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Packing Equal Circles in a Square I. — Problem Setting and Bounds for Optimal Solutions *

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Abstract. In the paper, a short review of the problem of finding the densest packing of n equal circles in a square is made. There will be new lower bounds for this problem defined on the basis of regular arrangements. Also, there will new upper bounds be established based on the computation of the areas of circle and minimum gap between circles and between circles and sides of the square. The paper also contains all the known exact values of optimal packings and the corresponding minimal polynomials.

Keywords: Optimal Packing, Minimal Polynomials, Regular Patterns.

1. Introduction

The general problem of finding the densest packing of n equal objects in a bounded space is a classical one which arises in many scientific and engineering fields. For the two-dimensional case, this problem has been studied for several different shapes of the bounded space; e.g. packing n equal circles in a circle [7, 8, 9, 12, 15, 20, 28, 42, 43], or circles in an equilateral triangle [13, 26, 27, 29, 38] or circles in a square where circles are non-overlapping and the radius of equal circles should be maximized. In the paper all the discussions will only concern the packing circles in a square problem.

The packing problem has a long history in the mathematical literature [3, 4, 5, 6], but the packing circles in a square problem is 40 years old only. In 1960, Leo Moser was the first to study it when he wrote the next conjecture [32]: eight points in or on a unit square determine a distance $\leq \frac{1}{2}\sec 15^o$. This problem for up to nine circles ($n = 2, \ldots, 9$) was solved in 1965 [44, 46], although the first proof for $n = 6$ was reported in 1970 [48] and for $n = 7$ in 1996 [34]. Between 1970 and 1990, at least ten papers have reported solutions for $n = 10$ [11, 16, 17, 18, 30, 31, 41, 45, 47, 49], but the optimal solution was given in 1990 [17]. R. Peikert *et al.* [40] proposed an elimination procedure and found optimal results for $n = 10, \ldots, 20$. Until recently the problem has been solved for $n \leq 27$ [35, 40] using computer-aided methods for

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optimality proofs [34, 40] and the $n = 36$ [19] case was proven by mathematical tools. Optimality proofs of packings obtained manually (without using a computer) exist up to 9 [25, 44, 46, 48] and for the $n = 14$ [51], 16 [50], 25 [52], and 36 [19] cases. Albeit Schaer and Meir in [44] claimed that for $n = 7$ they had found the optimality proof, but did not explain it further on, and until now it has not been published. A more detailed history of the packing equal circles in a square problem can be found in [53].

In addition to the above optimal solutions, there has been published good approximations to the problem solution for $n \leq 52$ and n =54, 55, 56, 60, 61, 62, 72, and 78. These numerical results were obtained by using several different strategies; for instance, using billiard simulation [14, 21, 22], minimization of the energy function [33], standard BFGS quasi-Newton algorithm [40], nonlinear programming solver (MINOS 5.3) [24] or the Cabri-Géomètre software in [31]. Although it is not known whether these solutions are optimal, in some cases these numerical results can help to find better solutions (if any) when they are used as lower bounds of the optimal (maximal) solutions. For instance, several strategies described in the packing literature, as that of R. Peikert *et al.* in [40], are based on the knowledge of a good lower bound of the solution. In this sense, lower bounds are useful for the problem solution.

The paper provides new results, lower and upper bounds on the radius of circles for up to one hundred circles. The paper is organized as follows: in Section 2 four equivalent definitions of the packing n circles in a square problem as well as notations and parameters used are given. Section 3 is devoted to set up several proposals for defining upper and lower bounds on the radius for the packing n circles in a unit square problem. Upper bounds are based on the computation of the circle areas as well as on the minimum gap areas between circles and between circles and the sides of the square. Our suggestion for lower bounds are based on specific regular patterns to arrange n equal circles in a unit square. In Section 4, for several specific values of n, theoretical solutions and exact results of the optimal values of the problem at hand are given.

2. Definitions and notation

The packing circles in a square problem can be described by the following equivalent problem settings:

1. Locate n points in a unit square, such that the minimum distance m_n between any two points is maximal.
2. Find the value of the maximum circle radius, r_n, such that n equal non-overlapping circles can be placed in a unit square.
3. Give the smallest square of side s_n, which contains n equal and non-overlapping circles where the radius of circles is 1.
4. Determine the smallest square of side σ_n that contains n points with mutual distance of at least 1.

