COMPOSITIONS AND FIBONACCI NUMBERS

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1. INTRODUCTION

A composition of n is an ordered partition of n; that is, a representation of n as the sum of positive integers with regard to order. For example, 4 has the eight compositions

$$4 = 3+1 = 1+3 = 2+2 = 2+1+1 = 1+2+1 = 1+1+2$$
$$= 1+1+1+1 \ .$$

Some elementary properties of compositions have been given by Riordan [12, 124-125], and a more extensive study has been made by MacMahon [9, 150-216]. Isolated examples of composition formulas involving Fibonacci numbers have appeared sporadically in the literature (see [11], [13], [14], [15], [16]). In an earlier paper [6] the authors established a general composition formula and its inversion of which the above are particular examples. This formula generalized a result of Moser and Whitney [11], and from it followed a number of further results. In this paper we review the previous results, continue their development, and show how these techniques can be used to prove certain Fibonacci identities.

2. PREVIOUS RESULTS

From direct expansion we find that the enumerator of compositions with exactly k parts is $(x + x^2 + \cdots)^k$. That is, the coefficient of x^n in the resulting series is the number of compositions of n with k parts. If a summand j is given weight w_j, then

$$(w_1x + w_2x^2 + \cdots)^k = [W(x)]^k$$

maybe termed the enumerator of weighted k-part compositions. To obtain an explicit representation, put

$$C(x,y;w) = \sum_{k=0}^{\infty}[W(x)]^k y^k = \frac{1}{1 - yW(x)} = \sum_{k,n=0}^{\infty} c_{nk}(w)x^n y^k , \tag{2.1}$$

where $w \equiv \{w_1, w_2, \cdots\}$. Using the formula for derivatives of composite functions (see [12, p. 36]),

$$c_{nk}(w) \equiv c_{nk} = \sum_{\pi_k(n)} \frac{k!}{k_1! \cdots k_n!} w_1^{k_1} \cdots w_n^{k_n} \qquad (n,k > 0) , \tag{2.2}$$

where the sum is extended over all k-part partitions of n; that is, over all solutions of $k_1 + 2k_2 + \cdots + nk_n = n$ such that $k_1 + \cdots + k_n = k$. Since the number of distinct compositions obtainable from the above partition is the coefficient in (2.2), the omission of the coefficient calls for summation over compositions. We write

$$c_{nk}(w) = \sum_{\gamma_k(n)} w_{a_1} w_{a_2} \cdots w_{a_k} \qquad (n,k > 0) \tag{2.3}$$

where $\gamma_k(n)$ indicates summation over all k-part compositions $a_1 + \cdots + a_k$ of n. Specialize this by letting

$$C(x) \equiv C(x,1;w) = \frac{1}{1 - W(x)} = \sum_{n=0}^{\infty} c_n(w)x^n \tag{2.4}$$

in which

$$c_n(w) \equiv c_n = \sum_{k=1}^{\infty} c_{nk}(w) = \sum_{\gamma(n)} w_{a_1} \cdots w_{a_k} \qquad (n > 0) , \tag{2.5}$$

where $\gamma(n)$ indicates summation over all compositions $a_1 + \cdots + a_k$ of n, the number of summands k in the composition being variable. Equations (2.4) and (2.5) were given by Moser and Whitney [11].

To obtain an inversion formula for (2.5), note that

$$1 - W(x) = C(x)^{-1} = 1 + \sum_{n=1}^{\infty}\left(\sum_{\gamma(n)}(-1)^k c_{a_1} \cdots c_{a_k}\right) x^n .$$

Hence

$$-w_n = \sum_{\gamma(n)}(-1)^k c_{a_1} \cdots c_{a_k} \qquad (n > 0) . \tag{2.6}$$

To help motivate the above, we note that it is shown in [5] and [7] that if a pair of rabbits produces w_n pairs of offspring at the n^{th} time point, and their offspring do likewise, then the total number of pairs born at the n^{th} time point is c_n. We shall see below in example (3d) that our results generalize the famous rabbit reproduction problem which led Leonardo of Pisa to discover the Fibonacci numbers originally.

3. EXAMPLES AND ILLUSTRATIONS

In this section we specialize the above results, obtaining the known instances of Fibonacci related composition formulas appearing in the literature, as well as some other results.

Define the Fibonacci numbers F_n by

$$F_1 = F_2 = 1, \qquad F_{n+2} = F_{n+1} + F_n \qquad (n \geq 1) ,$$

and the Lucas numbers L_n by

$$L_1 = 1, \qquad L_2 = 3, \qquad L_{n+2} = L_{n+1} + L_n \qquad (n \geq 1).$$

We make use of several standard generating functions for Fibonacci and Lucas numbers, for which we refer the reader to [2].

