6 + h_{4|5}(n)$, as $H \geq 3$. Observe that the number of empty triangles (or convex 4-holes) not generated by convex 5-holes is an invariant of the point set. As changing the order of the point set does not change the point set itself, $h_3(S)$ and $h_4(S)$ are of course independent of the order. Thus, also $h_{3|5}(S)$ and $h_{4|5}(S)$ (and of course $h_{3|5}(n)$ and $h_{4|5}(n)$) do not depend on the chosen order of the points. In other words, although the empty triangles and convex 4-holes generated by convex 5-holes may change with different orders, their numbers stay the same.

Proving $h_{3|5}(n) \geq 3 \cdot \left\lfloor \frac{n-4}{8} \right\rfloor$ and $h_{4|5}(n) \geq 6 \cdot \left\lfloor \frac{n-4}{8} \right\rfloor$, García presented the lower bounds $h_3(n) \geq n^2 - \frac{37n}{8} + \frac{23}{8}$ and $h_4(n) \geq \frac{n^2}{2} - \frac{11n}{4} - \frac{9}{4}$. We will improve both bounds. Showing that for each convex 5-hole counted in Lemma 1 we may count one empty triangle generated by convex 5-holes and two convex 4-holes generated by convex 5-holes will already give an improved bound for both $h_{3|5}(n)$ and $h_{4|5}(n)$. Using a slightly adapted version of the proof from Theorem 3 will improve the bound on $h_{4|5}(n)$ even further. To this end we have to first prove the base case, i.e., sets of 10, 11, and 12 points.

Having a close look at the example shown in Figure 3, one can see that as soon as the triangle $\triangle$ (or the convex 4-hole $\diamond$) is generated by more than one convex 5-hole, there must exist at least one convex 6-hole. Note that this has been proven in [12], using a similar approach and figure. To remain self-contained we restate this fact in more detail and prove it in the following lemma.

Lemma 4. *Let S be a set of $n \geq 6$ points in the plane in general position. Let $\triangle$ ($\diamond$) be an empty triangle (a convex 4-hole) of S. If $\triangle$ ($\diamond$) is generated by at least two convex 5-holes, $\pentagon_1$ and $\pentagon_2$, of S, then there exists at least one convex 6-hole, $\hexagon_1$, of S, containing $\pentagon_1$, and one convex 6-hole, $\hexagon_2$, of S, containing $\pentagon_2$, where $\hexagon_1 = \hexagon_2$ is possible.*

Proof. See Figure 3 (left). Assume that there exists at least one empty triangle, $\triangle = \langle p_i, p_j, p_k \rangle$, with p_k being the top vertex, that is generated by two different convex 5-holes. Let one of them, $\pentagon_1$, be spanned by the points p_i, p_j, p_L, p_k, p_R (the points are shown as full dots in the figure).

As $\triangle$ is generated by another convex 5-hole, $\pentagon_2$, there must be at least one additional point in one of the regions L_h, L_l, R_h, and R_l (as indicated in Figure 3). Otherwise, the new pentagon would not be empty, not be convex, or $\triangle$ would not be generated by it (recall that p_k must be the top vertex). W.l.o.g. assume that there exists at least one point p_{new} in R_l. It is easy to see that in this case there exists a convex 4-hole spanned by the points p_i, p_k, p_R, p'_R ($p'_R = p_{new}$ is possible, but not necessary). Together with p_j and

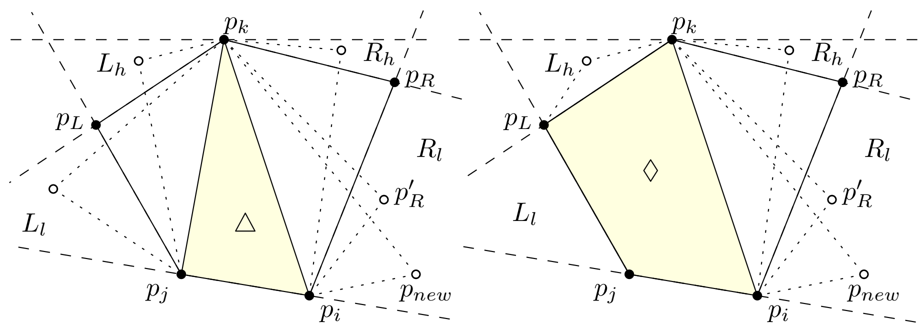

Figure 3: Auxiliary figure for the proof of Lemma 4. An empty triangle (left) or a convex 4-hole (right) generated by at least two convex 5-holes implies the existence of at least one convex 6-hole.

