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DETERMINING THE 95% CONFIDENCE INTERVAL OF ARBITRARY NON-GAUSSIAN PROBABILITY DISTRIBUTIONS

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Abstract − Measurements are nowadays permanent attendants of scientific research, health, medical care and treatment, industrial development, safety and even global economy. All of them depend on accurate measurements and tests, and many of these fields are under the legal metrology because of their severity. How the measurement is accurate, is expressed by uncertainty, which is obtained by multiplication of standard deviations by coverage factors to increase trustfulness in the measured results. These coverage factors depend on degree of freedom, which is the function of the number of implied repetitions of measurements, and therefore the reliability of the results is increased. The standard coverage factor is 1.96 for normal (Gaussian) distributions or near-Gaussian distributions, and the obtained expanded uncertainty has the 95% statistical probability. In general, it is not possible to achieve the 95% confidence interval by using the standard coverage factor 1.96, nevertheless of the degree of freedom. The present paper describes the method of estimating the expanded uncertainty by an algorithm of this model based on the 95% confidence interval of any probability distribution of any shape, dealing with the A-type or the B-type uncertainty. Furthermore the coverage factor is determined due to the 95% confidence interval of the actual probability distribution. The algorithm successfully copes with adding two or more uncertainties with mathematical properties of sums, and is established in accordance with the standards and guides. The model is introduced in procedures carried out in the calibration laboratory.

Keywords: coverage factor, distribution shape coefficient, uncertainty.

1. BASIC INFORMATION

The expanded uncertainty is calculated according to the standard EA-4/02 [1] as the multiplication of the standard coverage factor 1.96 with the standard uncertainty when the degree of freedom is approaching to infinite value. This standard also points out the necessity to determine the confidence interval of the 95% coverage probability of any distribution although it is not normal or Gaussian and furthermore it can be far from being normal that is being non-Gaussian. To see the problem, we must be aware of dealing with several kinds of distributions not only with the normal distribution. Namely, the distributions of the B-type uncertainties are mostly not normal, for instance the temperature drift, the time drift, and the resolution, all have the rectangular distribution. Sometimes also the A-type uncertainties do not match with the normal distribution as they are result of regulated quantities, or are affected by the resolution of measuring device. In the latter case the probability distribution consists of two Dirac functions, one at the lower measurement reading and the other at the upper measurement reading regarding the resolution. If the distribution is unknown the coverage factor is calculated from Chebyshev's inequality [2] and is 4.472 for the 95% confidence interval. The uncertainty, the extended or the standard one, does not stand by itself, but contributes its portion to a combined uncertainty. The probability distribution, which corresponds to the combined uncertainty, is the convolution of the contributing probability distributions. The convolution of two rectangular distributions gives the trapezoidal distribution, or in some cases the triangular distribution, and the next rectangular distribution convoluted to the trapezoidal or to the triangular distribution results in the distribution with the square dependent tails, and the further rectangular distributions lead to the distribution with the polynomial dependent tails – an arrow-point shape, as defined in this paper. By further convolutions, the resulted distribution tends toward the normal distribution. Nevertheless, there are some not "well-behaved probability distributions" as quoted by the standard [1] to cause the coverage probability of less than 95% by using standard coverage factor. Hence, very often we are to deal with distributions with the large probability around the mean values of measured quantity with some excessive, but still reliable values. The tails of such distributions are fatter than the tails of the normal distribution and the coverage factor must be greater than 1.96 to achieve the 95% confidence interval.

There are several sources of uncertainties with the rectangular probability distributions, and when combined the resulted probability is either trapezoidal, triangular. The rectangular, trapezoidal and triangular probability are discussed in the standard EA-4/02 [1], and further on U-shaped distribution is dealt in NIS3003 [3] used with sinus wave measuring signal, but the other distributions and further convolutions of these distributions are rarely described in literature. There is an algorithm of combining the normal distributions and the rectangular distributions only, described in the literature [4]. The symmetrical impulse measuring signal is very common in measuring systems, for instance the measurement of the contact resistance, the temperature measurement of the resistance temperature sensors with the DC current and several measurements where the influence of hysteresis is being avoided. This measuring signal gives the symmetrical Dirac shaped distribution.