(3a) Letting $w_n = 1$ $(n \geq 1)$, so that

$$W(x) = x + x^2 + \cdots = x/(1 - x),$$

and using the convention $\binom{n}{k} = 0$ if $k > n$, we have

$$C(x,y;w) = 1 + \frac{yW(x)}{1 - yW(x)} = \frac{xy}{1 - x(1 + y)}$$

(3.1)
$$= \sum_{n=0}^{\infty} x^n (1 + y)^n xy = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n}{k} x^{n+1} y^{k+1} ,$$

so that

(3.2)
$$c_{nk}(w) = \sum_{\gamma_k(n)} 1 = \binom{n-1}{k-1}$$

is the number of compositions of n into k parts. This appears in [12], and can be verified by combinatorial arguments.

It follows that

(3.3)
$$\sum_{\gamma(n)} 1 = c_n(w) = \sum_{k=0}^{\infty} c_{nk}(w) = \sum_{k=1}^{\infty}\binom{n-1}{k-1} = (1+1)^{n-1} = 2^{n-1}$$

is the total number of compositions of n. For example, 4 has the $8 = 2^{4-1}$ compositions mentioned in the Introduction.

(3b) Put $w_n = n$, which gives

$$W(x) = x + 2x^2 + 3x^3 + \cdots = x/(1 - x)^2 .$$

In this case,

$$C(x) - 1 = \frac{x}{1 - 3x + x^2} = \sum_{n=1}^{\infty} F_{2n}x^n .$$

Then (2.5) yields

$$(3.4) \qquad c_n = \sum_{\gamma(n)} a_1a_2 \cdots a_k = F_{2n} ,$$

which has been given by Moser and Whitney [11], and proposed as a problem in this Quarterly [16]. As an example, for $n = 4$ we have

$$c_4 = 4 + 2(3 \cdot 1) + 2 \cdot 2 + 3(2 \cdot 1 \cdot 1) + 1 \cdot 1 \cdot 1 \cdot 1 = 21 = F_8.$$

(3c) Set

$$w_1 = w_2 = 1, \quad w_n = 0 \quad (n \geq 3) ,$$

so that $W(x) = x + x^2$. Then

$$C(x) - 1 = \frac{x + x^2}{1 - x - x^2} = \sum_{n=1}^{\infty} F_{n+1}x^n ,$$

and using (2.5) we get

$$(3.5) \qquad c_n = \sum_{\gamma(n);\, a_j \leq 2} 1 = F_{n+1} ,$$

since in any composition with $a_j > 2$, $w_{a_j} = 0$ annihilates the summand. Thus the number of compositions of n into 1's and 2's is F_{n+1}. This was proposed by Moser as Problem B-5 [14].

(3d) Let $w_1 = 0$ and $w_n = 1$ $(n \geq 2)$, giving

$$W(x) = x^2(1 + x + \cdots) = x^2/(1 - x) .$$

Then

$$C(x) - 1 = \frac{x^2}{1 - x - x^2} = \sum_{n=1}^{\infty} F_{n-1}x^n ,$$

so that by (2.5) we have

$$c_n = \sum_{\gamma(n);a_j \geq 2} 1 = F_{n-1} . \tag{3.6}$$

Thus the number of compositions of n into parts greater than a unity is F_{n-1}. In this case we have

$$C(x,y;w) - 1 = \frac{x^2y}{1 - x(1 + xy)} = x^2y\sum_{j=0}^{\infty} x^j(1 + xy)^j$$

$$= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\binom{j}{i}x^{i+j+2}y^{i+1} = \sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\binom{n-k-1}{k-1}x^ny^k ,$$

so that by (2.3)

$$c_{nk}(w) = \sum_{\gamma_k(n);a_j>1} 1 = \binom{n-k-1}{k-1} \tag{3.7}$$

is the number of compositions of n into k parts, each of which is greater than one. Then (2.5) shows

$$F_{n-1} = \sum_{k=1}\binom{n-k-1}{k-1} , \tag{3.8}$$

which was first shown by Lucas [8, p. 186].

(3e) If

$$W(x) = x + x^3 + x^5 + \cdots = x/(1 - x^2),$$

then a calculation similar to that in (3d) shows

$$(3.9) \qquad c_{nk}(w) = \sum_{\gamma_k(n);a_j \text{ odd}} 1 = \binom{\frac{1}{2}(n+k) - 1}{k-1}$$

to be the number of k-part compositions of n into odd parts. Since

$$(3.10) \qquad c_n(w) = \sum_{\gamma(n);a_j \text{ odd}} 1 = \sum_{k=1} \binom{\frac{1}{2}(n+k) - 1}{k-1} = F_n,$$

we may state that the number of compositions of n into odd parts is F_n.