p_L this forms a convex 6-hole which contains ⬠$_1$. Repeating the argument with $\triangle$ being generated by ⬠$_2$ proves that also ⬠$_2$ is contained in a convex 6-hole.

The argument is analogous for a convex 4-hole, $\diamond$, that is generated by two different convex 5-holes. See Figure 3 (right). The only difference to the previous case is that the additional point p_{new} can not exist in either L_l or R_l, depending on which convex 4-hole (either $\diamond = \langle p_i, p_j, p_L, p_k \rangle$ or $\diamond = \langle p_i, p_j, p_k, p_R \rangle$) is considered. The former situation is depicted in Figure 3 (right). $\square$

Using Lemma 4 we are able to provide the base cases $10 \leq n \leq 12$ for $h_{3|5}(n)$ and $h_{4|5}(n)$. Note that the statement of the following lemma has already been proven in [12] for the case $n = 12$.

Lemma 5. *Every set of 10, 11, or 12 points in the plane in general position contains (i) at least 1, 2, and 3, respectively, different empty triangles generated by convex 5-holes (that is,* $h_{3|5}(10) = 1$*,* $h_{3|5}(11) = 2$*, and* $h_{3|5}(12) = 3$*) and (ii) at least 2, 4, and 6, respectively, different convex 4-holes generated by convex 5-holes (that is,* $h_{4|5}(10) = 2$*,* $h_{4|5}(11) = 4$*, and* $h_{4|5}(12) = 6$*).*

Proof. First assume that the set contains a convex 6-hole ⬡. Let $S_{⬡} = S \cap ⬡$ be the vertex set of ⬡. From [12] we know that

$$h_3(S_{⬡}) = n^2 - 5n + H + 4 + h_{3|5}(S_{⬡}) = 16 + h_{3|5}(S_{⬡}) \quad \text{and}$$
$$h_4(S_{⬡}) = \frac{n^2}{2} - \frac{7n}{2} + H + 3 + h_{4|5}(S_{⬡}) = 6 + h_{4|5}(S_{⬡}) \quad \text{as} \quad n = H = 6.$$

As $h_k(S) = \binom{|S|}{k}$ for S in convex position, we get $h_{3|5}(S_{\hexagon}) = \binom{6}{3} - 16 = 4$ and $h_{4|5}(S_{\hexagon}) = \binom{6}{4} - 6 = 9$. Hence, the 6 convex 5-holes contained in $\hexagon$ generate 4 different empty triangles and 9 different convex 4-holes, respectively. This proves the lemma if a convex 6-hole exists.

Now assume that no convex 6-hole exists. By Lemma 4, every empty triangle and every convex 4-hole is generated only once in this case. Hence, we have $h_{3|5}(n) = h_5(n)$ and $h_{4|5}(n) = 2 \cdot h_5(n)$, for $n \in \{10, 11, 12\}$, which proves the lemma. (Recall that $h_5(10) = 1$, $h_5(11) = 2$, and $h_5(12) = 3$, by [14, 8, 4].) □

Using these base cases we derive a statement similar to Lemma 1.

Lemma 6. *Every set S of n points in the plane in general position with $n = 7 \cdot m + 9 + t$ (for any natural number $m \geq 0$ and $t \in \{1, 2, 3\}$) contains at least $h_{3|5}(n) \geq \frac{3n-27+4t}{7}$ empty triangles generated by convex 5-holes and at least $h_{4|5}(n) \geq 2 \cdot \frac{(3n-27+4t)}{7}$ convex 4-holes generated by convex 5-holes.*

Proof. The proof follows the lines of the proof of Lemma 1. Obviously, $h_{3|5}(n) \geq h_{3|5}(n-1)$ and $h_{4|5}(n) \geq h_{4|5}(n-1)$. Hence, using Lemma 5, we have $h_{3|5}(10) = 1$, $h_{4|5}(10) = 2$, $h_{3|5}(11) = 2$, $h_{4|5}(11) = 4$, $h_{3|5}(12..16) \geq 3$, and $h_{4|5}(12..16) \geq 6$ as base cases.