2. THE CONTRIBUTING DISTRIBUTIONS

The measured signal is continuous function depended on one independent variable, such as time or a counter. Its range has the supremum and the infimum, which are the bounds of the domain of definition of the corresponding distribution. The upper and the lower bounds are finite values. Amplitude A is defined as the maximum of the absolute values of the difference between the mean value of the measured signal throughout its whole definition interval and the lower and the upper bounds respectively. The corresponding kind of distributions follows by the following equation:

$$\int_{-\infty}^{+\infty} p(X)\cdot dX = \int_{-A}^{+A} p(X)\cdot dX = 1 \tag{1}$$

We are looking for the coverage factor $\hat{K}$ of any distribution, applied to the standard deviation, which gives the 95% confidence interval as follows:

$$\int_{-\hat{K}\cdot\sigma}^{+\hat{K}\cdot\sigma} p(X)\cdot dX \geq 0.95 \tag{2}$$

at the infinite degree of freedom.
There are upper limits of this coverage factor as follows:

- any, even unknown distribution corresponds to Chebyshev's inequality [2], therefore:

$$\hat{K} \leq \left. \sqrt{\frac{1}{1-P}} \right|_{P=0.95} = \sqrt{20} \tag{3}$$

- any, even unknown distribution with the finite upper and lower bounds corresponds to (1), therefore according to (1) and (2), the upper limit of the coverage factor is:

$$\hat{K} \leq \frac{A}{\sigma} \tag{4}$$

where the equality is present only, when the cover factor of the 100% confidence level is considered.

The coverage factor of each known distribution is calculated by using the (2). The results of the calculation for the dealt distributions are presented in Fig. 1 as two functions: the first one is the function (C), which is calculated for the 95% probability level by (2), and the second one is the function (A) for the 100% probability level calculated by (1) and (4). The upper limits are also shown in this figure: the maximum of the function (C) for the 95% probability level of an unknown distribution due to Chebyshev's inequality due to (3). The functions (A) and (C) in Fig. 1 make the boundary of the area of the probability levels from 95% up to 100%. There is also the position of Gaussian distribution marked in Fig. 1.

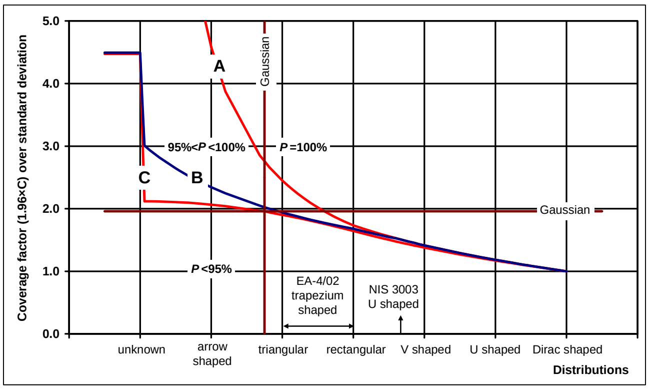

Fig. 1: The cover factors of several distributions and statistical confidence of the acquired data interval with the following legend:
A ... the cover factors of the 100% confidence level determined by (1) and (4),
B ... the cover factors of the 95% confidence level using the presented model and so far calculated by (9),
C ... the cover factors of the 95% confidence level calculated by integration (2).

There is no problem to determine the coverage factor of the probability distribution defined by the probability density $p(X)$ described by the analytic function, by the tabled values or by the geometrical definition. But, determining the coverage factor out of statistically acquired data, we have to establish the model, which solution gives results within the mentioned area with the probability levels from 95% up to 100%. The solution given by the presented model is one of many possible solutions, and it is shown in Fig. 1 as the function (B).

3. THE KURTOSIS AND MODELLING THE COVERAGE FACTOR

The kurtosis is the parameter of the descriptive statistics, which gives information about the probability distributions of acquired data that are created by our measurements. It is the classical measure of non-gaussianity, and it expresses the similarity to the normal or Gaussian distribution. The distribution shape is quantified by it, but the mapping of the set of the shapes to the set of their kurtosis numerical values is surjection. There is no rule to get the distribution shape out of the kurtosis. From the kurtosis, it can be concluded only that:

- a certain distribution is peaked around its mean and have the fat tails – leptokurtic distributions;
- it is flat (could be the rectangular distribution) or even concave (could be the U-shaped distribution) with the thin tails or without them – platykurtic distributions;
- it could be very similar to the normal distribution – mesokurtic distributions.