(3f) Put

$$W(x) = \frac{x^2}{1 - 3x + x^2} = \sum_{n=1}^{\infty} F_{2n-2}\, x^n,$$

so that

$$C(x) - 1 = \frac{x^2}{1 - 3x} = \sum_{n=2}^{\infty} 3^{n-2} x^n.$$

Then by (2.5)

$$(3.11) \qquad c_n(w) = \sum_{\gamma(n);a_j>1} F_{2a_1-2} \cdots F_{2a_k-2} = 3^{n-2} \quad (n \geq 2).$$

The inverse relation given by (2.6) is

$$-F_{2n-2} = \sum_{\gamma(n);a_j > 1} (-1)^k 3^{a_1-2} \cdots 3^{a_k-2} = \sum_{\gamma(n);a_j > 1} (-1)^k 3^{n-2k} \qquad (n \geq 2) .$$

Since the summand depends only on the number of integers in the composition, we may use the value of $c_{nk}(w)$ in (3d) to get

$$F_{2n} = \sum_{k=1} (-1)^{k-1} \binom{n-k}{k-1} 3^{n+1-2k} \qquad (n \geq 1) , \tag{3.12}$$

which was proposed as Problem H-83 in this Quarterly [17].

(3g) We shall establish some further Fibonacci identities via composition formulas. Let

$$W(x) = x^2 + 4x^3 + 4^2x^4 + \cdots = x^2/(1 - 4x) ,$$

so that

$$C(x) - 1 = \frac{x^2}{1 - 4x - x^2} = \frac{1}{2}\sum_{n=1}^{\infty} F_{3n-3}x^n .$$

Then with (2.5) we get

$$\frac{1}{2} F_{3n-3} = \sum_{\gamma(n);a_j > 1} 4^{a_1-2} \cdots 4^{a_k-2} = \sum_{\gamma(n);a_j > 1} 4^{n-2k} . \tag{3.13}$$

Again using the value for $c_{nk}(w)$ in (3d), we find

$$\frac{1}{2} F_{3n-3} = \sum_{k=1} \binom{n-k-1}{k-1} 4^{n-2k} . \tag{3.14}$$

We can generalize this as follows. First let s be odd, and set

$$W(x) = \frac{x^2}{1 - L_s x} = \sum_{n=2}^{\infty} L_s^{n-2} x^n .$$

Then

$$C(x) - 1 = \frac{x^2}{1 - L_s x - x^2} = \frac{1}{F_s} \sum_{n=1}^{\infty} F_{s(n-1)} x^n .$$

We then get

$$F_{s(n-1)}/F_s = \sum_{\gamma(n); a_j > 1} L_s^{n-2k} ,$$

so using (3d) we have

$$F_{s(n-1)}/F_s = \sum_{k=1} \binom{n-k-1}{k-1} L_s^{n-2k} \qquad (s \text{ odd}) .$$

For even s, a similar calculation with

$$W(x) = \frac{-x^2}{1 - L_s x} = \sum_{n=2}^{\infty} -L_s^{n-2} x^n$$

shows

$$F_{s(n-1)}/F_s = \sum_{k=1} (-1)^{k-1} \binom{n-k-1}{k-1} L_s^{n-2k} \qquad (s \text{ even}) .$$

The even and odd cases can be combined into

$$F_{s(n-1)}/F_s = \sum_{k=1} (-1)^{(k-1)(s-1)} \binom{n-k-1}{k-1} L_s^{n-2k} . \tag{3.15}$$

This result was recently posed as a problem [18].

We conclude this section by noting that Hoggatt [5], in connection with a study of the reproduction patterns of mathematical Fibonacci rabbits, has exhibited a number of generating functions $W(x)$ which have particularly convenient corresponding generating functions $C(x)$. Each of these has the natural combinatorial interpretation provided by (2.5) and (2.6).

4. RELATIONS INVOLVING FIBONACCI GENERALIZATIONS

In this section we consider composition formulas involving three distinct generalizations of the Fibonacci numbers. Most of these reduce to results contained in Section 3.

