MODULO ONE UNIFORM DISTRIBUTION OF THE SEQUENCE OF LOGARITHMS OF CERTAIN RECURSIVE SEQUENCES

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Let $\{x_j\}_1^\infty$ be a sequence of real numbers with corresponding fractional parts $\{\beta_j\}_1^\infty$, where $\beta_j = x_j - [x_j]$ and the bracket denotes the greatest integer function. For each $n \geq 1$, we define the function F_n on $[0,1]$ so that $F_n(x)$ is the number of those terms among $\beta_1, \cdots, \beta_n$ which lie in the interval $[0,x)$, divided by n. Then $\{x_j\}_1^\infty$ is said to be uniformly distributed modulo one iff $\lim_{n\to\infty} F_n(x) = x$ for all $x \in [0,1]$. In other words, each interval of the form $[0,x)$ with $x \in [0,1]$, contains asymptotically a proportion of the β_n's equal to the length of the interval, and clearly the same will be true for any subinterval (α,β) of $[0,1]$. The classical Weyl criterion ([1], p. 76) states that $\{x_j\}_1^\infty$ is uniformly distributed mod 1 iff

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} e^{2\pi i\nu x_j} = 0 \qquad \nu \geq 1 . \tag{1}$$

An example of a sequence which is uniformly distributed mod 1 is $\{n\xi\}_{n=0}^\infty$ where ξ is an arbitrary irrational number (see [1], p. 81 for a proof using Weyl's criterion).

The purpose of this paper is to show that the sequence $\{\ln V_n\}_1^\infty$ is uniformly distributed mod 1, where $\{V_n\}_1^\infty$ is defined by a linear recurrence of the form

$$V_{n+k} = a_{k-1}V_{n+k-1} + \cdots + a_0V_n \qquad n \geq 1 , \tag{2}$$

the initial terms $V_1, V_2, \cdots, V_k$ being given positive numbers. In (2), we assume that the coefficients are non-negative rational numbers with $a_0 \neq 0$, and that the associated polynomial $x^k - a_{k-1}x^{k-1} - \cdots - a_1x - a_0$, has roots $\beta_1, \beta_2, \cdots, \beta_k$ which satisfy the inequality $0 < |\beta_1| < \cdots < |\beta_k|$. Additionally, we require that $|\beta_j| \neq 1$ for $j = 1, 2, \cdots, k$.

In particular, our result implies that any sequence $\{U_n\}_1^\infty$ which satisfies the Fibonacci recurrence $U_{n+2} = U_{n+1} + U_n$ for $n \geq 1$ with $U_1 = k_1$ and $U_2 = k_2$ arbitrary positive initial terms (not necessarily integers) will have the property that $\{\ln U_n\}_1^\infty$ is uniformly distributed mod 1. With $k_1 = 1$, $k_2 = 1$, we obtain this conclusion for the classical Fibonacci sequence (see [2], Theorem 1), while for $k_1 = 1$, $k_2 = 3$, an analogous result is seen to hold for the Lucas sequence.

Before proving the main theorem, we prove two lemmas:

Lemma 1. If $\{x_j\}_1^\infty$ is uniformly distributed mod 1 and $\{y_j\}_1^\infty$ is such that $\lim_{j\to\infty} (x_j - y_j) = 0$, then $\{y_j\}_1^\infty$ is uniformly distributed mod 1.

Proof. From the hypothesis and the continuity of the exponential function, it follows that

$$\lim_{j\to\infty} (e^{2\pi i\nu x_j} - e^{2\pi i\nu y_j}) = 0 .$$

But it is well known ([3], Theorem B, p. 202) that if $\{\gamma_n\}$ is a sequence of real numbers converging to a finite limit L, then

$$\lim_{n\to\infty} \frac{1}{n}\sum_1^n \gamma_j = L .$$

Taking $\gamma_j = e^{2\pi i\nu x_j} - e^{2\pi i\nu y_j}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^n (e^{2\pi i\nu x_j} - e^{2\pi i\nu y_j}) = 0 .$$

Since

$$\lim_{n\to\infty} \frac{1}{n} \sum_1^n e^{2\pi i\nu x_j} = 0$$

by Weyl's criterion, we also have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{1}^{n} e^{2\pi i\nu y_j} = 0$$

and the sufficiency of Weyl's criterion proves the sequence $\{y_j\}_1^\infty$ to be uniformly distributed mod 1.

Lemma 2. If α is a positive algebraic number not equal to one, then $\ln \alpha$ is irrational.