The kurtosis is the fourth standardized moment k_4 about the mean, and is defined as the quotient of the fourth moment m_4 about the mean and the fourth power of the standard deviation σ, and it is:

$$k_4 = \frac{m_4}{\sigma^4} \quad (5)$$

The kurtosis of the normal distribution is taken as 2.45, and the skewness of the normal distribution or any symmetrical distributions is zero.

The basis of the present modelling of the coverage factor of the 95% confidence interval is the kurtosis, because it is the statistical parameter that quantifies the shape of the analysed probability distribution. The coverage factor, including all contributed coefficients of the presented model, is basically the square root of the ratio between the kurtosis k_4 of the analyzed distribution and the kurtosis of the normal distribution, corrected by the empirical coefficient κ and multiplied with the coverage factor of the normal distribution at the infinite degree of freedom:

$$\hat{K}_{basic} = 1.96 \cdot \left(\kappa \cdot \sqrt{\frac{k_4}{2.45}} \right) \quad (6)$$

where the empirical coefficient κ is a function of the absolute value of skewness k_3, which is determined empirically:

$$\begin{aligned} \kappa = 1 &\Leftarrow |k_3| \le \frac{1}{3} \\ \kappa > 1 &\Leftarrow |k_3| > \frac{1}{3} \end{aligned} \quad (7)$$

The empirical coefficient is necessary to correct the basic coverage factors of skewed distributions with the finite bounds of their domain. This is not the case with the normal distribution or near-normal distributions or any approximately symmetrical distribution, where this empirical coefficient is unit.

Due to (6) the coverage factor of the probability distribution with the finite bounds of its domain actually is:

$$\hat{K} = K(\nu) \cdot C \quad (8)$$

and further on taking into account the upper limits of the coverage factor due to (3) and (4) the ***shape coefficient C*** of the probability distribution is:

$$C = \min\left(\kappa \cdot \sqrt{\frac{m_4}{2.45 \cdot \sigma^4}} \ , \ \frac{A}{1.96 \cdot \sigma} \ , \ \frac{\sqrt{20}}{1.96} \right) \quad (9)$$

This coverage factor at the infinite degree of freedom is graphically shown as the function (B) in Fig. 1. Its values are in the lower range of the area indicating the confidence interval from 95% up to 100%, which is very good. The advantage of this coverage factor is, that it consists of two multiplicands as in (8): the first one – $K(\nu)$ is dependent on degree of freedom, as it is generally known as the coverage factor, and the other - C depends on the shape of the probability distribution, mainly on the kurtosis and we named it as the shape coefficient.

So far, the shape coefficient is defined, but if it should be useful, a mathematic operation(s) between several shape coefficients must be established.

4. THE CONVOLUTION AND THE ADDITION ALGORITHM

When combining several probability distributions in uncertainty calculations the resulted probability distribution is the convolution of all participant distributions. The convolution of two probability distributions is:

$$\begin{aligned} p_{12}(X) &= p_1(X) \otimes p_2(X) = \\ &= \int_{-\infty}^{+\infty} p_1(X - \tau) \cdot p_2(\tau) \cdot d\tau \end{aligned} \quad (10)$$

and further on for the N convoluted distribution written just symbolically as:

$$p_\Sigma(X) = p_1(X) \otimes \ldots \otimes p_i(X) \otimes \ldots \otimes p_N(X) \quad (11)$$

Each distribution contributes the amplitude A_i, the standard deviation σ_i about the mean and the shape coefficient C_i to the resulting shape coefficient C_Σ that is obtained by the following ***addition algorithm***:

$$C_\Sigma = \min\left(\sqrt{\frac{\sum_{i=1}^{N} C_i^2 \cdot \sigma_i^4 + 2.45 \cdot \sum_{i=1}^{N-1} \left(\kappa_i \cdot \sigma_i^2 \cdot \sum_{j=i+1}^{N} \kappa_j \cdot \sigma_j^2 \right)}{\left(\sum_{i=1}^{N} \sigma_i^2 \right)^2}} \ , \ \frac{\sum_{i=1}^{N} A_i}{1.96 \cdot \sqrt{\sum_{i=1}^{N} \sigma_i^2}} \ , \ \frac{\sqrt{20}}{1.96} \right) \quad (12)$$

The addition of the shape coefficients is commutative and associative and the resulted shape coefficient is a member of the same set of values as the participant shape coefficients in the evaluating process, which all are the necessary mathematic conditions for the applied methods of determining the combined uncertainties as it is prescribed by the standard [7] as universality, internal consistency and transferability.