Consider the case in which there exists a $p \in (\partial \operatorname{CH}(S) \cap S)$ that is a point of a convex 5-hole. Let p be the top vertex of that convex 5-hole (choose the sort order accordingly). Then $h_{3|5}(n) \geq 1 + h_{3|5}(S \setminus \{p\}) \geq 1 + h_{3|5}(n-1)$ and $h_{4|5}(n) \geq 2 + h_{4|5}(S \setminus \{p\}) \geq 2 + h_{4|5}(n-1)$ (as $h_{3|5}(n-1)$ and $h_{4|5}(n-1)$ are independent of the sort order). In this case, the lemma is true by induction (see previous paragraph for base cases).

For the second case, let $p \in (\partial \operatorname{CH}(S) \cap S)$ be a point which is not part of a convex 5-hole and do the same clockwise partitioning of $S \setminus \{p\}$ around p as in the proof of Lemma 1 (recall Figure 1). Additionally, let the order of $S \setminus \{p\}$ be counterclockwise around p, i.e., such that the last point of S_{rem} has lowest index and the first point of S_0 has highest index in this order. When processing the sets of (at most) 12 points (see the proof of Lemma 1), they overlap only in 4 points. Thus, the top vertex (point with highest index in the chosen order) of a convex 5-hole is never shared by two such sets. Hence, applying Lemma 5 for sets of 10, 11, and 12 points, the number of empty triangles generated by convex 5-holes is at least the number of convex 5-holes counted in Lemma 1, and the number of convex 4-holes generated by convex 5-holes is at least twice the number of convex 5-holes counted in Lemma 1.

Note that this counting is possible, as p is not part of any convex 5-hole, and as the total order of $S\setminus\{p\}$ around p is well defined; see also the first paragraph of Section 2. □

As mentioned in the beginning of this section, this lemma already improves the bounds for $h_{3|5}(n)$ and $h_{4|5}(n)$. We will further improve the bound for $h_{4|5}(n)$ in Theorem 8. Therefore, we state only the bound for $h_{3|5}(n)$ in the following theorem.

Theorem 7. *Every set S of $n \geq 12$ points in the plane in general position contains at least $h_{3|5}(n) \geq 3 \cdot \left\lfloor \frac{n-12}{7} \right\rfloor + 3 + f(|S_{rem}|) \geq \left\lceil \frac{3n-27}{7} \right\rceil$ empty triangles generated by convex 5-holes. The point set $S_{rem} \subset S$ is the remainder set with $0 \leq |S_{rem}| \equiv (n-12) \mod 7 \leq 6$, and $f(0\ldots4) = 0$, $f(5) = 1$, and $f(6) = 2$.*

Proof. The first inequality in the bound, $h_{3|5}(n) \geq 3 \cdot \left\lfloor \frac{n-12}{7} \right\rfloor + 3 + f(|S_{\text{rem}}|)$, is simply a reformulation of the bound in Lemma 6. The second inequality results from taking the minimum of the first inequality over all possible values for $|S_{\text{rem}}|$, which is obtained by $|S_{\text{rem}}| = 4$. □

The basic principles of the proof of the following theorem are the same as for the proof of Theorem 3. The main difference is a slightly different counting (as compared to just counting convex 5-holes). To avoid over-counting, only five out of the six possible convex 4-holes generated by convex 5-holes that have at least one point in $S_L \cap S''$ may be counted for Case 1 (see Figure 4 and the proof below for details).

Theorem 8. *Every set S of $n \geq 12$ points in the plane in general position contains at least $h_{4|5}(n) \geq \frac{5n}{4} - \frac{383}{303} \cdot n^{\text{ld}\,\frac{19}{10}} + \frac{55}{24} = \frac{5n}{4} - o(n)$ convex 4-holes generated by convex 5-holes.*

Proof. Again, we follow the lines of the proof of Theorem 3. The difference stems only from a slightly modified counting.