From these statements it is easy to prove that for the defined parameters m_n, r_n, s_n and σ_n the following relations hold:

$$r_n = \frac{m_n}{2(m_n+1)}, \qquad m_n = \frac{2r_n}{1-2r_n}, \qquad s_n = \frac{1}{r_n}, \qquad \sigma_n = \frac{1}{m_n}. \tag{1}$$

The problem of maximizing the minimal pairwise distance of n points being in a unit square (Problem 1) can be formulated as the following continuous global optimization problem:

$$\begin{gathered} \mu(x,y) = \min_{1\le i<j\le n} \sqrt{(x_i-x_j)^2+(y_i-y_j)^2} \\ \frac{\max \mu(x,y)}{x,y \in [0,1]^n,\ n>1 \text{ integer},} \end{gathered} \tag{2}$$

where x_i, y_i are the coordinates of the i-th point. The goal is not only to obtain the maximum of the minimum distance between any two points ($\max \mu(x,y)$), but also to find the respective locations of the n points in the unit square (the coordinates x_i, y_i; $1 \le i \le n$).

Definition 1. Given an $\epsilon \ge 0$ (tolerance error), we say that *two circles i, j* (Problem 1) located at (x_i, y_i) and (x_j, y_j) *are in contact* if:

$$\sqrt{(x_i-x_j)^2+(y_i-y_j)^2} \le m_n + \epsilon.$$

Similarly to this, a circle i is in contact with one side of the square if $x_i \le \epsilon$, $y_i \le \epsilon$, $x_i \ge 1-\epsilon$, or $y_i \ge 1-\epsilon$.

The exact mathematical definition for the contact of two circles is that m_n is equal to $\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}$, and also when a point is on the side of the square. We have considered the above exact form of definition in the theoretical results as well as in Definition 1 in numerical results [1], since usually a finite arithmetic is adapted to computers. For the study we set $\epsilon = 10^{-10}$.

Definition 2. Let us suppose that there is a given solution of the problem (2). We say that *a circle is free* if its center can be moved towards a positive distance point without causing the others' overlapping.

We can point out that when a packing contains one or more free circles, then the solution is obviously not unique, moreover, the locations of the center of any free circles form a non-empty interior and connected set. In the paper the number of contacts will be denoted by c_n, while the number of free circles by f_n. In all the figures a contact will be represented by a short line section and free circles will be indicated by dark shading.

Definition 3. The density d_n of a packing can be given by:

$$d_n = n\pi r_n^2 = \frac{n\pi m_n^2}{4(m_n+1)^2}.$$

3. Lower and upper bounds of the maximized minimum distance between n points in a unit square

An approximate solution of (2) can be used as a lower bound m_n^l of the optimal solution m_n. Let us suppose that it is possible to determine a good value for the upper bound m_n^u of the packing problem, so an approximate solution m_n of (2) can be considered a good solution of the packing problem if it is close enough to m_n^u.

On the other hand, it has been proved in [6] that the densest packing of circles in the plane is a regular (hexagonal) arrangement (see Figure 1) and the density of this packing is $\sqrt{3}\pi/6 \approx 0.9068996821$. In addition, it can also be proved that this value can provide an upper bound for the solution of (2), although upper bound values better than that can be proposed. A hexagonal pattern can give an asymptotic formula for m_n, where $m_n \approx \sqrt{\frac{2}{\sqrt{3}n}}$ [3].

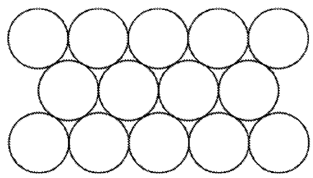

Figure 1. The arrangement of the densest packing in the plane.

It is also known that for several values of n, the optimal solutions of the packing circles in a unit square problem are regular arrangements. Keeping these ideas, in mind it is easy to guess that for problem (2), other regular arrangements may also provide at least a good lower bound of the optimal solution. In the coming section improved lower and upper bounds are discussed.