Proof. Assume, to the contrary, $\ln \alpha = (p/q)$, where p and q are non-zero integers. Then $e^{p/q} = \alpha$, so that $e^p = \alpha^q$. But α^p is algebraic, since the algebraic numbers are closed under multiplication ([1], p. 84). Thus e^p is algebraic, in turn implying e is algebraic. But e is known to be transcendental ([1], p. 25) so that a contradiction is obtained.

Theorem. Let $\{V_n\}_1^\infty$ be a sequence generated by the recursion relation,

$$V_{n+k} = a_{k-1}V_{n+k-1} + \cdots + a_1V_{n+1} + a_0V_n \qquad (n \geq 1), \tag{2}$$

where $a_0, a_1, \cdots, a_{k-1}$ are non-negative rational coefficients with $a_0 \neq 0$, k is a fixed integer, and

$$V_1 = \gamma_1, \quad V_2 = \gamma_2, \cdots, \quad V_k = \gamma_k \tag{3}$$

are given positive values for the initial terms. Further, we assume that the polynomial $x^k - a_{k-1}x^{k-1} - \cdots - a_1x - a_0$ has k distinct roots $\beta_1, \beta_2, \cdots, \beta_k$ satisfying $0 < |\beta_1| < \cdots < |\beta_k|$ and such that none of the roots has magnitude equal to 1. Then $\{\ln V_n\}_1^\infty$ is uniformly distributed mod 1.

Proof. The general solution of the recurrence (2) is

$$V_n = \sum_{j=1}^{k} \alpha_j \beta_j^n \qquad (n \geq 1), \tag{4}$$

where the arbitrary constants $\alpha_1, \alpha_2, \cdots, \alpha_k$ are determined by the specification of the initial terms in (3). [It is easily checked that the determinant of the k x k matrix (β_j^i) does not vanish, so that determination of the α_j's is unique.] Since the initial terms were not all zero, at least one of the α_j's is non-zero. Let p be the largest value of j for which $\alpha_j \neq 0$, so that $p \geq 1$. Then

$$V_n = \sum_1^p \alpha_j \beta_j^n$$

and

$$\left| 1 - \frac{V_n}{\alpha_p \beta_p^n} \right| = \left| \sum_1^{p-1} \frac{\alpha_j \beta_j^n}{\alpha_p \beta_p^n} \right| \leq \sum_1^{p-1} \left| \frac{\alpha_j}{\alpha_p} \right| \left| \frac{\beta_j}{\beta_p} \right|^n \quad .$$

But

$$\left| \frac{\beta_j}{\beta_p} \right| < 1$$

for $j = 1, 2, \cdots, p-1$, and hence,

$$\lim_{n \to \infty} \left(\frac{V_n}{\left| \alpha_p \beta_p^n \right|} \right) = 1 \quad ,$$

or equivalently,

$$\lim_{n \to \infty} \left[\ln V_n - \ln \left| \alpha_p \beta_p \right|^n \right] = 0 \tag{5}$$

Since β_p is algebraic, it is easily verified that $|\beta_p|$ is also algebraic. Moreover, $|\beta_p| \neq 1$ by hypothesis so that $\ln|\beta_p|$ is irrational by application of Lemma 2. But the sequence $\{n\xi\}_1^\infty$ is uniformly distributed mod 1 whenever ξ is irrational; therefore, the sequence

$$\{n \ln |\beta_p|\}_1^\infty = \{\ln |\beta_p|^n\}_1^\infty$$

is uniformly distributed mod 1 and the same is true for the sequence

$$\{\ln |\alpha_p||\beta_p|^n\}_1^\infty .$$

From (5) and Lemma 1, it is then clear that $\{\ln V_n\}_1^\infty$ is uniformly distributed mod 1 as asserted. q.e.d.

The specialization to sequences satisfying the Fibonacci recurrence, $U_{n+2} = U_{n+1} + U_n$ $(n \geq 1)$, is immediate since the relevant polynomial in this case is $x^2 - x - 1$, and there are two distinct roots of unequal magnitude, namely

$$\frac{1 \pm \sqrt{5}}{2} .$$

From the theorem, we conclude $\{\ln U_n\}_1^\infty$ is uniformly distributed mod 1 independently of the (non-zero) values specified for U_1 and U_2.

Lastly, we give an example to show that our assumption on the roots of the associated polynomial cannot be relaxed. Consider the recurrence $V_{n+2} = V_n$ for $n \geq 1$ with $V_1 = 1$, $V_2 = 1$. Then clearly $V_n = 1$ for all $n \geq 1$ so that $\{\ln V_n\}_1^\infty$ is a sequence of zeroes and hence not uniformly distributed mod 1. The associated polynomial in this case is $x^2 - 1$ which has two distinct roots, ± 1; however, the roots have magnitude unity, and therefore, the conditions of our theorem are not met.

REFERENCES

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