For the base case, $n \leq 23$, we use $h_{4|5}(12\ldots16) \geq 6$ and $h_{4|5}(17\ldots23) \geq 8$ by Lemma 6.

If $n \geq 24$ consider an (almost) halving line ℓ of S which splits S into S_L ($|S_L| = \lceil \frac{n}{2} \rceil$) and S_R ($|S_R| = \lfloor \frac{n}{2} \rfloor$) and does not contain any point of S. See Figure 2 in the proof of Theorem 3. Furthermore, consider a line ℓ' that intersects ℓ and cuts off a set $S' \subseteq S$, consisting of eight points from S_L and four points from S_R. If a convex 5-hole is intersected by ℓ, then also at least

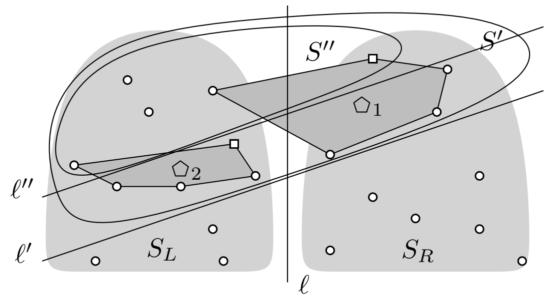

Figure 4: Depicting the differences between the proofs of Theorem 3 and Theorem 8. If a convex 4-hole generated by a convex 5-hole, $\pentagon_1$, is completely contained in S_R, then the other convex 4-hole generated by $\pentagon_1$ is intersected by the halving line ℓ resulting in Case 2. The top vertices of $\pentagon_1$ and $\pentagon_2$ are shown as squared dots.

one of the two convex 4-holes generated by this convex 5-hole is intersected by ℓ. This convex 4-hole is not counted again during recursion and thus there is no over-counting. In other words, at most one convex 4-hole generated by a convex 5-hole can be completely contained in $S' \cap S_R$. But then the other convex 4-hole generated by the same convex 5-hole is certainly intersected by ℓ. See Case 2 below and $\pentagon_1$ in Figure 4.

Let a line ℓ'' be parallel to ℓ' and split $S' \cap S_L$ into two groups of four points. Let $S'' \subset S'$ be the set which is cut off by ℓ'', such that $|S'' \cap S_L| = 4$. Note that neither ℓ' nor ℓ'' contain any points of S. This is the same splitting as in the proof of Theorem 3, depicted in Figure 2. Observe that at most one convex 4-hole generated by a convex 5-hole can be completely contained in $S_L \cap (S' \setminus S'')$. See $\pentagon_2$ in Figure 4. To avoid over-counting, we may only count five out of the at least six convex 4-holes generated by the at least three convex 5-holes of the 12 points in S'. See Case 1 below.

Apart from this slightly different counting and the resulting change in values, the remainder of this proof is an adapted copy of the proof of Theorem 3.

Again we distinguish two cases: *Case 2* (at least one convex 4-hole generated by a convex 5-hole is intersected by ℓ) appears c_L times. *Case 1* (at least 5 convex 4-holes generated by the at least three convex 5-holes of the 12 points in S' have at least one point in $S_L \cap S''$) appears at least $\left\lfloor \frac{1}{4} \cdot \left(\left\lceil \frac{n}{2} \right\rceil - 8c_L \right) \right\rfloor - 1$ times in S_L. Equivalently, Case 2 appears c_R times and Case 1 appears at least $\left\lfloor \frac{1}{4} \cdot \left(\left\lfloor \frac{n}{2} \right\rfloor - 8c_R \right) \right\rfloor - 1$ times in S_R.