3.1. Lower bounds of m_n by using patterns

THEOREM 1 *Consider the packing circles in a unit square problem according to Problem 1. The maximum of the minimal distance between n points in the unit square m_n is not smaller than*

$$m_n^l = \max(L_1(n), L_2(n), L_{3a}(n), L_{3b}(n), L_4(n), L_5(n))$$

with

$$
\begin{aligned}
L_1(n) &= \frac{1}{\lceil\sqrt{n}\rceil - 1},\\
L_2(n) &= \frac{1}{\lceil\sqrt{n+1}\rceil - 3 + \sqrt{2+\sqrt{3}}},\\
L_{3a}(n) &= \frac{1}{\lceil\sqrt{n+2}\rceil - 2 + \frac{1}{2}\sqrt{3}},\\
L_{3b}(n) &= \frac{1}{\lceil\sqrt{n+2}\rceil - 5 + 2\sqrt{2+\sqrt{3}}},\\
L_4(n) &= \frac{k^2 - k - \sqrt{2k}}{k^3 - 2k^2}, \text{ if } n = k(k+1), \text{ and } 0, \text{ otherwise},\\
L_5(n) &= \sqrt{\frac{1}{p^2} + \frac{1}{q^2}}, \text{ if } n = \left\lceil \frac{(p+1)(q+1)}{2} \right\rceil, \; p^2 \le 3q^2, \text{ and } q^2 \le 3p^2;\\
&\quad \text{and } 0, \text{ otherwise.}
\end{aligned}
$$

(Here $\lceil . \rceil$ denotes the smallest integer not smaller than the argument)

Proof: The new lower bounds are based on the use of some patterns proposed in [14, 23, 33], also, on our own new patterns and at the assumption that $m_{n+1} \le m_n$. Thus m^l_{n+1} or any values of the m_{n+1} of (2) are also lower bounds of m_n. In this way lower bounds for all the values of n ($2 \le n \le 100$) can be obtained. The set of patterns we will take into account for defining lower bounds are (see them illustrated in Figure 3):

PAT1 For $n = k^2$ ($k \ge 2,\ k \in \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers) the packings have an obvious square grid pattern of $k \times k$ points [14], for which the minimum distance between the points is $L_1(n) = \frac{1}{k-1}$. This lower bound can be applied also for n values not conforming the equation $n = k^2$ with the $k = \lceil\sqrt{n}\rceil$ substitution.

PAT2 For $n = k^2 - 1 = (k-1)^2 + 2(k-1)$ ($k \ge 2,\ k \in \mathbb{N}$) the pattern can be considered as a square pattern such as PAT1 of $(k-1)^2$ circles into which one row and one column of shifted circles are inserted [14]. In this case $L_2(n) = \frac{1}{k-3+\sqrt{2+\sqrt{3}}}$, and $k = \lceil\sqrt{n+1}\rceil$, because $m_n(k - 3 + 2\cos(15^o)) = 1$.

PAT3 For $n = k^2 - 2 = (k-1)^2 + 2(k-2) + 1$ the pattern is similar to PAT2, but in this case there are two shifted columns and two shifted rows of fitting length [14]. In these cases
a) $L_{3a}(n) = \frac{1}{k-2+\frac{1}{2}\sqrt{3}}$, ($k \ge 3,\ k \in \mathbb{N}$),
where $k = \lceil\sqrt{n+2}\rceil$, since $m_n(k - 2 + \cos(30^o)) = 1$, and
b) $L_{3b}(n) = \frac{1}{k-5+2\sqrt{2+\sqrt{3}}}$, ($k \ge 5,\ k \in \mathbb{N}$),
where $k = \lceil\sqrt{n+2}\rceil$, since $m_n(k - 5 + 2\cos(15^o) + 2\cos(15^o)) = 1$.

PAT4 For $n = k(k+1)$ $(k \geq 2,\ k \in I\!N)$ the pattern consists of $(k-1)$ alternating columns with k circles each [14]. In Figure 3 this pattern is demonstrated for $n = 20$. In this case

$$L_4(n) = \frac{k^2 - k - \sqrt{2k}}{k^3 - 2k^2},$$

as $m_n(k - 1 + \sin(\alpha)) = 1$, where $\cos(\alpha) = \frac{1}{m_n k}$.

