ON PARTLY ORDERED PARTITIONS OF A POSITIVE INTEGER

C. C. CADOGAN
University of Waterloo, Waterloo, Ontario, Canada

1. INTRODUCTION

The following problem is discussed. Let

$$V_1 = (n, \underbrace{0, \cdots, 0}_{n-1}),$$

where n is a finite positive integer. From V_1 are generated

$$V_{i+1} = (n - i, i, \underbrace{0, \cdots, 0}_{n-2}), \qquad 1 \le i < n .$$

From V_2 are generated

$$V_{n+j} = (n - 1 - j, 1, j, \underbrace{0, \cdots, 0}_{n-3}), \qquad 1 \le j < n - 1 ,$$

and so on, until the entire list of non-null vectors V_i has been considered.

Suppose the first k $(0 \le k \le n)$ components from left to right in each vector V_i are fixed, with $k = 0$ meaning that none is fixed, and the remaining components are arranged from left to right in descending order of magnitude. The positive integers in each vector V_i form a partition of n and on arranging the components as above, we obtain what we define as <u>partly ordered partitions</u> of the integer n.

Let $\phi_k(n)$ denote the number of distinct non-null vectors V_i in the system generated above in which the first k components are kept fixed. The primary object of this paper is to derive a recurrence relation for $\phi_k(n)$. Several other interesting results are obtained.

2. IMMEDIATE RESULTS

Let p(n) denote the number of distinct partitions of the positive integer n. Several values of p(n) can be found in [1], page 35.

---

*This paper was written while the author was on an N.R.C. postdoctoral fellowship at the University of Waterloo.

Let V_i' be the vector obtained from V_i $(i = 1, 2, \cdots)$ by removing all zero components of V_i and let $[V]$, $[V']$ denote the set of non-null vectors V_i, V_i', respectively. There is a one-one correspondence between V_i and V_i' and hence between $[V]$, $[V']$. We have,

<u>Theorem 1.</u> $\phi_0(n) = p(n)$.

<u>Proof.</u> The components of V_i' constitute a partition of n. Suppose the components of each vector in $[V']$ are arranged from left to right in descending order of magnitude. Then each V_j' $(j \neq i)$ which has the same components as V_i' after rearrangement, hence the distinct vectors in $[V']$ are those vectors V_i' whose components are distinct partitions of n, hence

$$\phi_0(n) = p(n) .$$

<u>Theorem 2.</u> $\phi_k(n) = 2^{n-1}$, $k = n$ or $n - 1$, $(n \geq 1)$.

<u>Proof.</u> We show first that $\phi_{n-1}(n) = \phi_n(n)$.

$$V_i' = (1, \underbrace{1, \cdots,}_{n} 1)$$

is the only vector in $[V']$ which has more than $n - 1$ components, hence keeping $n - 1$ components fixed in $[V']$ is equivalent to keeping all n components fixed; that is,

$$\phi_{n-1}(n) = \phi_n(n) .$$

Now the system $[V']$ contains all the compositions of the integer n, hence by a result of [2, page 124], $\phi_n(n) = 2^{n-1}$.

This proves the theorem.

We come now to the more significant results.

3. MAIN RESULTS

<u>Theorem 3.</u> $\phi_k(n) = \phi_k(n-1) + \phi_{k-1}(n - 1)$, $(k \geq 1)$.

<u>Proof.</u> $\phi_k(n)$ is obtained from $\phi_{k-1}(r)$, $1 \leq r \leq n - 1$, in the following way:

Let $[U]$ be the system of distinct non-null vectors generated for a particular value of r $(1 < r \leq n - 1)$ in which the first $k - 1$ components in each vector are fixed and the other components are arranged in descending order. Let

$$U = (u_1, u_2, \cdots, u_r) \qquad [U] .$$

Define

$$U' = (n - r, u_1, u_2, \cdots, u_r) .$$

There is a one-one correspondence between U, U' and as U runs through the vectors in [U] we obtain a system of distinct non-null vectors in which the non-zero components sum to n and the first k components are fixed. As r runs through all integral values from 1 to n - 1 we obtain collectively all the distinct non-null vectors in $\phi_k(n)$ except

$$V = (n, \underbrace{0, 0, \cdots}_{n-1}, 0) ,$$

hence,

$$\begin{aligned} \phi_k(n) &= 1 + \sum_{r=1}^{n-1} \phi_{k-1}(r) , \\ &= \left(1 + \sum_{r=1}^{n+2} \phi_{k-1}(r)\right) + \phi_{k-1}(n - 1) , \\ &= \phi_k(n - 1) + \phi_{k-1}(n - 1) . \end{aligned}$$

Using this result and the values for $\phi_0(n)$ which are to be taken as initial values we obtain Table 1 for $1 \le n \le 10$. We take $\phi_0(0) = 0$, and for $k > n$ and finite we may also put $\phi_k(n) = \phi_n(n)$ since this simply entails expanding the vectors in [V] by adding a further $k - n$ zero components on the right in each vector. These values of $\phi_k(n)$ fall below the leading diagonal in the table and are omitted.