Summing up the convex 4-holes generated by convex 5-holes counted for

S_L, S_R, and ℓ we get

$$h_{4|5}(S) \geq 5\left(\left\lfloor\frac{1}{4}\left(\left\lceil\frac{n}{2}\right\rceil - 8c_L\right)\right\rfloor - 1\right) + 5\left(\left\lfloor\frac{1}{4}\left(\left\lfloor\frac{n}{2}\right\rfloor - 8c_R\right)\right\rfloor - 1\right) + \frac{c_L + c_R}{2}$$

which can be transformed into a first lower bound of

$$h_{4|5}(S) \geq \frac{5n}{4} - 19 \cdot \frac{c_L + c_R}{2} - \frac{35}{2} \ .$$

Identifying this bound with the second lower bound from the recursion

$$h_{4|5}(S) \geq \frac{c_L + c_R}{2} + 2 \cdot h_{4|5}\left(\left\lceil\frac{n-1}{2}\right\rceil\right)$$

leads to

$$\frac{c_L + c_R}{2} \quad = \quad \frac{n}{16} - \frac{7}{8} - \frac{1}{10} \cdot h_{4|5}\left(\left\lceil\frac{n-1}{2}\right\rceil\right)$$

and consequently to the combined bound

$$h_{4|5}(n) \geq \frac{n}{16} - \frac{7}{8} + \frac{19}{10} \cdot h_{4|5}\left(\left\lceil\frac{n-1}{2}\right\rceil\right) \ .$$

We claim that this recursion resolves to $h_{4|5}(n) \geq \frac{5n}{4} - \frac{383}{303} n^{\mathrm{ld}\,\frac{19}{10}} + \frac{55}{24}$, and prove it by induction. The base case ($n < 24$) can be checked directly, because we know that $h_{4|5}(12), \ldots, h_5(16) \geq 6$ and $h_{4|5}(17), \ldots, h_5(23) \geq 8$. For $n \geq 24$ we insert the claim into the recursive formula:

$$\begin{aligned}
h_{4|5}(n) \quad &\geq \quad \frac{n}{16} - \frac{7}{8} + \frac{19}{10} \cdot h_{4|5}\left(\left\lceil\frac{n-1}{2}\right\rceil\right) \\
&\geq \quad \frac{n}{16} - \frac{7}{8} + \frac{19}{10} \cdot \left(\frac{5}{4} \cdot \frac{n-1}{2} - \frac{383}{303} \cdot \left(\frac{n-1}{2}\right)^{\mathrm{ld}\,\frac{19}{10}} + \frac{55}{24}\right) \\
&= \quad \frac{5n}{4} + \frac{55}{24} - \frac{19}{10} \cdot \frac{1}{2^{\mathrm{ld}\,\frac{19}{10}}} \cdot \frac{383}{303}(n-1)^{\mathrm{ld}\,\frac{19}{10}} \\
&\geq \quad \frac{5n}{4} - \frac{383}{303} n^{\mathrm{ld}\,\frac{19}{10}} + \frac{55}{24} \ .
\end{aligned}$$

The last inequality is true because $(n-1)^{\mathrm{ld}\,\frac{19}{10}} < n^{\mathrm{ld}\,\frac{19}{10}}$. This concludes the proof as we have:

$$h_{4|5}(n) \quad \geq \quad \frac{5n}{4} - 1.2641 n^{0.926} + 2.2916 = \frac{5n}{4} - o(n) \ .$$

□

Remark. To use the principles of the above proof also for empty triangles generated by convex 5-holes, a very disadvantageous splitting is necessary to avoid over-counting. This would lead to a bound inferior to the one from Theorem 7.

Recall that García [12] recently proved $h_3(S) \geq n^2 - 5n + H + 4 + h_{3|5}(n)$ and $h_4(S) \geq \frac{n^2}{2} - \frac{7n}{2} + H + 3 + h_{4|5}(n)$. Combining these results with Theorem 7 and Theorem 8, we can state the following corollary on the number of empty triangles and convex 4-holes in a point set.