PAT5 Focusing on the new pattern class, we can see that the centers of the circles are located in a grid which is built by composing two shifted rectangular grids. An example of this pattern is demonstrated in Figure 2 and in Figure 3. The number of non-overlapping circles (n) which can be arranged in a square by using this pattern is a function of two integers $p, q \in I\!N,\ p \neq q$, subject to the following constraints:

$$p^2 \leq 3q^2,$$
$$q^2 \leq 3p^2.$$

For the sake of clarity, three cases will be distinguished:

- p and q are both even; then $p = 2p^*;\ q = 2q^*$, and $n = 2p^*q^* + p^* + q^* + 1$
- p is even and q is odd; then $p = 2p^*;\ q = 2q^* + 1$, and $n = (2p^* + 1)(q^* + 1)$
- both p and q are odd, then $p = 2p^* + 1;\ q = 2q^* + 1$, and $n = 2(p^* + 1)(q^* + 1)$

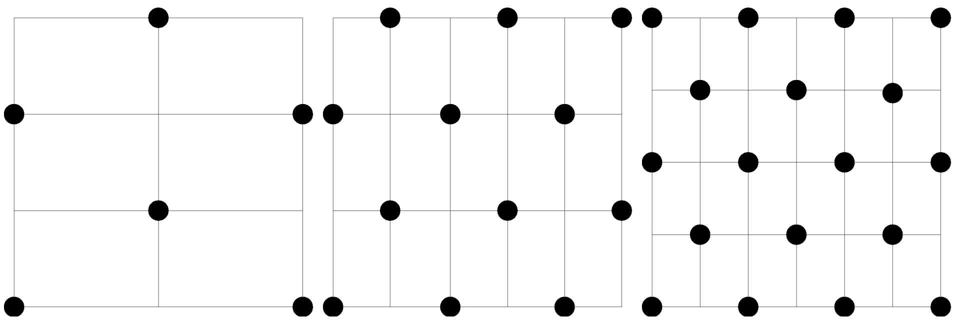

Figure 2. Examples (p,q,n) using PAT5, (2,3,6), (5,3,12) and (6,4,18).

and for all the cases $L_5(n) = \sqrt{\frac{1}{p^2} + \frac{1}{q^2}}$ holds for this lower bound.

The maximum of the derived lower bounds provides the statement of the theorem with the suggested substitutions, wherever necessary. ■

Further on, Table 1 will summarize the results obtained by applying the discussed patterns. With patterns PAT1 to PAT5 only 48 out of 99 values of n can directly