We note also that the binomial coefficients also satisfy a similar recurrence relation.

Table 1

n	0	1	2	3	4	5	6	7	8	9	10
ϕ_0	0	1	2	3	5	7	11	15	22	30	42
ϕ_1		1	2	4	7	12	18	30	45	67	97
ϕ_2			2	4	8	15	27	46	76	121	188
ϕ_3				4	8	16	31	58	104	180	301
ϕ_4					8	16	32	63	121	225	405
ϕ_5						16	32	64	127	248	473
ϕ_6							32	64	128	255	503
ϕ_7								64	128	256	511
ϕ_8									128	256	512

Here ϕ_i stands for $\phi_i(n)$ $(0 \le i \le 8)$.

<u>Corollary 1.</u> $\phi_{n-2}(n) = 2^{n-1}, \quad (n \ge 2)$.

<u>Proof.</u> By Theorem 3,

$$\sum_{s=0}^{n-3} (\phi_{n-2-s}(n-s) - \phi_{n-3-s}(n-s-1)) = \sum_{s=0}^{n-3} \phi_{n-2-s}(n-s-1),$$

that is,

$$\phi_{n-2}(n) - \phi_0(2) = \sum_{s=1}^{n-2} 2^s,$$

by Theorem 2, hence,

$$\phi_{n-2}(n) = 2(2^{n-2} - 1) + \phi_0(2),$$

$$= 2^{n-1} \quad .$$

The following result can also be obtained by using similar difference methods.

Corollary 2. $\qquad \phi_{n-3}(n) = 2^{n-1} - 1, \quad n \geq 3\,.$

Before we state a general expression for $\phi_{n-j}(n)$, $3 \leq j \leq n-1$, we prove the following lemmas.

Lemma 1.

$$\sum_{r=0}^{n-j-1} \binom{j-3+r}{r} = \binom{n-3}{n-j-1}, \quad 3 \leq j \leq n-1, \quad n \geq 4\,.$$

Proof.

$$\sum_{r=0}^{n-j-1} \binom{n-3+r}{r} = \sum_{r=1}^{n-j-1} \left[\binom{j-2+r}{r} - \binom{j-3+r}{r-1}\right] + 1$$

$$= \binom{n-j}{n-j-1} = 1 + 1$$

$$= \binom{n-3}{n-j-1} \quad .$$

Lemma 2.

$$\sum_{r=0}^{q-2} \binom{p+r}{r} 2^{q-r} = \sum_{r=0}^{q-3} \binom{p+r+1}{r} 2^{q-r-1} + 4\binom{p+q-1}{q-2}, \quad q \geq 2\,.$$

Proof.

$$\sum_{r=0}^{q-2} \binom{p+r}{r} 2^{q-r} = \binom{p+1}{0} 2^{q-1} + \left[\binom{p+1}{0} + \binom{p+1}{1}\right] 2^{q-1}$$

$$+ \sum_{r=2}^{q-2} \binom{p+r}{r} 2^{q-r}\,,$$

$$= \binom{p+1}{0} 2^{q-1} + \binom{p+2}{1} 2^{q-2} + \left[\binom{p+2}{1} + \binom{p+2}{2}\right] 2^{q-2}$$

$$+\sum_{r=3}^{q-2}\binom{p+r}{r}2^{q-r} \quad ,$$

$$\vdots$$

$$= \sum_{r=0}^{q-3}\binom{p+r+1}{r}2^{q-r-1} + 4\binom{p+q-1}{q-2} \quad .$$

<u>Theorem 4.</u>

$$\phi_{n-j}(n) = \sum_{r=0}^{n-j-1}\binom{j-3+r}{r}2^{n-j-r+1} + \sum_{r=3}^{j}\binom{n-r-1}{j-r}\phi_0(r) \quad ,$$

$$3 \le j \le n-1, \quad n \ge 4 .$$

<u>Proof.</u> When $j = 3$, the right-hand side is

$$\sum_{r=0}^{n-4} 2^{n-r-2} + \phi_0(3)$$

$$= 2^{n-1} - 4 + 3$$

$$= 2^{n-1} - 1 \quad .$$

By Corollary 2 above, theorem is true for $j = 3$. Assuming it is true for j, we have, by Theorem 3,