Corollary 9. *Every set S of $n \geq 12$ points in the plane in general position and with H points on the boundary of its convex hull contains at least $h_3(S) \geq n^2 - 5n + H + 4 + \left\lceil \frac{3n-27}{7} \right\rceil$ empty triangles and at least $h_4(S) \geq \frac{n^2}{2} - \frac{9n}{4} - \frac{383}{303} \cdot n^{\mathrm{ld}\,\frac{19}{10}} + H + \frac{127}{24}$ convex 4-holes. Consequently, $h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7}$ and $h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - 1.2641\, n^{0.926} + \frac{199}{24}$.*

3. Conclusion

In this paper we improve the lower bounds on the number of empty triangles (to $h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7}$), of convex 4-holes (to $h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - 1.2641\, n^{0.926} + \frac{199}{24}$), and of convex 5-holes (to $h_5(n) \geq \frac{3n}{4} - n^{0.87447} + 1.875$) that every set S of $n \geq 12$ points contains. To improve the bounds on $h_3(n)$ and $h_4(n)$ we use a recent result by García [12] and provide better bounds for the number of empty triangles and convex 4-holes which are "generated by convex 5-holes". The question whether there exists a super-linear lower bound for the number of convex 5-holes remains unsettled, though.

Still, we are able to answer several questions, which Dehnhardt [8] asked in 1987. Already in [4] a set of 12 points containing only three convex 5-holes has been presented. This implies $h_5(12) = 3$ and therefore disproves Dehnhardt's conjecture of $h_5(12) = 4$.

Consider the set S_{12} with $n = 12$ points and $H = 3$, depicted in Figure 5. It can be easily checked that this point set contains only the 3 shown convex 5-holes and no convex 6-hole. Hence, $h_{3|5}(S_{12}) = 3$ and $h_{4|5}(S_{12}) = 6$, as by Lemma 4. Using the equations $h_3(S) = n^2 - 5n + H + 4 + h_{3|5}(S)$ and $h_4(S) = \frac{n^2}{2} - \frac{7n}{2} + H + 3 + h_{4|5}(S)$ [12], we get $h_3(S_{12}) = 144 - 60 + 3 + 4 + 3 = 94$ and $h_4(S_{12}) = 72 - 42 + 3 + 3 + 6 = 42$. Of course, $h_3(S_{12})$ and $h_4(S_{12})$ can also be derived by counting all empty triangles and convex 4-holes in S_{12}. As, by Lemma 5, $h_{3|5}(12) = 3$ and $h_{4|5}(12) = 6$, the set S_{12} realizes the

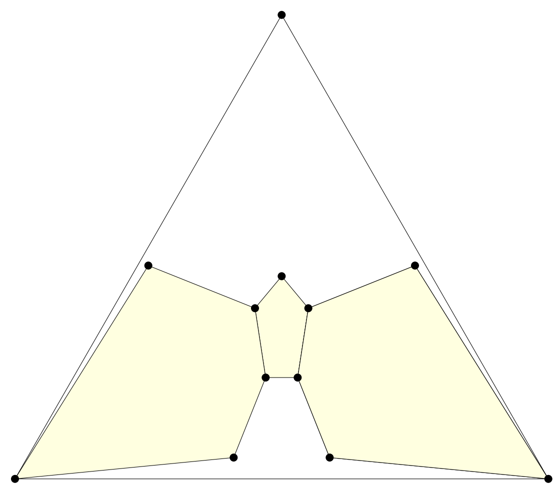

Figure 5: Set of 12 points with triangular convex hull, generating the minimal number of 3-holes (94), convex 4-holes (42), and convex 5-holes (3). The coordinates (x, y) of the 12 points are: (0, 0); (100, 0); (50, 87); (50, 38); (55, 32); (53, 19); (47, 19); (45, 32); (41, 4); (59, 4); (25, 40); (75, 40).

minimum number of empty triangles and convex 4-holes for sets with 12 points. ($h_3(12) \geq 94$ and $h_4(12) \geq 42$ has also been proven in [8].) Thus, the set S_{12} disproves two conjectures of Dehnhardt in [8], namely $h_3(12) = 95$ and $h_4(12) = 44$.

Furthermore, Dehnhardt's question for a set of n points that minimizes at least one of $h_3(n)$, $h_4(n)$, and $h_5(n)$, but not all of them is answered by the set of 12 points presented in [4], which has only 3 convex 5-holes but contains 95 empty triangles and 43 convex 4-holes, where both values are non-minimal for $n = 12$.

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References

[1] O. Aichholzer, [Empty] [colored] k-gons - Recent results on some Erdős-Szekeres type problems, in: Proc. XIII Encuentros de Geometría Computacional (EGC 2009), Zaragoza, Spain, pp. 43–52.