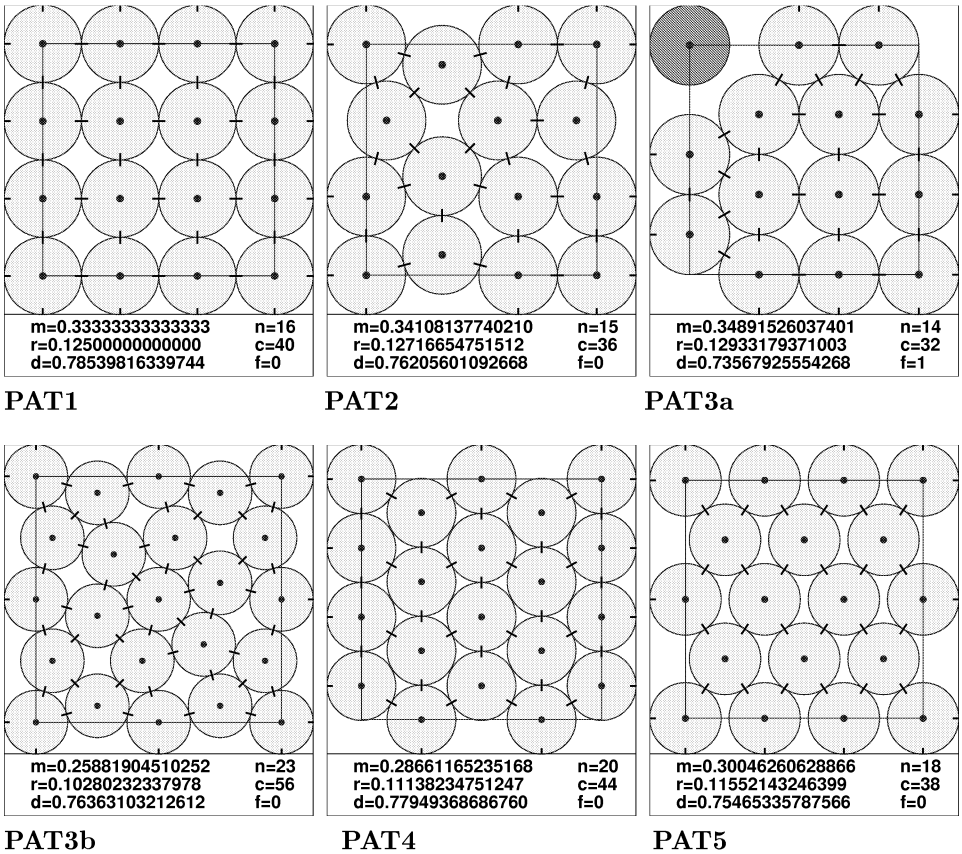

Figure 3. Examples of patterns for $n = 16$ (PAT1), $n = 15$ (PAT2), $n = 14$ (PAT3a), $n = 23$ (PAT3b), $n = 20$ (PAT4) and $n = 18$ (PAT5). These examples are optimal packings.

be modelled. In columns headers n, pattern and m_n^l denote the number of circles, the pattern used and the value of the lower bound of m_n, respectively. If there are two different patterns for the same value of n, then the better one has been chosen. In column n, boldface means that this packing is known as the optimal solution for the given packing circles in a square problem. All the values of m_n^l in Table 1 are those calculated by the respective formula of the specific pattern.

In [36], using a similar pattern as in PAT5, it was proved that if $N_p(\sigma)$ is the maximum number of points with mutual distance of at least 1, that can be placed into a square of side σ, then

$$\sigma^2 + \frac{1-\sqrt{3}}{2}\sigma \leq \frac{\sqrt{3}}{2}N_p(\sigma).$$

Resulting from this inequality we can provide the following lower bound of m_n:

$$\frac{1-\sqrt{3}+\sqrt{4-2\sqrt{3}+8\sqrt{3}n}}{2\sqrt{3}n} \le m_n \tag{3}$$

After an analysis of (3) and m_n^l, it can be seen that m_n^l is always greater than the lower bound given by (3) for $2 \le n \le 100$.

Table 1. Lower bounds of m_n obtained by using repeated patterns. The proven optimal solutions are denoted by boldface in columns n.

n	pattern	m_n^l	n	pattern	m_n^l	n	pattern	m_n^l
2	PAT5	1.414213	21	PAT5	0.260341	52	PAT5	0.165386
3	PAT2	1.035276	**23**	PAT3b	0.258819	56	PAT4	0.156156
4	PAT1	1.000000	**24**	PAT2	0.254333	59	PAT5	0.150231
5	PAT5	0.707106	**25**	PAT1	0.250000	63	PAT5	0.146772
6	PAT5	0.600925	**27**	PAT5	0.235849	64	PAT1	0.142857
7	PAT3a	0.535898	30	PAT4	0.224502	65	PAT5	0.138888
8	PAT2	0.517638	32	PAT5	0.208333	72	PAT4	0.135416
9	PAT1	0.500000	34	PAT3b	0.205604	75	PAT5	0.132089
10	PAT5	0.416666	35	PAT2	0.202763	80	PAT5	0.129576
12	PAT5	0.388730	**36**	PAT1	0.200000	81	PAT1	0.125000
13	PAT5	0.353553	39	PAT5	0.194365	83	PAT5	0.122890
14	PAT3a	0.348915	42	PAT4	0.184277	88	PAT5	0.120185
15	PAT2	0.341081	44	PAT5	0.174379	90	PAT4	0.119501
16	PAT1	0.333333	47	PAT3b	0.170540	94	PAT4	0.117924
18	PAT5	0.300462	48	PAT5	0.169329	99	PAT4	0.116018
20	PAT4	0.286611	49	PAT1	0.166666	100	PAT1	0.111111