(4a) Define the Fibonacci polynomials $f_n(t)$ by

$$f_1(t) = 1, \quad f_2(t) = t, \quad \text{and} \quad f_{n+2}(t) = tf_{n+1}(t) + f_n(t) \quad (n \geq 1) .$$

It follows that $f_n(1) = F_n$. It can also be easily verified that the generating function for these polynomials is

$$\frac{x}{1 - tx - x^2} = \sum_{n=1}^{\infty} f_n(t)\, x^n . \tag{4.1}$$

Letting $W(x)$ equal to (4.1), we find

$$C(x) - 1 = \frac{x}{1 - (t + 1)x - x^2} = \sum_{n=0}^{\infty} f_n(t + 1)x^n .$$

Then (2.5) yields

$$f_n(t+1) = \sum_{\gamma(n)} f_{a_1}(t) \cdots f_{a_k}(t) \quad . \tag{4.2}$$

As a special case given in [6] we get for $t = 1$ that

$$P_n = \sum_{\gamma(n)} F_{a_1} \cdots F_{a_k} \quad , \tag{4.3}$$

where $P_n = f_n(2)$ is the Pell sequence discussed by Lucas [8].

(4b) Miles [10] has investigated the properties of the r-generalized Fibonacci numbers $f_{n,r}$ defined for $r \geq 1$ by

$$\begin{gathered} f_{n,r} = 0 \quad (0 \leq n \leq r-2), \quad f_{r-1,r} = 1 \, , \\ f_{n,r} = \sum_{j=1}^{r} f_{n-j,r} \quad (n \geq r) \quad . \end{gathered} \tag{4.4}$$

If follows that $f_{n,1} = 1$ and $f_{n,2} = F_n$. The numbers $f_{n,3}$ are the so-called Tribonacci numbers studied by Feinberg [1]. It is not difficult to see that the generating function for the $f_{n,r}$ is

$$\frac{x^{r-1}}{1 - x - x^2 - \cdots - x^r} = \sum_{n=r-1}^{\infty} f_{n,r} x^n \quad . \tag{4.5}$$

For our first result, let $W(x) = x + x^2 + \cdots + x^r$. Then

$$C(x) = 1 + \sum_{n=1}^{\infty} f_{n+r-1} \, x^n \quad .$$

But it follows from (2.5) that $c_n(w)$ is the number of compositions of n into parts not greater than r. Thus we see

(4.6)
$$\sum_{\gamma(n);a_j\leq r} 1 = f_{n+r-1,r} ,$$

which reduces to (3c) by putting $r = 2$. By letting $r = 3$ we also obtain a partial solution to Problem B-96 in this Quarterly [15].

We may get $f_{n,r}$ in terms of a composition formula involving the $f_{i,r-1}$ in the following manner. Let

$$W(x) = \frac{x^r}{1 - x - \cdots - x^{r-1}} = \sum_{n=2}^{\infty} f_{n-2,r-1}x^n .$$

Then

$$C(x) - 1 = \frac{x^r}{1 - x - \cdots - x^r} = \sum_{n=1}^{\infty} f_{n-1,r-1}x^n .$$

Then from (2.5) we get

(4.7)
$$f_{n,r} = \sum_{\gamma(n+1)} f_{a_1-2,r-1} \cdots f_{a_k-2,r-1} ,$$

where $f_{n,r} = 0$ if $n < 0$. We note that for $r = 2$, (4.7) becomes (3.6). The inversion relation (2.6) gives

(4.8)
$$-f_{n-1,r-1} = \sum_{\gamma(n+1)} (-1)^k f_{a_1-1,r} \cdots f_{a_k-1,r} ,$$

giving a formula for $f_{n,r-1}$ in terms of the $f_{i,r}$.

(4c) If $w_j = 0$ $(1 \leq j < r)$, $w_j = 1$ $(j \geq r)$, then $W(x) = x^r/(1 - x)$. Now Hoggatt [4] has shown

(4.9)
$$\frac{1}{1 - x - x^p} = \sum_{n=0}^{\infty} u(n;p-1,1)x^n ,$$

where the $u(n;p,q)$ are the generalized Fibonacci numbers introduced by Harris and Styles [3] defined by

(4.10)
$$u(n;p,q) = \sum_{i=0}^{\left[\frac{n}{p+q}\right]} \binom{n-ip}{iq} \qquad (n \geq 0) .$$

Then

$$C(x) - 1 = \frac{x^r}{1 - x - x^r} = \sum_{n=r}^{\infty} u(n-r;r-1,1)x^n ,$$

so that

(4.11)
$$c_n(w) = \sum_{\gamma(n);a_j \geq r} 1 = u(n-r;r-1,1)$$

is the number of compositions of n into parts greater than or equal to r. It follows from (4.9) that $u(n;1,1) = F_{n+1}$, so that setting $r = 2$ in (4.11) yields (3.6).

On the other hand, letting $W(x) = x + x^p$, $C(x)$ becomes (4.9) and we see

(4.12)
$$\sum_{\gamma(n);a_j=1,p} 1 = u(n;p-1,1)$$

is the number of compositions of n into 1's and p's. This reduces to (3.5) by letting $r = 2$.

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