The expanded uncertainty is obtained by multiplying the standard deviation or the combined uncertainty, which is appropriate, by the coverage factor [1] and the shape coefficient, so that the expanded uncertainty is estimated to have at least the ***95% confidence level***:

$$\begin{aligned} U\big|_{P=95\%} &= K(\nu) \cdot C \cdot \sigma \\ U_c\big|_{P=95\%} &= K(\nu_{eff}) \cdot C_\Sigma \cdot u_c \end{aligned} \quad (13)$$

therefore the expanded uncertainty intervals $\pm U$ or $\pm U_c$ about the mean of the measured or calculated results are the 95% confidence interval.

5. THE CASE STUDY

Although the presented method was used in the calibration laboratory and it was considered as very effective, the authors decide to demonstrate the usage of this method in a very specialized scientific field of electron impacts to gas molecules and to steady material such as an electrode, and further on the voltages needed for ionization by these two kinds of the impacts were determined probabilistically with the appropriate expanded uncertainties and further on the resultant expanded uncertainty. Nevertheless, what the scientific problem was, we have two probability density distributions named as the first and the second probability distribution in Fig. 2. The domains of these two distributions were to be added together, hence these two distributions were convoluted and a resultant distribution, also in Fig. 2, is named as the convolution probability distribution.

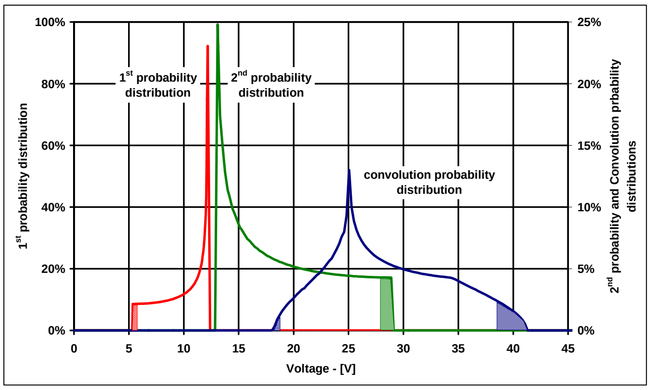

Fig. 2: The 1st, 2nd and convolution probability densities with the intervals oudside 95% conidence interval.

For the first and the second distributions, and further on, as their convolution is practically carried out, also for the convolution distribution, the shape coefficients $\boldsymbol{C}$ are calculated by (9) and the uncertainty intervals on the basis $U = 1.96{\cdot}C{\cdot}\sigma$ are established for all three distributions, and the confidence levels of these uncertainty intervals are obtained by the integration of these probability density distributions throughout the uncertainty intervals. The results are shown in the second, third and fourth column of Table 1.

Table 1: Comparison of results achieved by convolution and by addition algorithm.

	1st distribution	2nd distribution	convolution	addition algorithm
C	**0.948**	**0.918**	**0.945**	**0.993**
confidence of $\pm U$	**96.18%**	**96.12%**	**95.41%**	**96.93%**
mean $-1.96{\cdot}C{\cdot}\sigma$	5.754	10.496	18.672	18.597
mean	9.663	19.241	28.631	28.905
mean $+1.96{\cdot}C{\cdot}\sigma$	13.573	27.987	38.591	39.212
minimum in view of $\pm U$	outside	inside	outside	outside
maximum in view of $\pm U$	inside	outside	outside	outside

Further on, the shape coefficient $\boldsymbol{C}_\Sigma$ is determined by addition algorithm (12) of the presented method and also the uncertainty interval $\pm \boldsymbol{U}_c$ defined by (13) on the basis of $K(\nu_{eff}) = 1.96$ are recalculated for the convolution probability distribution. The confidence level of this uncertainty interval is determined too. These data, which are the results of the addition algorithm, are presented in the fifth column of Table 1.