3.2. An upper bound of $\boldsymbol{m_n}$

THEOREM 2 *Consider the packing circles in a unit square problem according to Problem 2. The maximum of the minimal distance between n points in the unit square m_n is not greater than*

$$m_n^u = \min(U_1(n), U_2(n))$$

with

$$U_1(n) = \frac{2}{\sqrt{n\pi + C_n(\sqrt{3}-\frac{\pi}{2}) + (4\lfloor\sqrt{n}\rfloor - 2)(2-\frac{\pi}{2}) - 2}}, \text{ where}$$

$$C_n = n-2 \quad \text{if } 3 \le n \le 6,$$
$$C_n = n-1 \quad \text{if } 7 \le n \le 9, \text{ and}$$
$$C_n = 3\left\lfloor\frac{n}{2}\right\rfloor - 5 + n \bmod 2, \text{ otherwise.}$$

$$U_2(n) = \frac{1+\sqrt{1+(n-1)\frac{2}{\sqrt{3}}}}{n-1}.$$

(Here $\lfloor . \rfloor$ denotes the greatest integer not greater than the argument.)

Proof: Let us describe an analytical expression for the upper bounds of the packing problem. It is based on the area of n circles and on the area of their neighbourhood within the unit square:

- U_1: The area of n circles ($nr^2\pi$) plus the gaps' area between circles and between circles and the sides of the square must be less than 1. The convex hull of the circles' centers divides the square into two regions. Let us compute now the area of the minimal gaps in these regions.

  Circles-to-side case. When packing n circles in a square, it is evident that the minimum gap between circles and the sides of the square can be gained when $4\lfloor\sqrt{n}\rfloor - 2$ gaps (with the respective corrections at the corners) are in contact with the sides of the square of $r_n^2(2-\pi/2)$ areas per gaps (see Figure 4).

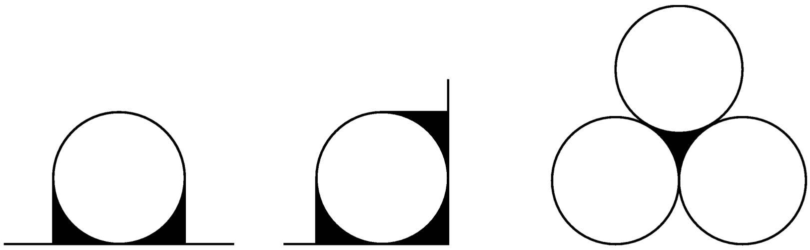

Figure 4. The minimal gaps between a circle and the sides of a square and among three circles.

  Circles-to-circles case. In a similar way, the minimal gap between circles occurs when there are three circles in the packing which have all the possible three contacts, as it is shown in Figure 4. For a packing of n circles the minimal number of circle-to-circle gaps is given by:

$$C_n = \begin{cases} n-2 & 3 \le n \le 6 \\ n-1 & 7 \le n \le 9 \\ 3\left\lfloor \frac{n}{2} \right\rfloor - 5 + n \bmod 2, & \text{otherwise} \end{cases}$$

  when the circles are in the densest (hexagonal) arrangement. The area of this minimum gap between three circles is $r_n^2(\sqrt{3}-\frac{\pi}{2})$, thus the sum of the area of the circles and gaps will be:

$$S = r_n^2\left(n\pi + C_n(\sqrt{3}-\frac{\pi}{2}) + (4\lfloor\sqrt{n}\rfloor - 2)(2-\frac{\pi}{2})\right).$$

  Due to $S \le 1$,

$$r_n^u \le \left(n\pi + C_n(\sqrt{3}-\frac{\pi}{2}) + (4\lfloor\sqrt{n}\rfloor - 2)(2-\frac{\pi}{2})\right)^{-\frac{1}{2}},$$

$$m_n^u \le U_1(n) = \frac{2}{\sqrt{n\pi + C_n(\sqrt{3} - \frac{\pi}{2}) + (4\lfloor\sqrt{n}\rfloor - 2)(2 - \frac{\pi}{2}) - 2}}.$$

- U_2: From Oler theorem [37] the following statement implies [10]: if X is a compact convex subset of the plane, then the number of points with a mutual distance of at least 1 can be at most

  $$\frac{2}{\sqrt{3}}A(X) + \frac{1}{2}P(X) + 1, \tag{4}$$

  where $A(X)$ is the area and $P(X)$ is the perimeter of X. If X is a σ side of square, then $A(X) = \sigma^2$ and $P(X) = 4\sigma$.
  Based on (4) and (1), we can conclude that

  $$m_n \le U_2(n) = \frac{1 + \sqrt{1 + (n-1)\frac{2}{\sqrt{3}}}}{n-1}.$$

■

By using the values of the lower and upper bounds of m_n, described beforehand and our own numerical results, obtained by the algorithm described in [1], Figure 5 has been built.