[2] O. Aichholzer, R. Fabila-Monroy, T. Hackl, C. Huemer, A. Pilz, B. Vogtenhuber, Lower bounds for the number of small convex k-holes, in: Proc. 24^{th} Annual Canadian Conference on Computational Geometry (CCCG 2012), Charlottetown, PEI, Canada, pp. 247–252.

[3] O. Aichholzer, T. Hackl, B. Vogtenhuber, On 5-holes and 5-gons, in: Proc. XIV Encuentros de Geometría Computacional (EGC 2011), Alcalá de Henares, Spain, pp. 7–10.

[4] O. Aichholzer, T. Hackl, B. Vogtenhuber, On 5-holes and 5-gons, in: A. Marquez, P. Ramos, J. Urrutia (Eds.), Computational Geometry: XIV Spanish Meeting on Computational Geometry, EGC 2011, Festschrift Dedicated to Ferran Hurtado on the Occasion of His 60th Birthday, Alcalá de Henares, Spain, June 27-30, 2011, Revised Selected Papers, volume 7579 of *Lecture Notes in Computer Science (LNCS)*, Springer, 2012, pp. 1–13.

[5] I. Bárány, Z. Füredi, Empty simplices in Euclidean space, Canadian Mathematical Bulletin 30 (1987) 436–445.

[6] I. Bárány, G. Károlyi, Problems and results around the Erdős–Szekeres convex polygon theorem, in: J. Akiyama, M. Kano, M. Urabe (Eds.), Discrete and Computational Geometry, volume 2098 of *Lecture Notes in Computer Science (LNCS)*, Springer, 2001, pp. 91–105.

[7] I. Bárány, P. Valtr, Planar point sets with a small number of empty convex polygons, Studia Scientiarum Mathematicarum Hungarica 41 (2004) 243–266.

[8] K. Dehnhardt, Leere konvexe Vielecke in ebenen Punktmengen, Ph.D. thesis, TU Braunschweig, Germany, 1987. In German.

[9] H. Edelsbrunner, Algorithms in Combinatorial Geometry, Springer, 1987.

[10] P. Erdős, Some more problems on elementary geometry, Australian Mathematical Society Gazette 5 (1978) 52–54.

[11] A. García, A note on the number of empty triangles, in: Proc. XIV Encuentros de Geometría Computacional (EGC 2011), Alcalá de Henares, Spain, pp. 101–104.

[12] A. García, A note on the number of empty triangles, in: A. Marquez, P. Ramos, J. Urrutia (Eds.), Computational Geometry: XIV Spanish Meeting on Computational Geometry, EGC 2011, Festschrift Dedicated to Ferran Hurtado on the Occasion of His 60th Birthday, Alcalá de Henares, Spain, June 27-30, 2011, Revised Selected Papers, volume 7579 of *Lecture Notes in Computer Science (LNCS)*, Springer, 2012, pp. 249–257.

[13] T. Gerken, Empty convex hexagons in planar point sets, Discrete and Computational Geometry 39(1–3) (2008) 239–272.

[14] H. Harborth, Konvexe Fünfecke in ebenen Punktmengen, Elemente der Mathematik 33 (1978) 116–118. In German.

[15] J. Horton, Sets with no empty convex 7-gons, Canadian Mathematical Bulletin 26(4) (1983) 482–484.

[16] C. Nicolás, The empty hexagon theorem, Discrete and Computational Geometry 38(2) (2007) 389–397.

[17] R. Pinchasi, R. Radoičić, M. Sharir, On empty convex polygons in a planar point set, Journal of Combinatorial Theory, Series A 113 (2006) 385–419.

[18] P. Valtr, On empty pentagons and hexagons in planar point sets, in: Proc. 18th Computing: Australasian Theory Symposium (CATS 2012), Melbourne, Australia, pp. 47–48.

[19] B. Vogtenhuber, Combinatorial Aspects of [Colored] Point Sets in the Plane, Ph.D. thesis, Institute for Software Technology, Graz University of Technology, Graz, Austria, 2011.