So far, the fourth and the fifth columns of Table 1 represent the data of the same distribution, namely of the convolution probability distribution, and hence are comparable, although the data in the fourth column are obtained by the convolution integral and the data of the fifth column by the addition algorithm. There is quite a large difference between the shape coefficients, but it is not significant in this case to the combined uncertainty and its confidence level. As we see, the confidence levels attained by the convolution and by the addition algorithm match very well, but the latter one is a little bit greater, but it belongs also to the wider uncertainty interval. In Fig. 2 are seen the intervals, which are outside the uncertainty intervals with the confidence levels for the fist, the second and the convolution distributions, as stated in Table 1. But the positions of minima and maxima of these distributions domains in view of the uncertainty intervals are described also in Table 1.

The uncertainty intervals of the 95% confidence level is established by presented method with the shape coefficients, obtained by kurtosis, which is the fourth standardized moment about the mean, without calculating the probability integrals as in (2), or previously calculating the convolution integral of the participatory probability distributions as in (10) when the sum of the uncertainties are required.

6. CONCLUSIONS

The presented method of evaluating the expanded uncertainty of the measurand on the basis of the 95% confidence interval has the following features:

- it determinines uncertainty intervals of the 95% confidence level of arbitrary non-Gausian probability distributions without resolving its borders by the integration of their densities or calculating convolution integrals when adding them together;
- it is universal [7], because this algorithm is applicable to all kinds of measurements, to the A and B-type of the uncertainty evaluation and to all type of input data distribution;
- it is internally consistent [7], which mathematically means being commutative and associative, so that combined uncertainty is independent of grouping and decomposing the contributing components;
- it is transferable [7], which mathematically means that the resulted shape coefficient and the participant shape coefficients are the members of the same set of values or are fitting the same function, so the one result can be directly used as a component in evaluating the uncertainty of another measuring process;
- the convolution of many normal distributions gives normal distribution and so does the resulted shape coefficient; further on, the convolution of great number of whatever distributions leads to mesokurtic distribution and even to normal distribution and so also does the resulted shape coefficient, hence the central limit theorem is met by this method [1];

- the expanded uncertainty depends on its effective degrees of freedom as in (13) so that the proper reliability is achieved [1];
- the expanded uncertainty estimated by this method, as in (13), takes into account the effective degree of freedom of the output estimates and the non-normality or non-gaussianity of the probability distributions and so far meets regulations [1] about the 95% confidence interval.

REFERENCES

[1] *EA-4/02: Expression of the Uncertainty of Measurement in Calibration*, European co-operation for Accreditation, Publication Reference, December 1999.

[2] G. A. Korn, T. M. Korn: *Mathematical Handbook for scientists and engineers*, McGraw-Hill Inc., 2nd edition, 1968.

[3] *NIS3003: The expression of Uncertainty and Confidence in Measurement for Calibrations*, NAMAS Executive, National Physical Laboratory, Teddington, Middlesex, TW110LW, England, Edition 8, May 1995.

[4] M. J. Korczynski, A. Hetman: *Direct method of coverage interval calculation: advantages and disadvantages*, International Workshop: From Data Acquisition to Data Processing and Retrieval, Republic of Slovenia, Ministry of Education, Science and Sport, Metrology Institute and European Thematic Network, Advanced Mathematical and Computational Tools in Metrology, Ljubljana, Sep. 13-15, 2004.

[5] F. Pavlovčič, D. Groselj: *An establishment and a propagation of a measuring uncertainty of a medium by processing Gaussian random signal* (in Slovenian); V: ZAJC, Baldomir (ed.). *Zbornik šeste Elektrotehniške in računalniške konference ERK '97, 25. - 27. september 1997, Portorož, Slovenija*. Ljubljana: IEEE Region 8, IEEE Slovenian section, 1997, pp. 415-418.

[6] F. Pavlovčič: *A dynamic modelling of measuring uncertainty of industrial and environmental media using a signal processing technique*; In: ILIĆ, Damir (ed.), BORŠIĆ, Mladen (ed.), BUTORAC, Josip (ed.). XVII IMEKO World Congress, June 22-27, 2003, Dubrovnik, Croatia. *Metrology in the 3rd millennium: proceedings*. Zagreb: HMD - Croatian Metrology Society, 2003, p. 1663.

[7] *Guide to the expression of uncertainty in measurement*, International Organisation for Standardisation, 1st edition, 1995.