4. Minimal polynomials of optimal packings and exact results

Actually, we know the exact values for the solution of the packing circles in a unit square problem for several values of n. The set of known exact values is given in Table 2.

Table 2. Exact values of m_n for some n.

n	m_n	n	m_n
2	$\sqrt{2}$	15	$(1+\sqrt{2}-\sqrt{3})/2$
3	$\sqrt{6}-\sqrt{2}$	16	$1/3$
4	1	18	$\sqrt{13}/12$
5	$\sqrt{2}/2$	20	$(6-\sqrt{2})/16$
6	$\sqrt{13}/6$	23	$(\sqrt{6}-\sqrt{2})/4$
7	$2(2-\sqrt{3})$	24	$4+2\sqrt{3}-\sqrt{26+15\sqrt{3}}$
8	$(\sqrt{6}-\sqrt{2})/2$	25	$1/4$
9	$1/2$	27	$\sqrt{89}/40$
12	$\sqrt{34}/15$	36	$1/5$
14	$2(4-\sqrt{3})/13$		

In case we are familiar with the optimal solution for the problem of packing n circles in a square, the definition of the corresponding minimal polynomial may be

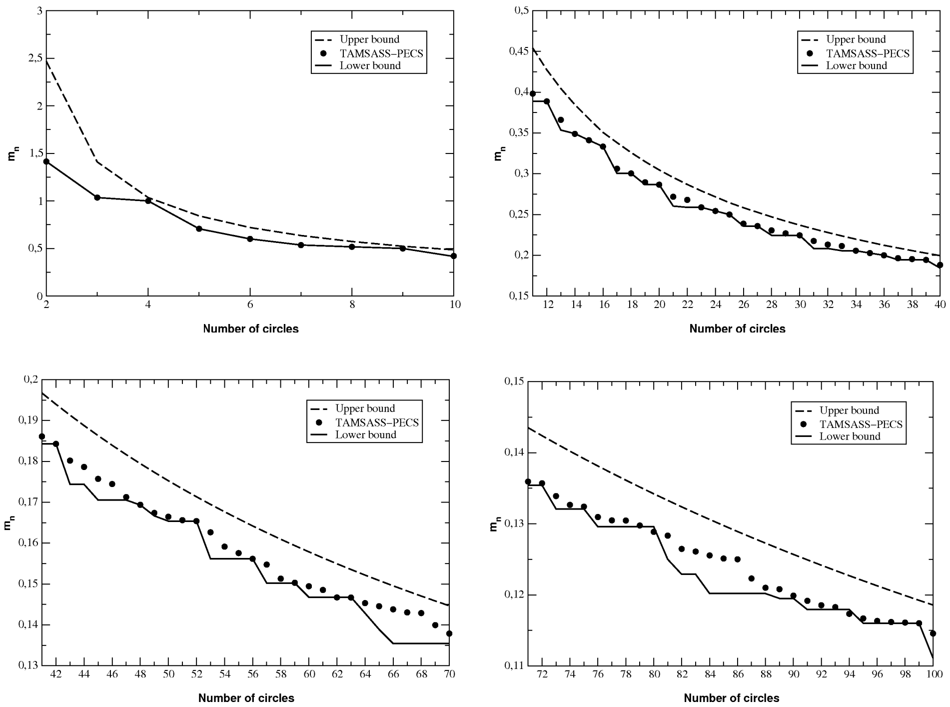

Figure 5. Lower (m_n^l) and upper bound (m_n^u) on the m_n values and the best m_n values found by our program [1] (denoted by circles) are displayed.

given. The procedure to obtain this minimal polynomial involves the generation of a system of equations representing the corresponding optimal packing. The variables of the system of equations are the coordinates of the centers of the circles and the maximum minimal distance, m_n. So the number of variables for the system is $2n + 1$. For every two points (centers of the circles) of which distance is equal to m_n, an equation can be written. In addition, it is also well-known that every point located at the border of the square has got at least one coordinate which is equal to 0 or 1, so the corresponding equation(s) can be written. After reducing the system of equations, a symbolic algebra package e.g. MAPLE, MATHEMATICA, etc., can be used to calculate the Gröbner basis of the system [40] and the minimal polynomial for which the smallest positive root is m_n.