$$\sum_{s=0}^{n-j-2}(\phi_{n-j-s-1}(n-s) - \phi_{n-j-s-2}(n-s-1)) = \sum_{s=0}^{n-j-2}\phi_{n-j-s-1}(n-s-1) ,$$

$$= \sum_{s=0}^{n-j-2}\left(\sum_{r=0}^{n-j-s-2}\binom{j-3+r}{r}2^{n-j-r-s} + \sum_{r=3}^{j}\binom{n-r-s-2}{j-r}\phi_0(r)\right) \quad ,$$

$$= \sum_{r=0}^{n-j-2} \binom{j-3+r}{r} 2^{n-j-r} \sum_{s=0}^{n-j-r-2} 2^{-s} + \sum_{r=3}^{j} \phi_0(r) \sum_{s=0}^{n-j-2} \binom{n-r-s-2}{j-r},$$

$$= \sum_{r=0}^{n-j-2} \binom{j-3+r}{r} (2^{n-j-r+1} - 4) + \sum_{r=3}^{j} \binom{n-1-r}{j-r+1} \phi_0(r), \quad \text{by Lemma 1,}$$

$$= \sum_{r=0}^{n-j-1} \binom{j-3+r}{r} 2^{n-j-r+1} - 4\left\{\binom{n-4}{n-j-1} + \sum_{r=0}^{n-j-2} \binom{j-3+r}{r}\right\}$$

$$+ \sum_{r=3}^{j} \binom{n-r-1}{j-r+1} \phi_0(r),$$

$$= \sum_{r=0}^{n-j-2} \binom{j-2+r}{r} 2^{n-j-r} + 4\binom{n-3}{n-j-1} - 4\left\{\binom{n-4}{n-j-1} + \binom{n-4}{n-j-2}\right\}$$

$$+ \sum_{r=3}^{j} \binom{n-r-1}{j-r+1} \phi_0(r),$$

by Lemmas 1 and 2,

$$= \sum_{r=0}^{n-j-2} \binom{j-2+r}{r} 2^{n-j-r} + \sum_{r=3}^{j} \binom{n-r-1}{j-r+1} \phi_0(r).$$

Hence,

$$\phi_{n-j-1}(n) = \sum_{r=0}^{n-j-2} \binom{j-2+r}{r} 2^{n-j-r} + \sum_{r=3}^{j} \binom{n-r-1}{j-r+1} \phi_0(r) + \phi_0(r+1),$$

$$= \sum_{r=0}^{n-j-2} \binom{j-2+r}{r} 2^{n-j-r} + \sum_{r=3}^{j+1} \binom{n-r-1}{j-r+1} \phi_0(r).$$

Thus, if true for j, also true for $j + 1$. This proves the theorem.

This proves the theorem.

Further reductions on the result of Lemma 2 give the following:

Theorem 5.

$$\sum_{r=0}^{q-2} \binom{p+r}{r} 2^{q-r} = 4 \sum_{r=0}^{q-2} \binom{p+q-1}{r} .$$

Theorem 4 can now be stated in the following way:

Lemma 3.

$$\phi_{n-j}(n) = 4 \sum_{r=0}^{n-j-1} \binom{n-3}{r} + \sum_{r=3}^{j} \binom{n-r-1}{j-r} \phi_0(r) .$$

Two special cases which are easily obtained from Lemma 3 are stated in

Theorem 6.

$$\phi_{\frac{n-1}{2}}(n) = 2^{n-2} + 2\binom{n-3}{\frac{n-3}{2}} + \sum_{r=3}^{\frac{n+1}{2}} \binom{n-r-1}{\frac{n+1}{2}-r} \phi_0(r), \quad n \text{ odd} \qquad (\geq 5) ,$$

$$\phi_{\frac{n-2}{2}}(n) = 2^{n-2} + \sum_{r=3}^{\frac{n+2}{2}} \binom{n-r-1}{\frac{n+2}{2}-r} \phi_0(r), \quad n \text{ even} \qquad (\geq 4) .$$

The author is indebted to R. N. Burns of the University of Waterloo for his many helpful suggestions.

REFERENCES

1. Marshall Hall, Jr., Combinatorial Theory, Blaisdell Publishing Company, Toronto, 1967.
2. J. Riordan, An Introduction to Combinatorial Analysis, John Wiley and Sons, Inc., London, 1958.