The minimal polynomial can be obtained for the most cases, but e.g. for $n = 13$ this method was unable [40] to provide the minimal polynomial. In Table 3 the minimal polynomials for several values of n are shown [17, 39]. The degrees of these polynomials indicate how difficult it might be to find the relevant solution of the packing problem. When the degree of the minimal polynomial became less than five, we were able to calculate the exact value of m_n, which values can be found in Table 2. For $n = 10$, 11, 17 and 19, we were able to provide close bounds of m_n

Table 3. The minimal polynomials of certain optimal solutions. Here * denotes cases with unknown minimal polynomials.

n	Minimal polynomials
2	$m^2 - 2$
3	$m^4 - 16m^2 + 16$
4	$m - 1$
5	$2m^2 - 1$
6	$36m^2 - 13$
7	$m^2 - 8m + 4$
8	$m^4 - 4m^2 + 1$
9	$2m - 1$
10	$1180129m^{18} - 11436428m^{17} + 98015844m^{16} - 462103584m^{15} + 1145811528m^{14} +$ $-1398966480m^{13} + 227573920m^{12} + 1526909568m^{11} - 1038261808m^{10} - 2960321792m^9 +$ $+7803109440m^8 - 9722063488m^7 + 7918461504m^6 - 4564076288m^5 + 1899131648m^4 -$ $-563649536m^3 + 114038784m^2 - 14172160m + 819200$
11	$m^8 + 8m^7 - 22m^6 + 20m^5 + 18m^4 - 24m^3 - 24m^2 + 32m - 8$
12	$225m^2 - 34$
13	$5322808420171924937409m^{40} + 5867739593380498861732 32m^{39} +$ $+1302444884533227120326 6928m^{38} - 1298840956705690999017 0432m^{37} + ... +$ $+29600757197947367587 84m^2 - 1741035320946091622 40m + 4756927106410086400$
14	$13m^2 - 16m + 4$
15	$2m^4 - 4m^3 - 2m^2 + 4m - 1$
16	$3m - 1$
17	$m^8 - 4m^7 + 6m^6 - 14m^5 + 22m^4 - 20m^3 + 36m^2 - 26m + 5$
18	$-144m^2 + 13$
19	$242m^{10} - 1430m^9 - 8109m^8 + 58704m^7 - 78452m^6 - 2918m^5 + 43315m^4 + 39812m^3 -$ $-53516m^2 + 20592m - 2704$
20	$128m^2 - 96m + 17$
21	*
22	*
23	$16m^4 - 16m^2 + 1$
24	$m^4 - 16m^3 + 20m^2 - 8m + 1$
25	$4m - 1$
26	*
27	$1600m^2 - 89$
36	$5m - 1$

using a reliable algorithm for finding the first root of a function [2]. The mentioned algorithm is based on a branch and bounds technique with interval arithmetic. The numerical results for these four values of n are given in interval form in Table 4.

Table 4. Reliable numerical results for $n = 10, 11, 17, 19$.

n	Lower and upper bounds for m_n
10	[0.421279541378085, 0.421279544064904]
11	[0.398207310236837, 0.398207310236850]
17	[0.306153985300327, 0.306153985300338]
19	[0.289541991994965, 0.289541991994996]

5. Summary

First of all, in this series of papers a short introduction has been given to the problem of equal circles packing in a unit square. Then, in a nutshell the most important results have been summarized. Finally, new and improved lower and upper bounds on the maximal distance of the centers of the circles are demonstrated. These bounds are based on geometrical considerations and on an extended set of patterns of the related circles, moreover, they can be applied directly in some computational procedures for finding the best packings. Last but not least, they can also be well-utilised in the evaluation of the approximate solutions provided by our numerical algorithm discussed in the second paper of this series. The known exact optimal values are comprised in a table together with the related minimal polynomials for some cases.

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